# ~ $T_3$ - ESPACIOS EUCLÍDEOS ~

! $\langle a, b \rangle$ prod escalar euclídeo

Recordar: $(x_1, y_1) \cdot (x_2, y_2) = x_1 x_2 + y_1 y_2$

Un ESP. EUCLÍDEO es un esp. vectorial y una aplicación:

$\langle , \rangle : V \times V \longrightarrow \mathbb{R}$ que verifica

$(\bar{u}, \bar{v}) \longmapsto \langle \bar{u}, \bar{v} \rangle$

(1) Propiedad simétrica: $\langle \bar{u}, \bar{v} \rangle = \langle \bar{v}, \bar{u} \rangle$

(2) Prop. distributiva: $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$

(3) $\forall \lambda \in \mathbb{R}$ $\quad \langle \lambda u, v \rangle = \lambda \langle u, v \rangle$

(4) $\langle \bar{u}, \bar{u} \rangle \geq 0$ y sólo es $0$ si $\vec{u} = \vec{0}$

↳ ($\vec{u}$ multiplicado por sí mismo)

• EJEMPLO 1 $\quad V = \mathbb{R}$

$$\langle (x_1, y_1), (x_2, y_2) \rangle = (x_1, y_1) \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}}_{\text{matriz simétrica}} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

((1) ✓)

comprobamos propiedad

¿PROPIEDAD 4?

$$\langle (x_1, y_1), (x_1, y_1) \rangle = (x_1, y_1) \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = (x_1, y_1) \begin{pmatrix} x_1 + y_1 \\ x_1 + 2y_1 \end{pmatrix} = x_1(x_1 + y_1) + y_1(x_1 + 2y_1) \geq 0$$

↳ sólo es $0$ si $\vec{u} = \vec{0}$ ☑

• Def: La LONGITUD de $\vec{u} \in E$ se define: $\|\vec{u}\| = \sqrt{\langle \vec{u}, \vec{u} \rangle}$

↳ en el ejemplo: $\vec{u} = (0, 1)$

$\|\vec{u}\| = \sqrt{\langle \vec{u}, \vec{u} \rangle}$

(la matriz del espacio)

↳ $\langle \vec{u}, \vec{u} \rangle = (0, 1) \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (0, 1) \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 2$

$\Rightarrow \boxed{\|\vec{u}\| = \sqrt{2}}$

• EJEMPLO 2 $\quad E =$ espacio de funciones continuas en el intervalo $[a, b]$.

$f: [a, b] \longrightarrow \mathbb{R}$

$g: [a, b] \longrightarrow \mathbb{R}$

$\longrightarrow \langle f, g \rangle = \int_a^b f(t) g(t) \, dt$

¿PROPIEDAD 4?

$\langle f, f \rangle = \int_a^b f(t) \cdot f(t) \, dt$ $\quad \geq 0$, sólo $0$ si $f(t) = 0$ ☑ se cumple

PROD. ESCALAR DE $g \cdot f$ EN INTERV. $[-\pi,\pi]$

$f(x) = \sin(x)$
$g(x) = \cos(x)$
$[a,b] = [-\pi,\pi]$ ← interv.

$\mapsto \langle f,g\rangle = \int_{-\pi}^{\pi} \sin(t)\cos(t)\,dt = \left.\frac{\sin^2(t)}{2}\right|_{-\pi}^{\pi} = 0$

$\boxed{\sin(x) \perp \cos(x) \text{ en } [-\pi,\pi] \text{ (su prod. esc. } = 0)}$

¿LONG.? $\|f\| = \sqrt{\langle f,f\rangle} \Rightarrow \begin{cases} \|\sin x\| = ? \to \langle \sin x, \sin x\rangle = \int_{-\pi}^{\pi} \sin^2(t)\,dt = I \\ \|\cos x\| = ? \to \langle \cos x, \cos x\rangle = \int_{-\pi}^{\pi} \cos^2(t)\,dt = J \end{cases}$

$I + J = \int_{-\pi}^{\pi} (\cos^2(t) + \sin^2(t))\,dt = \int_{-\pi}^{\pi} 1\,dt = 2\pi$

$J - I = \int_{-\pi}^{\pi} (\cos^2(t) - \sin^2(t))\,dt = \int_{-\pi}^{\pi} \cos(2t)\,dt = \frac{\sin(2t)}{2}$

Recordar: $e^{i\theta} = \cos\theta + i\sin\theta$

$e^{i2\theta} = \cos(2\theta) + i\sin(2\theta)$
$\|$
$e^{i\theta}\cdot e^{i\theta} = (\cos^2\theta - \sin^2\theta) + i(2\sin\theta\cos\theta)$

$\begin{cases} I + J = 2\pi \\ J - I = 0 \end{cases} \to \begin{cases} 2J = 2\pi \Rightarrow \\ 2I = 2\pi \Rightarrow \end{cases} \boxed{\begin{matrix} J = \pi \\ I = \pi \end{matrix}}$

