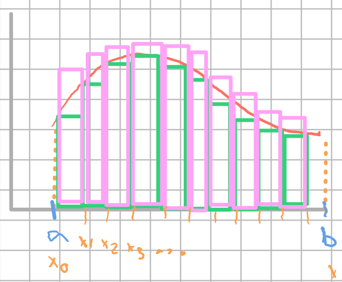
→ INTEGRAL RIEMANN Calcular áreas.

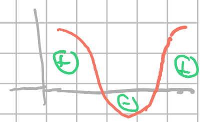


MIN → suma sup.: $\sum_{i=1}^{n} m_i(x_i - x_{i-1}) \to \int_a^b f = \sup$ (INT. INFERIOR) ↖ "mín por arriba"

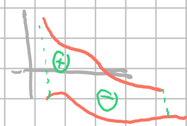
MÁX → suma inf.: $\sum_{i=1}^{n} M_i(x_i - x_{i-1}) \to \int_a^b f = \inf$ (INT. SUP.)

- Si $f[a,b]$ es integrable y $f(x) \geq 0 \to$ área $= \int_a^b f(x)\,dx$
- Si f cambia de signo en $[a,b] \to$ área $= \int_a^b f(x)\,dx$

PARTIMOS INTEGRAL



- Si f y g son integrables $\to$ área $= \int_a^b (f(x) - g(x))\,dx$



→ PROPIEDADES

(INTEGRALES)

↳ *TODA Fn CONTÍNUA ES INTEGRABLE

Si f y g son integrables en $[a,b]$:

- $f+g$ es integrable y $\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b f(x)\,dx$
- $\int_a^b x f(x)\,dx = x \int_a^b f(x)\,dx$
- Para x entre a y b: $\int f(x)\,dx = \int_a^x f(x) + \int_x^b f(x)\,dx$

  $(a < x < b)$

- Si $f(x) \leq g(x)$: $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$, $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b g(x)\,dx$
- El valor medio de f es: $\left[\frac{1}{b-a}\int_a^b f(x)\,dx\right] = f(c)$
- Si $f(x) = g'(x) \to \int_a^b f(t)\,dt = g(b) - g(a)$

  $\left(= \int f(b) - \int f(a)\right)$

EJEMPLOS

(1) $\displaystyle\int_2^3 x^5\,dx = \int f(3) - \int f(2) = \left[\frac{3^6}{6} - \frac{2^6}{6}\right]$ $\quad \left(g(x) = \frac{x^6}{6}\right)$

(2) $\displaystyle\int_0^{\pi} \operatorname{sen} x\,dx = -\cos x \Big|_0^{\pi} = -\cos(\pi) - (-\cos 0) = [2]$

(3) $\displaystyle\int_0^3 2x e^{x^2}\,dx = e^{x^2}\Big|_0^3 = e^9 - e^0 = [e^9 - 1]$

(4) $\displaystyle F(x) = \int_0^{x^2} (\operatorname{sen} t^2)\ln(1+t^2)\,dt$ $\qquad H(x) = \int_{g(x)}^{f(x)} F(t)\,dt$

(REGLA CADENA)

$H'(x) = F(f(x))\cdot f'(x) - F(g(x))\cdot g'(x)$

→ INTEGRACIÓN POR PARTES

$$\int_a^b f(x)g(x) = f(b)G(b) - f(a)g(a) - \int_a^b f(x)G(x)\,dx$$

Ej.:

$$\int_a^b \underset{f(x)}{x}\,\underset{g(x)}{e^x}\,dx = \left[x e^x\Big|_a^b - \int_a^b 1 e^x\,dx\right]$$

$f(x) = 1$ $\quad (f(x) = f'(x))$

$G(x) = e^x$ $\quad (g(x) = G'(x))$

- $\displaystyle\int_{\pi/2}^{\pi} x \operatorname{sen} x\,dx \longrightarrow -x\cos x\Big|_{\pi/2}^{\pi} + \operatorname{sen} x\Big|_{\pi/2}^{\pi} = \pi - 0 + 0 - 1 = \boxed{\pi - 1}$

$f(x) = x \longrightarrow f = 1\,dx$

$g(x) = \operatorname{sen} x \rightarrow G = -\cos x$

→ CAMBIO DE VARIABLE

Ej.

