# Convex symmetric dual quadratic programs

Anne-Marie Bausch - Dénes Heins

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## 1 Introduction

A quadratic program (QP) deals with the minimization or maximization of a quadratic objective function subject to some linear inequality and inequality constraints on the variables. QPs find applications in many different areas, such as economics, applied sciences and engineering [1]. And although quadratic programming is much harder to solve than linear programming, most interior point methods that work in the linear case can be generalized to the special case of convex quadratic programming, which we treat in this paper. Requiring the objective function and constraint set to be convex (or concave in the case of maximization problems), means that the matrix of the quadratic term of the objective function must be positive semi-definite (negative semi-definite for maximization problems). We start this paper by giving a short recapitulation on linear programming and then introduce convex, symmetric quadratic programming. Our main goal is to give a detailed proof of the duality theorem in QP. To finish, we describe the special case of self-duality in linear programming, as well as quadratic programming. Everything presented on the following pages is based on [2].

# 2 Introductory example: convex quadratic program

In the following we will give an example of a convex quadratic program and derive according to our definitions in Section 4 the dual program.

This example should give you an idea of the duality between a primal program and its dual. For simplicity the dimensions $n, m$ in our example are equal to 1. This means that all matrices and vectors in the definition of the primal and dual program in Section 4 will be scalars in this example.

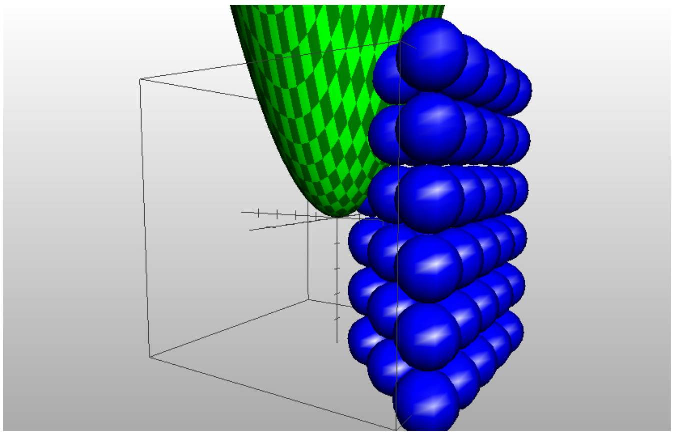

Figure 1: Primal program

Figure 1 shows an example of a convex quadratic optimization problem according to an inequality constraint. The equations which discribe the primal program are given by

$$\begin{aligned} &\text{Minimize} \quad F(x,y) = \frac{1}{2}(x^2+y^2) \\ &\text{subject to} \quad x+y-1 \geq 0 \\ &\text{and} \qquad\qquad x \geq 0. \end{aligned}$$

Comparing this primal program with the given defintion of the primal program in Section 4 we get that:

$$\begin{aligned} &A = 1, \quad D = 1 \quad \text{and} \quad C = 1 \\ &b = -1 \quad \text{and} \quad p = 0. \end{aligned}$$

If we now use the definition of the dual program given in Section 4 we derive that it is of the form:

$$\begin{aligned} &\text{Maximize} \quad G(u,v) = -\frac{1}{2}(u^2+v^2)+v \\ &\text{subject to} \quad u - v \geq 0 \\ &\text{and} \qquad\quad\ v \geq 0. \end{aligned}$$

Figure 2 illustrates the dual program which is given by above equations.

