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SELF-DUAL QUADRATIC PROGRAMS*

W. S. DORN†

1. Introduction. Two programming problems, one a maximization and one a minimization, are said to be dual if (1) the existence of an optimal solution to one implies the existence of an optimal solution to the other, and (2) the optimal values of the objective functions are identical. The notion of duality for a class of quadratic programs has been discussed previously [1, 2]. The present note is concerned with quadratic programming problems which are self-dual.

2. Definitions and terminology. If constraints are added to (or subtracted from) a program in such a way that the solution (both the optimal value of the objective function and the optimal values of the variables) is unchanged, the new program thus constructed is called *equivalent* to the original program. A program is called *self-dual* if it is equivalent to its dual. Consider the following quadratic programming problem: Minimize $f(x_1, x_2, \cdots, x_n) = \sum_{i=1}^n \sum_{j=1}^n c_{ij}x_ix_j + \sum_{i=1}^n p_ix_i$ where

$$\sum_{j=1}^n a_{ij}x_j \geqq b_i \qquad (i = 1, 2, \cdots, m)$$

$$x_j \geqq 0 \qquad (j = 1, 2, \cdots, n)$$

and

$$\sum_{i=1}^n \sum_{j=1}^n c_{ij}x_ix_j \geqq 0 \qquad (\text{for all } x_i).$$

It has been shown [1] that if the above problem has a finite solution then the following problem also has a solution: Maximize

$$g(u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_m) = -\sum_{i=1}^n \sum_{j=1}^n c_{ij}u_iu_j + \sum_{i=1}^m b_iv_i$$

where

$$\sum_{i=1}^m a_{ij}v_i - \sum_{i=1}^n (c_{ij} + c_{ji})u_i \leqq p_j \qquad (j = 1, \cdots, n)$$

$$v_i \geqq 0 \qquad (i = 1, 2, \cdots, m).$$

Moreover,

$$\min f(x_1, x_2, \cdots, x_n) = \max g(u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_m).$$

The two problems are, therefore, dual.

3. Self-dual quadratic programs. Consider now the following quadratic programming problem: Minimize

$$(1) \qquad f(x_1, x_2, \cdots, x_n) = \sum_{i=1}^n \sum_{j=1}^n a_{ij}x_ix_j + \sum_{i=1}^n p_ix_i$$

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† IBM Research Center, Yorktown Heights, New York.

where

$$\sum_{j=1}^{n} a_{ij}x_j + p_i \geqq 0 \qquad (i = 1, 2, \cdots, n) \tag{2}$$

$$x_i \geqq 0 \qquad (i = 1, 2, \cdots, n) \tag{3}$$

and the a_{ij} are such that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}x_ix_j \geqq 0 \tag{4}$$

for all x_i, and equality holds in (4) if and only if $x_i = 0$ for all[1] $i = 1, 2, \cdots, n$. It will be shown that this problem always has a finite solution[2] and that it is self-dual. Moreover the minimum value of f is 0.

THEOREM 1. *The minimum problem posed in* (1)–(4) *possesses a finite solution.*

Proof. From one of the theorems of the alternative for matrices[3] either

(i) there exists y_i and w_j such that

$$\sum_{i=1}^{n} a_{ij}y_i \leqq 0, \qquad \sum_{j=1}^{n} a_{ij}w_j \geqq 0$$

$$y_i \geqq 0, \qquad w_i \geqq 0$$

$$\sum_{i=1}^{n} y_i = 1, \qquad \sum_{i=1}^{n} w_i = 1,$$

or

(ii) there exist y_i such that

$$\sum_{i=1}^{n} a_{ij}y_i < 0$$

$$y_i \geqq 0$$

$$\sum_{i=1}^{n} y_i = 1,$$

or

(iii) there exist y_i such that

$$\sum_{j=1}^{n} a_{ij}y_j > 0$$

$$y_j \geqq 0$$

$$\sum_{j=1}^{n} y_j = 1,$$

and the three alternatives exclude one another.

Suppose now that (i) holds. Then multiplying the first inequality by y_j and summing on j

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}y_iy_j \leqq 0.$$

From the assumption (4) regarding the a_{ij} this implies that $y_i \equiv 0$.

[1] The matrix whose elements are a_{ij} is, therefore, a positive definite matrix (but is not necessarily symmetric).

[2] The author is indebted to the referee for pointing out this fact.

[3] See, e.g., Theorem 16: E, on page 142 of [3].

However,

$$\sum_{i=1}^{n} y_i = 1$$

and, therefore, y_i satisfying (i) cannot exist.