También se puede sustituir lím usando "$x^2+1 = t$": $\int_2^5$

(1) $\displaystyle\int_1^2 \frac{x}{\sqrt{x^2+1}}\,dx \longrightarrow \int \frac{dt}{2\sqrt{t}} = \frac{1}{2}\cdot\frac{t^{-1/2+1}}{-1/2+1} = t^{1/2} \rightsquigarrow \sqrt{x^2+1}\Big|_1^2 = \boxed{\sqrt{5} - \sqrt{2}}$

↳ $x^2+1 = t$ $\quad (\Rightarrow dt = 2x\,dx)$

(2) $\displaystyle\int_{1/2}^{1} \frac{1}{x^2} e^{1/x}\,dx \longrightarrow -\int e^t\,dt = e^2 - e$

↳ $t = 1/x \rightsquigarrow dt = \underbrace{-\frac{1}{x^2}\,dx}$

$\Rightarrow \frac{1}{x} = t \Rightarrow x^{-1} = t \Rightarrow -1x^{-2}\,dx = dt$

(3) $\displaystyle\int_1^2 \frac{4x^3+6x}{x^4+3x^2}\,dx \longrightarrow \int_4^{28} \frac{dt}{t} = \ln t \Big|_4^{28} = \ln 28 - \ln 4 \longrightarrow \ln \frac{28}{4} = \boxed{\ln 7}$

↳ $t = x^4+3x^2$

$(4x^3+6x)dx = dt$

→ DESCOMPOSICIÓN EN FRAC. SIMPLES

Ej:

(1) $\displaystyle\int \frac{2x+3}{(x-2)(x+5)(x-1)}\,dx = \frac{A_1}{(x-2)} + \frac{A_2}{(x+5)} + \frac{A_3}{(x-1)}$

(mcm) ↳ $2x+3 = A_1(x+5)(x-1) + A_2(x-2)(x-1) + A_3(x-2)(x+5)$

para $x=1$:

$5 = A_3(-1)(6) \rightarrow A_3 = -5/6$

para $x=2$:

$7 = A_1(7)(1) \rightarrow A_1 = 1$

para $x=-5$:

$-7 = A_2(-7)(-6) \rightarrow A_2 = -1/6$

(2) $\displaystyle\int \frac{(x^2+2x+3)}{(x-1)(x+1)^2}\,dx \Rightarrow \frac{(x^2+2x+3)}{(x-1)(x+1)^2} = \frac{A_1}{(x-1)} + \frac{A_2}{(x+1)} + \frac{A_3}{(x+1)^2} =$

← Para términos con exp., ponerlos varias veces.

Ej.) $\frac{A}{B^3} \rightarrow \frac{A_1}{B} + \frac{A_2}{B^2} + \frac{A_3}{B^3}$

$$= \frac{A_1(x+1)^2 + A_2(x-1)(x+1) + A_3(x-1)}{(x-1)(x+1)^2}$$

$\Rightarrow x^2+2x+3 = A_1(x+1)^2 + A_2(x-1)(x+1) + A_3(x-1)$

⇝ para $x=1$: $1^2+2+3 = A_1(2^2) \Rightarrow A_1 = 3/2$

⇝ para $x=-1$: $2 = A_3(-2) \Rightarrow A_3 = -1$

↳ despejo $A_2 = -1/2$

$\displaystyle ⇝ \frac{3}{2}\int \frac{dx}{(x-1)} - \frac{1}{2}\int \frac{dx}{(x+1)} - \int \frac{dx}{(x+1)^2} = \left[\frac{3}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| - \left(-\frac{1}{x+1}\right) + C\right]$

(3) $\int \frac{(2x^2-x+1)}{(x-1)(x^2+x+1)}$

$$\frac{(2x^2-x+1)}{(x-1)(x^2+x+1)} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+x+1)} \Rightarrow 2x^2-x+1 = A(x^2+x+1)+(Bx+C)(x-1)$$

$\rightsquigarrow x=1$: $2 = A\cdot 3$

PROBAR $\begin{cases} \rightsquigarrow x=0: & 1 = A-C \\ \rightsquigarrow x=2: & 7 = A + (2B+C) \rightarrow . \end{cases}$

$$\frac{2}{3}\int \frac{dx}{x-1} + \frac{1}{3}\int \underbrace{\frac{(8x-1)}{(x^2+x+1)}dx}$$

$$= 4\int \frac{2x - 1/4}{(x^2+x+1)}dx = 4\int \frac{2x - 1/4 + 1 - 1}{x^2+x+1}dx = 4\int \frac{2x+1}{x^2+x+1}dx + 4\int \frac{-5/4}{x^2+x+1}dx$$

** $F(x) = \int_{h_1(x)}^{h_2(x)} g(t)dt \longrightarrow \left[F'(x) = g(h_2(t))h_2'(x) - g(h_1(x))h_1'(x)\right]$

INTEGRALES TRIG.

