Tema 2 CTi

V.a. continue

Pb 1) $f(x) = \begin{cases} A\cos x, & x \in [0; \frac{\pi}{2}] \\ 0, & \text{în rest} \end{cases}$

$A \in \mathbb{R}$

a) f densitate de probabilitate $\Leftrightarrow \begin{cases} 1)\ f(x) \geq 0, \forall x \in \mathbb{R} \\ 2)\ \int_{-\infty}^{\infty} f(x)\,dx = 1 \end{cases}$

$$1)\ f(x) \geq 0, \forall x \in \mathbb{R} \Leftrightarrow \left.\begin{matrix} A\cos x \geq 0, \forall x \in [0; \frac{\pi}{2}] \\ x \in [0; \frac{\pi}{2}] \Rightarrow \cos x \geq 0 \end{matrix}\right\} \Rightarrow A \geq 0$$

$$2)\ \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{\frac{\pi}{2}} A\cos x\,dx + \int_{\frac{\pi}{2}}^{\infty} 0\,dx =$$

$$\left.\begin{matrix} = \int_{0}^{\frac{\pi}{2}} A\cos x\,dx \\ \int_{-\infty}^{\infty} f(x)\,dx = 1 \end{matrix}\right\} \Rightarrow \underbrace{\int_{0}^{\frac{\pi}{2}} A\cos x\,dx}_{i} = 1 \Leftrightarrow i = 1$$

VI $u = A \Rightarrow u' = 0$

$v' = \cos x \Rightarrow v = \sin x$

$$i = A \sin x \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 0\,dx = A \cdot 1 - A \cdot 0 = A$$

$$\left.\begin{matrix} i = 1 \\ i = A \end{matrix}\right\} \Rightarrow A = 1 \geq 0$$

VII $$\left.\begin{matrix} i = A\int_0^{\frac{\pi}{2}} \cos x\,dx = A \cdot \sin x \Big|_0^{\frac{\pi}{2}} = A \cdot 1 - A \cdot 0 = A \\ i = 1 \end{matrix}\right\} \Rightarrow A = 1 \geq 0$$

$$f(x) = \begin{cases} \cos x, & x \in [0; \frac{\pi}{2}] \\ 0, & \text{în rest} \end{cases}$$

b) $\mathbb{P}\left(X<\frac{\pi}{3}\right)=\int_{-\infty}^{\frac{\pi}{3}} f(x)\,dx=\int_{-\infty}^{0} f(x)\,dx+\int_{0}^{\frac{\pi}{3}} f(x)\,dx=$

$=\int_{-\infty}^{0} 0\,dx+\int_{0}^{\frac{\pi}{3}} \cos x\,dx=0+\sin x\Big|_{0}^{\frac{\pi}{3}}=\frac{\sqrt{3}}{2}-0=\frac{\sqrt{3}}{2}$

$\mathbb{P}\left(X<\frac{\pi}{4} \,\middle|\, X>\frac{\pi}{6}\right)=\frac{\mathbb{P}\left(\left(X<\frac{\pi}{4}\right)\cap\left(X>\frac{\pi}{6}\right)\right)}{\mathbb{P}\left(X>\frac{\pi}{6}\right)}=\frac{\mathbb{P}\left(\frac{\pi}{6}<X<\frac{\pi}{4}\right)}{\mathbb{P}\left(X>\frac{\pi}{6}\right)}=\frac{\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} f(x)dx}{\int_{\frac{\pi}{6}}^{\infty} f(x)dx}=\frac{I_1}{I_2}$

$\left(\frac{\pi}{6};\frac{\pi}{4}\right)\subset\left[0;\frac{\pi}{2}\right]$

$I_1=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos x\,dx=\sin x\Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}}=\frac{\sqrt{2}}{2}-\frac{1}{2}=\frac{\sqrt{2}-1}{2}$

$I_2=\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos x\,dx+\int_{\frac{\pi}{2}}^{\infty} 0\,dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos x\,dx=\sin x\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

$=\sin\frac{\pi}{2}-\sin\frac{\pi}{6}=1-\frac{1}{2}=\frac{1}{2}$

$\mathbb{P}\left(X<\frac{\pi}{4} \,\middle|\, X>\frac{\pi}{6}\right)=\frac{\frac{\sqrt{2}-1}{2}}{\frac{1}{2}}=\frac{(\sqrt{2}-1)\cancel{2}}{\cancel{2}}=\sqrt{2}-1$

c) $\mathbb{E}(X)=\int_{-\infty}^{\infty} x\cdot f(x)\,dx=\int_{-\infty}^{0} x\cdot 0\,dx+\int_{0}^{\frac{\pi}{2}} x\cdot\cos x\,dx+$

$+\int_{\frac{\pi}{2}}^{\infty} x\cdot 0\,dx=\int_{0}^{\frac{\pi}{2}} x\cdot\cos x\,dx$

$u=x\Rightarrow u'=1$

$v'=\cos x\Rightarrow v=\sin x$

$\mathbb{E}(X)=x\sin x\Big|_{0}^{\frac{\pi}{2}}-\int_{0}^{\frac{\pi}{2}} \sin x\,dx=\frac{\pi}{2}-0+\cos x\Big|_{0}^{\frac{\pi}{2}}=\frac{\pi}{2}+(0-1)$

$=\frac{\pi}{2}-1=\frac{\pi-2}{2}$

$Var(X)=\mathbb{E}(X^2)-\mathbb{E}(X)^2$

$\mathbb{E}(X^2)=\int_{-\infty}^{\infty} x^2 f(x)\,dx=\int_{-\infty}^{0} x^2\cdot 0\,dx+\int_{0}^{\frac{\pi}{2}} x^2\cos x\,dx+$

$+\int_{\frac{\pi}{2}}^{\infty} x^2\cdot 0\,dx=\int_{0}^{\frac{\pi}{2}} x^2\cdot\cos x\,dx$

$u = x^2 \Rightarrow u' = 2x$

$v' = \cos x \Rightarrow v = \sin x$

$$\mathbb{E}(X^2) = x^2 \sin x \Big|_0^{\frac{\pi}{2}} - 2\underbrace{\int_0^{\frac{\pi}{2}} x \cdot \sin x \, dx}_{i_1} = \frac{\pi^2}{4} \cdot 1 - 2 i_1$$

$$i_1 = \int_0^{\frac{\pi}{2}} x \cdot \sin x \, dx$$

$u_1 = x \Rightarrow u_1' = 1$

$v_1' = \sin x \Rightarrow v_1 = -\cos x$

$$i_1 = -x \cos x \Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x \, dx = -\frac{\pi}{2} \cdot 0 + 0 \cdot 1 + \sin x \Big|_0^{\frac{\pi}{2}} =$$

$$= 1 - 0 = 1$$

$$\mathbb{E}(X^2) = \frac{\pi^2}{4} - 2 = \frac{\pi^2 - 8}{4}$$

$$Var(X) = \frac{\pi^2 - 8}{4} - \frac{(\pi-2)^2}{4} = \frac{\cancel{\pi^2} - 8 - \cancel{\pi^2} + 4\pi - 4}{4} = \frac{4\pi -}{4} = \pi - 3$$

$$\sigma = \sqrt{Var(X)} = \sqrt{\pi - 3}$$

d) $$f(t) = \begin{cases} 0, & t < 0 \\ \cos t, & t \in [0; \frac{\pi}{2}] \\ 0, & t > \frac{\pi}{2} \end{cases}$$

$$F(x) = \int_{-\infty}^{x} f(t) \, dt$$

$\underline{I}$ $x < 0 \Rightarrow F(x) = \int_{-\infty}^{x} 0 \, dt = 0$

$\underline{II}$ $x \in [0; \frac{\pi}{2}] \Rightarrow F(x) = \int_{-\infty}^{0} dt + \int_0^x \cos t \, dt = 0 + \sin t \Big|_0^x =$

$$= \sin x$$

$\underline{III}$ $x > \frac{\pi}{2} \Rightarrow F(x) = \int_{-\infty}^{0} 0 \, dt + \int_0^{\frac{\pi}{2}} \cos t \, dt + \int_{\frac{\pi}{2}}^{x} 0 \, dt =$

$$= \sin t \Big|_0^{\frac{\pi}{2}} = 1 - 0 = 1$$

$$F(x)=\begin{cases}0, & x<0\\ \sin x, & x\in[0;\frac{\pi}{2}]\\ 1, & x>\frac{\pi}{2}\end{cases}$$

e) $\left.\begin{array}{l} F(m_e)=\frac{1}{2} \Rightarrow m_e \notin (-\infty;0)\cup(\frac{\pi}{2};+\infty) \\ m_e\in[0;\frac{\pi}{2}] \Rightarrow \sin m_e=\frac{1}{2}\end{array}\right\} \Rightarrow m_e=\frac{\pi}{6}\in[0;\frac{\pi}{2}]$

x	$-\infty$	0	$\frac{\pi}{2}$	$+\infty$
$f'(x)$	$+ + + + + +$	| $- - - - - -$	$+ + + + + +$	
$f(x)$	$0 \longrightarrow$	| $\longrightarrow 0$	$\longrightarrow 0$	

$\left.\begin{array}{l} l_s(0)=1 \\ l_d(0)=0 \\ f(0)=1\end{array}\right\} \Rightarrow f$ nu este continuă în $0 \Rightarrow f$ nu este derivabilă în 0

$$f'(x)=\begin{cases}-\sin x, & x\in(0;\frac{\pi}{2})\\ 0, & \text{în rest}\end{cases}$$

$x\in(0;\frac{\pi}{2}) \Rightarrow \sin x>0, \forall x\in(0;\frac{\pi}{2}) \Rightarrow -\sin x<0, \forall x\in(0;\frac{\pi}{2})$

$m_o=0$ pentru ramura $\cos x$; în rest $m_o=\emptyset$

Pb2) $f(x)=\begin{cases}k(e^{-\frac{1}{2}x}+e^{\frac{1}{2}x}), & x\in[0;2]\\ 0, & \text{în rest}\end{cases}$

$k\in\mathbb{R}$

a) f densitate de probabilitate $\Leftrightarrow \begin{cases}1)\ f(x)\geq 0, \forall x\in\mathbb{R}\\ 2)\ \int_{-\infty}^{\infty} f(x)\,dx=1\end{cases}$

1) $f(x)\geq 0, \forall x\in\mathbb{R} \Leftrightarrow k(\underbrace{e^{-\frac{1}{2}x}+e^{\frac{1}{2}x}}_{>0 \text{ (sumă de funcții exponențiale)}})\geq 0, \forall x\in[0;2] \Rightarrow$

$\Rightarrow k\geq 0$

2) $\int_{-\infty}^{\infty} f(x)\,dx=1 \Leftrightarrow \int_{-\infty}^{0} 0\,dx+\int_{0}^{2} f(x)\,dx+\int_{2}^{\infty} 0\,dx=1$

$$\Leftrightarrow \int_0^2 k(e^{-\frac{1}{2}x}+e^{\frac{1}{2}x})dx = 1 \Leftrightarrow \underbrace{\int_0^2 ke^{-\frac{1}{2}x}dx}_{i_1} + \underbrace{\int_0^2 ke^{\frac{1}{2}x}}_{i_2}dx = 1$$

$$i_1 = \int_0^2 k\cdot e^{-\frac{1}{2}x}\,dx$$

$$-\frac{1}{2}x = t \Rightarrow x = -2t \Rightarrow dx = -2\,dt$$

$$x = 0 \Rightarrow t = 0$$

$$x = 2 \Rightarrow t = -1$$

$$i_1 = \int_0^{-1} k\cdot e^t\cdot(-2)\,dt = -2\int_0^{-1} ke^t\,dt = 2\int_{-1}^0 ke^t\,dt$$

V I $u = k \Rightarrow u' = 0$

$v' = e^t \Rightarrow v = e^t$

$$i_1 = 2\left(ke^t\Big/_{-1}^{0} - \int_{-1}^0 0\,dt\right) = 2(k - ke^{-1}) = 2k - \frac{2k}{e}$$

V II $i_1 = 2k\int_{-1}^0 e^t\,dt = 2k(e^0 - e^{-1}) = 2k - \frac{2k}{e}$

$$i_1 = 2k - \frac{2k}{e}$$

$$i_2 = \int_0^2 k\cdot e^{\frac{1}{2}x}\,dx$$

$$\frac{1}{2}x = t \Rightarrow x = 2t \Rightarrow dx = 2\,dt$$

$$x = 0 \Rightarrow t = 0$$

$$x = 2 \Rightarrow t = 1$$

$$i_2 = \int_0^1 k\cdot e^t\cdot 2\,dt = 2\int_0^1 ke^t\,dt$$

V I $u = k \Rightarrow u' = 0$

$v' = e^t \Rightarrow v = e^t$

$$i_2 = 2\left(ke^t\Big/_0^1 - \int_0^1 0\,dt\right) = 2(ke - k) = 2ke - 2k$$

V II $i_2 = 2k\int_0^1 e^t\,dt = 2k(e-1) = 2ke - 2k$

$$i_2 = 2ke - 2k$$

$i_1 + i_2 = 1 \Rightarrow 2K - \frac{2K}{e} + 2Ke - 2K = 1 \Rightarrow 2Ke - \frac{2K}{e} = 1 \Rightarrow 2K\left(e - \frac{1}{e}\right) = 1 \Rightarrow$

$\Rightarrow K = \frac{1}{2\left(e - \frac{1}{e}\right)} \Rightarrow K = \frac{e}{2e^2 - 2} \Rightarrow K = \frac{e}{2(e^2-1)} \Rightarrow K = \frac{e}{2(e-1)(e+1)} > 0$

$$f(x) = \begin{cases} \frac{e}{2(e^2-1)} \left(e^{-\frac{1}{2}x} + e^{\frac{1}{2}x}\right), & x \in [0;2] \\ 0, & \text{în rest} \end{cases}$$

b) $$f(t) = \begin{cases} 0, & t < 0 \\ \frac{e}{2(e^2-1)} \left(e^{-\frac{1}{2}t} + e^{\frac{1}{2}t}\right), & t \in [0;2] \\ 0, & t > 2 \end{cases}$$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

