T.5 - Ec. DIFERENCIALES

Son ecuaciones que involucran derivadas.

ej) ORDEN 2 $y'' - 2xy' + 2py = 0$ 3 $y''' - 3y'' + 3y' - y = 0$ (BUSCAMOS $y = f(x)$ QUE VERIFIQUE EC.)

→ Ec. SEPARABLES

Son de la forma:

$$y' = g(x)h(y)$$

$y = y(x)$ $\quad \frac{dy}{dx} = g(x)h(y) \longrightarrow \int \frac{dy}{h(y)} = \int g(x)\,dx$

EJEMPLO

(1) $y' = x(1-x)$

$\frac{dy}{dx} = x(1-x) \Rightarrow dy = x(1-x)\,dx \Rightarrow \int dy = \int x(1-x)\,dx$

$y = \int (x - x^2)\,dx = \boxed{\frac{x^2}{2} - \frac{x^3}{3} + C = y(x)}$

- Si además doy condiciones iniciales → e.g.: QUIERO QUE $y(0) = -2$

$y(0) = -2 \longrightarrow y(0) = 0 - 0 + C \Rightarrow \boxed{C = -2} \rightarrow \left[y(x) = \frac{x^2}{2} - \frac{x^3}{3} - 2 \right]$

(2) $y' = xy \longrightarrow \frac{dy}{dx} = xy \longrightarrow \frac{dy}{y} = x\,dx \Rightarrow \int \frac{dy}{y} = \int x\,dx$

[$y(0) = 1$]

$\ln|y| = \frac{x^2}{2} + C$

$|y| = e^{x^2/2 + C} = e^{x^2/2}\, e^C$

e^C → e^C y C es lo mismo, le llamo C_1

he podido quitar el val. abs. por C_1

→ $y = e^{x^2/2} \cdot C_1 \longrightarrow \boxed{y(x) = e^{x^2/2} \cdot C_1}$ SOL GRAL.

→ CONDICIÓN $y(0) = 1 \longrightarrow 1 = C_1 e^0 \Rightarrow \boxed{C_1 = 1} \rightarrow \left[y(x) = e^{x^2/2} \right]$

(3) $\frac{dy}{dx} = \overbrace{K(y-a)}^{\text{ctes.}} \longrightarrow \frac{dy}{y-a} = k\,dx \rightarrow \int \frac{dy}{y-a} = \int k\,dx$

[$y(1) = 1$]

$\ln|y-a| = Kx + C$

$|y-a| = e^{kx + c}$

$y - a = e^{kx} \cdot C_1 \rightarrow \boxed{y(x) = e^{kx} C_1 + a}$

→ $y(1) = 1 = e^k C_1 + a \Rightarrow \boxed{C_1 = \frac{1-a}{e^k}} \rightarrow \left[y(x) = a + (1-a)e^{-x} \cdot e^{kx} \right]$

(3) $\frac{dy}{dt} = 2(y-1)(y+2) \longrightarrow \underbrace{\int \frac{dy}{(y-1)(y+2)}} = 2\int dt$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \text{"} 2t$

$$\frac{1}{(y-1)(y+2)} = \frac{A_1}{y-1} + \frac{A_2}{y+2} \Rightarrow 1 = A_1(y+2) + A_2(y-1)$$

Para $y=-2$ $\quad 1 = -3A_2$

Para $y=1$ $\quad 1 = 3A_1$

$$\Rightarrow \frac{1/3}{y-1} - \frac{1/3}{y+2}$$

$$\rightsquigarrow = \frac{1}{3}\left(\int \frac{dx}{y-1} - \int \frac{dy}{y+2}\right) = \frac{1}{3}\left(\ln|y-1| - \ln|y+2|\right)$$

$$\frac{1}{3}\ln\frac{|y-1|}{|y+2|} = 2t + C \Rightarrow \ln\left|\frac{y-1}{y+2}\right| = 6t + C$$