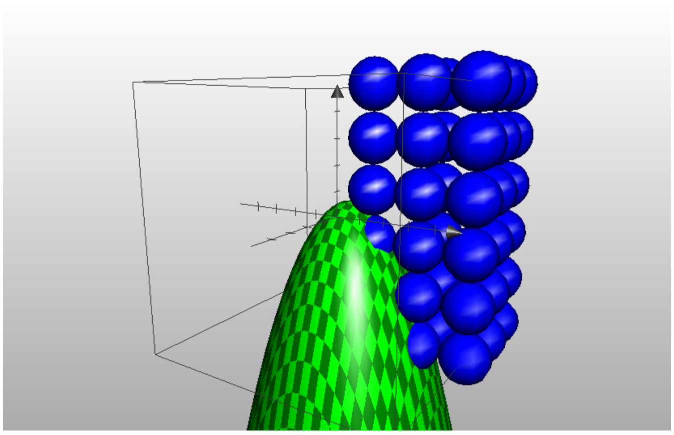

Figure 2: Dual program

Theorem 4.5 tells us that if the primal or the dual program is solvable then one can find a joint solution which is optimal for both programs. The two optimization problems here have both the unique solution, which is equal to $(\frac{1}{2}, \frac{1}{2})$.

As we will show in Theorem 4.2 the extremal values of $F$ and $G$ are equal. By plugging the unique solution into $F$ and $G$ we get that the extremal values for both are equal to $\frac{1}{4}$.

# 3 Linear Programming

We start this paper by giving a quick recapitulation on linear programming.

**Definition 3.1.** A *standard linear programming problem* is that of finding non-negative numbers $x_1, .., x_n$ which either maximize or minimize a given linear function, that is,

$$\sum_{i=1}^{n} c_i x_i$$

with respect to a set of linear inequalities.

First we introduce the definition of the primal program and its corresponding dual program.

**Definition 3.2.** A *primal program* can be expressed as

$$\begin{array}{ll} \text{Minimize} & p'x \\ \text{subject to} & Ax \geq -b \\ \text{and} & \quad x \geq 0. \end{array}$$

**Definition 3.3.** The corresponding *dual program* can be expressed as

$$\begin{array}{ll} \text{Maximize} & -b'v \\ \text{subject to} & -A'v \geq -p \\ \text{and} & v \geq 0, \end{array}$$

where $x \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$ represent the vectors of variables (to be determined), $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ are vectors of (known) coefficients and $A \in \mathbb{R}^{m \times n}$ is a (known) matrix of coefficients.

**Remark 1.** Every linear program, referred to as a primal problem, can be converted into a *dual program.* We will see later that the primal problem provides an upper bound to the optimal value of the dual problem.

One of the most important theorems in linear programming is the Duality Theorem of Linear Programming. We omit the proof here, but will prove its extension to quadratic programming in a few pages.

**Theorem 3.1.** *(Duality Theorem of Linear Programming)*
*If a standard maximum or minimum problem and its dual are both feasible then they both have optimal solutions and both have the same value. If either problem is not feasible then neither has an optimal solution.*

# 4 Quadratic Programming

Quadratic programming is the simplest form of non-linear programming. It deals with optimizing (maximizing or minimizing) a quadratic function of several variables with respect to linear constraints on these variables.

**Idea:**
If we have two dual programs and one of them has an optimal solution, then the Duality Theorem of Quadratic Programming, which we shall prove later, tells us that they share an optimal solution.

We will consider the following two programs:

*Primal program* $(P)$*:*

$$\begin{aligned} &\text{Minimize} \quad F(x,y) = \frac{1}{2}y'Dy + \frac{1}{2}x'Cx + p'x \\ &\text{subject to} \quad Dy + Ax \geq -b \\ &\text{and} \qquad\quad x \geq 0, \end{aligned}$$

with the *constraint set*

$$P = \{(x,y) \mid Dy + Ax \geq -b, \quad x \geq 0\}.$$

*Dual program* $(P^\star)$*:*

$$\begin{aligned} &\text{Maximize} \quad G(u,v) = -\frac{1}{2}v'Dv - \frac{1}{2}u'Cu - b'v \\ &\text{subject to} \quad -A'v + Cu \geq -p \\ &\text{and} \qquad\quad v \geq 0, \end{aligned}$$

with the *constraint set*

$$P^\star = \{(u,v) \mid -A'v + Cu \geq -p, \quad v \geq 0\}.$$

**Remark 2.** We suppose that $A \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^{n\times n}$ and $D \in \mathbb{R}^{m\times m}$, where $C$ and $D$ are symmetric, as well as positive semi-definite, and the vectors $b, y, v$ are m-vectors and $p, x, u$ are n-vectors.
An inequality between vectors means that the inequality holds between each of the corresponding components.