I $x < 0 \Rightarrow F(x) = \int_{-\infty}^{x} 0\,dt = 0$

II $x \in [0;2] \Rightarrow F(x) = \int_{-\infty}^{0} 0\,dt + \int_{0}^{x} \frac{e}{2(e^2-1)} \left(e^{-\frac{1}{2}t} + e^{\frac{1}{2}t}\right) dt$

$$= \frac{e}{2(e^2-1)} \Big(\underbrace{\int_0^x e^{-\frac{1}{2}t}\,dt}_{i_1} + \underbrace{\int_0^x e^{\frac{1}{2}t}\,dt}_{i_2} \Big) =$$

$$= \frac{e}{2(e^2-1)} (i_1 + i_2)$$

$$i_1 = \int_0^x e^{-\frac{1}{2}t}\,dt$$

$-\frac{1}{2}t = y \Rightarrow t = -2y \Rightarrow dt = -2\,dy$

$t = 0 \Rightarrow y = 0$

$t = x \Rightarrow y = -\frac{x}{2}$

$$i_1 = -2\int_0^{-\frac{x}{2}} e^y\,dy = 2\int_{-\frac{x}{2}}^{0} e^y\,dy = 2\left(1 - \frac{1}{e^{\frac{x}{2}}}\right) = 2 - \frac{2}{e^{\frac{x}{2}}}$$

$i_2 = \int_0^x e^{\frac{1}{2}t}\,dt$

$\frac{1}{2}t = y \Rightarrow t = 2y \Rightarrow dt = 2dy$

$t = 0 \Rightarrow y = 0$

$t = x \Rightarrow y = \frac{x}{2}$

$i_2 = 2\int_0^{\frac{x}{2}} e^y\,dy = 2e^{\frac{x}{2}} - 2$

$F(x) = \frac{e}{2(e^2-1)}\left(2 - \frac{2}{e^{\frac{x}{2}}} + 2e^{\frac{x}{2}} - 2\right) = \frac{e}{2(e^2-1)}\left(2e^{\frac{x}{2}} - \frac{2}{e^{\frac{x}{2}}}\right) =$

$= \frac{e}{e^2-1}\left(e^{\frac{x}{2}} - \frac{1}{e^{\frac{x}{2}}}\right) = \frac{e}{e^2-1}\left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)$

$\underline{III}$ $x > 2 \Rightarrow F(x) = \int_{-\infty}^0 0\,dt + \int_0^2 \frac{e}{2(e^2-1)}\left(e^{-\frac{1}{2}t} + e^{\frac{1}{2}t}\right)dt +$

$+ \int_2^x 0\,dt = \frac{e}{2(e^2-1)}\left(\underbrace{\int_0^2 e^{-\frac{1}{2}t}dt}_{i_1} + \underbrace{\int_0^2 e^{\frac{1}{2}t}dt}_{i_2}\right) =$

$= \frac{e}{2(e^2-1)}(i_1 + i_2)$

$i_1 = \int_0^2 e^{-\frac{1}{2}t}\,dt$

$-\frac{1}{2}t = y \Rightarrow t = -2y \Rightarrow dt = -2dy$

$t = 0 \Rightarrow y = 0$

$t = 2 \Rightarrow y = -1$

$i_1 = 2\int_{-1}^0 e^y\,dy = 2 - 2e^{-1} = 2 - \frac{2}{e}$

$i_2 = \int_0^2 e^{\frac{1}{2}t}\,dt$

$\frac{1}{2}t = y \Rightarrow t = 2y \Rightarrow dt = 2dy$

$t = 0 \Rightarrow y = 0$

$t = 2 \Rightarrow y = 1$

$i_2 = 2\int_0^1 e^y\,dy = 2e - 2$

$$F(x) = \frac{e}{2(e^2-1)}\left(x - \frac{2}{e} + 2e - x\right) = \frac{e}{2(e-1)(e+1)}\left(xe - \frac{x}{e}\right) =$$

$$= \frac{e}{(e-1)(e+1)}\left(\frac{e^2-1}{e}\right) = \frac{e^2-1}{e^2-1} = 1$$

$$F(x) = \begin{cases} 0, & x<0 \\ \frac{e}{e^2-1}\left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right), & x \in [0;2] \\ 1, & x>2 \end{cases}$$

$$\mathbb{P}\left(X < \tfrac{1}{2} \,\middle|\, X > \tfrac{1}{4}\right) = \frac{\mathbb{P}\left(\left(X<\frac{1}{2}\right) \cap \left(X > \frac{1}{4}\right)\right)}{\mathbb{P}\left(X > \frac{1}{4}\right)} = \frac{\mathbb{P}\left(\frac{1}{4} < X < \frac{1}{2}\right)}{\mathbb{P}\left(X > \frac{1}{4}\right)} = \frac{\int_{\frac{1}{4}}^{\frac{1}{2}} f(x)dx}{\int_{\frac{1}{4}}^{\infty} f(x)dx} =$$

$$= \frac{i_1}{i_2}$$

$$\left(\tfrac{1}{4}; \tfrac{1}{2}\right) \subset [0;2]$$

$$i_1 = \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{e}{2(e^2-1)}\left(e^{-\frac{1}{2}x} + e^{\frac{1}{2}x}\right) dx = \frac{e}{2(e^2-1)}\Big(\underbrace{\int_{\frac{1}{4}}^{\frac{1}{2}} e^{-\frac{1}{2}x} dx}_{i_3} +$$

$$+ \underbrace{\int_{\frac{1}{4}}^{\frac{1}{2}} e^{\frac{1}{2}x} dx}_{i_4}\Big) = \frac{e}{2(e^2-1)}(i_3 + i_4)$$

$$i_3 = \int_{\frac{1}{4}}^{\frac{1}{2}} e^{-\frac{1}{2}x} dx$$

$$-\frac{1}{2}x = t \Rightarrow x = -2t \Rightarrow dx = -2dt$$

$$x = \frac{1}{4} \Rightarrow t = -\frac{1}{8}$$

$$x = \frac{1}{2} \Rightarrow t = -\frac{1}{4}$$

$$i_3 = 2\int_{-\frac{1}{4}}^{-\frac{1}{8}} e^t dt = 2\left(e^{-\frac{1}{8}} - e^{-\frac{1}{4}}\right)$$

$$i_4 = \int_{\frac{1}{4}}^{\frac{1}{2}} e^{\frac{1}{2}x} dx$$

$$\frac{1}{2}x = t \Rightarrow x = 2t \Rightarrow dx = 2dt$$

$$x = \frac{1}{4} \Rightarrow t = \frac{1}{8}$$

$$x = \frac{1}{2} \Rightarrow t = \frac{1}{4}$$

$$i_4 = 2\int_{\frac{1}{8}}^{\frac{1}{4}} e^t dt = 2(e^{\frac{1}{4}} - e^{\frac{1}{8}})$$

$$i_1 = \frac{e}{2(e^2-1)} \cdot 2(e^{-\frac{1}{8}} - e^{-\frac{1}{4}} + e^{\frac{1}{4}} - e^{\frac{1}{8}}) = \frac{e}{e^2-1}(e^{-\frac{1}{8}} - e^{\frac{1}{8}} + e^{\frac{1}{4}} - e^{-\frac{1}{4}})$$

$$i_2 = \int_{\frac{1}{4}}^{\infty} f(x)\,dx = \int_{\frac{1}{4}}^{2} \frac{e}{2(e^2-1)}(e^{-\frac{1}{2}x} + e^{\frac{1}{2}x})\,dx + \int_2^{\infty} 0\,dx =$$

$$= \frac{e}{2(e^2-1)} \int_{\frac{1}{4}}^{2} (e^{-\frac{1}{2}x} + e^{\frac{1}{2}x})\,dx = \frac{e}{2(e^2-1)}\Big(\underbrace{\int_{\frac{1}{4}}^{2} e^{-\frac{1}{2}x}\,dx}_{i_5} + \underbrace{\int_{\frac{1}{4}}^{2} e^{\frac{1}{2}x}\,dx}_{i_6}\Big)$$

$$= \frac{e}{2(e^2-1)}(i_5 + i_6)$$

$$i_5 = \int_{\frac{1}{4}}^{2} e^{-\frac{1}{2}x}\,dx$$

$$-\frac{1}{2}x = t \Rightarrow x = -2t \Rightarrow dx = -2dt$$

$$x = \frac{1}{4} \Rightarrow t = -\frac{1}{8}$$

$$x = 2 \Rightarrow t = -1$$

$$i_5 = \int_{-\frac{1}{8}}^{-1} e^t \cdot (-2)\,dt = 2\int_{-1}^{-\frac{1}{8}} e^t\,dt = 2(e^{-\frac{1}{8}} - e^{-1})$$

$$i_6 = \int_{\frac{1}{4}}^{2} e^{\frac{1}{2}x}\,dx$$

$$\frac{1}{2}x = t \Rightarrow x = 2t \Rightarrow dx = 2dt$$

$$x = \frac{1}{4} \Rightarrow t = \frac{1}{8}$$

$$x = 2 \Rightarrow t = 1$$

$$i_6 = 2\int_{\frac{1}{8}}^{1} e^t dt = 2(e - e^{\frac{1}{8}})$$

$$i_2 = \frac{e}{2(e^2-1)} \cdot 2\,(e^{-\frac{1}{8}} - e^{-1} + e - e^{\frac{1}{8}}) = \frac{e}{e^2-1}(e^{-\frac{1}{8}} - e^{\frac{1}{8}} + e - e^{-1})$$

$$\mathbb{P}\left(X < \frac{1}{2} \,\Big|\, X > \frac{1}{4}\right) = \frac{i_1}{i_2} = \frac{\frac{e}{e^2-1}(e^{-\frac{1}{8}} - e^{\frac{1}{8}} + e^{\frac{1}{4}} - e^{-\frac{1}{4}})}{\frac{e}{e^2-1}(e^{-\frac{1}{8}} - e^{\frac{1}{8}} + e - e^{-1})} =$$

$$= \frac{e^{-\frac{1}{8}} - e^{\frac{1}{8}} + e^{\frac{1}{4}} - e^{-\frac{1}{4}}}{e^{-\frac{1}{8}} - e^{\frac{1}{8}} + e - e^{-1}} \simeq \frac{0,255}{2,1} \simeq 0,121$$

c) $\mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx = \int_{-\infty}^{0} x \cdot 0\, dx + \int_0^2 x \cdot \frac{e}{2(e^2-1)} \left(e^{-\frac{1}{2}x} + e^{\frac{1}{2}x}\right)$

$dx + \int_2^{\infty} x \cdot 0\, dx = \frac{e}{2(e^2-1)} \left(\int_0^2 x \cdot e^{-\frac{1}{2}x} dx + \int_0^2 x e^{\frac{1}{2}x} dx\right)$

$= \frac{e}{2(e^2-1)} (i_1 + i_2)$

$i_1 = \int_0^2 x \cdot e^{-\frac{1}{2}x}\, dx$

$-\frac{1}{2}x = t \Rightarrow x = -2t \Rightarrow dx = -2\, dt$

$x = 0 \Rightarrow t = 0$

$x = 2 \Rightarrow t = -1$

$i_1 = -2 \int_0^{-1} -2t \cdot e^t\, dt = 4 \int_0^{-1} t \cdot e^t dt$

$u = t \Rightarrow u' = 1$

$v' = e^t \Rightarrow v = e^t$

$i_1 = 4\left(t e^t \Big|_0^{-1} - \int_0^{-1} e^t dt\right) = 4\left(-e^{-1} - e^t \Big|_0^{-1}\right) =$

$= 4(-e^{-1} - e^{-1} + 1) = -8e^{-1} + 4$

$i_2 = \int_0^2 x \cdot e^{\frac{1}{2}x}\, dx$

$\frac{1}{2}x = t \Rightarrow x = 2t \Rightarrow dx = 2dt$

$x = 0 \Rightarrow t = 0$

$x = 2 \Rightarrow t = 1$

$i_2 = 2\int_0^1 2t \cdot e^t\, dt = 4\int_0^1 t \cdot e^t dt$

$u = t \Rightarrow u' = 1$

$v' = e^t \Rightarrow v = e^t$

$$i_2 = 4\left(te^t \Big|_0^1 - \int_0^1 e^t\,dt\right) = 4(e - e + 1) = 4$$

$$\mathbb{E}(X) = \frac{e}{2(e^2-1)}(-8e^{-1}+4+4) = \frac{e}{(e-1)(e+1)}4(-e^{-1}+1) = \frac{4\cancel{(e-1)}}{\cancel{(e-1)}(e+1)} =$$

$$= \frac{4}{e+1}$$

$$Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_{-\infty}^{0} x^2 \cdot 0\,dx + \int_0^2 x^2 \frac{e}{2(e^2-1)}(e^{-\frac{1}{2}x}+e^{\frac{1}{2}x})$$

$$dx + \int_2^{\infty} x^2 \cdot 0\,dx = \frac{e}{2(e^2-1)}\int_0^2 x^2(e^{-\frac{1}{2}x}+e^{\frac{1}{2}x})\,dx =$$

$$= \frac{e}{2(e^2-1)}\left(\int_0^2 x^2 \cdot e^{-\frac{1}{2}x}\,dx + \int_0^2 x^2 \cdot e^{\frac{1}{2}x}\,dx\right) = \frac{e}{2(e^2-1)}(i_3+i_4)$$

$$i_3 = \int_0^2 x^2 \cdot e^{-\frac{1}{2}x}\,dx$$

$$-\frac{1}{2}x = t \Rightarrow x = -2t \Rightarrow dx = -2\,dt$$

$$x = 0 \Rightarrow t = 0$$

$$x = 2 \Rightarrow t = -1$$

$$i_3 = -2\int_0^{-1} 4t^2 \cdot e^t\,dt = 8\int_{-1}^{0} t^2 \cdot e^t\,dt$$

$$u = t^2 \Rightarrow u' = 2t$$

$$v' = e^t \Rightarrow v = e^t$$

$$i_3 = 8\left(t^2e^t\Big|_{-1}^0 - 2\underbrace{\int_{-1}^0 te^t\,dt}_{i_5}\right)$$