$\cos^2 x + \sin^2 x = 1$

$[\cos(2x) = \cos^2 x - \sin^2 x] < \begin{matrix} \sin^2 x = \frac{1-\cos(2x)}{2} \\ \cos^2 x = \frac{1+\cos(2x)}{2} \end{matrix}$

$\cos^2 = 1 - \sin^2 x$

$\sin^2 x + \cos^2 x = 1$

$tg^2 x + 1 = \sec^2 x$

$\cosh^2 x - \sinh^2 x = 1$

$t = \sinh x$

- $\int \sin^2 x\, dx = \int \frac{1-\cos(2x)}{2} dx$

- $\int \cos^3 x\, dx = \int \cos x\,(1-\sin^2 x) = \int \cos x\, dx - \int \cos x \sin^2 x\, dx$

$\hookrightarrow \sin x - \int t^2 dt = \sin x - \frac{t^3}{3} = \left[\sin x - \frac{\sin^3 x}{3} + C\right]$

$t = \sin x$
$dt = \cos x\, dx$

POR PARTES $\int \sin^3 dx = \int \sin^2 x \sin x\, dx \longrightarrow \sin^2 x(-\cos x) + \int \cos x\, 2\sin x \cos x\, dx =$

$f = \sin^2 x \quad f' = 2\sin x \cos x\, dx$

$g = \sin x\, dx \quad G = -\cos x$

=

CON RAÍCES

• $\int \frac{1}{x\sqrt{4-x^2}}\,dx \xrightarrow[\substack{x=2\,\text{sen}\,t\\ dx=2\cos t\,dt}]{} 2\int \frac{\cos t}{2\,\text{sen}\,t\sqrt{4-4\,\text{sen}^2 t}}\,dt = \int \frac{\cos t}{\text{sen}\,t\cdot 2\underbrace{\sqrt{1-\text{sen}^2 t}}_{\cos^2 t}}\,dt = \frac{1}{2}\int \frac{1}{\text{sen}\,t}\,dt =$

$= \frac{1}{2}\int \text{cosec}\,t \cdot \frac{(\text{cosec}\,t + \text{cotg}\,t)}{(\text{cosec}\,t + \text{cotg}\,t)}\,dt =$

$= \xrightarrow[\substack{u=\text{cosec}\,t+\text{cotg}\,t\\ du=(\text{cosec}^2 t+\text{cosec}\,t\,\text{cotg}\,t)dx}]{} \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \left[\frac{1}{2}\ln|\text{cosec}\,t + \text{cotg}\,t| + C\right]$

• $\int \frac{1}{\sqrt{1+x^2}}$

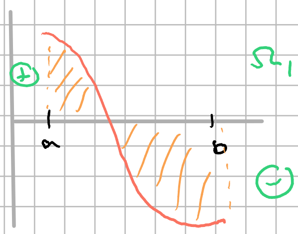
→ ÁREAS, VOLÚMENES Y LONG.

• El área que encierra $f(x)$ entre a y b (Ω) es:

$$\Omega = \int_a^b f(x)\,dx$$

↳ Si cambia de signo:

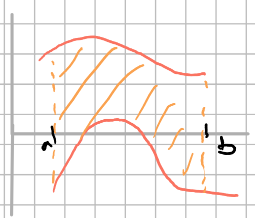
$$\Omega = \int_a^b f(x)\,dx = \text{área}\,\Omega_1 - \text{área}\,\Omega_2$$



$\Omega_1 - (-\Omega_2) = \Omega$

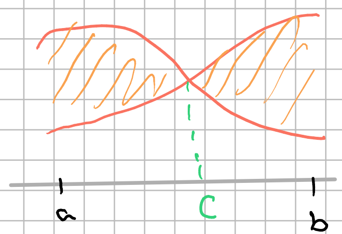
• Entre dos Fn, $f(x)$ y $g(x)$:

$$\Omega = \int_a^b (f(x) - g(x))\,dx$$



↳ Si se cortan:

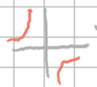
$$\Omega = \int_a^c (f(x) - g(x))\,dx + \int_c^b ((g(x) - f(x))\,dx$$

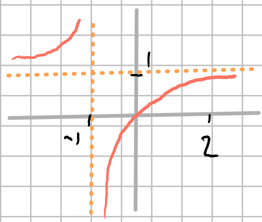


EJEMPLOS

(1) Área entre $f(x)=\frac{x}{x+1}$, el eje x y las rectas $x=0$ y $x=2$.