$$\rightsquigarrow \left|\frac{y-1}{y+2}\right| = e^{6t}e^{C} \Rightarrow \left[\frac{y-1}{y+2} = e^{6t}\cdot C_1\right]$$

$$y(0)=2 \longrightarrow \frac{2-1}{2+2} = C_1 e^0 \Rightarrow \boxed{C_1 = \frac{1}{4}} \longrightarrow \frac{y-1}{2+2} = \frac{1}{4}e^{6t}$$

$$\Rightarrow \left[\, y(t) = \frac{2e^{6t}+4}{4-e^{6t}} \,\right]$$

→ Ec. LINEALES

→ 1er ORDEN

$y' + a(x)y = b(x)$ [$a(x)$ y $b(x)$ son funciones.]

MÉTODO: Si tenemos una función $\underbrace{P(x)}_{\text{Factor de integración}}$ tal que $[P(x)y]' = P(x)b(x)$

Podemos despejar "y" integrando: $P(x)y = \int P(x)b(x)dx$

EJEMPLO

$$2y' - 2(x^2+1)y - 2x^2 = 2$$

① Lo escribo adecuadamente:

$$y' - (x^2+1)y = x^2+1 \quad \begin{cases} a(x) = -(x^2+1) \\ b(x) = (x^2+1) \end{cases}$$

$$\int a(x)dx = \int -(x^2+1)\,dx = \frac{-x^3}{3} - x + C$$

$$P(x) = e^{-x^3/3 - x}$$

$$\int P(x)b(x)dx = \int e^{-(x^3/3 + x)} (x^2+1)\,dx$$

$$t = \frac{x^3}{3} + x$$
$$dt = (x^2+1)dx$$

→ Ec. LINEALES 2DO ORDEN con coef. ctes.

Son ec. de la forma: $[ay'' + by' + cy = f(x)]$

HOMOGÉNEAS $(f(x)=0)$ ⊛ $(ay'' + by' + cy = 0)$

① Si $y_1(x)$, $y_2(x)$ son soluciones de ⊛ → $y_1 + y_2$ también es solución, y también multiplicado: $C_1 y_1(x) + C_2 y_2(x)$

② Si $y_1(x) = e^{\alpha x}$ (cualquier exp.) es solución → $ay_1'' + by_1' + cy_1 = a(\alpha^2 e^{\alpha x}) + b(\alpha e^{\alpha x}) + ce^{\alpha x} = 0$

↳ $[a\alpha^2 + b\alpha + c = 0]$

por tanto, conviene estudiar: $[ar^2 + br + c = 0]$ (llamo r a α)

↓

$\boxed{\text{MÉTODO}}$

Consideramos la ec. característica: "$ar^2 + br + c = 0$" y buscamos sol. para "r".

→ CASO 1: $r_1 \neq r_2$ y son $\mathbb{R}$ ⟶ SOL GRAL: $[y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}]$

→ CASO 2: $r_1 = r_2$ y son $\mathbb{R}$ ⟶ SOL GRAL: $[y(x) = C_1 e^{r_1 x} + C_2 e^{r_1 x}]$ $(r_1 = r_2)$

(EJ CON EXP)

$$ay''(x) + by'(x) + cy(x) = a(2r_1 e^{r_1 x} + r_1^2 x e^{r_1 x}) + b(e^{r_1 x} + r_1 x e^{r_1 x}) + c(xe^{r_1 x}) =$$

$$= e^{r_1 x}\underbrace{(2ar_1 + b)}_{0} + xe^{r_1 x}\underbrace{(ar_1^2 + br_1 + c)}_{0}$$

→ CASO 3: r_1, r_2 no son $\mathbb{R}$ $\begin{cases} r_1 = \alpha + \beta i \\ r_2 = \alpha - \beta i \end{cases}$ y SOL. GRAL.: $[y(x) = C_1 e^{\alpha x}\cos(\beta x) + C_2 e^{\alpha x}\operatorname{sen}(\beta x)]$