**Remark 3.** We could leave away the assumption of symmetry, because by redefining $D$ as $\frac{1}{2}(D + D')$ and $C$ as $\frac{1}{2}(C + C')$, we could make any matrices symmetric without altering the program.

**Definition 4.1.** In optimization problems the *duality gap* is the difference between the primal and dual solutions. Here it would be the difference between the extremal values of $F$ and $G$.

**Remark 4.** As $C$ and $D$ are positive semi-definite matrices, it follows that the quadratic program is a convex optimization problem. We will show later in Theorem 4.2 that in this case the duality gap is zero and therefore the extremal values of $F$ and $G$ are equal.

**Remark 5.** By setting the matrices $C$ and $D$ equal to zero in the defintion of the primal and the dual program for quadratic programming you can see that the quadratic programs can be reduced to linear programs.

## 4.1 Solving LP and QP problems

LP problems are often solved via the *Simplex method.* This method, originally developed by Dantzig in 1948, has been dramatically enhanced in the last decade, using advanced methods from numerical linear algebra. Now it is possible to solve linear programs with up to hundreds of thousands of decision variables and constraints. Two other common methods are called the *Interior Point* and *Newton-Barrier-method.*

As QP problems are special cases of smooth nonlinear problems, they can be solved by smooth nonlinear methods such as *GRG* (Generalized reduced gradient method) or *SQP* (Sequential quadratic programming method). Still, a faster way to solve QP problems appears to be an extension of the Simplex, Interior point or Barrier method. When assuming convexity, the last-mentioned methods are even easier to generalize than for QP.

## 4.2 Definitions

To be able to continue with the theory, we first need to write down a few definitions which will be used in the following theorems and lemmas.

**Definition 4.2.** An element of P or $P^\star$ is said to be a *feasible solution* of $(P)$ or $(P^\star)$, respectively.

**Definition 4.3.** A program is *infeasible* if its constraint set is empty.

**Definition 4.4.** A program is *solvable* if its constraint set contains an element for which its *objective function* ($F$ in $(P)$ or G in $(P^\star)$) attains the desired extremum. Such an element is called an *optimal solution* of the program.

The most important notion is that of duality between $(P)$ and $(P^\star)$. We will spend a great deal of time proving step by step using different lemmas and theorems that there exists indeed a duality relation between the primal and the dual program for quadratic programming.

**Definition 4.5.** We say that there is a relation of *duality* between $(P)$ and $(P^\star)$ if the following conditions are satisfied:

i) $\sup_{P^\star} G \leq \inf_P F$,
ii) the solvability of one problem implies that of the other, and the extremal values of $F$ and $G$ are equal,
iii) if one problem is feasible while the other is not, then on its constraint set, the objective function of the feasible program is unbounded in the direction of extremization.

## 4.3 Duality

To ultimately prove the existence of a duality relation between $(P)$ and $(P^\star)$ we start by proving the points of the Definition 4.5 one by one, starting with the first one.

**Theorem 4.1.** $\sup_{P^\star} G \leq \inf_P F$.

*Proof.* We use the convention that

$$\text{if} \quad P^\star = \emptyset, \quad \sup_{P^\star} G = -\infty, \tag{1}$$

$$\text{if} \quad P = \emptyset, \quad \sup_{P} F = \infty. \tag{2}$$

As one can see using (1) and (2), the above inequality holds if either one of the programs or both are not feasible. The only case remaining to prove is when both programs are feasible and thus, when $P \neq \emptyset$ and $P^\star \neq \emptyset$. Consequently, we can choose $(x, y) \in P, (u, v) \in P^\star$.