$f(x)=1-\frac{1}{x+1}$

$\left(\frac{1}{x}:\right.$  $\left.\right)$



$$\Omega=\int_0^2\left(1-\frac{1}{x+1}\right)=$$

$$=\left(x-\ln|x+1|\right)\Big|_0^2=\boxed{2-\ln 3}$$

(2) Área entre $f(x)=x^3-x$ y el eje x. $(y=0)$

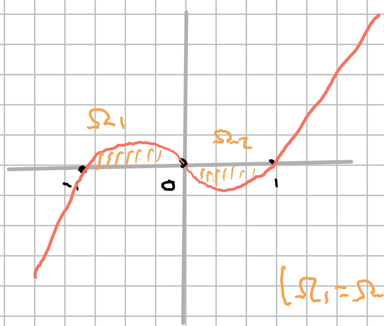
VER DONDE CRECE O DECRECE

$f(x)=x(x^2-1)$ ⇝ PT. CORTE $f(x)=0 < \begin{matrix}+1\\-1\\0\end{matrix}$

↳ IMPAR

$f'(x)$	+ ↗	$-1/\sqrt{3}$	− ↘	$1/\sqrt{3}$	+ ↗

$\left(f'(x)=0 < \begin{matrix}1/\sqrt{3}\\-1/\sqrt{3}\end{matrix}\right)$



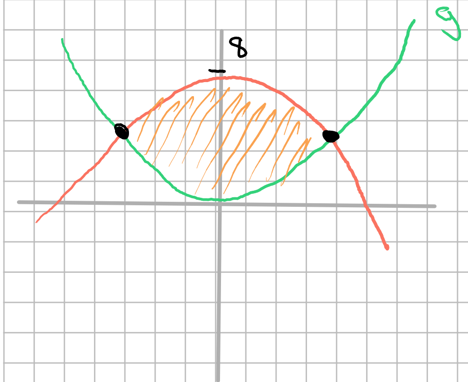
$(\Omega_1=\Omega_2 \to A=2\Omega_1)$

(si no sé cuándo poner el menos, hacer un val. abs. al final y ya)

$$\to \quad \Omega=-2\int_0^1 x^3-x=-2\left[\left(\frac{x^4}{4}-\frac{x^2}{2}\right)\Big|_0^1\right]=$$

$$=-2\left(\frac{1}{4}-\frac{1}{2}\right)=\boxed{1/2}$$

(3) Área entre $f(x)=8-x^2$ y $g(x)=x^2$



→ HALLAR PT. CORTE (cuando se cumplen las dos)

$f(x)=g(x)$

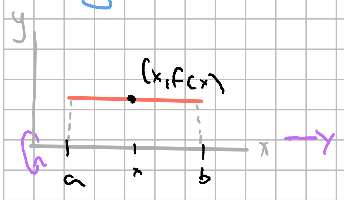
$8-x^2=x^2 \Rightarrow x=\pm 2$

$$\Omega=\int_{-2}^{2}\left(f(x)-g(x)\right)=\int_{-2}^{2}8-2x^2=$$

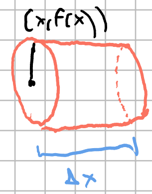
$$=\left(8x-\frac{2}{3}x^3\right)\Big|_{-2}^{2}=\boxed{\frac{64}{3}}$$

$\boxed{\text{VOLUMEN}}$

Al girar un área Ω alrededor de un eje se obtiene un sólido "T".



$\longrightarrow$

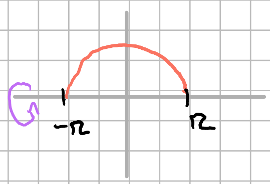


$$\text{Vol} \sim \pi R^2 \Delta x = \pi (f(x))^2 \Delta x$$

$$\hookrightarrow \left[V(T) = \int_a^b \pi (f(x))^2 \right]$$

es:

(1) Vol. de esfera de radio R.



$$x^2+y^2=R$$

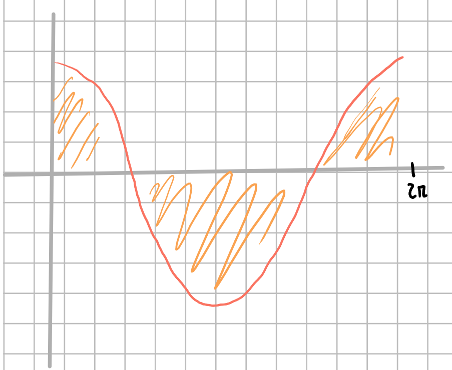
$$y^2=R^2-x^2 \longrightarrow f(x)=\sqrt{R^2-x^2}$$