$$i_5 = \int_{-1}^0 te^t\,dt$$

$$u_1 = t \Rightarrow u_1' = 1$$

$$v_1' = e^t \Rightarrow v_1 = e^t$$

$$i_5 = te^t\Big|_{-1}^0 - \int_{-1}^0 e^t\,dt = +e^{-1} - 1 + e^{-1} = -1 + 2e^{-1}$$

$$i_3 = 8(-e^{-1} + 2 - 4e^{-1}) = 16 - 40e^{-1}$$

$$i_4 = \int_0^2 x^2 \cdot e^{\frac{1}{2}x}\, dx$$

$$\frac{1}{2}x = t \Rightarrow x = 2t \Rightarrow dx = 2dt$$

$$x = 0 \Rightarrow t = 0$$

$$x = 2 \Rightarrow t = 1$$

$$i_4 = 2\int_0^1 4t^2 \cdot e^t\, dt = 8\int_0^1 t^2 e^t\, dt$$

$$u = t^2 \Rightarrow u' = 2t$$

$$v' = e^t \Rightarrow v = e^t$$

$$i_4 = 8\Big(t^2 e^t\Big|_0^1 - 2\underbrace{\int_0^1 t \cdot e^t\, dt}_{i_6}\Big)$$

$$i_6 = \int_0^1 t e^t\, dt$$

$$u_1 = t \Rightarrow u_1' = 1$$

$$v_1' = e^t \Rightarrow v_1 = e^t$$

$$i_6 = t e^t\Big|_0^1 - \int_0^1 e^t\, dt = e - e + 1 = 1$$

$$i_4 = 8(e - 2) = 8e - 16$$

$$\mathbb{E}(x^2) = \frac{e}{2(e^2-1)}(\cancel{16} - 40e^{-1} + 8e - \cancel{16}) = \frac{e}{2(e^2-1)}(8e - 40e^{-1}) = \frac{4e}{e^2-1}(e - 5e^{-1}) =$$

$$= \frac{4e}{e^2-1}\left(e - \frac{5}{e}\right) = \frac{4\cancel{e}}{e^2-1}\left(\frac{e^2-5}{\cancel{e}}\right) = \frac{4e^2-20}{e^2-1}$$

$$\mathrm{Var}(x) = \frac{4e^2-20}{e^2-1} - \frac{16}{(e+1)^2} = \frac{(4e^2-20)(e+1) - 16(e-1)}{(e+1)^2(e-1)} =$$

$$= \frac{4e^3 + 4e^2 - 20e - 20 - 16e + 16}{(e+1)^2(e-1)} = \frac{4e^3 + 4e^2 - 36e - 4}{(e+1)^2(e-1)} =$$

$$= \frac{4(e^3 + e^2 - 9e - 1)}{(e^2+2e+1)(e-1)} = \frac{4(e^3 + e^2 - 9e - 1)}{e^3 - e^2 + 2e^2 - 2e + e - 1} = \frac{4(e^3 + e^2 - 9e - 1)}{e^3 + e^2 - e - 1} \approx$$

$\simeq \frac{8.04}{23.76} \simeq 0,34$

Pb 3) $f:\mathbb{R}\to\mathbb{R}$, $f(x) = \begin{cases} k\cdot x^{a-1}\cdot(1-x)^{b-1}, & x\in(0;1) \\ 0, & \text{în rest} \end{cases}$

$a,b>0$; $k\in\mathbb{R}$

a) f densitate de probabilitate $\Leftrightarrow \begin{cases} 1)\ f(x)\geq 0,\ \forall x\in\mathbb{R} \\ 2)\ \int_{-\infty}^{\infty} f(x)\,dx = 1 \end{cases}$

1) $f(x)\geq 0,\ \forall x\in\mathbb{R} \Leftrightarrow k\cdot x^{a-1}\cdot(1-x)^{b-1}\geq 0,\ \forall x\in(0;1)$

$a>0$ ①

$b>0$ ②

$x\in(0;1) \Rightarrow 0<x<1 \Rightarrow -1<-x<0 \Rightarrow 0<1-x<1 \Rightarrow$

$\Rightarrow (1-x)\in(0;1)$ ③

Din ①, ②, ③ $\Rightarrow \begin{rcases} x^{a-1}>0,\ \forall x\in(0;1),\ a>0,\ b>0 \\ (1-x)^{b-1}>0,\ \forall x\in(0;1),\ a>0,\ b>0 \end{rcases} \Rightarrow$

$\Rightarrow k\geq 0$

2) $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Leftrightarrow \int_{-\infty}^{0} 0\,dx + \int_0^1 k\,x^{a-1}\cdot(1-x)^{b-1}\,dx +$

$+ \int_1^{\infty} 0\,dx = 1 \Leftrightarrow \int_0^1 k\cdot x^{a-1}\cdot(1-x)^{b-1}\,dx = 1 \Leftrightarrow$ $\forall a,b>0$

$\Leftrightarrow k\int_0^1 x^{a-1}\cdot(1-x)^{b-1}\,dx = 1 \Leftrightarrow k\cdot\beta(a,b) = 1 \Rightarrow k = \frac{1}{\frac{\Gamma(a)\cdot\Gamma(b)}{\Gamma(a+b)}}$,

$\forall a,b>0 \Rightarrow k = \frac{\Gamma(a+b)}{\Gamma(a)\cdot\Gamma(b)} > 0,\ \forall a,b>0$

$f(x) = \begin{cases} \frac{\Gamma(a+b)}{\Gamma(a)\cdot\Gamma(b)}\cdot x^{a-1}\cdot(1-x)^{b-1}, & x\in(0;1) \\ 0, & \text{în rest} \end{cases}$

b) $\mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx = \int_{-\infty}^{0} x \cdot 0\, dx + \int_0^1 x \cdot \frac{\Gamma(a+b)}{\Gamma(a)\cdot\Gamma(b)} \cdot x^{a-1} \cdot$
$\cdot (1-x)^{b-1}\, dx + \int_1^{\infty} x \cdot 0\, dx = \frac{\Gamma(a+b)}{\Gamma(a)\cdot\Gamma(b)} \int_0^1 x^a \cdot (1-x)^{b-1} dx$

$= \frac{\Gamma(a+b)}{\Gamma(a)\cdot\Gamma(b)} \cdot \beta(a+1,b) = \frac{\Gamma(a+b)}{\Gamma(a)\cdot\cancel{\Gamma(b)}} \cdot \frac{\Gamma(a+1)\cdot\cancel{\Gamma(b)}}{\Gamma(a+b+1)} =$

$\left.\begin{matrix} a>0 \\ b>0 \end{matrix}\right\} \Rightarrow \begin{cases} a+1>1 \\ a+b>0 \Rightarrow a+b+1>1 \end{cases}$

$= \frac{\cancel{\Gamma(a+b)}}{\cancel{\Gamma(a)}} \cdot \frac{\cancel{\Gamma(a)}\cdot a}{\cancel{\Gamma(a+b)}\cdot(a+b)} = \frac{a}{a+b}$

$Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$

$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_{-\infty}^{0} x^2 \cdot 0\, dx + \int_0^1 x^2 \frac{\Gamma(a+b)}{\Gamma(a)\cdot\Gamma(b)} \cdot$
$\cdot x^{a-1} \cdot (1-x)^{b-1} dx + \int_1^{\infty} x^2 \cdot 0\, dx = \frac{\Gamma(a+b)}{\Gamma(a)\cdot\Gamma(b)} \cdot \int_0^1 x^{a+1} \cdot$
$\cdot (1-x)^{b-1} dx = \frac{\Gamma(a+b)}{\Gamma(a)\cdot\Gamma(b)} \cdot \beta(a+2,b) = \frac{\Gamma(a+b)}{\Gamma(a)\cdot\cancel{\Gamma(b)}} \cdot \frac{\Gamma(a+2)\cdot\cancel{\Gamma(b)}}{\Gamma(a+b+2)}$

$\left.\begin{matrix} a>0 \\ b>0 \end{matrix}\right\} \Rightarrow \begin{cases} a+2>2 \\ a+b>0 \Rightarrow a+b+2>2 \end{cases}$

$= \frac{\Gamma(a+b)}{\Gamma(a)} \cdot \frac{\Gamma(a+1)\cdot(a+1)}{(a+b+1)\Gamma(a+b+1)} = \frac{\cancel{\Gamma(a+b)}}{\cancel{\Gamma(a)}} \cdot \frac{a\,\cancel{\Gamma(a)}}{(a+b)\cdot\cancel{\Gamma(a+b)}} \cdot \frac{a+1}{a+b+1}$

$= \frac{a(a+1)}{(a+b)(a+b+1)}$

$Var(X) = \frac{a(a+1)}{(a+b)(a+b+1)} - \frac{a^2}{(a+b)^2} = \frac{(a^2+a)(a+b) - a^2(a+b+1)}{(a+b)^2(a+b+1)} =$

$= \frac{\cancel{a^3} + \cancel{a^2b} + \cancel{a^2} + ab - \cancel{a^3} - \cancel{a^2b} - \cancel{a^2}}{(a+b)^2(a+b+1)} = \frac{ab}{(a+b)^2(a+b+1)}$

$$f'(x) = \begin{cases} \left(\frac{\Gamma(a+b)}{\Gamma(a)\cdot\Gamma(b)} \cdot x^{a-1}\cdot(1-x)^{b-1}\right)', x\in(0;1) \\ 0, \text{ în rest} \end{cases}$$

$$\left(\frac{\Gamma(a+b)}{\Gamma(a)\cdot\Gamma(b)} \cdot x^{a-1}\cdot(1-x)^{b-1}\right)' = \frac{\Gamma(a+b)}{\Gamma(a)\cdot\Gamma(b)}\left(x^{a-1}\cdot(1-x)^{b-1}\right)'$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\cdot\Gamma(b)}\left\{(a-1)\cdot x^{a-2}\cdot(1-x)^{b-1} + x^{a-1}\cdot[-(b-1)\cdot(1-x)^{b-2}]\right\}$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\cdot\Gamma(b)}\left[(a-1)\cdot x^{a-2}\cdot(1-x)^{b-1} - (b-1)(1-x)^{b-2}x^{a-1}\right] =$$

$$= \underbrace{\frac{\Gamma(a+b)}{\Gamma(a)\cdot\Gamma(b)} x^{a-2}\cdot(1-x)^{b-2}}_{\geq 0, \forall x\in(0;1); a,b>0}\left[(a-1)(1-x) - (b-1)x\right]$$

$x\in(0;1)$
$a>0$
$b>0$

$\underline{\text{I}}$ $a\in(0;1)$; $b\in(0;1) \Rightarrow \begin{cases} a-1<0 \\ b-1<0 \\ 1-x>0 \\ x>0 \end{cases} \Rightarrow \begin{cases} (a-1)(1-x)<0 \\ (b-1)\, x<0 \Rightarrow -(b-1)x>0 \end{cases}$

$(a-1)(1-x)-(b-1)\cdot x = a-ax-1+x-bx+x = 2x+a-1-x(b+a)$

$0<a<1 \Rightarrow -1<a-1<0$

$0<x<1 \Rightarrow 0<2x<2$

$\left.\begin{matrix} 0<a<1 \\ 0<b<1 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} 0<b+a<2 \\ 0<x<1 \end{matrix}\right\} \Rightarrow 0<x(b+a)<2 \Rightarrow -2<-x(b+a)<0$

$(a-1)(1-x)-(b-1)x<0$

x	$-\infty$		0		1		$+\infty$
$f'(x)$		$+++++$		$-----$		$+++++$	
$f(x)$	0	$\nearrow$	0	$\searrow$	0	$\nearrow$	0

$m_o = 0$

II $a \in (0;1)$, $b \in (1;\infty) \Rightarrow \left.\begin{cases} a-1<0 \\ b-1>0 \\ 1-x>0 \\ x>0 \end{cases}\right\} \Rightarrow \left.\begin{cases} (a-1)(1-x)<0 \\ (b-1)x>0 \Rightarrow -(b-1)\cdot x<0 \end{cases}\right\} \Rightarrow$

$\Rightarrow (a-1)(1-x)-(b-1)x<0$

x	$-\infty$		0		1		$+\infty$
$f'(x)$		$+++++$		$----$		$+++++$	
$f(x)$	0	$\nearrow$	0	$\searrow$	0	$\nearrow$	0

$m_\theta = 0$

III $a \in (1;\infty)$; $b \in (0;1) \Rightarrow \left.\begin{cases} a-1>0 \\ b-1<0 \\ 1-x>0 \\ x>0 \end{cases}\right\} \Rightarrow \left.\begin{cases} (a-1)(1-x)>0 \\ (b-1)x<0 \Rightarrow -(b-1)x>0 \end{cases}\right\} \Rightarrow$

$\Rightarrow (a-1)(1-x)-(b-1)x>0$

x	$-\infty$		0		1		$+\infty$
$f'(x)$		$------$		$+++++$		$------$	
$f(x)$	0	$\searrow$	0	$\nearrow$	0	$\searrow$	0

$m_\theta = 1$

IV $a \in (1;\infty)$; $b \in (1;\infty) \Rightarrow \left.\begin{cases} a-1>0 \\ b-1>0 \\ 1-x>0 \\ x>0 \end{cases}\right\} \Rightarrow \begin{matrix} (a-1)(1-x)>0 \\ (b-1)x>0 \Rightarrow -(b-1)x<0 \end{matrix}$

$(a-1)(1-x)-(b-1)x = 2x+(a-1)-x(b+a)$

$1<a<\infty \Rightarrow 0<a-1<\infty$

$0<x<1 \Rightarrow 0<2x<2$

$\left.\begin{matrix} 1<a<\infty \\ 1<b<\infty \end{matrix}\right\} \Rightarrow 2<a+b<\infty \Rightarrow 0<x(a+b)<\infty \Rightarrow -\infty<-x(a+b)<0$

Dacă $a<b \Rightarrow (a-1)(1-x)-(b-1)x<0$

x	$-\infty$		0		1		$+\infty$
$f'(x)$		$++++$		$----$		$++++$	
$f(x)$		$\nearrow$	0	$\searrow$	0	$\nearrow$	

$m_\theta = 0$

Dacă $a > b \Rightarrow (a-1)(1-x) - (b-1)x > 0$

x	$-\infty$ 0 1 $+\infty$
$f'(x)$	- - - - - - ++++ - - - - -
$f(x)$	$\searrow$ 0 $\nearrow$ 0 $\searrow$

$m_0 = 1$

Dacă $a = b \Rightarrow (a-1)(1-x) - (b-1)x = 0 \Rightarrow f'(x) = 0, \forall x \in (0;1)$

$f'(x) = 0, \forall x \in (-\infty; 0] \cup [1; +\infty)$

$\Rightarrow f'(x) = 0, \forall x \in \mathbb{R} \Rightarrow f(x) = c, c \in \mathbb{R}$

x	$-\infty$ 0 1 ∞
$f'(x)$	- - - - 0++ 0+++
$f(x)$	0 $\searrow$ 0 $\nearrow$ 0 $\nearrow$ 0

Pe $(-\infty;0)$ am considerat 0_- ca în cazul limitelor.
Pe $(0;\infty)$ am considerat 0_+ ca în cazul limitelor.