Next, we multiply the constraint inequality $Dy + Ax \geq -b$ of $(P)$ by the row vector $v'$ from the left side and, using the second constraint inequality $v \geq 0$ of $(P^\star)$ in addition, we find

$$v'Dy + v'Ax \geq -v'b,$$

which is equivalent to

$$-b'v - y'Dv \leq x'A'v, \tag{3}$$

as all of the above terms are scalars, we can take the transpose without changing anything and $D$ is symmetric ($D' = D$).
In a similar way, we multiply the constraint inequality $-A'v + Cu \geq -p$ of $(P^\star)$ by the row vector $x'$ from the left side and, using the second constraint inequality $x \geq 0$ of $(P)$ in addition, we find

$$-x'A'v + x'Cu \geq -x'p,$$

which is equivalent to

$$x'A'v \leq p'x + x'Cu, \tag{4}$$

as all of the above terms are scalars, we can take the transpose without changing anything.
Assembling (3) and (4) into one equation, we get

$$-b'v - y'Dv \leq x'A'v \leq p'x + x'Cu. \tag{5}$$

Moreover, we use the symmetry and positive semi-definiteness of $D$ to find

$$\begin{aligned} y'Dy + v'Dv - 2y'Dv &= y'Dy + v'Dv - y'Dv - v'D'y \\ &\overset{\text{D symmetric}}{=} y'Dy + v'Dv - y'Dv - v'Dy \\ &= (y-v)'D(y-v) \\ &\overset{\text{D positive semi-definite}}{\geq} 0, \end{aligned}$$

which is equivalent to

$$2y'Dv \leq y'Dy + v'Dv. \tag{6}$$

Analogously, we can repeat the same computations with $C$, as it is symmetric and positive semi-definite as well. This time we find

$$2x'Cu \leq x'Cx + u'Cu. \tag{7}$$

We apply (6) and (7) to (5) in order to find

$$\begin{aligned} -b'v - \frac{1}{2}y'Dy - \frac{1}{2}v'Dv &\overset{(6)}{\leq} -b'v - y'Dv \\ &\overset{(5)}{\leq} p'x + x'Cu \\ &\overset{(7)}{\leq} p'x + \frac{1}{2}x'Cx + \frac{1}{2}u'Cu. \end{aligned}$$

Rearranging some terms, we come to our final result

$$G(u,v) = -\frac{1}{2}v'Dv - \frac{1}{2}u'Cu - b'v \leq \frac{1}{2}y'Dy + \frac{1}{2}x'Cx + p'x = F(x,y),$$

which implies the theorem. □

Next, we introduce and prove an auxiliary lemma, which we need to show the second point in Definition 4.5.

**Lemma 4.1.** *If $(x_0, y_0)$ is an optimal solution of $(P)$, then it is an optimal solution of the linear program $(L_0)$:*

$$\begin{aligned} &\text{Minimize} \quad f(x,y) = (y_0'D)y + (x_0'C)x + p'x \\ &\text{subject to} \quad Dy + Ax \geq -b \\ &\text{and} \qquad\qquad x \geq 0. \end{aligned}$$

*Proof.* According to the given definition of the primal program $(P)$ in Section 4 we realize that the constraint set of $(P)$ is equal to that of $(L_0)$.

We will show this lemma by contradiction. Hence, we assume that there exists $(x_1, y_1) \in P$ such that $f(x_1, y_1) - f(x_0, y_0) < 0$. In other words, we assume that for $(L_0)$ there exists a better solution than $(x_0, y_0)$.

If this is not the case we are already done, as then $(x_0, y_0)$ would also be the optimal solution for $(L_0)$.