$$\hookrightarrow V = \pi \int_{-R}^{R} \left(\sqrt{R^2-x^2}\right)^2 dx = \pi \left(R^2x - \frac{x^3}{3}\right)\Big|_{-R}^{R} =$$

$$= \pi \left(R^3 - \frac{R^3}{3} - \left(-R^3 - \frac{(-R)^3}{3}\right)\right) =$$

$$= \boxed{\frac{4}{3}\pi R^3}$$

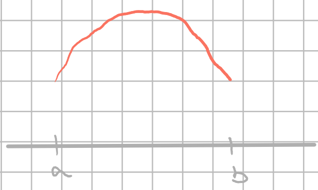
(2) Girar área de $f(x) = \cos x$ alrededor de eje x para $[0, 2\pi]$



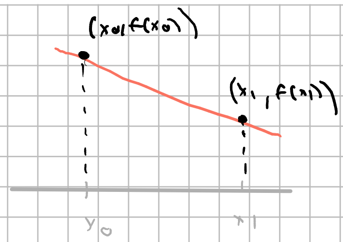
$$V = \pi \int_0^{2\pi} \cos^2 x = \pi \left(\frac{x}{2} + \frac{\operatorname{sen}(2x)}{4}\right)\Big|_0^{2\pi} = \boxed{\pi^2}$$

- **LONGITUDES**

$$\boxed{L = \int_a^b \sqrt{1+(f'(x))^2}}$$



$\Rightarrow$



→ INTEGRALES IMPROPIAS

$$\left[\int_1^{\infty} x^{\alpha} dx = \lim_{x\to\infty} \int_1^N x^{\alpha} dx\right]$$

! OJO $\int x^{\alpha} dx$ se separa $\begin{cases} \frac{x^{\alpha+1}}{\alpha+1} \\ \ln|x| \end{cases}$

(ya que si $\alpha = -1$, la integral es $\ln|x|$)

Ej: • $\int_1^{+\infty} x^{\alpha} dx$

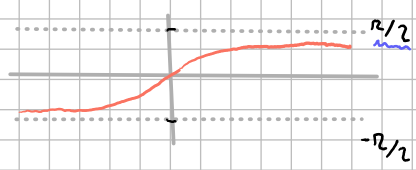
$$\int_1^N x^{\alpha} dx = \begin{cases} \left.\frac{x^{\alpha+1}}{\alpha+1}\right|_1^N & \text{si } \alpha \neq -1 \\ \ln|x|\Big|_1^N & \text{si } \alpha = -1 \end{cases} = \begin{cases} \frac{N^{\alpha+1}}{\alpha+1} & \text{si } \alpha \neq -1 \\ \ln N & \text{si } \alpha = -1 \end{cases}$$

La integral converge (no da ∞) si $\alpha+1<0 \Rightarrow \alpha < -1$

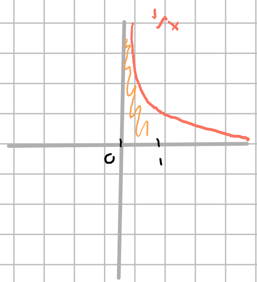
↳ En ese caso: $\int_1^{+\infty} x^{\alpha} dx = \lim_{N\to\infty} \int_1^N x^{\alpha} dx = \lim_{N\to\infty} \frac{N^{\alpha+1}-1}{1+\alpha} = \frac{-1}{1+\alpha}$

• $\int_0^{\infty} \frac{1}{x^2+1} dx \longrightarrow \int_0^N \frac{1}{x^2+1} dx = \operatorname{arctg}\Big|_0^N = \operatorname{arctg}(N) - \cancel{\operatorname{arctg}(0)}^{\,0}$

$\Rightarrow \lim_{N\to\infty} \operatorname{arctg} N = \boxed{\pi/2}$



- **TIPO 2**



¿Integral entre 0 y 1?

"Hacer una hasta cerca del cero y después hacer su lim."

