Q1:

(a) $f'_x(x,y) = 2a_1y^2x + a_4y + a_5$

$f''_x(x,y) = 2a_1y^2$

$f'_y(x,y) = 2a_1x^2y + a_4x$

$f''_y(x,y) = 2a_1x^2$

(b) $f'_x(x,y) = 2a_1y^2x + 2a_2yx + a_3y^2 + a_4y + a_5$

$f''_x(x,y) = 2a_1y^2 + 2a_2y$

$f'_y(x,y) = 2a_1x^2y + a_2x^2 + 2a_3xy + a_4x + a_6$

$f''_y(x,y) = 2a_1x^2 + 2a_3x$

(c) $\sigma(x) = \frac{1}{1+e^{-x}} = \frac{e^x}{1+e^x} = e^x(1+e^x)^{-1}$

$\sigma'(x) = e^x(e^x+1)^{-1} - e^{2x}(e^x+1)^{-2}$

$= e^x(e^x+1)^{-2}$

$= e^{-x}(1+e^{-x})^{-2}$

$= \frac{1}{1+e^{-x}} \cdot \frac{e^x}{1+e^{-x}}$

$= \sigma(x)\cdot(1-\sigma(x))$

(d) $y_1 = 4x^2 - 3x + 3$, $y_1' = 8x - 3$, $y_1' = 0$,

$\cancel{y_1' = 12x^3 - 6x}$ $x = \frac{3}{8}$, $y = \frac{39}{16}$, it's local minimun

---

$y_2 = 3x^4 - 2x^3$, $y_2' = 12x^3 - 6x^2$, $y_2' = 0$ (x_a, y_a) is a saddle point

$x_a = 0$, $y_a = 0$, $x_b = \frac{1}{2}$, $y_b = \frac{1}{16}$, only (x_b, y_b) is local minimun

---

$y_3 = 4x + (1-x)^{\frac{1}{2}}$

$y_3' = 4 - \frac{1}{2\sqrt{1-x}} = 0$

$x_a = \frac{63}{64}$, $y_a = \frac{65}{16}$

(x_a, y_a) is local maximun

$y_4 = x + x^{-1}$

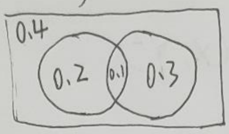
$y_4' = 1 - x^{-2} = 0$

$\begin{cases} x_a = 1 \\ y_a = 2 \end{cases}$ $\begin{cases} x_b = -1 \\ y_b = -2 \end{cases}$

(x_a, y_a) is a local minimun

(x_b, y_b) is a local maximun

Q2: (a) drawing the venn diagram



thus, $P(A)=0.3$, $P(B)=0.4$

$P(A\cup B)=0.6$, $P(\bar{A}\cap\bar{B})=P(\overline{A\cup B})=0.4$

(b) (i) $r=1-\frac{1}{6}-\frac{1}{6}-\frac{1}{6}-\frac{1}{12}-\frac{1}{12}=\frac{1}{3}$

(ii) $P(X=2, Y=3)=\frac{1}{6}$

(iii) $P(X=3)=r=\frac{1}{3}$, $P(X=3|Y=2)=\dfrac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{12}}=\frac{4}{5}$

(iv) $E[X]=1\cdot(\frac{1}{6}+\frac{1}{12}+\frac{1}{12})+2\cdot(\frac{1}{6}+\frac{1}{6})+3\cdot\frac{1}{3}=2$

$E[Y]=1\cdot(\frac{1}{6}+\frac{1}{6})+2\cdot(\frac{1}{12}+\frac{1}{3})+3\cdot(\frac{1}{12}+\frac{1}{6})=\frac{23}{12}$

$E[XY]=1\cdot P(XY=1)+2\cdot P(XY=2)+3\cdot P(XY=3)+4\cdot P(XY=4)$
$+6\cdot P(XY=6)+9\cdot P(XY=9)=\frac{47}{12}$

(v) $E[X^2]=1\cdot P(X^2=1)+4\cdot P(X^2=4)+9\cdot P(X^2=9)$
$=1\cdot\frac{1}{3}+4\cdot\frac{1}{3}+9\cdot\frac{1}{3}=\frac{14}{3}$

$E[Y^2]=1\cdot P(Y^2=1)+4\cdot P(Y^2=4)+9\, P(Y^2=9)$
$=1\cdot\frac{1}{3}+4\cdot\frac{5}{12}+9\cdot\frac{1}{4}=\frac{17}{4}$