- Teorema: DESIGUALDAD DE CAUCHY-SCHWARZ $[\,|\langle \vec{u},\vec{v}\rangle| \leq \|\vec{u}\|\,\|\vec{v}\|\,]$

$\cos\theta(u,v) := \frac{\langle u,v\rangle}{\|u\|\cdot\|v\|}$

(1) $\vec{u}$ y $\vec{v}$ son ortogonales si $\langle u,v\rangle = 0$

(2) El subespacio ortogonal ($W^\perp$) de $W$ es: ($W \subset E$ : $W$ subespacio de $E$)

$W^\perp = \{\vec{v} \in E \mid \langle u,v\rangle = 0, \vec{u} \in W\}$ = complemento ortogonal a $W$

EJEMPLO $W = \langle \vec{u_1} = (1,0,1), \vec{u_2} = (2,-1,1)\rangle$

$W^\perp = \{u = (x,y,z) \mid \langle (x,y,z), u_1\rangle = 0, \langle (x,y,z), u_2\rangle = 0\}$

$= \{(x,y,z) \mid x + z = 0,\ 2x - y + z = 0\} \to \begin{cases} x + z = 0 \\ 2x - y + z = 0 \end{cases} \mapsto \boxed{\begin{matrix} x = -z \\ y = -z \\ z = z \end{matrix}}$

$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right)$

## DISTANCIA ENTRE PUNTOS

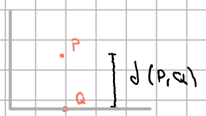
$\mathrm{dist}(P,Q) = \|\vec{PQ}\|$ ← la dist. es el vector que une P y Q

$= \sqrt{\langle \vec{PQ}, \vec{PQ}\rangle}$

**EJEMPLO** $E = \mathbb{R}^2$ ¿$d(P,Q)$?

$\langle \bar{u}, \bar{v} \rangle = (x_1, y_1) \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

$P = (1,1)$
$Q = (1,0)$




$\vec{PQ} = (1,0) - (1,1) = (0,-1)$

$d(P,Q) = \sqrt{(0,-1)\cdot(0,-1)} = \sqrt{1} = \boxed{1}$ ~~esto es en un esp. no euclídeo~~

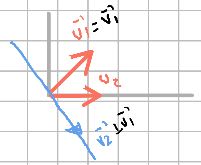
$d(P,Q) = \sqrt{\langle \vec{PQ}, \vec{PQ} \rangle} = \boxed{\sqrt{2}}$

**Objetivo:** buscar bases formadas por vectores ortogonales dos a dos y de long. 1.

**EJEMPLO.** $V = \mathbb{R}^2$, prod. escalar habitual.

Base inicial
$u_1 = (1,1)$
$u_2 = (1,0)$

$u_1 \longrightarrow v_1 = u_1$
$u_2 \longrightarrow v_2 = ?$
$(v_1 \perp v_2)$




$v_2 = u_2 + \lambda u_1 \Rightarrow 0 = \langle v_1, v_2 \rangle = \langle v_1, u_2 + \lambda_1 \rangle = \langle u_1, u_2 \rangle + \lambda u_1$

$\longrightarrow \left[ \lambda = \frac{\langle u_1, u_2 \rangle}{\|u_1\|^2} \right]$

$\rightarrow v_2 = \vec{v_2}$.

## ALGORITMO DE GRAM-SCHMIDT

**HALLAR VECT. ORTONORMALES**

**Observ.** - Ventajas de tener base ortonormal (vectores $\perp$ dos a dos y longitud $=1$)

$V = \mathbb{R}^n$, $B = \{ \bar{e}_1, \dots, \bar{e}_n \}$, $\bar{e}_i \perp \bar{e}_j$ si $i \neq j$, $\|\bar{e}_i\| = 1$

$\overset{\in}{\vec{v}} = \lambda_1 \bar{e}_1 + \dots + \lambda_n \bar{e}_n \Rightarrow \langle \bar{v}, e_1 \rangle = \langle \lambda_1 e_1 + \dots + \lambda_n e_n, e_1 \rangle = \lambda_1 \langle e_1, e_1 \rangle + \dots + \lambda_n \langle e_n, e_1 \rangle = \lambda_1 \overset{=1}{\|e_1\|^2} = [\lambda_1]$

En consecuencia: $\left[ \bar{v} = \langle \bar{v}, \bar{e}_1 \rangle \bar{e}_1 + \dots + \langle \bar{v}, \bar{e}_n \rangle \bar{e}_n \right]$

(2 vect.) **Caso $n = 2$** $V = \mathbb{R}^2$

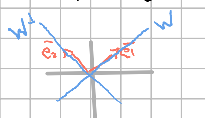
$B = \{ \bar{e}_1, \bar{e}_2 \}$ $e_1 \perp e_2$

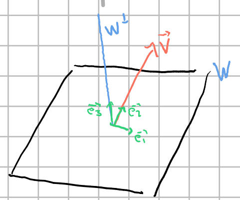
$\|e_1\| = 1$ $\|e_2\| = 1$

(proyecc. = comp. horizontal)