Nu se poate determina m_0 deoarece funcția crește la infinit și nu are un punct de maxim local.

$\underline{\text{V}}$ $a = 1; b = 1 \Rightarrow \begin{cases} a-1=0 \\ b-1=0 \\ 1-x>0 \\ x>0 \end{cases} \Rightarrow (a-1)(1-x) - (b-1)x = 0 \Rightarrow$

$\left.\begin{array}{l} \Rightarrow f'(x) = 0, \forall x \in (0;1) \\ f'(x) = 0, \forall x \in (-\infty; 0] \cup [1; +\infty) \end{array}\right\} \Rightarrow f'(x) = 0, \forall x \in \mathbb{R} \Rightarrow f(x) = c,$

$c \in \mathbb{R}$

x	$-\infty$ 0 1 $+\infty$
$f'(x)$	- - - 0 ++ 0 +++++
$f(x)$	0 $\searrow$ 0 $\nearrow$ 0 $\nearrow$ 0

Pe $(-\infty;0)$ am considerat 0_- ca în cazul limitelor.
Pe $(0;\infty)$ am considerat 0_+ ca în cazul limitelor.

Nu se poate determina m_0 deoarece funcția crește la infinit și nu are un punct de maxim local.

$\underline{\text{VI}}$ $a = 1; b \in (0;1) \Rightarrow \begin{cases} a-1=0 \\ b-1<0 \\ 1-x>0 \\ x>0 \end{cases} \Rightarrow -(b-1)x > 0$

x	$-\infty$		0		1		$+\infty$
$f'(x)$		$- - - -$		$+ + + +$		$- - - -$	
$f(x)$	0	$\searrow$	0	$\nearrow$	0	$\searrow$	0

$m_\theta = 1$

VII $a = 1;\ b \in (1;\infty) \Rightarrow \left\{\begin{array}{l} a-1=0 \\ b-1>0 \\ 1-x>0 \\ x>0 \end{array}\right\} \Rightarrow -(b-1)x < 0$

x	$-\infty$		0		1		$+\infty$
$f'(x)$		$+ + + +$		$- - - -$		$+ + + +$	
$f(x)$		$\nearrow$	0	$\searrow$	0	$\nearrow$	

$m_\theta = 0$

VIII $a \in (0;1);\ b = 1 \Rightarrow \left\{\begin{array}{l} a-1<0 \\ b-1=0 \\ 1-x>0 \\ x>0 \end{array}\right\} \Rightarrow (a-1)(1-x) < 0$

x	$-\infty$		0		1		$+\infty$
$f'(x)$		$+ + + +$		$- - -$		$+ + +$	
$f(x)$	0	$\nearrow$	0	$\searrow$	0	$\nearrow$	0

$m_\theta = 0$

IX $a \in (1;\infty);\ b = 1 \Rightarrow \left\{\begin{array}{l} a-1>0 \\ b-1=0 \\ 1-x>0 \\ x>0 \end{array}\right\} \Rightarrow (a-1)(1-x) > 0$

x	$-\infty$		0		1		$+\infty$
$f'(x)$		$- - - -$		$+ + +$		$- - - -$	
$f(x)$	0	$\searrow$	0	$\nearrow$	0	$\searrow$	0

$m_\theta = 1$

$$m_r = \int_{-\infty}^{\infty} x^r \cdot f(x)\,dx = \int_{-\infty}^{0} x^r \cdot 0\,dx + \int_0^1 x^r \cdot \frac{\Gamma(a+b)}{\Gamma(a)\cdot\Gamma(b)} \cdot$$

$$\cdot x^{a-1} \cdot (1-x)^{b-1}\,dx + \int_1^{\infty} x^r \cdot 0\,dx = \frac{\Gamma(a+b)}{\Gamma(a)\cdot\Gamma(b)} \int_0^1 x^{a+r-1} \cdot (1-x)^{b-1}\,dx$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\cdot\Gamma(b)} \cdot \beta(a+r, b) = \frac{\Gamma(a+b)}{\Gamma(a)\cdot\cancel{\Gamma(b)}} \cdot \frac{\Gamma(a+r)\cdot\cancel{\Gamma(b)}}{\Gamma(a+b+r)} = \frac{\Gamma(a+r)\,\Gamma(a+b)}{\Gamma(a)\,\Gamma(a+b+r)},$$

$$\forall r \in \mathbb{N}^* \Leftrightarrow m_r = \frac{(a+r-1)(a+r-2)\cdot\ldots\cdot(a+r-r)\,\cancel{\Gamma(a)}\,\cancel{\Gamma(a+b)}}{\cancel{\Gamma(a)}\,(a+b+r-1)(a+b+r-2)(a+b+r-3)\ldots(a+b+r-r)}$$

$$\cdot \frac{1}{\cancel{\Gamma(a+b)}} = \frac{(a+r-1)(a+r-2)\cdot\ldots\cdot(a+r-r)}{(a+b+r-1)(a+b+r-2)(a+b+r-3)\ldots(a+b+r-r)}, \forall r \in \mathbb{N}^*$$

c) $a=2$; $b=3$

$$f(x) = \begin{cases} \frac{\Gamma(5)}{\Gamma(2)\cdot\Gamma(3)} \cdot x \cdot (1-x)^2, & x \in (0;1) \\ 0, & \text{în rest} \end{cases}$$

$$\frac{\Gamma(5)}{\Gamma(2)\cdot\Gamma(3)} = \frac{4\,\Gamma(4)}{1\,\Gamma(1)\,2\,\Gamma(2)} = \frac{4\cdot 3\cdot 2\cdot 1\,\Gamma(1)}{1\cdot 2\cdot 1\,\Gamma(1)} = 12$$

$$f(x) = \begin{cases} 12\,x(1-x)^2, & x \in (0;1) \\ 0, & \text{în rest} \end{cases}$$

$$f(t) = \begin{cases} 0, & t \le 0 \\ 12t(1-t)^2, & t \in (0;1) \\ 0, & t \ge 1 \end{cases}$$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

I $x < 0 \Rightarrow F(x) = \int_{-\infty}^{x} 0\,dt = 0$

II $x \in (0;1) \Rightarrow F(x) = \int_{-\infty}^{0} 0\,dt + \int_0^x 12t(1-t)^2\,dt = \int_0^x 12t(1-t)^2\,dt$

$$= 12\int_0^x t(t^2-2t+1)\,dt = 12\int_0^x (t^3-2t^2+t)\,dt = 12\Big(\int_0^x t^3\,dt$$

$$- 2\int_0^x t^2\,dt + \int_0^x t\,dt\Big) = 12\left(\frac{t^4}{4}\Big|_0^x - 2\,\frac{t^3}{3}\Big|_0^x + \frac{t^2}{2}\Big|_0^x\right) =$$

$$= 3t^4 \Big|_0^x - 8t^3 \Big|_0^x + 6t^2 \Big|_0^x = 3x^4 - 8x^3 + 6x^2 = x^2(3x^2 - 8x + 6)$$

III $x \geq 1 \Rightarrow F(x) = \int_{-\infty}^{0} 0\,dt + \int_0^1 12t(1-t)^2\,dt + \int_1^x 0\,dt =$

$$= 12\int_0^1 t(t^2 - 2t + 1)\,dt = 12\int_0^1 (t^3 - 2t^2 + t)\,dt = 12\int_0^1 t^3\,dt - 24\int_0^1 t^2\,dt$$

$$+ 12\int_0^1 t\,dt = 12\frac{t^4}{4}\Big|_0^1 - 24\frac{t^3}{3}\Big|_0^1 + 12\frac{t^2}{2}\Big|_0^1 = 3(1-0) - 8(1-0) + 6(1-0)$$

$$= 9 - 8 = 1$$

$$F(x) = \begin{cases} 0, & x \leq 0 \\ x^2(3x^2 - 8x + 6), & x \in (0;1) \\ 1, & x \geq 1 \end{cases}$$

$$\mathbb{P}(X < \tfrac{1}{2}) = \int_{-\infty}^{\frac{1}{2}} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_0^{\frac{1}{2}} 12x(1-x)^2\,dx =$$

$$= 12\int_0^{\frac{1}{2}} (x^3 - 2x^2 + x)\,dx = 12\frac{x^4}{4}\Big|_0^{\frac{1}{2}} - 24\frac{x^3}{3}\Big|_0^{\frac{1}{2}} + 12\frac{x^2}{2}\Big|_0^{\frac{1}{2}} =$$

$$= 3\left(\frac{1}{16} - 0\right) - 8\left(\frac{1}{8} - 0\right) + 6\left(\frac{1}{4} - 0\right) = \frac{3}{16} - \frac{8}{8} + \frac{6}{4} =$$

$$= \frac{3 - 16 + 24}{16} = \frac{11}{16} \approx 0{,}6875$$

$$\mathbb{P}(X > \tfrac{1}{3}) = \int_{\frac{1}{3}}^{\infty} f(x)\,dx = \int_{\frac{1}{3}}^{1} 12x(1-x)^2\,dx + \int_1^{\infty} 0\,dx =$$

$$= 12\int_{\frac{1}{3}}^{1} (x^3 - 2x^2 + x)\,dx = 12\frac{x^4}{4}\Big|_{\frac{1}{3}}^{1} - 24\frac{x^3}{3}\Big|_{\frac{1}{3}}^{1} + 12\frac{x^2}{2}\Big|_{\frac{1}{3}}^{1}$$

$$= 3\left(1 - \frac{1}{81}\right) - 8\left(1 - \frac{1}{27}\right) + 6\left(1 - \frac{1}{9}\right) = \frac{80}{27} - 8\cdot\frac{26}{27} +$$

$$+ 6\cdot\frac{8}{9} = \frac{80 - 208 + 144}{27} = \frac{16}{27} \approx 0{,}5926$$

$$\mathbb{P}(X \leq \tfrac{1}{2} / X > \tfrac{1}{4}) = \frac{\mathbb{P}((X \leq \frac{1}{2}) \cap (X > \frac{1}{4}))}{\mathbb{P}(X > \frac{1}{4})} = \frac{\mathbb{P}(\frac{1}{4} < X \leq \frac{1}{2})}{\mathbb{P}(X > \frac{1}{4})} = \frac{\int_{\frac{1}{4}}^{\frac{1}{2}} f(x)\,dx}{\int_{\frac{1}{4}}^{\infty} f(x)\,dx}$$

$= \frac{i_1}{i_2}$ $\quad [\frac{1}{4}; \frac{1}{2}] \subset [0;1)$

$i_1 = \int_{\frac{1}{4}}^{\frac{1}{2}} 12x(1-x)^2 \, dx = 12\int_{\frac{1}{4}}^{\frac{1}{2}} (x^3 - 2x^2 + x)\, dx = 12 \cdot \frac{x^4}{4}\Big|_{\frac{1}{4}}^{\frac{1}{2}} -$

$- 24 \frac{x^3}{3}\Big|_{\frac{1}{4}}^{\frac{1}{2}} + 12 \frac{x^2}{2}\Big|_{\frac{1}{4}}^{\frac{1}{2}} = 3\left(\frac{1}{16} - \frac{1}{256}\right) - 8\left(\frac{1}{8} - \frac{1}{64}\right) + 6\left(\frac{1}{4} - \frac{1}{16}\right) =$

$= \frac{45}{256} - \frac{56}{64} + \frac{18}{16} = \frac{45 - 224 + 288}{256} = \frac{109}{256}$

$i_2 = \int_{\frac{1}{4}}^{1} 12x(1-x^2)\, dx + \int_{1}^{\infty} 0 \, dx = 12\int_{\frac{1}{4}}^{1} (x^3 - 2x^2 + x)\, dx =$

$= 3x^4\Big|_{\frac{1}{4}}^{1} - 8x^3\Big|_{\frac{1}{4}}^{1} + 6x^2\Big|_{\frac{1}{4}}^{1} = 3\left(1 - \frac{1}{256}\right) - 8\left(1 - \frac{1}{64}\right) + 6\left(1 - \frac{1}{16}\right)$

$= 3 \cdot \frac{255}{256} - 8 \cdot \frac{63}{64} + 6 \cdot \frac{15}{16} = \frac{3 \cdot 255 - 8 \cdot 63 \cdot 4 + 6 \cdot 15 \cdot 16}{256} = \frac{189}{256}$

$\mathbb{P}\left(X \le \frac{1}{2} / X > \frac{1}{4}\right) = \frac{109}{\cancel{256}} \cdot \frac{\cancel{256}}{189} = \frac{109}{189} \simeq 0,577$

Pb 4) $f: \mathbb{R} \to \mathbb{R}, \; f(x) = \begin{cases} k \cdot x \cdot e^{-\frac{x^2}{2a^2}}, & x \ge 0, \\ 0, & x < 0 \end{cases}$

$k \in \mathbb{R}, a > 0$

a) f densitate de probabilitate $\Leftrightarrow \begin{cases} 1)\; f(x) \ge 0, \forall x \in \mathbb{R} \\ 2)\; \int_{-\infty}^{\infty} f(x)\, dx = 1 \end{cases}$

1) $f(x) \ge 0, \forall x \in \mathbb{R} \Leftrightarrow k \cdot \underbrace{x \cdot e^{-\frac{x^2}{2a^2}}}_{\ge 0} \ge 0, \forall x \ge 0 \Rightarrow k \ge 0$

2) $\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{0} 0 \, dx + \int_{0}^{\infty} k x e^{-\frac{x^2}{2a^2}} dx =$