$(r_2$ es $\bar{r}_1)$

(ES) Si lo escribimos como caso (1):

$$y(x) = C_1 e^{(\alpha+\beta i)x} + C_2 e^{(\alpha-\beta i)x}$$

$$= C_1 e^{\alpha x} e^{\beta x i} + C_2 e^{\alpha x} e^{-\beta x i}$$

$$= C_1 e^{\alpha x}(\cos(\beta x) + i\,\text{sen}(\beta x)) + C_2 e^{\alpha x}(\cos(-\beta x) + i\,\text{sen}(-\beta x))$$

$$= \underbrace{(C_1+C_2)}_{D_1} e^{\alpha x}\cos(\beta x) + \underbrace{(C_1-C_2)i}_{D_2} e^{\alpha x}\,\text{sen}(\beta x)$$

EJEMPLOS

(1) $2y'' + y' - y = 0$

$$2r^2 + r - 1 = 0 \Rightarrow y = \left\langle \begin{array}{l} -1 = r_1 \\ 1/2 = r_2 \end{array} \right\} \quad \text{SOL GR.:} \quad \boxed{y(x) = C_1 \overset{y_1}{e^{-x}} + C_2 \overset{y_2}{e^{\frac{1}{2}x}}}$$

⇝ y CON CONDIC. INICIALES: $[y(0) = 0 \ , \ y'(0) = 1]$

$$0 = C_1 e^{-x} + C_2 e^{1/2 x} \Rightarrow C_1 + C_2 = 0$$

$$y' = -C_1 e^{-x} + \frac{C_2}{2} e^{x/2} = 1 \Rightarrow -C_1 + \frac{C_2}{2} = 1$$

$$\hookrightarrow \left\{ \begin{array}{l} C_1 + C_2 = 0 \\ -C_1 + \frac{C_2}{2} = 1 \end{array} \right\} \longrightarrow \boxed{y(x) = -\frac{2}{3}e^{-x} + \frac{2}{3}e^{x/2}}$$

$\hookrightarrow [C_1 = -2/3]$

$\hookrightarrow [C_2 = 2/3]$

(2) $y'' - 4y' + 13y = 0$

$$r^2 - 4r + 13 = 0 \Rightarrow r = \left\langle \begin{array}{l} \overset{\alpha}{2} + \overset{\beta}{3}i = r_1 \\ 2 - 3i = r_2 \end{array} \right\} \quad \boxed{y(x) = C_1 e^{2x}\cos(3x) + C_2 e^{2x}\,\text{sen}(3x)}$$

(también: $D_1 e^{r_1 x} + D_2 e^{r_2(x)} = y(x)$)

⇝ CON: $[y(0) = 1 \ , \ y'(0) = 8]$

$$1 = C_1 e^0 \cos(0) + C_2 e^0\,\text{sen}(0) \longrightarrow [C_1 = 1]$$

$$y'(x) = 2C_1 e^{2x}\cos(3x) + 3C_1 e^{2x}(-\text{sen}(3x)) + 2C_2 e^{2x}\,\text{sen}(3x) + 3C_2 e^{2x}\cos(3x)$$

$y(0) = 0 \rightsquigarrow 2C_1 + 3C_2 = 8 \ \Big\langle$ $[C_1 = 1]$ $[C_2 = 2]$

$$\hookrightarrow \boxed{y(x) = e^{2x}\cos(3x) + 2e^{2x}\,\text{sen}(3x)}$$

BUSCAR PARECIDA

[PROBAR
• POLIN: $y_p = Ax^2 + Bx + C$
• TRIG: $y_p = A\cos(x) + B\,\text{sen}(x)$]

¡ O !

→ No Homogéneas $(f(x) \neq 0)$

Sol. particular

La solución general es: $\left[\, y(x) = y_p + \text{sol. gral homogénea} \,\right]$

$(ay'' + by' + cy = 0)$

¿1° sol. como homog.