Similarly no y_i satisfying (ii) can exist. It follows then from (iii) that there exist y_i such that

$$\sum_{j=1}^{n} a_{ij} y_j = q_i > 0$$

$$y_j \geqq 0.$$

Now let

$$x_j = \lambda y_j$$

where

$$\lambda = \max_{p_i<0} \left\{ -\frac{p_i}{q_i}, 1 \right\}.$$

Then

$$x_j \geqq 0$$

and

$$\sum_{j=1}^{n} a_{ij} x_j = \lambda q_i \geqq -p_i .$$

It is possible, therefore, to find at least one set of x_i satisfying the constraints (2) and (3).

Consider now all x_i satisfying (2) and (3). Multiplying (2) by x_i and summing over i

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j + \sum_{i=1}^{n} p_i x_i \geqq 0,$$

i.e.,

$$f(x_1, \cdots, x_n) \geqq 0. \tag{5}$$

Since f is bounded from below and the constraints possess at least one solution, the minimization problem has a finite solution.

THEOREM 2. *The minimum problem posed in* (1)–(4) *is self-dual.*

Proof. The dual problem from §2 is: Maximize

$$g(u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n) = -\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} u_i u_j - \sum_{i=1}^{n} p_i v_i \tag{6}$$

where

$$-\sum_{j=1}^{n} (a_{ij} + a_{ji}) u_j + \sum_{j=1}^{n} a_{ji} v_j - p_i \leqq 0 \qquad (i = 1, 2, \cdots, n) \tag{7}$$

$$v_i \geqq 0 \qquad (i = 1, 2, \cdots, n) \tag{8}$$

and

$$\min f(x_1, \cdots, x_n) = \max g(u_1, \cdots, u_n, v_1, \cdots, v_n). \tag{9}$$

Multiplying (7) by v_i and summing over i

$$-\sum_{i=1}^n p_i v_i \leqq \sum_{i=1}^n \sum_{j=1}^n (a_{ij} + a_{ji}) u_j v_i - \sum_{i=1}^n \sum_{j=1}^n a_{ji} v_j v_i$$
$$= \sum_{i=1}^n \sum_{j=1}^n [a_{ij} u_j v_i + a_{ij} u_i v_j - a_{ij} v_i v_j].$$

Thus (6) becomes

$$\begin{aligned} &g(u_1, \cdots, u_n, v_1, \cdots, v_n) \\ (10) \qquad &\leqq \sum_{i=1}^n \sum_{j=1}^n a_{ij}(-u_i u_j + v_i u_j + u_i v_j - v_i v_j) \\ &= -\sum_{i=1}^n \sum_{j=1}^n a_{ij}(u_i - v_i)(u_j - v_j) \leqq 0 \end{aligned}$$

where the last inequality results from the assumption (4). It follows that

$$\max g(u_1, \cdots, u_n, v_1, \cdots, v_n) \leqq 0.$$

From (5), however,

$$\min f(x_1, \cdots, x_n) \geqq 0.$$

Combining these with (9) then

$$(11) \quad \max g(u_1, \cdots, u_n, v_1, \cdots, v_n) = \min f(x_1, \cdots, x_n) = 0.$$

From (10) and the assumption regarding a_{ij}, however, $g(u_i, v_i) = 0$ if and only if $u_i - v_i = 0$ for all $i = 1, 2, \cdots, n$. The constraint

$$(12) \qquad u_i = v_i$$

may, therefore, be added to (7) and (8) and an equivalent problem is obtained. This equivalent problem, defined by (6), (7), (8) and (12) may be reduced by elimination of u_i to: Maximize

$$g(v_1, v_2, \cdots, v_n) = -\sum_{i=1}^n \sum_{j=1}^n a_{ij} v_i v_j - \sum_{i=1}^n p_i v_i = -f(v_1, \cdots, v_n),$$

where

$$-\sum_{j=1}^n a_{ij} v_j - p_i \leqq 0 \qquad (i = 1, 2, \cdots, n)$$
$$v_i \geqq 0, \qquad (i = 1, 2, \cdots, n)$$

which is precisely the original problem posed by (1), (2) and (3).

4. Remarks. Note that (11) implies that for each i ($i = 1, 2, \cdots, n$) equality is satisfied in either (2) or (3). Thus the function $f(x_1, \cdots, x_n)$ takes on its minimal value at an *extreme point* of the convex set defined by (2) and (3) in contrast to the general quadratic programming problem.

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