$= k\int_{0}^{\infty} x \cdot e^{-\frac{x^2}{2a^2}} dx$

$t = -\frac{x^2}{a^2} \Rightarrow -x^2 = 2a^2 t \Rightarrow -2x\,dx = 2a^2 dt \Rightarrow$

$\Rightarrow -x\,dx = a^2 dt \Rightarrow x\,dx = -a^2 dt$

$t=0 \Rightarrow t=0$

$t=\infty \Rightarrow t=-\infty$

$K\cdot\int_0^{-\infty} -a^2\cdot e^t\,dt = K\cdot(-a^2)\int_0^{-\infty} e^t\,dt = K(-a^2)\,e^t\Big|_0^{-\infty}$

$\left.\begin{array}{l} = K(-a^2)(e^{-\infty}-e^0) = (-a^2)\cdot K\cdot(-1) = Ka^2 \\ \int_{-\infty}^{\infty} f(x)\,dx = 1 \end{array}\right\} \Rightarrow Ka^2 = 1 \Rightarrow$

$\left.\begin{array}{l} \Rightarrow K = \frac{1}{a^2} \\ a>0 \end{array}\right\} \Rightarrow K = \frac{1}{a^2} > 0$

$f(x) = \begin{cases} \frac{1}{a^2}\cdot x\cdot e^{-\frac{x^2}{2a^2}}, & x \geq 0 \\ 0, & x<0 \end{cases}$

b) $f(t) = \begin{cases} 0, & t<0 \\ \frac{1}{a^2}\cdot t\cdot e^{-\frac{t^2}{2a^2}}, & t\geq 0 \end{cases}$

$F(x) = \int_{-\infty}^{x} f(t)\,dt$

I $x<0 \Rightarrow F(x) = \int_{-\infty}^{x} 0\,dt = 0$

II $x\geq 0 \Rightarrow F(x) = \int_{-\infty}^{0} 0\,dt + \int_0^x \frac{1}{a^2}\,t\cdot e^{-\frac{t^2}{2a^2}}\,dt$

$= \frac{1}{a^2}\int_0^x t\cdot e^{-\frac{t^2}{2a^2}}\,dt$

$-\frac{t^2}{2a^2} = y \Rightarrow -t^2 = y\cdot 2a^2 \Rightarrow -2t\,dt = 2a^2\,dy \Rightarrow$

$\Rightarrow t\,dt = -a^2\,dy$

$t=0 \Rightarrow y=0$

$t=x \Rightarrow y=-\frac{x^2}{2a^2}$

$F(x)=\frac{1}{a^2}\int_0^{-\frac{x^2}{2a^2}} -a^2 \cdot e^y dy = -\int_0^{-\frac{x^2}{2a^2}} e^y\, dy =$

$= -(e^{-\frac{x^2}{2a^2}} - e^0) = 1-e^{-\frac{x^2}{2a^2}}$

$F(x) = \begin{cases} 0, & x<0 \\ 1-e^{-\frac{x^2}{2a^2}}, & x\geq 0 \end{cases}$

c) $\mathbb{E}(X)=\int_{-\infty}^{\infty} x\cdot f(x)dx = \int_{-\infty}^{0} x\cdot 0\, dx + \int_0^{\infty} x\cdot\frac{1}{a^2}\cdot x\cdot$

$\cdot e^{-\frac{x^2}{2a^2}} dx = \frac{1}{a^2}\int_0^{\infty} x^2\cdot e^{-\frac{x^2}{2a^2}} dx = \frac{1}{a^2}\int_0^{\infty} x\cdot x\cdot e^{-\frac{x^2}{2a^2}} dx$

$u=x \Rightarrow u'=1$

$v'=x\cdot e^{-\frac{x^2}{2a^2}} \Rightarrow v=\underbrace{\int x\cdot e^{-\frac{x^2}{2a^2}}dx}_{i}$

$i=\int x\cdot e^{-\frac{x^2}{2a^2}} dx$

$t=-\frac{x^2}{2a^2} \Rightarrow 2a^2 t = -x^2 \Rightarrow 2a^2 dt = -2x dx$

$\Rightarrow x dx = -a^2 dt$

$i=\int -a^2\cdot e^t dt = -a^2\cdot e^t + \mathcal{C} =$

$=-a^2 e^{-\frac{x^2}{2a^2}} + \mathcal{C},\ \mathcal{C}\in\mathbb{R}$

$v=-a^2 e^{-\frac{x^2}{2a^2}}$

$\mathbb{E}(X)=\frac{1}{a^2}\left(-a^2 x e^{-\frac{x^2}{2a^2}}\Big|_0^{\infty} - \int_0^{\infty} -a^2 e^{-\frac{x^2}{2a^2}} dx\right) = -x e^{-\frac{x^2}{2a^2}}\Big|_0^{\infty} +$

$+\int_0^{\infty} e^{-\frac{x^2}{2a^2}} dx = 0 + \int_0^{\infty} e^{-\frac{x^2}{2a^2}} dx = \int_0^{\infty} e^{-\frac{x^2}{2a^2}} dx =$

$$= \int_0^\infty e^{-\left(\frac{x}{\sqrt{2}a}\right)^2} dx$$

$$\frac{x}{\sqrt{2}a} = t \Rightarrow x = \sqrt{2}\, a t \Rightarrow dx = \sqrt{2}\, a\, dt$$

$$x = 0 \Rightarrow t = 0$$
$$x = \infty \Rightarrow t = \infty$$

$$\mathbb{E}(X) = \int_0^\infty \sqrt{2}a\, e^{-t^2} dt = \sqrt{2}\, a \underbrace{\int_0^\infty e^{-t^2} dt}_{\text{Integrala Euler-Poisson}} =$$

$$= \sqrt{2}\, a \, \frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}\, a}{\sqrt{2}}$$

$$Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx = \int_{-\infty}^{0} x^2 \cdot 0\, dx + \int_0^\infty x^2 \cdot \frac{1}{a^2} \cdot x \cdot$$

$$\cdot e^{-\frac{x^2}{2a^2}} dx = \frac{1}{a^2} \int_0^\infty x^3 \cdot e^{-\frac{x^2}{2a^2}} dx = \frac{1}{a^2} \int_0^\infty x^2 \cdot x \cdot e^{-\frac{x^2}{2a^2}} dx$$

$$x^2 = t \Rightarrow 2x\, dx = dt \Rightarrow x\, dx = \frac{1}{2} dt$$

$$x = 0 \Rightarrow t = 0$$
$$x = \infty \Rightarrow t = \infty$$

$$\mathbb{E}(X^2) = \frac{1}{2a^2} \int_0^\infty t \cdot e^{-\frac{t}{2a^2}} dt$$

$$-\frac{t}{2a^2} = y \Rightarrow -t = 2a^2 y \Rightarrow -dt = 2a^2 dy \Rightarrow$$

$$\Rightarrow dt = -2a^2 dy$$

$$t = 0 \Rightarrow y = 0$$
$$t = \infty \Rightarrow y = -\infty$$

$$\mathbb{E}(X^2) = \frac{1}{2a^2} \int_0^{-\infty} -2a^2 y \cdot e^{y} \cdot (-2a^2) dy = 2a^2 \int_0^{-\infty} y e^y dy$$

$$u = y \Rightarrow u' = 1$$
$$v' = e^y \Rightarrow v = e^y$$

$\mathbb{E}(X^2) = 2a^2 \left(y e^y \Big/_0^{-\infty} - \int_0^{-\infty} e^y \, dy \right) = 2a^2$

$Var(X) = 2a^2 - \frac{\pi a^2}{2} = \frac{4a^2 - \pi a^2}{2} = \frac{a^2(4-\pi)}{2}$

$m_r = \int_{-\infty}^{\infty} x^r \cdot f(x) dx = \int_{-\infty}^{0} x^r \cdot 0 \, dx + \int_0^{\infty} x^r \cdot \frac{1}{a^2} \cdot$

$\cdot x \cdot e^{-\frac{x^2}{2a^2}} dx = \frac{1}{a^2} \int_0^{\infty} x^r \cdot x \cdot e^{-\frac{x^2}{2a^2}} dx$

$\frac{x^2}{2a^2} = t \Rightarrow x^2 = 2a^2 t \Rightarrow 2x dx = 2a^2 dt \Rightarrow$

$\Rightarrow x dx = a^2 dt$

$x = 0 \Rightarrow t = 0$

$x = \infty \Rightarrow t = \infty$

$x = \sqrt{2a^2 t}$ (din $x^2 = 2a^2 t$)

$m_r = \frac{1}{\cancel{a^2}} \int_0^{\infty} \cancel{a^2} (\sqrt{2a^2 t})^r \cdot e^{-t} dt = \int_0^{\infty} (\sqrt{2})^r a^r (\sqrt{t})^r \cdot$

$\left.\begin{array}{l} \sqrt{a^2} = |a| \\ a > 0 \end{array}\right\} \Rightarrow \sqrt{a^2} = a$

$\cdot e^{-t} dt = \int_0^{\infty} (\sqrt{2})^r a^r (t^{\frac{1}{2}})^r \cdot e^{-t} dt =$

$= 2^{\frac{r}{2}} a^r \int_0^{\infty} t^{\frac{r}{2}} \cdot e^{-t} dt$

$a - 1 = \frac{r}{2} \Rightarrow a = \frac{r}{2} + 1$

$m_r = 2^{\frac{r}{2}} a^r \Gamma\left(\frac{r}{2} + 1\right), r \in \mathbb{N}^*$

d) $\left.\begin{array}{l} \mathbb{P}(X < 2a) = \int_{-\infty}^{2a} f(x) dx \\ a > 0 \end{array}\right\} \Rightarrow \mathbb{P}(X < 2a) = \int_{-\infty}^{0} 0 \, dx +$

$$+\int_0^{2a} \frac{1}{a^2} x \cdot e^{-\frac{x^2}{2a^2}} dx = \frac{1}{a^2}\int_0^{2a} x \cdot e^{-\frac{x^2}{2a^2}} dx$$

$$-\frac{x^2}{2a^2} = t \Rightarrow -x^2 = 2a^2 t \Rightarrow -2x\,dx = 2a^2\,dt \Rightarrow$$

$$\Rightarrow x\,dx = -a^2\,dt$$

$$x = 0 \Rightarrow t = 0$$

$$x = 2a \Rightarrow t = -\frac{4a^2}{2a^2} = -2$$

$$\mathbb{P}(X<2a) = \frac{1}{a^2}\int_0^{-2} -\cancel{a^2} \cdot e^t\,dt = -\int_0^{-2} e^t\,dt =$$

$$= -e^t\Big/_0^{-2} = -e^{-2} + e^0 = -\frac{1}{e^2} + 1 = \frac{-1+e^2}{e^2} =$$

$$= \frac{e^2-1}{e^2} \simeq 0,865$$

$$\mathbb{P}(X>a) = \int_a^{\infty} f(x)\,dx = \int_a^{\infty} \frac{1}{a^2} x \cdot e^{-\frac{x^2}{2a^2}} dx =$$

$$\left.\begin{array}{l} a>0 \\ x\in[0;\infty) \end{array}\right\} \Rightarrow (a;\infty) \subset [0;\infty)$$

$$= \frac{1}{a^2}\int_a^{\infty} x \cdot e^{-\frac{x^2}{2a^2}} dx$$

$$-\frac{x^2}{2a^2} = t \Rightarrow -x^2 = 2a^2 t \Rightarrow -2x\,dx = 2a^2\,dt \Rightarrow$$

$$\Rightarrow x\,dx = -a^2\,dt$$

$$x = a \Rightarrow t = \frac{-a^2}{2a^2} = -\frac{1}{2}$$

$$x = \infty \Rightarrow t = -\infty$$

$$\mathbb{P}(X>a) = \frac{1}{\cancel{a^2}}\int_{-\frac{1}{2}}^{-\infty} -\cancel{a^2} \cdot e^t\,dt = -e^t\Big/_{-\frac{1}{2}}^{-\infty} =$$

$$= -0 + e^{-\frac{1}{2}} = \frac{1}{e^{\frac{1}{2}}} = \frac{1}{\sqrt{e}} = \frac{\sqrt{e}}{e} \simeq 0,61$$

$$\mathbb{P}(X \le 4a \mid X > 2a) = \frac{\mathbb{P}((X\le 4a)\cap(X>2a))}{\mathbb{P}(X>2a)} = \frac{\mathbb{P}(2a < X \le 4a)}{\mathbb{P}(X>2a)} = \frac{\int_{2a}^{4a} f(x)dx}{\int_{2a}^{\infty} f(x)dx}$$

$$\left.\begin{array}{l} a>0 \Rightarrow 4a>0 \\ \quad 2a>0 \\ x\in[0;\infty) \end{array}\right\} \Rightarrow \begin{cases} (2a;4a] \subset [0;\infty) \\ (2a;\infty) \subset [0;\infty) \end{cases}$$

$$= \frac{I_1}{I_2}$$

$$I_1 = \int_{2a}^{4a} \frac{1}{a^2}\cdot x \cdot e^{-\frac{x^2}{2a^2}}\,dx$$

$$-\frac{x^2}{2a^2} = t \Rightarrow -x^2 = 2a^2 t \Rightarrow 2x\,dx = -2a^2\,dt \Rightarrow$$

$$\Rightarrow x\,dx = -a^2\,dt$$

$$x = 2a \Rightarrow t = -\frac{4a^2}{2a^2} = -2$$

$$x = 4a \Rightarrow t = -\frac{16a^2}{2a^2} = -8$$

$$I_1 = \frac{1}{a^2}\int_{-2}^{-8} -a^2\cdot e^t\,dt = -e^t\Big|_{-2}^{-8} = -e^{-8} + e^{-2} = e^{-2} - e^{-8}$$

$$I_2 = \int_{2a}^{\infty} \frac{1}{a^2}\cdot x\cdot e^{-\frac{x^2}{2a^2}}\,dx$$

$$-\frac{x^2}{2a^2} = t \Rightarrow -x^2 = 2a^2 t \Rightarrow 2x\,dx = -2a^2\,dt \Rightarrow$$

$$\Rightarrow x\,dx = -a^2\,dt$$

$$x = 2a \Rightarrow t = -\frac{4a^2}{2a^2} = -2$$

$$x = \infty \Rightarrow t = -\infty$$

$$I_2 = \frac{1}{a^2}\int_{-2}^{-\infty} -a^2\cdot e^t\,dt = -e^t\Big|_{-2}^{-\infty} = -(e^{-\infty} - e^{-2}) =$$

$$= e^{-2}$$

$$\mathbb{P}(X \le 4a \mid X > 2a) = \frac{e^{-2} - e^{-8}}{e^{-2}} = \frac{e^{-2}(1 - e^{-6})}{e^{-2}} = 1 - e^{-6} = 1 - \frac{1}{e^6} =$$