(EJ)

(1) $y'' - 3y' + 2y = 2x^2 - 4x - 1$

① RESOLVER HOMOGÉNEA

$$y'' - 3y' + 2y = 0$$

$$r^2 - 3r + 2 = 0 \;\begin{cases} 1 \\ 2 \end{cases} \longmapsto y_{hom}(x) = C_1 e^x + C_2 e^{2x}$$

② PARTICULAR

[Probar con $y_p = Ax^2 + Bx + C$]

$y_p'(x) = 2Ax + B$ $\qquad 2x^2 - 4x - 1 = y'' - 3y' + 2y$

$$y_p''(x) = 2A \rightsquigarrow 2x^2 - 4x - 1 = \underbrace{(2A)}_{y''} - 3\underbrace{(2Ax + B)}_{y'} + 2\underbrace{(Ax^2 + Bx + C)}_{y}$$

$$\Rightarrow \underbrace{2}_{1}x^2 \underbrace{-4}_{2}x \underbrace{-1}_{3} = \underbrace{A}_{1}x^2 + \underbrace{(-6A + 2B)}_{2}x + \underbrace{4A - 3B + 2C}_{3}$$

$$\begin{cases} 2A = 2 & ① \\ -6A + 2B = -4 & ② \\ 2A - 3B + 2C = -1 & ③ \end{cases} \longmapsto \begin{matrix} A = 1 \\ B = 1 \\ C = 0 \end{matrix} \rightsquigarrow y_p(x) = x^2 + x$$

③ TOTAL

$$\boxed{y(x) = \underbrace{x^2 + x}_{\text{PART.}} + \underbrace{C_1 e^x + C_2 e^{2x}}_{\text{HOMOG.}}}$$

! SI NOS DAN CONDICIONES, HALLAR C_1 Y C_2 Y SUSTITUIR AQUÍ

(2) $y'' - 3y' + 2y = 20\,\text{sen}(2x)$

[HOM.] $y'' - 3y' + 2y = 0 \longrightarrow r^2 - 3r + 2 = 0 < \dots \longmapsto y_{hom}(x) = C_1 e^x + C_2 e^{2x}$

[PART.] [Probar $y_p(x) = A\cos(2x) + B\,\text{sen}(2x)$]

! Se puede probar, p. ej. con $y_p = A\,\text{sen}(2x)$, pero no es recom. hacerlo tan restrictivo.

$$\begin{cases} y_p(x) = A\cos(2x) + B\,\text{sen}(2x) \\ y_p'(x) = -2A\,\text{sen}(2x) + 2B\cos(2x) \\ y_p''(x) = -4A\,\text{sen}(2x) - 4B\,\text{sen}(2x) \end{cases}$$

$$\longrightarrow 20\,\text{sen}(2x) = (-4A\cos(2x) - 4B\,\text{sen}(2x)) - 3(-2A\,\text{sen}(2x) + 2B\cos(2x)) + 2(A\cos(2x) + B\,\text{sen}(2x))$$

$$\Rightarrow (-4A-6B+2A)\cos(2x) + (-4B+6A+2B)\,\text{sen}(2x) = 20\,\text{sen}(2x)$$

$$(-4A-6B+2A)\cos(2x) + (-4B+6A+2B-20)\,\text{sen}(2x) = 0$$

$\hookrightarrow x=0 \qquad -4A-6B+2A=0$

$\hookrightarrow x=\pi/4 \qquad -4B+6A+2B-20=0$

$$\left[\begin{array}{l} A=3 \\ B=-1 \end{array}\right]$$

$$\rightsquigarrow \quad y_p(x) = 3\cos(2x) - \text{sen}(2x)$$

TOTAL

$$\boxed{y(x) = \underbrace{3\cos(2x) - \text{sen}(2x)}_{} + \underbrace{C_1 e^{x} + C_2 e^{2x}}_{}}$$

(3) $y'' - y' - 4y = 6e^{-2x}$

HOM.