DEFINIMOS

$$\left[\int_a^b f(x)dx = \lim_{\varepsilon\to 0^+} \int_{a+\varepsilon}^b f(x)dx\right] \quad \left[\int_a^{d} f(x)dx = \lim_{\varepsilon\to 0^+} \int_a^{d-\varepsilon} f(x)\right]$$

indef. — nº cerca de a

Si ambas son finitas: $\left[\int_a^b f(x) = \int_a^{\alpha} f(x) + \int_{\alpha}^b f(x)\right]$

Ej:

$$\int_0^1 \frac{1}{x^2} \longrightarrow \int_{\varepsilon}^1 x^{-2} = \left.\frac{x^{-2+1}}{-2+1}\right|_{\varepsilon}^1 = \left.\frac{-1}{x}\right|_{\varepsilon}^1 = -1 + \frac{1}{\varepsilon} \longrightarrow \lim_{\varepsilon\to 0} -1 + \frac{1}{\varepsilon} = \boxed{\infty}$$

Es DIVERGENTE

• $\int_0^1 \frac{1}{x^{1/2}}\,dx \longrightarrow \int_\varepsilon^1 \frac{1}{x^{1/2}}\,dx = \frac{x^{-1/2+1}}{-1/2+1}\Big|_\varepsilon^1 = 2x^{1/2}\Big|_\varepsilon^1 = 2-2\varepsilon^{1/2} \longrightarrow \lim_{\varepsilon\to 0} 2-2\varepsilon^{1/2} = \boxed{2}$

• $\int_0^1 x^\beta\,dx$ con $\beta<0$

$(\beta<0)$

$$\int_\varepsilon^1 x^\beta\,dx = \begin{cases} \frac{x^{\beta+1}}{\beta+1}\Big|_\varepsilon^1 & \text{si } \beta\neq -1 \\ \ln|x|\Big|_\varepsilon^1 & \text{si } \beta=-1 \end{cases} = \begin{cases} \frac{1-\varepsilon^{\beta+1}}{\beta+1} & \text{si } \beta<0 \text{ y } \beta\neq -1 \\ -\ln|\varepsilon| & \text{si } \beta=-1 \end{cases}$$

$\lim_{\varepsilon\to 0} -\ln|\varepsilon| = -\infty$ (x)

$\lim_{\varepsilon\to 0} \frac{1-\varepsilon^{\beta+1}}{\beta+1} = \boxed{\frac{1}{\beta+1}}$

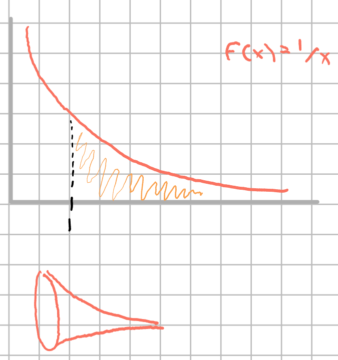
• $\int_0^1 \ln x\,dx \longrightarrow \int_\varepsilon^1 \ln x\,dx \rightsquigarrow x\ln|x|\Big|_\varepsilon^1 - \int_\varepsilon^1 x\frac{1}{x}\,dx = x\ln|x|\Big|_\varepsilon^1 - x\Big|_\varepsilon^1 = -\varepsilon\ln\varepsilon - 1 + \varepsilon$

$u=\ln x \to du=\frac{1}{x}dx$
$dv=dx \to v=x$

$\lim_{\varepsilon\to 0}(-\varepsilon\ln\varepsilon - 1 + \varepsilon) \longrightarrow = \boxed{-1}$

$\left(\lim \frac{1/\varepsilon}{-1/\varepsilon^2} = \lim_{\varepsilon\to 0}(-\varepsilon) = \boxed{0}\right)$

Ejemplo



¿Vol para $x\geq 1$?

(πR²)

$\int_1^\infty \pi\frac{1}{x}\,dx = \pi\int_1^\infty x^{-2}\,dx$

$\longrightarrow \int_1^N x^{-2}\,dx = \frac{x}{-2+1}\Big|_1^N = -\frac{1}{x}\Big|_1^N = -\frac{1}{N}+1$

$\lim_{N\to\infty} -\frac{1}{N}+1 = 1$

$V = \pi\left(\frac{1}{1}\right)^2 = \boxed{\pi}$

¿Área superficie?

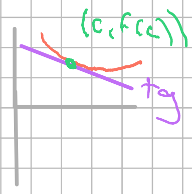
$2\pi\int_1^\infty \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx = 2\pi\int_1^\infty \frac{1}{x^3}\sqrt{x^4+1}\,dx = \boxed{\infty}$ Es DIVERGENTE, ÁREA $=\infty$

→ Aprox. de Fn: Desarrollo de Taylor

→ Ap. Lineal

aprox. lineal en $x=c$

Sea f definida en un intervalo, la tg en el punto $x=c$ es: $\left[\, y = f(c) + f'(c)(x-c) \,\right]$



Y cumple: $\left[\, \lim_{x\to c} \dfrac{f(x)-y}{x-c} = 0 \,\right]$

EJEMPLO

$f(x) = \ln(1+x) \xrightarrow{\text{P. Taylor}} p(x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3}$

¿Error para $x = 0'1$?

