Computer Vision

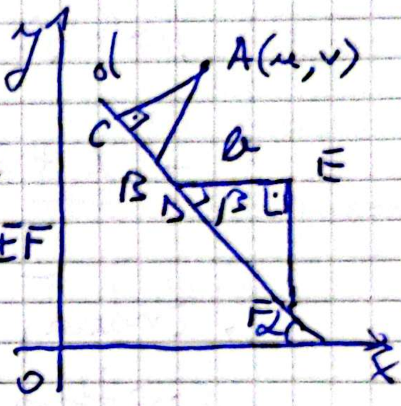
Homework 4

1. Prove that if $a^2+b^2=1$ then $d(A(u,v), ax+by+c=0) = \cancel{|au+bv+c|}$ $|au+bv+c|$

Two main parts:

a) Prove the distance formula

b) Show the equality when using $a^2+b^2=1$ constraint.




Consider the right picture. The distance of interest is $|\vec{AC}|$! To find this vector length, we can use a right triangle $\Delta DEF$ whose upper side is $|\vec{ED}| = b$.

The motivation is to make use of the slope in order to compute the other side, that is $|\vec{EF}|$

$\Rightarrow tg\beta = \frac{|\vec{EF}|}{|\vec{DE}|}$, but $tg\beta = -tg\alpha = -\left(-\frac{a}{b}\right) = \frac{a}{b}$

$\Rightarrow |\vec{EF}| = a$. Using $\Delta DEF$ we can find a relation for $|\vec{AC}|$ by thinking about the similarity between these two triangles, meaning that $\frac{|\vec{AC}|}{|\vec{AB}|} = \frac{|\vec{DE}|}{|\vec{DF}|}$ due to the fact that $\measuredangle ACB = \measuredangle DEF = \frac{\pi}{2}$ and $\vec{AB} \parallel \vec{EF}$. So, $|\vec{AC}| = \frac{|\vec{DE}|\cdot|\vec{AB}|}{|\vec{DF}|} = \frac{b\cdot|\vec{AB}|}{\sqrt{a^2+b^2}}$

$\vec{AB} = (u-u)\vec{i} + (m-v)\vec{j} = (m-v)\vec{j}$

$\Rightarrow |\vec{AB}| = \sqrt{(m-v)^2} = |m-v|$, ~~then~~ $au+bm+c=0$

$\Leftrightarrow c = -au-bm \Leftrightarrow m = \frac{-c-au}{b}$

$$\Rightarrow |\vec{AC}| = \frac{b\cdot\left|\frac{-c-au}{b} - v\right|}{\sqrt{a^2+b^2}} = \frac{|au+bv+c|}{\sqrt{a^2+b^2}}$$

Using $a^2+b^2=1 \Rightarrow d(A(u,v), ax+by+c=0) = |au+bv+c|$ QED.

2. Let $\mathcal{C}$ be a 2D curve, $\mathcal{C}: \begin{cases} x(t) = \dfrac{1-t^2}{1+t^2} \\ y(t) = \dfrac{2t}{1+t^2} \end{cases}$

Show that $\mathcal{C}$ is a circular arc

Intuitive, $\mathcal{C}$ looks like a parametrization of a circle, in $t$ (the variable, also called the parameter). Recall the equation of a circle, in the $V = \mathbb{R}^2$ vector space, that is $(x-x_0)^2 + (y-y_0)^2 = r^2$, where $(x_0, y_0)$ are the coordinates of the origin, and $r$ is the radius.

Computing $x^2(t) + y^2(t)$ should result in a constant, so $x^2(t) + y^2(t) = \left(\dfrac{1-t^2}{1+t^2}\right)^2 + \left(\dfrac{2t}{1+t^2}\right)^2 = \dfrac{1-2t^2+t^4+4t^2}{(1+t^2)^2}$

$= \dfrac{1+2t^2+t^4}{(1+t^2)^2} = \dfrac{(1+t^2)^2}{(1+t^2)^2} = 1 = 1^2$, so $r = 1$, and

, indeed, $\mathcal{C}$ is a circular arc, as it can be also written as $\mathcal{C}: \begin{cases} x(t) = 1 \cdot \cos(2\arctan t) \\ y(t) = 1\sin(2\arctan t) \end{cases}$. This is

because $\cos\theta = \dfrac{1-tg^2\frac{\theta}{2}}{1+tg^2\frac{\theta}{2}}$ and $\sin\theta = \dfrac{2tg\frac{\theta}{2}}{1+tg^2\frac{\theta}{2}}$

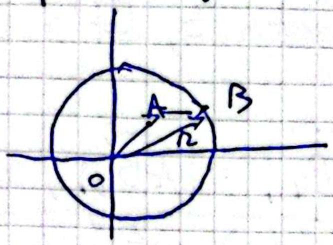
3/4. The equation (in $t$) for the closest point on $\mathcal{C}$ to some data point $A(x, y)$?