$y'' - y' - 4y = 0$

$r^2 - r - 4 = 0 \quad \begin{cases} \frac{1+\sqrt{17}}{2} \\ \frac{1-\sqrt{17}}{2} \end{cases} \quad \hookrightarrow \quad y_h(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$

PART. [pruebo con $y_p(x) = Ae^{-2x}$]

$y_p(x) = Ae^{-2x}$

$y_p'(x) = -2Ae^{-2x}$

$y_p''(x) = 4Ae^{-2x}$

$6e^{-2x} = y'' - y' - 4 \cdot y$

$$6e^{-2x} = (4Ae^{-2x}) - (2Ae^{-2x}) - 4(Ae^{-2x})$$

$$2A = 6 \Rightarrow [A=3]$$

$$\rightsquigarrow \quad y_p(x) = 3e^{-2}$$

TOT.

$$\boxed{y(x) = \underbrace{3e^{-2x}}_{} + \underbrace{C_1 e^{\left(\frac{1+\sqrt{17}}{2}\right)x} + C_2 e^{\left(\frac{1-\sqrt{17}}{2}\right)x}}_{}}$$

RESUMEN

$f(x)$	probar con
polinomio	polin. de $\geq$ grado
$\cos(cx)$, $\operatorname{sen}(cx)$	$A\cos(cx) + B\operatorname{sen}(cx)$
e^{cx}	Ae^{cx}
$(\text{polin}) \cdot e^{cx}$	$(\text{polin.} \geq \text{gr.}) \cdot e^{cx}$
$(\text{polin}) \cdot \cos(cx)$	$(\text{polin})\cos(cx) + (\text{polin.}_2)\operatorname{sen}(cx)$

! Si no funciona, probar a multiplicar todo por x. Si no, por x^2, x^3, x^4...

(4) $y'' - y = 6e^{-x}$

HOM. $r^2 - 1 = 0 < \begin{matrix} 1 \\ -1 \end{matrix} \mapsto y_h(x) = C_1e^{x} + C_2e^{-x}$

PART. pruebo: $y_p(x) = Ae^{-x}$

$$\begin{aligned} y_p &= Ae^{-x} \\ y_p' &= -Ae^{-x} \\ y_p'' &= Ae^{-x} \end{aligned} \mapsto 6e^{-x} = Ae^{-x} - Ae^{-x} \quad ? \text{ NO VALE}$$

$\rightsquigarrow$ pruebo $y_p(x) = Axe^{-x}$

$$\begin{aligned} y_p &= Axe^{-x} \\ y_p' &= Ae^{-x} - Axe^{-x} \\ y_p'' &= -Ae^{-x} - Ae^{-x} + Axe^{-x} = \\ &= -2Ae^{-x} + Axe^{-x} \end{aligned} \mapsto \begin{aligned} 6e^{-x} &= (-2Ae^{-x} + Axe^{-x}) - (Axe^{-x}) \\ 6e^{-x} &= -2Ae^{-x} \Rightarrow \boxed{A = -3} \end{aligned}$$

$\rightsquigarrow y_p(x) = -3xe^{-x}$

TOT. (GRAL)

$$\boxed{y(x) = -3xe^{-x} + C_1e^{x} + C_2e^{-x}}$$

REPASO

F Taylor en $x=0$

$$\sum \frac{f^{(n)}(0)}{n!} x^n$$

$$\sum_{n=0}^{\infty} a_n x^n \quad \text{con } n \neq 0$$

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$$

$$\sum_{n=0}^{\infty} b_n \quad b_n \neq 0$$

Si $\lim_{n\to\infty} \frac{|b_{n+1}|}{|b_n|} < 1$

$\rightsquigarrow$ Serie converge

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$|b_{n+1}| = \left| \frac{(-1)^{n+1} x^{2(n+1)}}{(2(n+1))!} \right| = \frac{|x|^{2n+2}}{(2n+2)!}$$

$$|b_n| = \left| \frac{(-1)^n x^{2n}}{(2n)!} \right| = \frac{|x|^{2n}}{(2n)!}$$

$$\frac{|b_{n+1}|}{|b_n|} = \frac{|x|^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{|x|^{2n}} = \frac{|x^2| \cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}} \xrightarrow[n\to\infty]{} 0 < 1$$