$|R(n)| = |f(n) - p(n)|$

• Sustituyo: $\begin{cases} p(0'1) = 0'1 - \dfrac{0'1^2}{2} + \dfrac{0'1^3}{3} = [\text{VALOR DE } p(0'1)] \\ f(0'1) = \ln(1+0'1) = [\text{VALOR DE } f(0'1)] \end{cases}$

$\hookrightarrow = \dfrac{|f^{(4)}(z)|}{4!}(0'1)^4$

• CALCULO 4ª DERIVADA DE $f(x) \rightarrow f^{(4)}(x) = \dfrac{-6}{(1+x)^4}$

$\Rightarrow \dfrac{6}{4!} \underbrace{\dfrac{1}{(1+z)^4}}_{} (0'1)^4$

buscar z para el que el error sea mayor

$0 \le z \le 0'1$

$\dfrac{1}{1'1} \le \dfrac{1+z}{1'1} \le \dfrac{1'1}{1'1} \Rightarrow z \le 1$, da el mayor error

→ $[z=1]$

↳ USAR EN P Y F Y RESTAR PARA HALLAR ERROR DE CADA UNO

$\left[\begin{array}{l} P_0 \pm R(P_0) \\ f(x) \pm R(f(x)) \end{array}\right]$

→ Series Potencias

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots$$

$\left[\text{EJ: } \sum_{n=0}^{\infty} x^n, \quad \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \quad \sum_{n=0}^{\infty} \frac{e}{n!}(x-1)^n \right]$

Para todo x real tal que $\left[\lim_{N\to\infty} \sum_{n=0}^{N} a_n (x-c)^n\right]$ sea finito, la serie converge a x y defino:

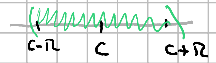
$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ (para los que no se cumple, la serie diverge)

$\left[\;!\;\; \text{SERIE CONOCIDA}: \dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n \right]$

(ej.: $\dfrac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n \cdot x^n$)

Dada una serie, existe un RADIO DE CONVERGENCIA entre 0 e ∞ tal que:

- Converge entre $(c-R,\ c+R)$
- diverge fuera de ese intervalo



↳ Si $R=0$ → sólo converge en $x=c$

↳ Si $R=\infty$ → converge en todo $\mathbb{R}$

→ CALCULAR R: criterio del cociente

Si existe la serie y $\left[\lim_{n\to\infty} \frac{|a_n|}{|a_{(n+1)}|}\right]$ con $a_n \neq 0$ existe o es $+\infty$, R es ese valor

↳ $1+x+x^2+x^3 \dots + \underset{[a_n]}{x^n} + \underset{[a_{(n+1)}]}{x^{n+1}}$

Ej:

(1) $\sum_{n=0}^{\infty} x^n$ $\quad a_n = 1$

$$\lim_{n\to\infty} \frac{|a_n|}{|a_{(n+1)}|} = \boxed{1}$$ converge para $|x|<1$, diverge para $|x|>1$ $\quad \boxed{R=1}$

(2) $\sum_{n=0}^{N} 1^n = 1+1+\dots+1 = \underline{N+1}$

↳ $\sum_{n=0}^{N} x^n = 1+x+x^2+\dots+x^N = S_N$

$x \cdot \sum_{n=0}^{N} = x+x^2+x^3+\dots+x^{N+1} = x \cdot S_N$

↳ $\underset{S_N(1-x)}{S_N - x\cdot S_N} = 1-x^{N+1}$

$$\Rightarrow \left[S_N = \frac{1-x^{(N+1)}}{1-x}\right] \xrightarrow[N\to\infty]{} \boxed{\frac{1}{1-x}}$$

(3) $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ $\quad a_n = \frac{1}{n!}$

$$\rightarrow \lim_{n\to\infty} \frac{|a_n|}{|a_{(n+1)}|} = \lim_{n\to\infty} \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} \frac{(n+1)\cancel{n!}}{\cancel{n!}} = \infty$$

[CONVERGE EN TODO $\mathbb{R}$]

Esta serie define e^x

→ DERIVADAS

Expresión general:

$$f'(x) = \sum_{n=0}^{\infty} n a_n (x-c)^{n-1}$$

$$f^{(2)}(x) = \sum_{n=0}^{\infty} n(n-1)(x-c)^{n-2}$$

$$\left[f^{(k)}(x) = n(n-1)\dots(n-k+1)a_n(x-c)^{n-k}\right]$$

Ej.) $f(x) = \sum_{n=0}^{\infty} x^n$ $\quad f'(x) = \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1}$

$f(x) = \frac{1}{1-x}$

Si para una función existen todas las derivadas, llamamos a la serie de potencias SERIE DE TAYLOR.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Y ambos tienen radio de convergencia R y $|x-c|<R$.