Now, by inserting the definition of $f$ from above into the previous inequality, we find that

$$\begin{aligned} &f(x_1, y_1) - f(x_0, y_0) < 0 \\ &\Leftrightarrow (y_0'D)y_1 + (x_0'C)x_1 + p'x_1 - (y_0'D)y_0 + (x_0'C)x_0 + p'x_0 < 0 \\ &\Leftrightarrow (y_0'D)(y_1 - y_0) + (x_0'C + p')(x_1 - x_0) < 0. \end{aligned} \tag{8}$$

Let $0 < \lambda < 1$, and define

$$\begin{aligned} x^\star &:= (1-\lambda)x_0 + \lambda x_1 = x_0 + \lambda(x_1 - x_0), \\ y^\star &:= (1-\lambda)y_0 + \lambda y_1 = y_0 + \lambda(y_1 - y_0). \end{aligned}$$

The variables $x^\star$ and $y^\star$ are just convex combinations of $x_0, x_1$ resp. $y_0, y_1$. Hence, as $P$ (the constraint set of the primal program $(P)$) is a convex set, we can conclude that $(x^\star, y^\star) \in P$.

Next, we will recall the definition of the objective function $F$ of the primal quadratic program $(P)$ given in Section 4.

$$F(x,y) = \frac{1}{2}y'Dy + \frac{1}{2}x'Cx + p'x$$

If we now plug the definition of $x^\star$ and $y^\star$ into $F$ we can calculate the following difference

$$\begin{aligned} F(x^\star, y^\star) - F(x_0, y_0) =& \lambda \overbrace{[(y_0' D)(y_1 - y_0) + (x_0' C + p')(x_1 - x_0)]}^{\overset{\text{by (8)}}{<} 0} \\ &+ \frac{\lambda^2}{2} \left[ \underbrace{(y_1 - y_0)' D (y_1 - y_0)}_{\geq 0} + \underbrace{(x_1 - x_0)' C (x_1 - x_0)}_{\geq 0} \right], \end{aligned} \tag{9}$$

where we used the fact that $C$ and $D$ are positive semi-definite.

Equation (8) implies that the right-hand side of (9) can be made negative by choosing a sufficiently small positive $\lambda$. This means that we choose $x^\star$ and $y^\star$ really close to $x_0$ resp. $y_0$.

Hence, the difference $F(x^\star, y^\star) - F(x_0, y_0)$ is neagtive which tells us that $(x^\star, y^\star)$ is a better solution for the primal quadratic program $(P)$ than $(x_0, y_0)$. But this would contradict our assumption from the lemma that $(x_0, y_0)$ is the optimal solution of $(P)$.

Consequently, there is no $(x_1, y_1) \in P$ such that $f(x_1, y_1) - f(x_0, y_0) < 0$. Finally we get that $(x_0, y_0)$ is also the optimal solution of $(L_0)$.

□

**Theorem 4.2.** *If $(P)$ is sovable, then $(P^\star)$ is solvable and the extremal values of $F$ and $G$ are equal.*

*Proof.* Let $(x_0, y_0)$ solve (P). Theorem 4.1 implies that if there exists $(u_0, v_0) \in P^\star$ such that

$$G(u_0, v_0) = F(x_0, y_0),$$

then $(u_0, v_0)$ solves $(P^\star)$. This is because all the extremal values of $G$ are necessarily smaller or equal than the extremal values of $F$ ($\sup_{P^\star} G \leq \inf_P F$) and thus, as soon as the extremal optimal value of $F$ is equal to the extremal value of $G$ in some other point, this other point has to solve $(P^\star)$.

According to Lemma 4.1 we know that if $(x_0, y_0)$ is optimal for $(P)$, then $(x_0, y_0)$ is also optimal for the linear program $(L_0)$.

Using the duality theorem of linear programming, we find the dual program $(L_0^\star)$ of $(L_0)$. Furthermore, we know that their extremal values have to be equal and thus we know that there exists a vector $v_0$ fulfilling the following equations:

$$\begin{aligned} &- b' v_0 = y_0' D y_0 + x_0' C x_0 + p' x_0, \qquad (10) \\ &- A' v_0 + C x_0 \geq -p, \\ &v_0 \geq 0 \quad \text{and} \quad v_0' D = y_0' D. \end{aligned}$$