(vi) $Cov(X,Y)=E[XY]-E[X]E[Y]=\frac{47}{12}-2\cdot\frac{23}{12}=\frac{1}{12}$

(vii) $Var(X)=E(X^2)-[E(X)]^2=\frac{14}{3}-4=\frac{2}{3}$
$Var(Y)=E(Y^2)-[E(Y)]^2=\frac{17}{4}-(\frac{23}{12})^2=\cancel{\frac{119}{144}}\ \cancel{\approx 3.65}\ 0.576$

(viii) $Corr(X,Y)=\dfrac{Cov(X,Y)}{\sqrt{Var(X)}\cdot\sqrt{Var(Y)}}\approx\cancel{0.028}\ 0.135$

(ix) $E[X+Y]=2\cdot P(X+Y=2)+3\cdot P(X+Y=3)+4\cdot P(X+Y=4)$
$+5\cdot P(X+Y=5)+6\cdot P(X+Y=6)$
$=2\cdot\frac{1}{6}+3\cdot(\frac{1}{6}+\frac{1}{12})+5\cdot(\frac{1}{3}+\frac{1}{6})+4\cdot\frac{1}{12}=\frac{47}{12}$

$E[X+Y^2]=E[X]+E[Y^2]=\frac{25}{4}$

$Var(X+Y)=Var(X)+Var(Y)+2Cov(X,Y)\approx 4.48$
$Var(X+Y^2)=Var(X)+Var(Y^2)+2Cov(X,Y^2)\approx$ Next Page

About last $Var(X+Y^2)$:

$Var(X+Y^2) = Var(X) + Var(Y^2) + 2Cov(X, Y^2)$

Suppose $Z = Y^2$, then $Var(Y^2) = Var(Z)$

$Cov(X, Y^2) = Cov(X, Z) = \cancel{Cov} \; E[XZ] - E[X]E[Z]$

Z	1	4	9
P	$\frac{1}{3}$	$\frac{5}{12}$	$\frac{1}{4}$

X	1	2	3
P	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

Z^2	1	16	81
P	$\frac{1}{3}$	$\frac{5}{12}$	$\frac{1}{4}$

First calculate $Var(Z) = E(Z^2) - (E(Z))^2$

$= (\frac{1}{3} + 16 \cdot \frac{5}{12} + 81 \cdot \frac{1}{4}) - (\frac{1}{3} + 4 \cdot \frac{5}{12} + 9 \cdot \frac{1}{4})^2$

$\approx 27.25 - 18.0625$

$= 9.1875$

Then $E[XZ] = 1 \cdot P(XZ=1) + 2 \cdot P(XZ=2) + 3 \cdot P(XZ=3)$

$+ 4 \cdot P(XZ=4) + 8 \cdot P(XZ=8) + 12 \cdot P(XZ=12)$

$+ 9 \cdot P(XZ=9) + 18 \cdot P(XZ=18) + 27 P(XZ=27)$

$= \frac{51}{6}$

Thus $Var(X+Y^2) = Var(X) + Var(Z) + 2(E(XZ) - E(X)E(Z))$

$= 0.667 + 9.1875 + 2 \cdot (8.5 - 2 \cdot 4.25)$

$= 9.8545$

Q3: (a)

$$A=\begin{bmatrix}1&3&1&0&2\\1&1&4&1&2\\1&1&1&5&2\end{bmatrix}=\begin{bmatrix}1&2&0&0&0\\1&0&3&1&0\\1&0&0&5&0\end{bmatrix}=\begin{bmatrix}0&2&0&0&0\\0&0&3&0&0\\1&0&0&0&0\end{bmatrix}$$

thus dimension of $A = r(A) = 3$

$b=[1\ 1\ 1\ 3\ 3\ 2]^T$, dimension of $b = r(b) = 1$

Because the transpose of a matrix has the same dimension.

Thus the dimension of $A^T = r(A^T) = r(A) = 3$.

(b) (i) the column of A is 3, the row of B is 2, can't match.