(la fn original y su deriv.)

→ INTEGRALES $\left(f(x)=\sum_{n=0}^{\infty} a_n (x-c)^n\right)$

$$\int_c^x f(t)\,dt = \int_c^x \sum_{n=0}^{\infty} a_n (t-c)^n\,dt = \sum_{n=0}^{\infty} a_n \int_c^x (t-c)^n\,dt = \sum_{n=0}^{\infty} a_n \frac{(t-c)^{n+1}}{n+1}\Big|_c^x = \left[\sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1}\right]$$

Y ambos tienen radio de convergencia R y $|x-c|<R$.
(la fn original y su integral)

Ej.) $\sum_{n=0}^{\infty} (-1)^n t^n$ converge en: $R=\lim_{n\to\infty} \frac{|(-1)^n|}{|(-1)^{n+1}|} = \lim 1 = \boxed{1}$

vimos que esta serie es $= \frac{1}{1+t}$

$$\Rightarrow \int_0^x \sum_{n=0}^{\infty} (-1)^n t^n\,dt = \sum_{n=0}^{\infty} (-1)^n \int_0^x t^n\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$$

$$\int_0^x \frac{1}{1+t} = \ln(1+x)$$

Por lo que la serie de $[\ln(1+x)]$ es $\left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}\right]$

NOS PIDEN SACAR LA SERIE DE TAYLOR DE LN (1+ x), Y COMO CONOCEMOS LA DE SU INTEGRAL, QUE ES $\left(\frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n t^n\right)$, USAMOS LA INTEGRAL EN AMBOS PARA SACARLA.

→ $$(f\circ p)(x) = \sum_{n=0}^{\infty} a_n (p(x))^n$$

ej.) (1) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $e^{x-1} = \sum \frac{(x-1)^n}{n!}$

$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!}$

(2) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum (-1)^n x^{2n}$

$$\int_0^x \frac{1}{1+x^2}\,dx = \int_0^x \sum (-1)^n x^{2n}\,dx$$

$$\left[\operatorname{arctg}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}\right]$$

- El radio de convergencia de $f(x)\cdot g(x)$, siendo cada una una serie con su propio R, será el min de R_1 y R_2.

EJEMPLO

$$h(x)=\frac{1+x^2}{1-x} \sim \frac{1}{1-x}=\sum x^k \quad \text{para } |x|<1$$

$$(1+x^2)(1+x+x^2+x^3+\dots)$$

$$=1+x+2x^2+2x^3+\dots=\left[1+x+\sum_{n=2}^{\infty}2x^n\right]$$

→ SERIES IMPORTANTES

SABEMOS ESTAS

- $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n \quad (R=1)$
- $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!} \quad (R=\infty)$
- $\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} \quad (R=\infty)$
- $\operatorname{sen} x=\sum \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad (R=\infty)$
- $(1+x)^\alpha=\sum_{n=1}^{\infty}\binom{\alpha}{n}x^n \quad (R=1)$

$$\binom{\alpha}{n}=\frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+1)}{n!}$$

- $\cosh=\frac{e^x+e^{-x}}{2}=\sum_{n=0}^{\infty}\frac{(2x)^{2n}}{(2n)!} \quad (R=\infty)$
- $\operatorname{senh} x=\sum_{n=0}^{\infty}\frac{x^{(2n+1)}}{(2n+1)!} \quad (R=\infty)$
- $\ln(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^n \quad (R=1)$
- $\operatorname{arctg} x=\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)}x^{2n+1} \quad (R=1)$

EJEMPLO

a) $\lim_{x\to 0}\frac{x^3}{1-\cos x}$

$$1-\cos x=1-\sum\frac{(-1)^n}{(2n)!}x^{2n}=\cancel{1}-\left(\cancel{1}-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots\right)=$$

$$=x^2\left(\frac{1}{2!}-\frac{x^2}{4!}+\frac{x^4}{6!}+\dots\right)$$

$$\frac{x^3}{1-\cos x}=\frac{x^3}{x^2\left[\frac{1}{2!}-\frac{x^2}{4!}+\dots\right]}=\frac{x}{\frac{1}{2!}-\frac{x^2}{4!}+\dots}\rightsquigarrow\frac{0}{\frac{1}{2!}}=\boxed{0}$$