Find the degree of it and the number of possible solutions!

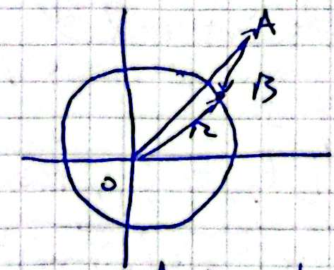
~~Let~~ If the point $A(x_A, y_A)$ is on the arc, there is a family of solutions: $\begin{cases} x(t) = \cos(2\arctan t + 2k\pi) \\ y(t) = \sin(2\arctan t + 2k\pi) \end{cases}$

, $k \in \mathbb{Z}$, which means $x(t) = x_A$ and $y(t) = y_A$ $\begin{cases} t = tg\left(\dfrac{\arccos x_A}{2} - k\pi\right) = tg\left(\dfrac{\arccos x_A}{2}\right) \\ t = tg\left(\dfrac{\arcsin y_A}{2} - k\pi\right) = tg\left(\dfrac{\arcsin y_A}{2}\right) \end{cases}$

If $A(x,y)$ is not on the arc, then there are 2 cases



A is inside the circle



A is outside the circle

To solve both cases, consider this vector $\vec{AB} = (x(t)-x)\vec{i} + (y(t)-y)\vec{j}$. Finding the closest point (which is B in this case, randomly chosen in both pictures), we need to minimize $|\vec{AB}|$ in terms of $t$.

$$|\vec{AB}| = \sqrt{(x(t)-x)^2 + (y(t)-y)^2} = \sqrt{x^2(t) - 2x(t)x + x^2 + y^2(t) - 2y(t)y + y^2} = \sqrt{x^2+y^2-2(x(t)x+y(t)y)}$$

The square root monotically increases, so finding the minimum of $f(t) = x^2+y^2-2(x(t)x+y(t)y)$ will suffice

$$f(t) = x^2+y^2-2\left(x\frac{1-t^2}{1+t^2} + y\frac{2t}{1+t^2}\right).$$

$$\frac{df(t)}{dt} = -2\left[x\frac{-2t(1+t^2)-2t(1-t^2)}{(1+t^2)^2} + y\frac{2(1+t^2)-4t^2}{(1+t^2)^2}\right]$$

$$= \frac{8xt}{(1+t^2)^2} + \frac{y(4t^2-4)}{(1+t^2)^2} = \frac{8xt+4yt^2-4y}{(1+t^2)^2}$$

$(1+t^2)^2 > 0, \forall t \in \mathbb{R}$

$$\Rightarrow \frac{df(t)}{dt} = 0 \iff 4yt^2+8xt-4y=0 \iff yt^2+2xt-y=0$$

2nd degree equation

if $x=y=0$ (the center of $\mathcal{C}$) $\Rightarrow$ there are infinitely many $t$ with equal distance (that is $r=1$, the radius)

so, considering $x \neq 0$ and $y \neq 0$, $t_1 = \frac{-2x+\sqrt{4x^2+4y^2}}{2y}$ and $t_2 = \frac{-2x-\sqrt{4x^2+4y^2}}{2y} \Rightarrow t_{1,2} = \frac{-x \pm \sqrt{x^2+y^2}}{2y}$

Obviously, the correct formula of $t$ will be chosen based on the interval of parameter $t$, let's call it $I = [t_s, t_f]$

If $t_1, t_2 \in [t_s, t_f]$, then $|\vec{AB_1}| = |\vec{AB_2}|$, meaning that $\sqrt{(x(t_1)-x)^2+(y(t_1)-y)^2} = \sqrt{(x(t_2)-x)^2+(y(t_2)-y)^2}$

So, the two families of solution are:

$$B_1 : \begin{cases} x_{B_1} = x(t_1) \\ y_{B_1} = y(t_1) \\ \text{if } t_1 \in I \end{cases} \quad \text{and} \quad B_2 : \begin{cases} x_{B_2} = x(t_2) \\ y_{B_2} = y(t_2) \\ \text{if } t_2 \in I \end{cases}$$

And, in the general case:

$$B_1 : \begin{cases} x_{B_1} = x(t_1) + 2k\pi \\ y_{B_1} = y(t_1) + 2k\pi \\ \text{if } t_1 \in I \end{cases} \quad \text{and} \quad B_2 : \begin{cases} x_{B_2} = x(t_2) + 2k\pi \\ y_{B_2} = y(t_2) + 2k\pi \\ \text{if } t_2 \in I \end{cases}$$

5 Verify these properties of rotation matrices:
a) their inverse is their transpose (orthogonality)
b) their determinant equals 1.

