J. SIAM Appl. Math.
Vol. 14, No. 3, May, 1966
Printed in U.S.A.

SELF-DUALITY IN MATHEMATICAL PROGRAMMING*

BERTRAM MOND† AND RICHARD W. COTTLE‡

1. Introduction. Self-duality in mathematical programming was investigated by Duffin [6] in the case of linear programming, by Dorn [5] and Cottle [1] in the case of quadratic programming and, recently, by Hanson [7] in the case of more general nonlinear programming. Problems of a self-dual type are discussed in [2] and [3].

In [5] and [7] self-duality requires the notion of *equivalence*. Dorn [5] defines this concept in the following way: "If constraints are added to (or subtracted from) a program in such a way that the solution (both the optimal value of the objective function and the optimal values of the variables) is unchanged, the new program thus constructed is called *equivalent* to the original program. A program is called *self-dual* if it is equivalent to its dual."

Self-duality in [1] and [6] is of a *formal* type, that is, a program is its own dual without adding (or subtracting) constraints. The main purpose of this note is to formulate a nonlinear program that is *formally self-dual*.

2. Notation and terminology. Small letters will generally denote vectors over the real number field; capital letters will denote matrices; $x \geqq y$ means that every component of x is greater than or equal to the corresponding component of y; a prime will denote transpose.

Let $f(x, y)$ be a real-valued differentiable function of vectors x and y of dimensions n and m, respectively. $\nabla_1 f(x, y)$ and $\nabla_2 f(x, y)$ will denote the vector-valued functions which are the gradients of f with respect to x and y, respectively.

Assume x and y have the same dimension n. The function $f(x, y)$ will be said to be *skewsymmetric* if

$$\begin{aligned} f(x, y) &= f(x_1, \cdots, x_n, y_1, \cdots, y_n) \\ &= -f(y_1, \cdots, y_n, x_1, \cdots, x_n) = -f(y, x) \end{aligned}$$

for all (x, y) in the domain of f.

3. Symmetric duality. Consider the following two programs:

Primal program (P):

* Received by the editors January 4, 1965, and in final revised form July 28, 1965.

† Applied Mathematics Research Laboratory, Aerospace Research Laboratory, Wright-Patterson Air Force Base, Ohio.

‡ Bell Telephone Laboratories, Incorporated, Holmdel, New Jersey. The research of the second-named author was partially supported by the Office of Naval Research under Contract Nonr-222(83) with the University of California.

$$\text{minimize} \quad H(x, y) \equiv f(x, y) - y'\nabla_2 f(x, y)$$

$$\text{subject to} \quad -\nabla_2 f(x, y) \geqq 0, \tag{1}$$

$$x \geqq 0, \tag{2}$$

$$y \geqq 0. \tag{3}$$

Dual program (P^*):

$$\text{maximize} \quad G(x, y) \equiv f(x, y) - x'\nabla_1 f(x, y)$$

$$\text{subject to} \quad \nabla_1 f(x, y) \geqq 0, \tag{4}$$

$$x \geqq 0, \tag{5}$$

$$y \geqq 0. \tag{6}$$

(P) and (P^*) will be said to be *dual programs* if, when either program has an optimal solution,

(i) (P) and (P^*) have a joint optimal solution, and

(ii) the minimum of (P) equals the maximum of (P^*).

The duality of (P) and (P^*) has been established under various hypotheses (see [4], [9], [10], [13], [14]). Our purpose in the theorem below is not to discuss the conditions under which this duality obtains, but to indicate the additional information contributed by the skewsymmetry of f.

4. Self-duality.

THEOREM. *Suppose f is differentiable and skewsymmetric. Then* (P) *and* (P^*) *are formally identical. If* (P) *and* (P^*) *are dual programs and* (x_0, y_0) *is a joint optimal solution, then so is* (y_0, x_0) *and*

$$H(x_0, y_0) = f(x_0, y_0) = 0.$$

Proof. Consider (P) and note that (P^*) can be written:

$$\begin{aligned} \text{minimize} \quad & -f(u, v) + u'\nabla_1 f(u, v) \\ \text{subject to} \quad & \nabla_1 f(u, v) \geqq 0, \\ & u \geqq 0, \\ & v \geqq 0. \end{aligned}$$

Since f is skewsymmetric, $\nabla_1 f(u, v) = -\nabla_2 f(v, u)$, and program ($P^*$) becomes:

$$\begin{aligned} \text{minimize} \quad & f(v, u) - u'\nabla_2 f(v, u) \\ \text{subject to} \quad & -\nabla_2 f(v, u) \geqq 0, \\ & v \geqq 0, \\ & u \geqq 0, \end{aligned}$$

which is just (P).

