

# Tema 13. Oscilații armonice libere

Oscilații  $\left\{ \begin{array}{l} \text{libere sau proprii} \\ \text{forțate} \end{array} \right.$

Relația dintre frecvență și perioadă

frecvență liniară  $\nu = \frac{N}{t} = \frac{N}{NT} = \frac{1}{T}$       frecvență ciclică  $\omega = 2\pi\nu = \frac{2\pi}{T}$

Pentru orice mărime fizică  $\xi$  care variază periodic

$$\xi(t+T) = \xi(t)$$

Dacă variația are loc conform unei legi armonice, atunci oscilațiile se numesc armonice și au loc în conformitate cu legea

$$\xi(t) = A \sin(\omega_0 t + \varphi_0)$$

$\xi_{\max} = A = \text{const} > 0$  – amplitudinea oscilației;

$\Phi(t) = \omega_0 t + \varphi_0$  – faza oscilației;

$\Phi(0) = \varphi_0$  – faza inițială a oscilației;

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Viteza și accelerația oscilațiilor

$$\left. \begin{aligned} \dot{\xi} &= A\omega_0 \cos(\omega_0 t + \varphi_0) = A\omega_0 \sin(\omega_0 t + \varphi_0 + \pi/2) \\ \ddot{\xi} &= -A\omega_0^2 \sin(\omega_0 t + \varphi_0) = A\omega_0^2 \sin(\omega_0 t + \varphi_0 + \pi) \end{aligned} \right\} \rightarrow \ddot{\xi} = -\omega_0^2 \xi$$

Ecuția diferențială a oscilațiilor armonice

$$\ddot{\xi} + \omega_0^2 \xi = 0$$

Considerăm oscilațiile mecanice ale unui punct material de-a lungul axei  $x$ .  $\xi(t) = x(t)$ . Atunci

$$\begin{aligned} v = \dot{x} &= A\omega_0 \cos(\omega_0 t + \varphi_0) = v_0 \cos(\omega_0 t + \varphi_0), \\ a = \ddot{x} &= -A\omega_0^2 \sin(\omega_0 t + \varphi_0) = -a_0 \sin(\omega_0 t + \varphi_0) = -\omega_0^2 x, \end{aligned}$$

$$F = ma \Rightarrow F = -m\omega_0^2 x \Rightarrow \vec{F} = -m\omega_0^2 x \cdot \vec{i}$$

Forțele proporționale cu abaterea de la poziția de echilibru (liniare) se numesc **quasielastice**

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Energia cinetică a unui punct material care oscilează

$$E_c = \frac{mv^2}{2} = \frac{mv_0^2 \cos^2(\omega_0 t + \varphi_0)}{2} = \frac{1}{2} mA^2 \omega_0^2 \cos^2(\omega_0 t + \varphi_0) \quad \rightarrow$$

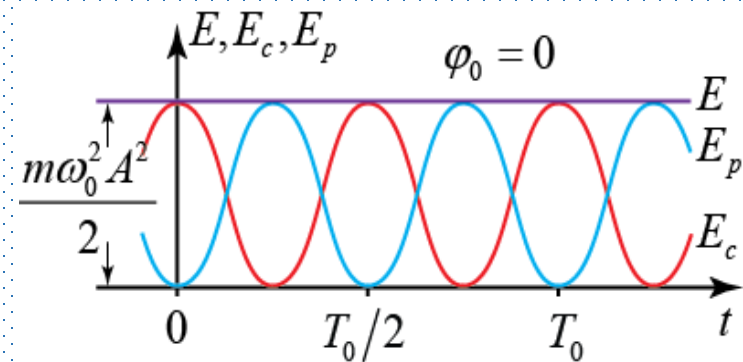
$$\rightarrow E_c = \frac{1}{4} mA^2 \omega_0^2 [1 + \cos(2\omega_0 t + 2\varphi_0)]$$