2D case

$$R(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, \text{ also called Givens matrix}$$

$$R^t(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$R(\alpha)R^t(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha+\sin^2\alpha & \cos\alpha\sin\alpha-\sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha-\sin\alpha\cos\alpha & \sin^2\alpha+\cos^2\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$\det(R(\alpha)) = \begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix} = \cos^2\alpha + \sin^2\alpha = 1$$

3D Case

$$R_1(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}, \quad R_2(\alpha) = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

$$, R_3(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1(\alpha) R_1^t(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos^2\alpha+\sin^2\alpha & 0 \\ 0 & 0 & \cos^2\alpha+\sin^2\alpha \end{bmatrix} = I_3$$

$$R_2(\alpha) R_2^t(\alpha) = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha+\sin^2\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos^2\alpha+\sin^2\alpha \end{bmatrix} = I_3$$

$$R_3(\alpha) R_3^t(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha+\sin^2\alpha & 0 & 0 \\ 0 & \cos^2\alpha+\sin^2\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

$$\det(R_1(\alpha)) = \det(R_2(\alpha)) = \det(R_3(\alpha))$$

$$= \begin{vmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{vmatrix}$$

$$= (-1)^{2k} \cdot \begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix} = 1.$$

General case

$$R_n(\alpha) = \begin{array}{c} \\ \\ k \\ \\ l \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} \quad & k & \quad & l & \quad \end{array} \\ \begin{bmatrix} I_1 & 0 & 0 & 0 & 0 \\ 0 & \cos\alpha & 0 & -\sin\alpha & 0 \\ 0 & 0 & I_2 & 0 & 0 \\ 0 & \sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 0 & I_3 \end{bmatrix} \end{array}$$

those zeros are vectors, not scalars.

where $I_1 \in M(\mathbb{R})_{k-1,k-1}$, $I_2 \in M(\mathbb{R})_{l-k-1,l-k-1}$, $I_3 \in M(\mathbb{R})_{n-l,n-l}$ are identities of different sizes.

To calculate the determinant and show that is equal to 1, we will use induction

Base case:

$$R_2 = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, \quad \det(R_2) = 1$$

Induction Hypothesis:

Suppose $\det(R_n) = 1$ is true. $(P(n))$

Induction Step $(P(n) \to P(n+1))$

We want to show that $\det(R_{n+1}) = 1$, using $\det(R_n) = 1$

There are 3 cases of $R_{n+1}$:

a) $R_{n+1} = \begin{bmatrix} 1 & 0 \\ 0 & R_n \end{bmatrix}$, so $\det(R_{n+1}) = \begin{vmatrix} 1 & 0 \\ 0 & R_n \end{vmatrix} = (-1)^{1+1} \det(R_n) = 1$

b) $R_{n+1} = \begin{bmatrix} R_{11_n} & 0 & R_{12_n} \\ 0 & 1 & 0 \\ R_{21_n} & 0 & R_{22_n} \end{bmatrix}$, where $R_{11n}, R_{12n}, R_{21n}, R_{22n}$

are matrices (blocks) from $R_n$ (up-left, up-right, down-left, down-right),

so $\det(R_{n+1}) = \det\left(\begin{bmatrix} R_{11n} & 0 & R_{12n} \\ 0 & 1 & 0 \\ R_{21n} & 0 & R_{22n} \end{bmatrix}\right) = \begin{vmatrix} R_{1n} & 0 & R_{12n} \\ 0 & 1 & 0 \\ R_{21n} & 0 & R_{22n} \end{vmatrix}$

$= (-1)^{k+k} \cdot \begin{vmatrix} R_{11n} & R_{12n} \\ R_{21n} & R_{22n} \end{vmatrix} = (-1)^{2k} \cdot \det(R_n) = 1$

c) $R_{n+1} = \begin{bmatrix} R_n & 0 \\ 0 & 1 \end{bmatrix}$, so $\det(R_{n+1}) = \begin{vmatrix} R_n & 0 \\ 0 & 1 \end{vmatrix} =$

$= (-1)^{2n+2} \det(R_n) = 1$ what we wanted to show.