Thus, (x_0, y_0) optimal for (P^*) implies (y_0, x_0) optimal for (P). By an analogous argument, (x_0, y_0) optimal for (P) implies (y_0, x_0) optimal for (P^*).

If (P) and (P^*) are dual programs and (x_0, y_0) is jointly optimal, then

$$-y_0'\nabla_2 f(x_0, y_0) = -x_0'\nabla_1 f(x_0, y_0). \tag{7}$$

From (1) and (3) or (6),

$$-y_0'\nabla_2 f(x_0, y_0) \geqq 0; \tag{8}$$

and, from (4) and (2) or (5),

$$-x_0'\nabla_1 f(x_0, y_0) \leqq 0. \tag{9}$$

Combining (7), (8), and (9) yields

$$0 \geqq -x_0'\nabla_1 f(x_0, y_0) = -y_0'\nabla_2 f(x_0, y_0) \geqq 0.$$

Thus

$$-y_0'\nabla_2 f(x_0, y_0) = 0.$$

Since (y_0, x_0) is also a joint optimal solution, one can show, in a similar manner, that

$$-x_0'\nabla_2 f(y_0, x_0) = 0.$$

Hence,

$$H(x_0, y_0) = H(y_0, x_0) = f(x_0, y_0) = f(y_0, x_0) = -f(x_0, y_0)$$

and, therefore,

$$H(x_0, y_0) = f(x_0, y_0) = 0.$$

5. Remarks. It was pointed out in [4] and [10] that if $f(x, y) = b'y - y'Ax + c'x$, then (P) and (P^*) reduce to a pair of dual linear programs; and if

$$f(x, y) = b'y - y'Ax - \tfrac{1}{2}y'Dy + \tfrac{1}{2}x'Cx + p'x,$$

then (P) and (P^*) become the symmetric dual quadratic programs of [1]. The condition for self-duality, $b = -c$ and $A = -A'$ in the linear case [6], and $b = -p$, $A = -A'$, and $C = D$ in the quadratic case [1], is just the requirement that $f(x, y)$ be skewsymmetric.

REFERENCES

[1] R. W. Cottle, *Symmetric dual quadratic programs*, Quart. Appl. Math., 21 (1963), pp. 237–243.

[2] ———, *Note on a fundamental theorem in quadratic programming*, this Journal, 12 (1964), pp. 663–665.

[3] G. B. Dantzig and R. W. Cottle, *Positive (semi-) definite matrices and mathematical programming*, ORC Report 63-18 (RR), University of California, 1963.

[4] G. B. Dantzig, E. Eisenberg, and R. W. Cottle, *Symmetric dual nonlinear programs*, Pacific J. Math., 15 (1965), pp. 809–812.

[5] W. S. Dorn, *Self-dual quadratic programs*, this Journal, 9 (1961), pp. 51–54.

[6] R. J. Duffin, *Infinite programs*, Linear Inequalities and Related Systems, Study 38, Princeton University Press, Princeton, 1956.

[7] M. A. Hanson, *Duality and self-duality in mathematical programming*, this Journal, 12 (1964), pp. 446–449.

[8] H. W. Kuhn and A. W. Tucker, *Nonlinear Programming*, Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, University of California Press, Berkeley, 1951, pp. 481–492.

[9] O. L. Mangasarian and J. Ponstein, *Minimax and duality in nonlinear programming*, J. Math. Anal. Appl., 11 (1965), pp. 504–518.

[10] B. Mond, *A symmetric dual theorem for nonlinear programs*, Quart. Appl. Math., 23 (1965), pp. 265–269.

[11] R. T. Rockafellar, *Convex functions and dual extremum problems*, Ph.D. thesis, Harvard University, 1963.

[12] J. Stoer, *Duality in nonlinear programming and the minimax theorem*, Numer. Math., 5 (1963), 371–379.

[13] ———, *Über einen Dualitätssatz der nichtlinearen Programmierung*, Ibid., 6 (1964), pp. 55–58.

[14] A. Whinston, *Conjugate functions and symmetric duality*, Cowles Foundation Discussion Paper 162, Yale University, 1963.