(ii) $$A\cdot C=\begin{bmatrix}1&3&4\\2&2&1\\6&4&3\end{bmatrix}\cdot\begin{bmatrix}7&3&3\\2&1&1\\2&2&2\end{bmatrix}=\begin{bmatrix}21&14&14\\20&10&10\\56&28&28\end{bmatrix}$$

$$C\cdot A=\begin{bmatrix}7&3&3\\2&1&1\\2&2&2\end{bmatrix}\cdot\begin{bmatrix}1&3&4\\2&2&1\\6&4&3\end{bmatrix}=\begin{bmatrix}31&39&40\\10&12&12\\18&18&16\end{bmatrix}$$

(iii) $$A\cdot D=\begin{bmatrix}1&3&4\\2&2&1\\6&4&3\end{bmatrix}\cdot\begin{bmatrix}4&2\\4&6\\1&3\end{bmatrix}=\begin{bmatrix}20&32\\17&19\\43&45\end{bmatrix}$$

$D\cdot A$ can't match because $row(D) \neq col(A)$

(iv) $D\cdot C$ don't exist

$$C\cdot D=\begin{bmatrix}7&3&3\\2&1&1\\2&2&2\end{bmatrix}\cdot\begin{bmatrix}4&2\\4&6\\1&3\end{bmatrix}=\begin{bmatrix}43&41\\13&13\\18&22\end{bmatrix}$$

$$D^T\cdot C=\begin{bmatrix}4&4&1\\2&6&3\end{bmatrix}\cdot\begin{bmatrix}7&3&3\\2&1&1\\2&2&2\end{bmatrix}=\begin{bmatrix}38&18&18\\32&18&18\end{bmatrix}$$

(V) $$B\cdot M=\begin{bmatrix}2&4\\1&1\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix}=\begin{bmatrix}14\\4\end{bmatrix}$$

$M\cdot B$ does not exist because $col(M) \neq row(B)$

(vi) $A \cdot u$ does not exist, $col(A) \neq row(u)$

(vii) $A \cdot v = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 2 & 1 \\ 6 & 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 18 \\ 13 \\ 31 \end{bmatrix}$

$v \cdot A$ does not exist, $col(v) \neq row(A)$

(viii) Because $col(B) \neq row(v)$, $B \cdot v$ can't exist.
The formula can't be computed.

[c] (i) $\|u\|_1 = |1| + |3| = 4$ $\quad \|u\|_2 = \sqrt{1^2+3^2} = \sqrt{10}$

$\|u\|_\infty = \max \|u\| = 3$ $\quad \|u\|_2^2 = u^T \cdot u = [1\ 3]\begin{bmatrix} 1 \\ 3 \end{bmatrix} = 10$

(ii) $\|v\|_1 = |2|+|4|+|1| = 7$ $\quad \|v\|_2 = \sqrt{2^2+4^2+1^2} = \sqrt{21}$

$\|v\|_\infty = \max \|v\| = 4$ $\quad \|v\|_2^2 = [2\ 4\ 1]\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} = 21$

(iii) $\|v+w\|_1 = |3|+|2|+|3| = 8$ $\quad \|v+w\|_2 = \sqrt{3^2+2^2+3^2} = \sqrt{22}$

$\|v+w\|_\infty = 3$

(iv) $A \cdot v = \begin{bmatrix} 18 \\ 13 \\ 31 \end{bmatrix}$, $\quad A \cdot (v-w) = \begin{bmatrix} 15 \\ 13 \\ 27 \end{bmatrix}$

$\|A \cdot v\|_2 = \sqrt{18^2+13^2+31^2} = \sqrt{1454} \approx 38.13$

$\|A \cdot (v-w)\|_\infty = 27$

[d] $\langle u, v \rangle = u^T \times v = 3$ $\quad \langle v, u \rangle = v^T \times u = 3$

$\langle u, w \rangle = u^T \times w = 0$ $\quad \langle w, u \rangle = w^T \times u = 0$

$\langle v, w \rangle = v^T \times w = -\frac{1}{2}$ $\quad \langle w, v \rangle = w^T \times v = -\frac{1}{2}$

[e] The dot product is
- positive: the angle between 2 vector space $< 90°$
- zero: the angle between 2 vector space $= 90°$
- negative the angle between 2 vector space $> 90°$

Q3

[f] $A = \begin{bmatrix} 1 & 3 \\ 4 & 1 \end{bmatrix}$

$A^{-1} = \frac{1}{1-3\cdot 4} \begin{bmatrix} 1 & -3 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{11} & \frac{3}{11} \\ \frac{4}{11} & -\frac{1}{11} \end{bmatrix}$

[g] $A = \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 4 & 0 \end{bmatrix}$

~~$A^{-1} =$~~

Because $|A| = 0$ and $r(A) = 1$, is not full.

So A does not have inverse matrix.

[h] Because $(X^T \cdot X)^T = X^T \cdot (X^T)^T = X^T X$

Thus, $X^T X$ is always symmetric.