Energia potențială a unui punct material care oscilează

$$E_p = \frac{m\omega_0^2 x^2}{2} = \frac{1}{2} mA^2 \omega_0^2 \sin^2(\omega_0 t + \varphi_0) \quad \rightarrow$$

$$\begin{aligned} \rightarrow E_p &= \frac{1}{4} mA^2 \omega_0^2 [1 - \cos(2\omega_0 t + 2\varphi_0)] = \\ &= \frac{1}{4} mA^2 \omega_0^2 [1 + \cos(2\omega_0 t + 2\varphi_0 + \pi)] \end{aligned}$$

$$E = E_c + E_p = \frac{mA^2 \omega_0^2}{2} = \text{const}$$

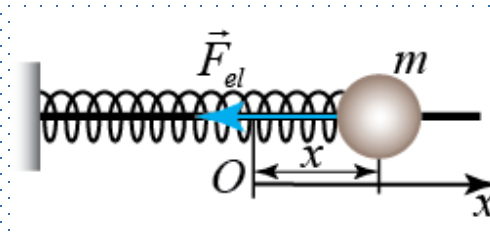


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## Pendulul elastic

$$ma = F_{el.} \quad \longrightarrow \quad m\ddot{x} = -kx \quad \longrightarrow \quad \ddot{x} + \frac{k}{m}x = 0$$

$$\omega_0^2 = \frac{k}{m} \quad \longrightarrow \quad \omega_0 = \sqrt{\frac{k}{m}}$$



Perioada pendulului elastic

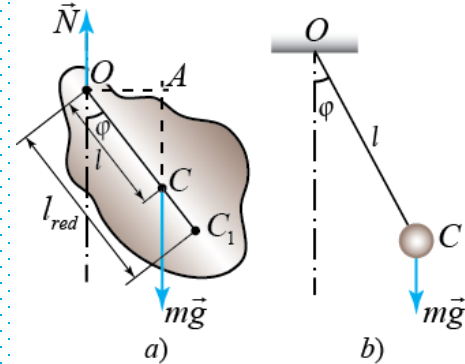
$$T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}$$

Punctul material care efectuează oscilații armonice de-a lungul unei drepte sub acțiunea forței elastice sau cvasielastice se numește **oscilator liniar armonic**.

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## Pendulul fizic

$$\left. \begin{aligned} M &= I\varepsilon \\ M &= -mgl \sin \varphi \\ \varepsilon &= \ddot{\varphi} \end{aligned} \right\} \begin{aligned} &\longrightarrow I\ddot{\varphi} = -mgl \sin \varphi \\ &\longrightarrow \ddot{\varphi} + \frac{mgl}{I} \sin \varphi = 0 \end{aligned} \longrightarrow$$



Pentru unghiuri mici  $\sin \varphi \approx \varphi \longrightarrow \ddot{\varphi} + \omega_0^2 \varphi = 0$

$$\omega_0 = \sqrt{\frac{mgl}{I}}$$

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi \sqrt{\frac{I}{mgl}}$$

Dacă masa pendulului fizic este concentrată în centrul său de greutate, atunci pendulul fizic se transformă într-un pendul matematic

$$T_0 = 2\pi \sqrt{\frac{I}{mgl}} = 2\pi \sqrt{\frac{l}{g}}$$

Lungimea unui pendul matematic care oscilează cu aceeași perioadă cu cea a pendulului fizic se numește **lungime redusă**

$$\frac{I}{mgl} = \frac{l_{red}}{g} \longrightarrow l_{red} = \frac{I}{ml}$$

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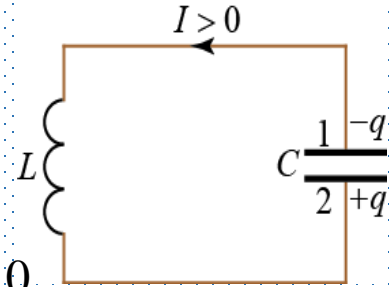
## Oscilatii armonice libere intr-un circuit oscilant electric