Looking at the two constraints of $(L_0^\star)$, we find that $(x_0, v_0) \in P^\star$, consequently we know that $(x_0, v_0)$ is feasible for $(P^\star)$. Next, we do an auxiliary calculation using the symmetry of $D$ and one of the equations from above:

$$v_0'Dv_0 = y_0'Dv_0 = v_0'Dy_0 = y_0'Dy_0. \tag{11}$$

Hence, by (10) and (11),

$$\begin{aligned} F(x_0, y_0) &= \frac{1}{2}y_0'Dy_0 + \frac{1}{2}x_0'Cx_0 + p'x_0 \\ &\overset{(10)}{=} -\frac{1}{2}y_0'Dy_0 - \frac{1}{2}x_0'Cx_0 - b'v_0 \\ &\overset{(11)}{=} -\frac{1}{2}v_0'Dv_0 - \frac{1}{2}x_0'Cx_0 - b'v_0 \\ &= G(x_0, v_0), \end{aligned}$$

we see that the extremal values of $F$ and $G$ are the same, which completes the proof.

□

Furthermore, we introduce, without proof, a few corollaries, that will be useful later on.

**Corollary 4.1.** *If $(x_0, y_0)$ solves $(P)$, there exists a vector $v_0$ such that $(x_0, v_0)$ solves $(P^\star)$ and $G(x_0, v_0) = F(x_0, y_0)$. Moreover, $Dy_0 = Dv_0$.*

**Corollary 4.2.** *If $(u_0, v_0)$ solves $(P^\star)$, there exists a vector $x_0$ such that $(x_0, v_0)$ solves $(P)$ and $G(u_0, v_0) = F(x_0, v_0)$. Moreover, $Cu_0 = Cx_0$.*

**Corollary 4.3.** *Non-negativity restrictions may be imposed on all the variables in $(P)$ and $(P^\star)$ without affecting the question of their solvability.*

Moreover, we need this theorem to prove part of the last point of Definition 4.5. We omit the proof, as it is not vital to the understanding of duality relations.

**Theorem 4.3.** *(Farka's Lemma)* [3, p. 46] *Exactly one of the following alternatives holds:*
*Either the inequality*

$$Ay \geq c$$

*has a soultion or the equations*

$$x'A = 0, \quad x'c = 1$$

*have a non-negative solution.*

**Theorem 4.4.** *If* $(P)$ *is feasible and* $(P^\star)$ *is infeasible, then* $\inf_P F = -\infty$.

*Proof.* As $(P)$ is feasible, we can choose $(x, y) \in P$. The assumption that $(P^\star)$ is not feasible, means that there exists no pair $(u, v)$ satisfying

$$\begin{pmatrix} -A' & C \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix} \geq \begin{pmatrix} -p \\ 0 \end{pmatrix}. \tag{12}$$

By Theorem 4.3, as (12) has no solution, there exist vectors $x^\star \geq 0$ and $y^\star \geq 0$ such that

$$\begin{pmatrix} -A & \mathbb{1} \\ C & 0 \end{pmatrix} \begin{pmatrix} x^\star \\ y^\star \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$(-p', 0) \begin{pmatrix} x^\star \\ y^\star \end{pmatrix} = 1.$$

Writing out the above equations, we find

$$\mathbb{1} y^\star = A x^\star \geq 0, \quad C x^\star = 0, \quad p' x^\star = -1. \tag{13}$$

Taking $\lambda \geq 0$ and using that $Ax^\star \geq 0$ it follows that $(x + \lambda x^\star, y) \in P$.

First recall the defintion of the objective function $F$ of the primal program $(P)$:

$$F(x, y) = \frac{1}{2} y' D y + \frac{1}{2} x' C x + p' x.$$

Using the above definition of $F$ and the two last equations from (13) we can compute that

$$\begin{aligned} F(x + \lambda x^\star, y) &= \frac{1}{2} y' D y + \frac{1}{2} (x + \lambda x^\star)' C (x + \lambda x^\star) + p'(x + \lambda x^\star) \\ &= \lambda \underbrace{p' x^\star}_{=-1} + \underbrace{\frac{1}{2} y' D y + \frac{1}{2} x' C x + p' x}_{=F(x,y)} + \frac{1}{2} \lambda \underbrace{x^{\star\prime} C}_{=0} x + \frac{1}{2} \lambda^2 x^{\star\prime} \underbrace{C x^\star}_{=0} \\ &= -\lambda + F(x, y). \end{aligned}$$