$= \frac{e^6 - 1}{e^6} \simeq 0,998$

e) $\mu_3 = \int_{-\infty}^{\infty} (x-m)^3 \cdot f(x)\, dx = \int_{-\infty}^{0} (x-m)^3 \, 0 \, dx +$

$m = \mathbb{E}(x) = \frac{\sqrt{\pi}\, a}{\sqrt{2}}$

$+ \int_{0}^{\infty} (x-m)^3 \cdot \frac{1}{a^2} \cdot x \cdot e^{-\frac{x^2}{2a^2}} dx =$

$= \frac{1}{a^2} \int_{0}^{\infty} (x^3 - 3x^2 m + 3x m^2 - m^3) \cdot x \cdot e^{-\frac{x^2}{a^2}} dx$

$= \frac{1}{a^2} \Big(\int_{0}^{\infty} x^3 \cdot x \cdot e^{-\frac{x^2}{2a^2}} dx - 3m \int_{0}^{\infty} x^2 \cdot x \cdot e^{-\frac{x^2}{2a^2}} dx +$

$+ 3m^2 \int_{0}^{\infty} x \cdot x \cdot e^{-\frac{x^2}{2a^2}} dx - m^3 \int_{0}^{\infty} x \cdot e^{-\frac{x^2}{2a^2}} dx \Big)$

$= \frac{1}{a^2} \Big(i_1 - 3m i_2 + 3m^2 i_3 - m^3 i_4 \Big)$

$i_1 = \int_{0}^{\infty} x^3 \cdot x \cdot e^{-\frac{x^2}{2a^2}} dx$

$u = x^3 \Rightarrow u' = 3x^2$

$v' = x \cdot e^{-\frac{x^2}{2a^2}} \Rightarrow v = \underbrace{\int x \cdot e^{-\frac{x^2}{2a^2}}}_{i}$

$i = \int x \cdot e^{-\frac{x^2}{2a^2}} dx$

$t = -\frac{x^2}{2a^2} \Rightarrow 2a^2 t = -x^2 \Rightarrow 2a^2 dt = -2x\, dx \Rightarrow$

$\Rightarrow x\, dx = -a^2 dt$

$i = \int -a^2 e^{t} dt = -a^2 \cdot e^{t} + \mathcal{C} = -a^2 \cdot e^{\frac{-x^2}{2a^2}} + \mathcal{C}, \mathcal{C} \in \mathbb{R}$

$v = -a^2 e^{-\frac{x^2}{2a^2}}$

$$i_1 = -a^2 x^3 e^{-\frac{x^2}{2a^2}} \Big/_0^\infty - \int_0^\infty -a^2 \cdot 3x^2 \cdot e^{-\frac{x^2}{2a^2}} dx$$

$$= 3a^2 \underbrace{\int_0^\infty x^2 \cdot e^{-\frac{x^2}{2a^2}} dx}$$

$$i_1' = \int_0^\infty x^2 \cdot \overset{i_1'}{e}^{-\frac{x^2}{2a^2}} dx = \int_0^\infty x \cdot x \cdot e^{-\frac{x^2}{2a^2}} dx$$

$$u = x \Rightarrow u' = 1$$

$$v' = x e^{-\frac{x^2}{2a^2}} \Rightarrow v = \underbrace{\int x e^{-\frac{x^2}{2a^2}} dx}_{i_v}$$

$$i_v = \int x e^{-\frac{x^2}{2a^2}} dx$$

$$t = -\frac{x^2}{2a^2} \Rightarrow 2a^2 t = -x^2 \Rightarrow 2a^2 dt = -2x dx \Rightarrow x dx =$$

$$= -a^2 dt$$

$$i_v = \int -a^2 e^t dt = -a^2 \cdot e^t + C = -a^2 \cdot e^{-\frac{x^2}{2a^2}} + C, \ C \in \mathbb{R}$$

$$v = -a^2 e^{-\frac{x^2}{2a^2}}$$

$$i_1' = -a^2 x \cdot e^{-\frac{x^2}{2a^2}} \Big/_0^\infty + a^2 \int_0^\infty e^{-\frac{x^2}{2a^2}} dx = 0 + a^2 \int_0^\infty e^{-\left(\frac{x}{\sqrt{2}a}\right)^2} dx$$

$$\frac{x}{\sqrt{2}a} = t \Rightarrow x = \sqrt{2} a t \Rightarrow dx = \sqrt{2} a dt$$

$$x = 0 \Rightarrow t = 0$$

$$x = \infty \Rightarrow t = \infty$$

$$i_1' = a^2 \int_0^\infty \sqrt{2} a \cdot e^{-t^2} dt = a^2 \cdot \sqrt{2} a \cdot \underbrace{\int_0^\infty e^{-t^2} dt}_{\text{Integrala Euler-Poisson}}$$

$$= \sqrt{2} a^3 \cdot \frac{\sqrt{\pi}}{2} = \frac{a^3 \sqrt{\pi}}{\sqrt{2}}$$

$$i_1 = \frac{3a^5 \sqrt{\pi}}{\sqrt{2}}$$

$$i_2 = \int_0^\infty x^2 \cdot x \cdot e^{-\frac{x^2}{2a^2}} \, dx$$

$t = x^2 \Rightarrow dt = 2x\,dx \Rightarrow x\,dx = \frac{1}{2}\,dt$

$x = 0 \Rightarrow t = 0$

$x = \infty \Rightarrow t = \infty$

$$i_2 = \frac{1}{2} \int_0^\infty t \cdot e^{-\frac{t}{2a^2}} \, dt$$

$-\frac{t}{2a^2} = y \Rightarrow -t = 2a^2 y \Rightarrow -dt = 2a^2 dy \Rightarrow dt = -2a^2 dy$

$t = 0 \Rightarrow y = 0$

$t = \infty \Rightarrow y = -\infty$

$$i_2 = \frac{1}{2} \int_0^{-\infty} (-2a^2 y) \cdot e^y (-2a^2) \, dy = 2a^4 \int_0^{-\infty} y e^y \, dy$$

$u = y \Rightarrow u' = 1$

$v' = e^y \Rightarrow v = e^y$

$$i_2 = 2a^4 \left(y e^y \Big/_0^{-\infty} - \int_0^{-\infty} e^y \, dy \right)$$

$$i_2 = 2a^4 (0 - 0 + 1) = 2a^4$$

$$i_3 = \int_0^\infty x \cdot x \cdot e^{-\frac{x^2}{2a^2}} \, dx$$

$u = x \Rightarrow u' = 1$

$v' = x \cdot e^{-\frac{x^2}{2a^2}} \Rightarrow v = \underbrace{\int x \cdot e^{-\frac{x^2}{2a^2}} \, dx}_{i}$

$$i = \int x \cdot e^{-\frac{x^2}{2a^2}} \, dx$$

$t = -\frac{x^2}{2a^2} \Rightarrow 2a^2 t = -x^2 \Rightarrow 2a^2 dt = -2x\,dx \Rightarrow$

$\Rightarrow x\,dx = -a^2 dt$

$i = \int -a^2 \cdot e^t \, dt = -a^2 \cdot e^t + C = -a^2 \cdot e^{-\frac{x^2}{2a^2}} + C, \; C \in \mathbb{R}$

$v = -a^2 \cdot e^{-\frac{x^2}{2a^2}}$

$$i_3 = -a^2 x e^{-\frac{x^2}{2a^2}} \Big|_0^{\infty} - \int_0^{\infty} -a^2 e^{-\frac{x^2}{2a^2}} dx = a^2 \int_0^{\infty} e^{-\frac{x^2}{2a^2}} dx$$

$$= a^2 \int_0^{\infty} e^{-\left(\frac{x}{\sqrt{2}a}\right)^2} dx$$

$$\frac{x}{\sqrt{2}a} = t \Rightarrow x = \sqrt{2}a\,t \Rightarrow dx = \sqrt{2}a\,dt$$

$$x = 0 \Rightarrow t = 0$$

$$x = \infty \Rightarrow t = \infty$$

$$i_3 = a^2 \int_0^{\infty} \sqrt{2}a\, e^{-t^2} dt = \sqrt{2}a^3 \underbrace{\int_0^{\infty} e^{-t^2} dt}_{\text{Integrala Euler-Poisson}} = \sqrt{2}a^3 \cdot \frac{\sqrt{\pi}}{2} =$$

$$= \frac{\sqrt{\pi}a^3}{\sqrt{2}}$$

$$i_4 = \int_0^{\infty} x \cdot e^{-\frac{x^2}{2a^2}} dx$$

$$-\frac{x^2}{2a^2} = t \Rightarrow -x^2 = 2a^2 t \Rightarrow -2x\,dx = 2a^2 dt \Rightarrow$$

$$\Rightarrow x\,dx = -a^2 dt$$

$$x = 0 \Rightarrow t = 0$$

$$x = \infty \Rightarrow t = -\infty$$

$$i_4 = \int_0^{-\infty} -a^2 e^t dt = -a^2 e^t \Big|_0^{-\infty} = a^2$$

$$\mu_3 = \frac{1}{a^2}\left(\frac{3a^5\sqrt{\pi}}{\sqrt{2}} - 6ma^4 + \frac{3m^2\sqrt{\pi}a^3}{\sqrt{2}} - m^3a^2\right)$$

$$= \frac{3a^3\sqrt{\pi}}{\sqrt{2}} - 6ma^2 + \frac{3m^2\sqrt{\pi}a}{\sqrt{2}} - m^3$$

$$= \frac{3a^3\sqrt{\pi}}{\sqrt{2}} - \frac{6\sqrt{\pi}a^3}{\sqrt{2}} + \frac{3\pi a^2\sqrt{\pi}a}{2\sqrt{2}} - \frac{\pi\sqrt{\pi}a^3}{2\sqrt{2}}$$

$$= -\frac{3a^3\sqrt{\pi}}{\sqrt{2}} + \frac{3a^3\pi\sqrt{\pi}}{2\sqrt{2}} - \frac{a^3\pi\sqrt{\pi}}{2\sqrt{2}}$$

$$= \frac{\cancel{2}a^3\pi\sqrt{\pi}}{\cancel{2}\sqrt{2}} - \frac{3a^3\sqrt{\pi}}{\sqrt{2}} = \frac{a^3\sqrt{\pi}(\pi-3)}{\sqrt{2}}$$

$F(m_e) = \frac{1}{2}$

Dacă $m_e \in (-\infty; 0) \Rightarrow F(m_e) = 0 \Rightarrow m_e \notin (-\infty; 0)$

$m_e \in [0; +\infty) \Rightarrow \cancel{F-(}e^{-\frac{m_e^2}{2a^2}} = \frac{1}{2} \Rightarrow e^{-\frac{m_e^2}{2a^2}} = 1 - \frac{1}{2} \Rightarrow$

$\Rightarrow e^{-\frac{m_e^2}{2a^2}} = \frac{1}{2} \Rightarrow -\frac{m_e^2}{2a^2} = \ln\frac{1}{2} \Rightarrow -m_e^2 = 2a^2 \ln 2^{-1} \Rightarrow$

$\Rightarrow m_e^2 = 2a^2 \ln 2 \Rightarrow \left.\begin{matrix} m_e = \pm\sqrt{2\ln 2 \cdot a^2} \\ m_e \in [0; +\infty) \end{matrix}\right\} \Rightarrow \left.\begin{matrix} m_e = \sqrt{2a^2 \ln 2} \\ a > 0 \end{matrix}\right\} \Rightarrow$

$\Rightarrow m_e = a\sqrt{2\ln 2}$

$l_s(0) = 0$

$l_d(0) = 0$

$f(0) = 0$

f continuă în $0 \Rightarrow f$ continuă pe $\mathbb{R}$

$$f'(x) = \begin{cases} \left(\frac{1}{a^2} \cdot x \cdot e^{-\frac{x^2}{2a^2}}\right)', & x > 0 \\ 0, & x < 0 \end{cases}$$

$$\frac{1}{a^2}\left(x \cdot e^{-\frac{x^2}{2a^2}}\right)' = \frac{1}{a^2}\left[e^{-\frac{x^2}{2a^2}} + x \cdot e^{-\frac{x^2}{2a^2}} \cdot \left(-\frac{x^2}{2a^2}\right)'\right]$$

$$= \frac{1}{a^2}\left[e^{-\frac{x^2}{2a^2}} + x\, e^{-\frac{x^2}{2a^2}} \left(-\frac{1}{2a^2}\right) \cdot (x^2)'\right] =$$

$$= \frac{1}{a^2}\left[e^{-\frac{x^2}{2a^2}} - \frac{x^2}{a^2} e^{-\frac{x^2}{2a^2}}\right] = \frac{e^{-\frac{x^2}{2a^2}}}{a^2}\left[1 - \frac{x^2}{a^2}\right] =$$

$$= -\frac{e^{-\frac{x^2}{2a^2}}}{a^4}(x^2 - a^2)$$

$$f'(x) = \begin{cases} -\frac{e^{-\frac{x^2}{2a^2}}}{4}(x^2 - a^2), & x > 0 \\ 0, & x < 0 \end{cases}$$

$x \in [0; \infty)$

$a > 0 \Rightarrow a \in (0; \infty)$

Dacă $x > a \Rightarrow -\underbrace{\frac{e^{-\frac{x^2}{2a^2}}}{a^4}}_{>0}\underbrace{(x^2 - a^2)}_{>0} < 0$

x	$-\infty$		0		$+\infty$
$f'(x)$	+ + + + + + +			− − − − − − −	
$f(x)$	0	$\nearrow$	0	$\searrow$	0

$m_0 = 0$

Dacă $x < a \Rightarrow -\underbrace{\frac{e^{-\frac{x^2}{2a^2}}}{a^4}}_{>0}\underbrace{(x^2 - a^2)}_{<0} > 0$

x	$-\infty$		0		$+\infty$
$f'(x)$	− − − − − −			+ + + + + + +	
$f(x)$	0	$\searrow$	0	$\nearrow$	0

Funcția crește de la 0 la infinit $\Rightarrow m_0$ nu se poate determina sau se poate considera $m_0 = 0$ deoarece la infinit valoarea lui $f(x)$ este 0.