$\text{Proyecc.}_W : \mathbb{R} \longrightarrow W$

$\langle \bar{v}, \bar{e}_1 \rangle \vec{e}_1 + \langle \bar{v}, \bar{e}_2 \rangle \vec{e}_2 = \vec{v} \longmapsto \langle \bar{v}, \bar{e}_1 \rangle \vec{e}_1$




$B = \{ \bar{e}_1, \bar{e}_2, \bar{e}_3 \}$ base ortonormal

$v = \underbrace{\langle v, e_1 \rangle e_1 + \langle v, e_2 \rangle e_2}_{\in W} + \underbrace{\langle v, e_3 \rangle e_3}_{\in W^\perp}$

$Pr_W(\bar{v}) = \langle v, e_1 \rangle e_1 + \langle v, e_2 \rangle e_2 \in W$

**Algoritmo:**

- [Input]: $\{u_1, \dots, u_n\}$ linealmente independientes
- [Output]: $\{v_1, \dots, v_n\}$ L.I. y $v_i \perp v_j$ si $i \neq j$

$v_1 := u_1$

$v_i := u_i - \left( \frac{\langle u_i, v_1 \rangle}{\|v_1\|^2} v_1 + \dots + \frac{\langle u_i, v_{i-1} \rangle}{\|v_{i-1}\|^2} v_{i-1} \right)$ para $i = 1, \dots, n$

## Caso particular $n=3$

[Input]: $\{u_1, u_2, u_3\}$ L.I.

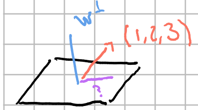
[Out]: $\{v_1, v_2, v_3\}$ L.I. $\quad v_1 \perp v_2,\ v_1 \perp v_3,\ v_2 \perp v_3$

$$v_1 := u_1$$

$$v_2 := u_2 - \frac{\langle u_2, v_1\rangle}{\|v_1\|^2} v_1 \qquad \left( \langle v_1, v_2\rangle = \left\langle u_1, u_2 - \frac{\langle u_2, v_1\rangle}{\|v_1\|^2} u_1 \right\rangle = \langle u_1, u_2\rangle - \left\langle u_1, \frac{\langle\ \rangle}{\|\ \|^2} u_1 \right\rangle = \langle u_1, u_2\rangle - \frac{\langle u_1, u_2\rangle}{\|u_1\|^2}\langle u_1, u_1\rangle = 0 \right)$$

$$v_3 := u_3 - \left( \frac{\langle u_3, v_1\rangle}{\|v_1\|^2} v_1 + \frac{\langle u_3, v_2\rangle}{\|v_2\|^2} v_2 \right)$$

**PROBLEMA:** Calcula proyecc. ortogonal de $(1,2,3)$ sobre el plano generado por $\vec{u}_1 = (1,0,-1)$, $\vec{u}_2 = (0,1,-1)$.




$$W = \langle u_1, u_2\rangle$$

PASO 1 Aplicar G-Smith a los $\vec{u}$ que generan plano.

$$v_1 = (1,0,-1)$$

$$v_2 = (0,1,-1) \frac{\langle (0,1,-1), (1,0,-1)\rangle}{2} (1,0,-1) =$$

$$= (0,1,-1) - \frac{1}{2}(1,0,1) = \left(\frac{1}{2}, 1, \frac{-1}{2}\right)$$

está generado por $\vec{v}_1 = (1,0,-1)$, $\vec{v}_2 = \left(\frac{1}{2}, 1, \frac{-1}{2}\right)$ $\quad W = \langle u_1, u_2\rangle = \langle v_1, v_2\rangle$

Los cambio para que sean de norma 1 (más fácil de calc.):

$$\langle v_1, v_2\rangle = \langle e_1, e_2\rangle \quad e_1 = \frac{v_1}{\|v_1\|},\ e_2 = \frac{v_2}{\|v_2\|} \longrightarrow \quad v_1 \longrightarrow \frac{(1,0,-1)}{\|v_1\|} = \left[\frac{(1,0,1)}{\sqrt{2}}\right]$$

$$v_2 \longrightarrow \left[\frac{(-1/2, 1, -1/2)}{\sqrt{3}/\sqrt{2}}\right]$$

La situación:

$$\vec{W} = (1,2,3) = \langle w, e_1\rangle \underbrace{\frac{1}{\sqrt{2}}(1,0,-1)}_{e_1} + \langle w, e_2\rangle \underbrace{\frac{\sqrt{2}}{\sqrt{3}}(-1/2, 1, -1/2)}_{e_2}$$

$$= \frac{1}{\sqrt{2}}(-2)\cdot\vec{e}_1 + \frac{\sqrt{2}}{\sqrt{3}}\left(1\cdot\left(\frac{-1}{2}\right) + 2\cdot 1 + 3\left(\frac{-1}{2}\right)\right)\vec{e}_2 = \frac{-2}{\sqrt{2}}\vec{e}_1 + 0\,\vec{e}_2 = \frac{-2}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}(1,0,1) =$$

$$= \boxed{(-1, 0, 1)}$$