Hence, $\det(R_n) = 1, \forall n \in \mathbb{N}^*, n \geq 2$

To show that the inverse of $R_n$ is $R_n^t$ $\forall n \in \mathbb{N}^*, n \geq 2$, we will use induction

Base case

$R_2 = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, $R_2^{-1} = R_2^t$ (it was already shown)

Induction Hypothesis:

Suppose $R_n \cdot R_n^t = R_n^t R_n = I_n$ $(P(n))$

Induction Step $(P(n) \to P(n+1))$

We want to show that $R_{n+1} R_{n+1}^t = R_{n+1}^t R_{n+1} = I_{n+1}$, by using $P(n)$.

There are 3 cases of $R_{n+1}$:

a) $R_{n+1} = \begin{bmatrix} 1 & 0_n \\ 0_n^t & R_n \end{bmatrix}$, so $R_{n+1}^t = \begin{bmatrix} 1 & (0_n^t)^t \\ 0_n^t & R_n^t \end{bmatrix} = \begin{bmatrix} 1 & 0_n \\ 0_n^t & R_n^t \end{bmatrix}$

$$R_{n+1}R_{n+1}^t = \begin{bmatrix} 1 & 0_n \\ 0_n^t & R_n \end{bmatrix}\begin{bmatrix} 1 & 0_n \\ 0_n^t & R_n^t \end{bmatrix} = \begin{bmatrix} 1 & 0_n \\ 0_n^t & R_n R_n^t \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0_n \\ 0_n^t & I_n \end{bmatrix} = I_{n+1}$$

b) $R_{n+1} = \begin{bmatrix} R_n & 0_n^t \\ 0_n & 1 \end{bmatrix}$, so $R_{n+1}^t = \begin{bmatrix} R_n^t & 0_n^t \\ (0_n^t)^t & 1 \end{bmatrix} = \begin{bmatrix} R_n^t & 0_n^t \\ 0_n & 1 \end{bmatrix}$

$$R_{n+1}R_{n+1}^t = \begin{bmatrix} R_n & 0_n^t \\ 0_n & 1 \end{bmatrix}\begin{bmatrix} R_n^t & 0_n^t \\ 0_n & 1 \end{bmatrix} = \begin{bmatrix} R_n R_n^t & 0_n^t \\ 0 & 1 \end{bmatrix} = I_{n+1}$$

c) $R_{n+1} = \begin{bmatrix} R_{11n} & 0 & R_{12n} \\ 0 & 1 & 0 \\ R_{21n} & 0 & R_{22n} \end{bmatrix}$, an observation would be that the columns of $R_{n+1}$ forms a basis in $\mathbb{R}^{n+1}$, so $R_{n+1}R_{n+1}^t = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases} \Rightarrow R_{n+1}R_{n+1}^t = I_{n+1}$

6 You are given 4 points, A, B, C, D in 3D. Write an algorithm which shows wether the projection of D on the plane defined by points A, B, and C falls inside the triangle (A, B, C).

It is known that 3 points can form a triangle if they are not colinear.

Assume that the cross product is implemented

string procedure (A, B, C, D)

{

$\vec{OA} = x_A\vec{i} + y_A\vec{j} + z_A\vec{k}$

$\vec{OB} = x_B\vec{i} + y_B\vec{j} + z_B\vec{k}$

$\vec{OC} = x_C\vec{i} + y_C\vec{j} + z_C\vec{k}$

$\vec{AB} = \vec{OB} - \vec{OA} = (x_B - x_A)\vec{i} + (y_B - y_A)\vec{j} + (z_B - z_A)\vec{k}$

$\vec{AC} = \vec{OC} - \vec{OA} = (x_C - x_A)\vec{i} + (y_C - y_A)\vec{j} + (z_C - z_A)\vec{k}$

$\vec{n} = \vec{AB} \times \vec{AC}$ // $(a, b, c)$

if $\vec{n} = (0, 0, 0)$

  return "A, B, C are colinear!"

$$var = \frac{|a(x_D - x_A) + b(y_D - y_A) + c(z_D - z_A)|}{\sqrt{a^2 + b^2 + c^2}}$$