This clearly implies $\lim_{\lambda \to +\infty} F(x + \lambda x^\star, y) = -\infty$ and thus we proved the theorem. □

The following corollaries finalize the proof of the duality relation between the programs $(P)$ and $(P^\star)$. We omit their proofs, as the first corollary can be proved analogue to Theorem 4.4 and the second corollary is a composition of the first corollary and Theorem 4.4.

**Corollary 4.4.** *If* $(P^\star)$ *is feasible and* $(P)$ *is infeasible, then* $\sup_{P^\star} G = +\infty$.

**Corollary 4.5.** *If either program* $(P)$ *or* $(P^\star)$ *is feasible , its objective function is bounded in the direction of extremization if and only if the other problem is feasible.*

Now that we have proved the existence of the duality relation for quadratic programs, we go one step further and show that not just the extremal values of $(P)$ and $(P^\star)$ are equal, but the optimal solution as well.

**Theorem 4.5.** *(Joint solution)*
*If* $(P)$ *or* $(P^\star)$ *is solvable, there exist vectors* $x_0$ *and* $v_0$ *such that* $(x_0, v_0)$ *is an optimal solution for both,* $(P)$ *and* $(P^\star)$.

*Proof.* We can assume without loss of generality that $(P)$ is solvable. Let $(x_0, y_0)$ solve $(P)$. Corollary 4.1 implies that there exists a vector $v_0$ such that $(x_0, v_0)$ solves $(P^\star)$ and

$$G(x_0, v_0) = F(x_0, y_0) \quad \text{and} \quad Dy_0 = Dv_0.$$

Consequently, $(x_0, v_0) \in P$ and $G(x_0, v_0) = F(x_0, v_0)$ and therefore, $(x_0, v_0)$ solves $(P)$ as well and we found the joint solution $(x_0, v_0)$. □

In linear programming, in the case that both, primal and dual program, are feasible, they are solvable as well, i.e., there exist optimal solutions for both of them. In the next theorem, we prove an analogue statement for QP.

**Theorem 4.6.** *If* $(P)$ *and* $(P^\star)$ *are feasible, then* $(P)$ *and* $(P^\star)$ *are solvable.*

*Proof.* As $P$ and $P^\star$ are non-empty, $F$ is bounded below on $P$ and $G$ is bounded above on $P^\star$. Modifying the result of Frank and Wolfe, we can show that $G$ must attain its supremum (over $P^\star$) on $P^\star$. The remainder of the proof is an application of Theorem 4.2 and Corollary 4.5. □

## 4.4 Special Case: Self-duality

To finish, we take a look at the special case of self-duality. However, first, me must give a few definitions.

**Definition 4.6.** [4] If constraints are added to (or subtracted from) a program in such a way that the solution (both the optimal value of the objective function and the extremal values of the variables) rests unchanged, the obtained program is called *equivalent* to the original problem.

**Definition 4.7.** [4] A program is called *self-dual* if it is equivalent to its dual program.

In linear programming, self-duality is quite straight-forward, as we see in the following illustration: Define the linear program $(L_1)$ as follows,

$$\begin{array}{ll} \text{Minimize} & q'z \\ \text{subject to} & A_1 z \geq -q \\ \text{and} & z \geq 0, \end{array}$$

where $A_1$ is skew symmetric. Then we clearly see that $(L_1)$ is self-dual, i.e., $(L_1)$ and $(L_1^\star)$ are equivalent. Moreover, if $L_1 \neq \emptyset$, i.e., the program is feasible, then $\min q'z = 0$. However, to get the same statement in quadratic programming, things are a little more complex.