Dacă $x = a \Rightarrow -\frac{e^{-\frac{x^2}{a^2}}}{a^4}\underbrace{(x^2 - a^2)}_{=0} = 0$

x	$-\infty$		0		$+\infty$
$f'(x)$	− − − − − −			+ + + + +	
$f(x)$	0	$\searrow$	0	$\nearrow$	0

Am considerat „−" de la „$-\infty$" la „0" și „+" de la „0" la „$+\infty$" ca în cazul limitelor, pe $(-\infty; 0)$ având o și pe $(0; +\infty)$ O_+.

Funcția crește de la 0 la infinit $\Rightarrow m_0$ nu se poate determina sau se poate considera $m_0 = 0$ deoarece valoarea lui $f(x)$, la infinit, este 0.

f) $\mathbb{P}(0 < X < a\sqrt{2\pi}) = \int_0^{a\sqrt{2\pi}} \frac{1}{a^2} x \cdot e^{-\frac{x^2}{2a^2}} dx =$

$a > 0 \Rightarrow (0; a\sqrt{2\pi}) \subset [0; \infty)$

$= \frac{1}{a^2} \int_0^{a\sqrt{2\pi}} x \cdot e^{-\frac{x^2}{2a^2}} dx$

$-\frac{x^2}{2a^2} = t \Rightarrow -x^2 = 2a^2 t \Rightarrow 2x\,dx = -2a^2 dt \Rightarrow$

$\Rightarrow x\,dx = -a^2 dt$

$x = 0 \Rightarrow t = 0$

$x = a\sqrt{2\pi} \Rightarrow t = -\frac{a^2 2\pi}{2a^2} = -\pi$

$\mathbb{P}(0 < X < a\sqrt{2\pi}) = \frac{1}{a^2} \int_0^{-\pi} -a^2 e^t dt = -e^t \Big|_0^{-\pi} =$

$= -e^{-\pi} + 1 = 1 - \frac{1}{e^\pi} \simeq 0{,}9568$

$A = X > 0$

$B = X < a\sqrt{2\pi}$

$\mathbb{P}(0 < X < a\sqrt{2\pi}) = \mathbb{P}((X > 0) \cap (X < a\sqrt{2\pi})) = \mathbb{P}(A \cap B)$

$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \Rightarrow \mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B)$

Vom folosi inegalitatea lui Boole pentru marginea inferioară

$0 \leq \mathbb{P}(A \cup B) \leq 1$

$\mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1$

$$\mathbb{P}(A) = \mathbb{P}(X>0) = \int_0^{\infty} f(x)\,dx = \int_0^{\infty} \frac{1}{a^2} \cdot x \cdot e^{-\frac{x^2}{2a^2}}\,dx =$$

$$= \frac{1}{a^2} \int_0^{\infty} x \cdot e^{-\frac{x^2}{2a^2}}\,dx$$

$$-\frac{x^2}{2a^2} = t \Rightarrow -x^2 = 2a^2 t \Rightarrow 2x\,dx = -2a^2\,dt \Rightarrow x\,dx = -a^2\,dt$$

$x = 0 \Rightarrow t = 0$

$x = \infty \Rightarrow t = -\infty$

$$\mathbb{P}(A) = \frac{1}{a^2}\int_0^{-\infty} -a^2 e^t\,dt = \frac{-a^2}{a^2} e^t \Big/_0^{-\infty} =$$

$$= -e^{-\infty} + e^0 = 1$$

$$\left.\begin{array}{l} \mathbb{P}(B) = \mathbb{P}(X < a\sqrt{2\pi}) \\ a > 0 \Rightarrow a\sqrt{2\pi} > 0 \end{array}\right\} \Rightarrow \mathbb{P}(B) = \int_{-\infty}^{a\sqrt{2\pi}} f(x)\,dx =$$

$$= \int_{-\infty}^{0} 0\,dx + \int_0^{a\sqrt{2\pi}} \frac{1}{a^2} x \cdot e^{-\frac{x^2}{2a^2}}\,dx =$$

$$= \frac{1}{a^2} \int_0^{a\sqrt{2\pi}} x e^{-\frac{x^2}{2a^2}}\,dx$$

$$-\frac{x^2}{2a^2} = t \Rightarrow -x^2 = 2a^2 t \Rightarrow 2x\,dx = -2a^2\,dt \Rightarrow x\,dx = -a^2\,dt$$

$x = 0 \Rightarrow t = 0$

$$x = a\sqrt{2\pi} \Rightarrow t = -\frac{a^2 2\pi}{2a^2} = -\pi$$

$$\mathbb{P}(B) = \frac{1}{a^2}\int_0^{-\pi} -a^2 e^t\,dt = -e^{-\pi} + e^0 = 1 - \frac{1}{e^{\pi}}$$

$$\mathbb{P}(A \cap B) \geq 1 + 1 - \frac{1}{e^{\pi}} - 1 \Leftrightarrow \mathbb{P}(A \cap B) \geq 1 - \frac{1}{e^{\pi}}$$

$\Rightarrow$ Marginea inferioară este $1 - \frac{1}{e^{\pi}}$.

Pb 5) $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \begin{cases} r \cdot x^{\frac{n}{2}-1} \cdot e^{-\frac{x}{3}}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

$r \in \mathbb{R}$, $n \in \mathbb{N}^*$

X_n v.a.

a) f_n densitate de probabilitate $\Leftrightarrow \begin{cases} 1) f(x) \geq 0, \forall x \in \mathbb{R} \\ 2) \int_{-\infty}^{\infty} f(x)\,dx = 1 \end{cases}$

1) $f(x) \geq 0$, $\forall x \in \mathbb{R} \Leftrightarrow r \cdot \underbrace{x^{\frac{n}{2}-1}}_{\geq 0} \cdot \underbrace{e^{-\frac{x}{3}}}_{>0} \geq 0$, $\forall x \geq 0 \Rightarrow$

$\Rightarrow r \geq 0$

2) $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} r \cdot x^{\frac{n}{2}-1} \cdot e^{-\frac{x}{3}}\,dx =$

$= r \int_0^{\infty} x^{\frac{n}{2}-1} \cdot e^{-\frac{x}{3}}\,dx$

$\frac{x}{3} = t \Rightarrow x = 3t \Rightarrow dx = 3dt$

$x = 0 \Rightarrow t = 0$

$x = \infty \Rightarrow t = \infty$

$\int_{-\infty}^{\infty} f(x)\,dx = r \int_0^{\infty} 3^{\frac{n}{2}-1} \cdot t^{\frac{n}{2}-1} \cdot e^{-t} \cdot 3\,dt$

$= r \cdot 3^{\frac{n}{2}} \cdot \int_0^{\infty} t^{\frac{n}{2}-1} e^{-t}\,dt = r \cdot 3^{\frac{n}{2}} \cdot \Gamma\left(\frac{n}{2}\right)$

$a - 1 = \frac{n}{2} - 1 \Rightarrow a = \frac{n}{2}$

$\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow r \cdot 3^{\frac{n}{2}} \cdot \Gamma\left(\frac{n}{2}\right) = 1 \Rightarrow r = \frac{1}{3^{\frac{n}{2}} \cdot \Gamma\left(\frac{n}{2}\right)}$

$, \forall n \in \mathbb{N}^* \Leftrightarrow r = \frac{1}{3^{\frac{n}{2}} \cdot \left(\frac{n}{2}-1\right)!}$, $\forall n \in \mathbb{N}^*$

b) $f_m, m \in \mathbb{N}^*$ densitate de probabilitate

Y este o v.a. continuă

X și Y sunt independente

$$f_n(x) = \begin{cases} \frac{1}{3^{\frac{n}{2}} \cdot \Gamma\left(\frac{n}{2}\right)} \cdot x^{\frac{n}{2}-1} \cdot e^{-\frac{x}{3}}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$f_m(x) = \begin{cases} \frac{1}{3^{\frac{m}{2}} \cdot \Gamma\left(\frac{m}{2}\right)} \cdot x^{\frac{m}{2}-1} \cdot e^{-\frac{x}{3}}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$f_{X+Y}(x) = \int f_{n+m}(x) = \int_{-\infty}^{\infty} f_n(x-t) \cdot f_m(t)\,dt$$

$$= \int_{-\infty}^{0} f_n(x-t) \cdot 0\,dt + \int_0^{\infty} f_n(x-t) \cdot r \cdot t^{\frac{m}{2}-1} \cdot e^{-\frac{t}{3}} dt$$

$$= r \int_0^{\infty} f_n(x-t) \cdot t^{\frac{m}{2}-1} \cdot e^{-\frac{t}{3}} dt$$

VI $x - t = y \Rightarrow -dt = dy \Rightarrow dt = -dy$

$t = 0 \Rightarrow y = x$ $\qquad x - t = y \Rightarrow t = x - y$

$t = \infty \Rightarrow y = -\infty$

$$f_{X+Y}(x) = -r \int_{x}^{-\infty} f_n(y) \cdot (x-y)^{\frac{m}{2}-1} \cdot e^{-\frac{x-y}{3}} dy$$

I $x < 0 \Rightarrow y < 0 \Rightarrow f_n(y) = 0 \Rightarrow f_{X+Y}(x) = 0$

II $x \geq 0 \Rightarrow y \geq 0 \Rightarrow f_n(y) = r\, y^{\frac{n}{2}-1} \cdot e^{-\frac{y}{3}} \Rightarrow$

$$\Rightarrow f_{X+Y}(x) = -r^2 \int_x^{-\infty} y^{\frac{n}{2}-1} \cdot e^{-\frac{y}{3}} \cdot (x-y)^{\frac{m}{2}-1} \cdot e^{-\frac{x-y}{3}} dy$$

$$= -r^2 \cdot e^{-\frac{x}{3}} \int_x^{-\infty} x^{\frac{n}{2}-1} \cdot \left(\frac{y}{x}\right)^{\frac{n}{2}-1} \cdot x^{\frac{m}{2}-1} \cdot \left(\frac{x-y}{x}\right)^{\frac{m}{2}-1} dy$$

$$= -r^2 \cdot e^{-\frac{x}{3}} \cdot x^{\frac{n}{2}+\frac{m}{2}-2} \int_x^{-\infty} \left(\frac{y}{x}\right)^{\frac{n}{2}-1} \cdot \left(1 - \frac{y}{x}\right)^{\frac{m}{2}-1} dy$$

$$-\frac{y}{x} = u \Rightarrow -y = u x \Rightarrow -dy = x\,du \Rightarrow -\frac{1}{x} dy = du$$

$y = x \Rightarrow u = -1$

$y = -\infty \Rightarrow u = +\infty$

$$f_{X+Y}(x) = -n^2 \cdot e^{-\frac{x}{3}} \cdot x^{\frac{n}{2}+\frac{m}{2}-1} \int_{-1}^{+\infty} (-u)^{\frac{n}{2}-1} \cdot (1+u)^{\frac{m}{2}-1} du$$

$$= -n^2 \cdot e^{-\frac{x}{3}} \cdot x^{\frac{n}{2}+\frac{m}{2}-1} \left(\int_{-1}^{0} 0\, du + \int_{0}^{\infty} (-u)^{\frac{n}{2}-1} \cdot \right.$$

$$\left. (1+u)^{\frac{m}{2}-1} du \right) = -n^2 \cdot e^{-\frac{x}{3}} \cdot x^{\frac{n}{2}+\frac{m}{2}-1} \cdot (-1)^{\frac{n}{2}-1} \int_0^{\infty} \frac{u^{\frac{n}{2}-1}}{(1+u)^{1-\frac{m}{2}}}$$

$$du = -n^2 \cdot e^{-\frac{x}{3}} \cdot x^{\frac{n}{2}+\frac{m}{2}-1} \cdot (-1)^{\frac{n}{2}-1}\, \beta\left(\frac{n}{2}, 1 - \frac{m+n}{2}\right)$$

$a - 1 = \frac{n}{2} - 1 \Rightarrow a = \frac{n}{2}$

$a + b = 1 - \frac{m}{2} \Rightarrow b = 1 - \frac{m}{2} - \frac{n}{2} = 1 - \frac{m+n}{2}$

$$f_{X+Y}(x) = \begin{cases} 0, & x < 0 \\ n^2 e^{-\frac{x}{3}} \cdot x^{\frac{n+m}{2}-1} \cdot (-1)^{\frac{n}{2}-1} \cdot \beta\left(\frac{n}{2}, 1 - \frac{m+n}{2}\right) & \end{cases}$$

X V II $f_{X+Y}(x) = n^2 \int_0^{\infty} (x-t)^{\frac{n}{2}-1} \cdot e^{-\frac{x-t}{3}} \cdot t^{\frac{m}{2}-1} \cdot e^{-\frac{t}{3}} dt$

$$= n^2 e^{-\frac{x}{3}} \int_0^{\infty} x^{\frac{n}{2}-1} \cdot \left(\frac{x-t}{x}\right)^{\frac{n}{2}-1} \cdot x^{\frac{m}{2}-1} \cdot \left(\frac{t}{x}\right)^{\frac{m}{2}-1} dt$$

$$= n^2 e^{-\frac{x}{3}} \cdot x^{\frac{n+m}{2}-2} \int_0^{\infty} \left(1 - \frac{t}{x}\right)^{\frac{n}{2}-1} \cdot \left(\frac{t}{x}\right)^{\frac{m}{2}-1} dt$$

$$-\frac{t}{x} = u \Rightarrow -t = u x \Rightarrow -dt = x\,du \Rightarrow -\frac{1}{x} dt = du$$

$t = 0 \Rightarrow u = 0$

$t = \infty \Rightarrow u = -\infty$

$$= -n^2 \cdot e^{-\frac{x}{3}} \cdot x^{\frac{n+m}{2}-1} \int_0^{-\infty} (1+u)^{\frac{n}{2}-1} \cdot (-u)^{\frac{m}{2}-1} du$$

Pb 1) X, Y v.a.