$O_{D'} = (x_D + a \cdot var,\ y_D + b \cdot var,\ z_D + c \cdot var)$

$\vec{OD'} = x_{D'} \vec{i} + y_{D'} \vec{j} + z_{D'} \vec{k}$ // D' is inside, $D' \in$ Plane A,B,C

$\vec{BC} = \vec{OC} - \vec{OB}$

$\vec{CA} = \vec{OA} - \vec{OC}$

$\vec{AD'} = \vec{OD'} - \vec{OA}$

$\vec{BD'} = \vec{OD'} - \vec{OB}$

$\vec{CD'} = \vec{OD'} - \vec{OC}$

$\vec{n_1} = (\vec{AB}, 0) \times (\vec{AD'}, 0)$

$\vec{n_2} = (\vec{BC}, 0) \times (\vec{BD'}, 0)$

$\vec{n_3} = (\vec{CA}, 0) \times (\vec{CD'}, 0)$ // check the sign

if $n_1[3] == n_2[3]$ && $n_2[3] == n_3[3]$

  return "Success, D is inside the $\Delta ABC$"

return "D is not inside the $\Delta ABC$"

}

7. Show that up to half of the elements in the neighborhood could be noise values and a median filter would still give the same (correct) answer when removing salt-and-pepper noise from a uniform (const.) background.

Let the window of filtering have size $= [n \times n]$ ($[m \times n]$)

Without noise, the hypothesis tells us that the background is constant and highlighted as $[ct, ct, \ldots, ct]$, $ct \in [0, 255]$

, namely a sorted list

Define the middle element as $middle = \begin{cases} \text{median, if } n \text{ is odd} \\ \text{mean of the middles, if } n \text{ is even} \end{cases}$

Based on the definition of salt-and-pepper, it is known that is a high intensity effect that lights or darkens a pixel, for simplicity let's assume that light = 255 and dark = 0 (both are noises).

The cases in which out of the $\frac{n^2}{2}$ there is both light and dark noise are simple $l = [\underbrace{0, 0, \dots, 0}_{k}, \underbrace{ct, ct \dots ct}_{\frac{n^2}{2}}, \underbrace{255 \dots 255}_{\frac{n^2}{2} - k}]$

if $n \% 2 == 1 \Rightarrow l[\frac{n^2}{2}] = ct \Rightarrow$ correct value

if $n \% 2 == 0 \Rightarrow \begin{array}{l} l[\frac{n^2}{2}] = ct \\ l[\frac{n^2}{2} + 1] = ct \end{array} \Bigg| \Rightarrow \frac{ct + ct}{2} = ct \Rightarrow$ correct value.

Now, consider that we have only light points or dark points. We employ the case in which there are only dark points on a side. The chance of getting dark for a pixel is $p = \frac{1}{2}$, and to get all of them on the same side is $p = \frac{1}{2^{\frac{n^2}{2}}}$

$l = [\underbrace{0\ 0 \dots 0}_{\lfloor \frac{n^2}{2} \rfloor}\ \underbrace{ct\ ct \dots ct}_{\lceil \frac{n^2}{2} \rceil}]$

$l[\frac{n^2}{2}] = ct \Rightarrow$ correct answer (for $n$ odd)

for $n$ even $\Rightarrow \begin{array}{l} l[\frac{n^2}{2}] = 0 \\ l[\frac{n^2}{2}] = ct \end{array} \Bigg| \Rightarrow \frac{ct}{2}$ incorrect answer (it darkens the correct background

8. Find $w$ which minimizes the function $J(w) = (Aw - b)^t (Aw - b)$

$$\nabla_w J = \nabla_w \left( (Aw)^t Aw - (Aw)^t b - b^t Aw + b^t b \right)$$

$$= \cancel{\nabla_w \left( w^t A^t Aw - w^t A^t b - (w^t A^t b)^t \right)}$$

$$= \nabla_w \left( w^t A^t Aw - w^t A^t b - (w^t A^t b)^t + b^t b \right)$$

$$= \nabla_w \left( w^t A^t Aw - w^t A^t b - b^t (w^t A^t)^t + b^t b \right), \quad b^t Aw = (A^t b)^t w$$

$$\cancel{= 2wA} \qquad \nabla_w \left( (A^t b)^t w \right) = A^t b$$

$$\Rightarrow \nabla_w J(w) = 2A^t Aw - 2A^t b = 0 \text{ (set it to 0)}$$

$$\Leftrightarrow w^* = (A^t A)^{-1} A^t b .$$