**Theorem 4.7.** *The quadratic program* $(P_1)$

$$\begin{array}{ll} \textit{Minimize} & \phi(x,y) = \frac{1}{2}y'C_1 y + \frac{1}{2}x'C_1 x + q'x \\ \textit{subject to} & C_1 y + A_1 x \geq -q \\ \textit{and} & x \geq 0, \end{array}$$

*where* $A_1$ *is skew symmetric and* $C_1$ *is symmetric positive semi-definite, is self-dual. Moreover, if* $(P_1)$ *is feasible, then* $\min \phi(x,y) = 0$.

*Proof.* First, we compute the dual of $(P_1)$, $(P_1^\star)$:

$$\begin{array}{ll} \text{Maximize} & \psi(u,v) = -\frac{1}{2}v'C_1 v - \frac{1}{2}u'C_1 u - q'v \\ \text{subject to} & -A_1'v + C_1 u \geq -q \\ \text{and} & v \geq 0. \end{array}$$

Using the fact that $A_1$ is skew symmetric, we notice that $(P_1)$ is self-dual.

Again by the skew symmetry of $A_1$, we find that $x'A_1x = 0$ for all $x$, because $x'A_1x = (x'A_1x)' = x'A_1'x = -x'A_1x = 0$ for all $x$.

Letting $(x, y) \in P_1$, multiplying the constraint inequality of $(P_1)$ by the row vector $x'$ and using that $x \geq 0$ and $x'A_1x = 0$, we get that

$$y'C_1x \geq -q'x.$$

Consequently,

$$\phi(x, y) = \frac{1}{2}y'C_1y + \frac{1}{2}x'C_1x + q'x \geq \frac{1}{2}(y-x)'C_1(y-x) \geq 0.$$

The program $(P_1)$ is solvable, since $\phi$ is bounded from below on $P_1$. Therefore, $(P_1^\star)$ is solvable as well. For all $(u, v) \in P_1^\star$, $\psi(u, v) \leq 0$. Now let $(x_0, y_0)$ be a joint solution of $(P_1)$ and $(P_1^\star)$, then $0 \geq \phi(x_0, y_0) = \psi(x_0, y_0) \leq 0$. The theorem follows. □

Finally, we show that out of every linear (and quadratic) program, we may obtain a self-dual program of the form $(L_1)$ (and $(P_1)$) by composing the program with its dual. So, imagine we have a linear primal program $(L)$ of the form

$$\begin{array}{ll} \text{Minimize} & p'x \\ \text{subject to} & A'x \geq -b \\ \text{and} & x \geq 0 \end{array}$$

and its dual $(L^\star)$

$$\begin{array}{ll} \text{Maximize} & -b'v \\ \text{subject to} & -A'v \geq -p \\ \text{and} & v \geq 0. \end{array}$$

We get a self-dual program of the form $(L_1)$ simply by taking

$$A_1 = \begin{pmatrix} 0 & A \\ -A' & 0 \end{pmatrix}, \quad q = \begin{pmatrix} b \\ p \end{pmatrix}, \quad z = \begin{pmatrix} v \\ x \end{pmatrix}.$$

In a similar manner, the composite of the programs $(P)$ and $(P^\star)$ yields a self-dual program of the form $(P_1)$ by taking

$$A_1 = \begin{pmatrix} 0 & A \\ -A' & 0 \end{pmatrix}, \quad C_1 = \begin{pmatrix} D & 0 \\ 0 & C \end{pmatrix}, \quad q = \begin{pmatrix} b \\ p \end{pmatrix}, \quad X = \begin{pmatrix} v \\ x \end{pmatrix}, \quad X = \begin{pmatrix} y \\ u \end{pmatrix}$$

and $\phi(x, y) = F(x, y) - G(x, y)$.

# 5 Conclusion

In conclusion, we can say that although quadratic programming is a lot harder to solve than linear programming, in the special case of convex QP there is no big difference. Furthermore, we hope that in the future, whenever the reader has to solve a QP (or a LP as a matter of fact), he or she will think of the dual and the duality relation and maybe that will make the problem easier to solve. Especially when keeping in mind all the different properties we proved in this paper.

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