X \ Y	-2	0	2	p_i
-1	0,2	0,3	0,1	0,6
2	0,1	0,1	0,2	0,4
q_j	0,3	0,4	0,3	1

a) $0{,}2 + x = 0{,}3 \Rightarrow x = 0{,}1$

$0{,}6 + p_2 = 1 \Rightarrow p_2 = 0{,}4$

$0{,}1 + 0{,}1 + y = 0{,}4 \Rightarrow y = 0{,}2$

$0{,}3 + 0{,}3 + z = 1 \Rightarrow z = 0{,}4$

$a + 0{,}1 = 0{,}4 \Rightarrow a = 0{,}3$

$b + 0{,}2 = 0{,}3 \Rightarrow b = 0{,}1$

b) $X : \begin{pmatrix} x_1 & x_2 \\ p_1 & p_2 \end{pmatrix}$ $\quad Y : \begin{pmatrix} y_1 & y_2 & y_3 \\ q_1 & q_2 & q_3 \end{pmatrix}$

$X : \begin{pmatrix} -1 & 2 \\ 0{,}6 & 0{,}4 \end{pmatrix}$ $\quad Y : \begin{pmatrix} -2 & 0 & 2 \\ 0{,}3 & 0{,}4 & 0{,}3 \end{pmatrix}$

c) X, Y independente $\Leftrightarrow \pi_{ij} = p_i \cdot q_j, \ \forall i = \overline{1,2}, j = \overline{1,3}$

$$\left.\begin{aligned}
i=1,\ j=1 &\Rightarrow \pi_{11} = p_1 \cdot q_1 = 0{,}6 \cdot 0{,}3 = 0{,}18 \neq 0{,}2 \\
j=2 &\Rightarrow \pi_{12} = p_1 \cdot q_2 = 0{,}6 \cdot 0{,}4 = 0{,}24 \neq 0{,}3 \\
j=3 &\Rightarrow \pi_{13} = p_1 \cdot q_3 = 0{,}6 \cdot 0{,}3 = 0{,}18 \neq 0{,}1 \\
i=2,\ j=1 &\Rightarrow \pi_{21} = p_2 \cdot q_1 = 0{,}4 \cdot 0{,}3 = 0{,}12 \neq 0{,}1 \\
j=2 &\Rightarrow \pi_{22} = p_2 \cdot q_2 = 0{,}4 \cdot 0{,}4 = 0{,}16 \neq 0{,}1
\end{aligned}\right\} \Rightarrow$$

$j=3 \Rightarrow \pi_{23} = p_2 \cdot q_3 = 0{,}4 \cdot 0{,}3 = 0{,}12 \neq 0{,}2$

$\Rightarrow$ X, Y nu sunt independente

d) $X + Y = \begin{pmatrix} x_1+y_1 & x_1+y_2 & x_1+y_3 & x_2+y_1 & x_2+y_2 & x_2+y_3 \\ \pi_{11} & \pi_{12} & \pi_{13} & \pi_{21} & \pi_{22} & \pi_{23} \end{pmatrix}$

$$= \begin{pmatrix} -3 & -1 & 1 & 0 & 2 & 4 \\ 0,2 & 0,3 & 0,1 & 0,1 & 0,1 & 0,2 \end{pmatrix} =$$

$$A: \begin{pmatrix} -3 & -1 & 0 & 1 & 2 & 4 \\ 0,2 & 0,3 & 0,1 & 0,1 & 0,1 & 0,2 \end{pmatrix}$$

$$X \cdot Y = \begin{pmatrix} x_1 \cdot y_1 & x_1 \cdot y_2 & x_1 \cdot y_3 & x_2 y_1 & x_2 \cdot y_2 & x_2 \cdot y_3 \\ \pi_{11} & \pi_{12} & \pi_{13} & \pi_{21} & \pi_{22} & \pi_{23} \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & -2 & -4 & 0 & 4 \\ 0,2 & 0,3 & 0,1 & 0,1 & 0,1 & 0,2 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & -2 & 0 & 2 & 4 \\ 0,1 & 0,1 & 0,4 & 0,2 & 0,2 \end{pmatrix}$$

$$B: \begin{pmatrix} -4 & -2 & 0 & 2 & 4 \\ 0,1 & 0,1 & 0,4 & 0,2 & 0,2 \end{pmatrix}$$

e) $\left.\begin{array}{l} \mathbb{E}(X) = \sum\limits_{i=1}^{n} x_i \cdot p_i \\ n = 2 \end{array}\right\} \Rightarrow \mathbb{E}(X) = \sum\limits_{i=1}^{2} x_i \cdot p_i = x_1 \cdot p_1 + x_2 \cdot p_2$

$= -0,6 + 0,8 = 0,2$

$\left.\begin{array}{l} \mathbb{E}(Y) = \sum\limits_{i=1}^{m} y_i \cdot q_i \\ m = 3 \end{array}\right\} \Rightarrow \mathbb{E}(Y) = \sum\limits_{i=1}^{3} y_i \cdot q_i = y_1 \cdot q_1 + y_2 \cdot q_2 +$

$+ y_3 \cdot q_3 = -0,6 + 0 + 0,6 = 0$

$Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$

$Var(Y) = \mathbb{E}(Y^2) - \mathbb{E}(Y)^2$

$X^2 : \begin{pmatrix} 1 & 4 \\ 0,6 & 0,4 \end{pmatrix}$

$\mathbb{E}(X^2) = \sum\limits_{i=1}^{2} x_i \cdot p_i = x_1 \cdot p_1 + x_2 \cdot p_2 = 0,6 + 1,6 = 2,2$

$Var(X) = 2,2 - 0,04 = 2,16$

$$Y^2 : \begin{pmatrix} 4 & 0 & 4 \\ 0,3 & 0,4 & 0,3 \end{pmatrix} \Leftrightarrow Y^2 : \begin{pmatrix} 0 & 4 \\ 0,4 & 0,6 \end{pmatrix}$$

$$\mathbb{E}(Y^2) = \sum_{i=1}^{2} y_i \cdot q_i = y_1 \cdot q_1 + y_2 \cdot q_2 = 0 + 2,4 = 2,4$$

$$Var(Y) = 2,4 - 0^2 = 2,4$$

f) $cov(X,Y) = \mathbb{E}(X \cdot Y) - \mathbb{E}(X) \cdot \mathbb{E}(Y)$

$$\mathbb{E}(X \cdot Y) = \sum_{i=1}^{5} r_i \cdot p_i = (-4) \cdot 0,1 + (-2) \cdot 0,1 + 0 \cdot 0,4 + 2 \cdot 0,2 + 4 \cdot 0,2 = -0,4 - 0,2 + 0 + 0,4 + 0,8 = 0,6$$

$$cov(X,Y) = 0,6 - 0,2 \cdot 0 = 0,6$$

$$\rho(X,Y) = \frac{cov(X,Y)}{\sqrt{Var(X) \cdot Var(Y)}} = \frac{0,6}{\sqrt{2,16 \cdot 2,4}} = \frac{0,6}{\sqrt{5,184}} \simeq \frac{0,6}{2,28} \simeq 0,26 \in [-1;1]$$

$|\rho(X,Y)| = 0,26 \in [0,25; 0,75] \Rightarrow X, Y$ sunt corelate

g) $$X|Y=2 : \begin{pmatrix} -1 & 2 \\ \frac{0,1}{0,3} & \frac{0,2}{0,3} \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 0,33 & 0,67 \end{pmatrix}$$

$$Y/X=-1 : \begin{pmatrix} -2 & 0 & 2 \\ \frac{0,2}{0,6} & \frac{0,3}{0,6} & \frac{0,1}{0,6} \end{pmatrix} = \begin{pmatrix} -2 & 0 & 2 \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{pmatrix} = \begin{pmatrix} -2 & 0 & 2 \\ 0,33 & 0,5 & 0,17 \end{pmatrix}$$

$$E(X/Y) = \sum_{x} x \, \mathbb{P}(X=x / Y=y)$$

Cum știm $X|Y=2$ și $Y/X=-1$

$$\mathbb{E}(X/Y=2) = \sum_{i=1}^{2} k_i \cdot p_i = k_1 \cdot p_1 + k_2 \cdot p_2 = -\frac{1}{3} + \frac{4}{3} = \frac{3}{3}$$

$$\mathbb{E}(Y/X=-1) = \sum_{i=1}^{3} r_i \cdot p_i = r_1 \cdot p_1 + r_2 \cdot p_2 + r_3 \cdot p_3 = -\frac{2}{3} + 0 + \frac{1}{3} = -\frac{1}{3}$$

Pb 2| X, Y v.a.

$\mathbb{P}(X=1, Y=-1) = K, \; K \in \mathbb{R}$

$$X: \begin{pmatrix} -1 & 1 \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} \qquad Y: \begin{pmatrix} -1 & 0 \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix}$$

a)

$X \backslash Y$	-1	0	r_i
-1	$\frac{1}{5} - K$	$\frac{1}{20} + K$	$\frac{1}{4}$
1	K	$\frac{3}{4} - K$	$\frac{3}{4}$
q_j	$\frac{1}{5}$	$\frac{4}{5}$	1

$\pi_{11} + K = \frac{1}{5} \Rightarrow \pi_{11} = \frac{1}{5} - K$

$\pi_{12} + \pi_{11} = \frac{1}{4} \Rightarrow \pi_{12} = \frac{1}{4} - \frac{1}{5} + K = \frac{1}{10} + K$

$K + \pi_{22} = \frac{3}{4} \Rightarrow \pi_{22} = \frac{3}{4} - K$

b) $\rho(X,Y) = \dfrac{cov(X,Y)}{\sqrt{Var(X) \cdot Var(Y)}}$ Vom considera X și Y dependente.

$cov(X,Y) = \mathbb{E}(X \cdot Y) - \mathbb{E}(X) \cdot \mathbb{E}(Y)$

$\mathbb{E}(X) = \sum_{i=1}^{2} x_i \cdot r_i = -1 \cdot \frac{1}{4} + 1 \cdot \frac{3}{4} = \frac{2}{4} = \frac{1}{2} = 0{,}5$

$\mathbb{E}(Y) = \sum_{i=1}^{2} y_i \cdot q_i = -1 \cdot \frac{1}{5} + 0 \cdot \frac{4}{5} = -\frac{1}{5} = -0{,}2$

$$X \cdot Y = \begin{pmatrix} 1 & 0 & -1 & 0 \\ \frac{1}{5} - K & \frac{1}{20} + K & K & \frac{3}{4} - K \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ K & \frac{4}{5} & \frac{1}{5} - K \end{pmatrix}$$

$\mathbb{E}(X \cdot Y) = \sum_{i=1}^{3} r_i \cdot q_i = r_1 \cdot q_1 + r_2 \cdot q_2 + r_3 \cdot q_3 =$

$= -K + 0 + \frac{1}{5} - K = \frac{1}{5} - 2K$

$cov(X,Y) = \frac{1}{5} - 2K + 0{,}1 = \frac{1}{5} - 2K + \frac{1}{10} = \frac{3}{10} - 2K$

$Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$

$X^2 = \begin{pmatrix} 1 & 1 \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\mathbb{E}(X^2) = 1$

$Var(X) = 1 - \left(\frac{1}{2}\right)^2 = 1 - \frac{1}{4} = \frac{3}{4} = 0,75$

$Var(Y) = \mathbb{E}(Y^2) - \mathbb{E}(Y)^2$

$Y^2 = \begin{pmatrix} 1 & 0 \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix}$

$\mathbb{E}(Y^2) = \sum_{i=1}^{2} y_i \cdot q_i = \frac{1}{5}$

$Var(Y) = \frac{1}{5} - \frac{1}{25} = \frac{4}{25} = 0,16$

$\rho(X,Y) = \frac{3-20K}{10\sqrt{\frac{3}{4}\cdot\frac{4}{25}}} = \frac{3-20K}{\frac{10}{5}\cdot\sqrt{3}} = \frac{3-20K}{2\sqrt{3}}$

c) X şi Y necorelate $\Rightarrow \rho = 0 \Rightarrow \frac{3-20K}{2\sqrt{3}} = 0 \Rightarrow 3-20K = 0 \Rightarrow$

$\Rightarrow 20K = 3 \Rightarrow K = \frac{3}{20} \in \mathbb{R}$

$X \backslash Y$	-1	0	p_i
-1	$\frac{1}{20}$	$\frac{1}{5}$	$\frac{1}{4}$
1	$\frac{3}{20}$	$\frac{3}{5}$	$\frac{3}{4}$
q_j	$\frac{1}{5}$	$\frac{4}{5}$	1

$\frac{1}{5} - K = \frac{1}{5} - \frac{3}{20} = \frac{4-3}{20} = \frac{1}{20}$

$\frac{1}{20} + K = \frac{1}{20} + \frac{3}{20} = \frac{4}{20} = \frac{1}{5}$

$\frac{3}{4} - K = \frac{3}{4} - \frac{3}{20} = \frac{12}{20} = \frac{3}{5}$

X, Y independente $\Leftrightarrow \pi_{ij} = p_i \cdot q_j,\ i = \overline{1,2},\ j = \overline{1,2}$

$$\left.\begin{array}{l} i=1,\ j=1 \Rightarrow \pi_{11} = r_1 \cdot q_1 = \frac{1}{4} \cdot \frac{1}{5} = \frac{1}{20}\ (A) \\ j=2 \Rightarrow \pi_{12} = r_1 \cdot q_2 = \frac{1}{4} \cdot \frac{4}{5} = \frac{1}{5}\ (A) \\ i=2,\ j=1 \Rightarrow \pi_{21} = r_2 \cdot q_1 = \frac{3}{4} \cdot \frac{1}{5} = \frac{3}{20}\ (A) \\ j=2 \Rightarrow \pi_{22} = r_2 \cdot q_2 = \frac{3}{4} \cdot \frac{4}{5} = \frac{3}{5}\ (A) \end{array}\right\} \Rightarrow$$

$\Rightarrow X, Y$ sunt independente