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n \quad \begin{bmatrix} R=1 \\ |t|<1 \end{bmatrix}$$

$t = x^3$

$$\frac{1}{1-x^3} = \sum_{n=0}^{\infty} (x^{3n}) \quad |x^3| < 1$$

es la deriv. de

$$\frac{1}{(1-t)^2} = \sum_{n=0}^{\infty} n t^{n-1} \quad |t|<1$$

$$= \sum_{n=1}^{\infty} n t^{n-1} \xrightarrow[k=n-1]{} \sum_{k=0}^{\infty} (k+1) t^k$$

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} \quad [R = \infty]$$

$[t = -x^3]$

$$e^{-x^3} = \sum_{n=0}^{\infty} \frac{(-3)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}!}{n!}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!}$$

$t = x^3$

$$\cos(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!}$$

$$\underbrace{-3x^2 \operatorname{sen}(x^3)}_{\text{Es su DERIV.}} = \sum_{k=0}^{\infty} (k+1) t^k$$

(1)(d) $y' = x^2y^2 + y^2$

$$y' = x^2(y^2+1) \longrightarrow \frac{dy}{dx} = x^2(y^2+1) \Rightarrow \int \frac{dy}{y^2+1} = \int x^2 dx$$

$$\Rightarrow \operatorname{arctg}(y) = \frac{x^3}{3} + C \longrightarrow y(x) = \operatorname{tg}\left(\frac{x^3}{3} + C\right)$$

(5)(c) $(x-1)y' + y = x^2 - 1$

$y' + a(x)y = b(x)$

$P(x) = e^{\int a(x)dx}$

$P(x)y(x) = \int P(x)b(x)dx$

$\mapsto y' + \frac{1}{(x-1)} y = \frac{x^2-1}{x-1}$ $\leftarrow (x-1)(x+1)$

$y' + \frac{1}{(x-1)} y = x+1$

$a(x) = \frac{1}{x-1}$, $b(x) = x+1$

$\int a(x)dx = \int \frac{1}{x-1} dx = \ln|x-1| + C$

$P(x) = e^{\ln|x-1|} = |x-1|$

$$\int P(x) b(x) dx = \int |x-1|(x+1) dx = \begin{cases} \int (x-1)(x+1)dx \text{ si } x>1 = \frac{x^3}{3} - x \\ \int -(x-1)(x+1) dx \text{ si } x<1 = \frac{-x^3}{3} + x \end{cases}$$

- Para $x>1$:

$y(x) = \frac{x^3/3 - x + C}{x-1}$

- Para $x<1$:

$y(x) = \frac{-(\frac{x^3}{3} - x) + C'}{-(x-1)}$

Dan la misma:

$$\boxed{y(x) = \frac{x^3/3 - x + C}{x-1}}$$

- $y'' + 4y = \cos x$

$y(0) = 1$, $y'(0) = 2$

HOM

$y'' + 4y = 0$

$r^2 + 4 = 0 \Rightarrow [r = \pm 2i]$

$r = \alpha + \beta i$

$\mapsto y_h(x) = C_1 e^{\alpha x} \cos(\beta x) + C_2 e^{\alpha x} \operatorname{sen}(\beta x)$ $(\alpha = 0)$

$y_h(x) = C_1 \cos(2x) + C_2 \operatorname{sen}(2x)$

PART.

$y_p(x) = A\cos x + B\operatorname{sen} x$

$y'_p(x) = -A\operatorname{sen} x + B\cos x$

$y''_p(x) = -A\cos x - B\operatorname{sen} x$

$\mapsto \cos x = (-A\cos x - B\operatorname{sen} x) + 4(A\cos x + B\operatorname{sen} x)$

$\mapsto B = 0$

$\mapsto A = 1/2$

$\rightsquigarrow y_p(x) =$

G-RAL

$y(x) = \frac{1}{3}\cos x + C_1 \cos(2x) + C_2 \operatorname{sen}(2x)$