$$\begin{aligned}
 & \left. \begin{aligned} IR &= \varphi_1 - \varphi_2 + \mathcal{E}_{ai} \\ \mathcal{E}_{ai} &= -L \frac{dI}{dt} = -L\dot{I} \\ I &= \frac{dq}{dt} = \dot{q}, \quad \varphi_1 - \varphi_2 = -\frac{q}{C} \end{aligned} \right\} \begin{aligned} & \longrightarrow IR = -\frac{q}{C} - L\dot{I} \longrightarrow \ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{LC}q = 0 \\ & \ddot{q} + \frac{1}{LC}q = 0 \end{aligned} \end{aligned}$$

$\omega_0 = \frac{1}{\sqrt{LC}}$

$T_0 = 2\pi\sqrt{LC}$

$$q = q_0 \sin(\omega_0 t + \varphi_0)$$



$$I = \dot{q} = q_0 \omega_0 \cos(\omega_0 t + \varphi_0) = I_0 \cos(\omega_0 t + \varphi_0) = I_0 \sin(\omega_0 t + \varphi_0 + \pi/2)$$

$$U = \varphi_2 - \varphi_1 = \frac{q}{C} = \frac{q_0}{C} \sin(\omega_0 t + \varphi_0) = U_0 \sin(\omega_0 t + \varphi_0) \quad I_0 = \frac{q_0}{\sqrt{LC}} = \frac{U_0}{\sqrt{L/C}}$$

$$W_e = \frac{q^2}{2C} = \frac{q_0^2}{2C} \sin^2(\omega_0 t + \varphi_0) = \frac{q_0^2}{4C} [1 - \cos(2\omega_0 t + 2\varphi_0)]$$

$$W_m = \frac{LI^2}{2} = \frac{LI_0^2}{2} \cos^2(\omega_0 t + \varphi_0) = \frac{LI_0^2}{2} [1 + \cos(2\omega_0 t + 2\varphi_0)] = \frac{LI_0^2}{2} [1 - \cos(2\omega_0 t + 2\varphi_0 + \pi)]$$

$$W_e + W_m = \frac{q_0^2}{2C} = \frac{LI_0^2}{2} = \text{const}$$

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## Compunerea oscilațiilor armonice colineare

$$\xi_1(t) = A_1 \sin[\Phi_1(t)] = A_1 \sin(\omega_0 t + \varphi_{01})$$

$$\xi_2(t) = A_2 \sin[\Phi_2(t)] = A_2 \sin(\omega_0 t + \varphi_{02})$$

$$\xi(t) = \xi_1(t) + \xi_2(t)$$

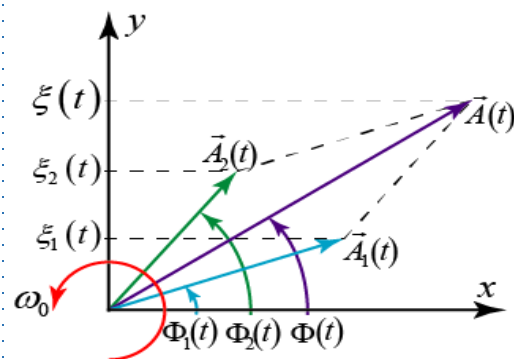
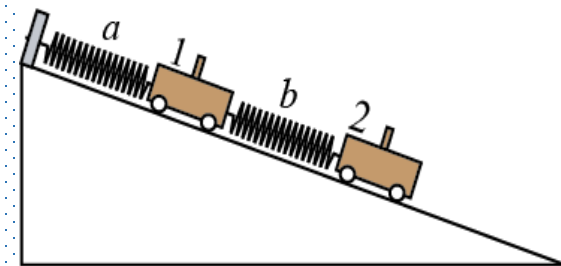
$$\xi(t) = A \sin[\Phi(t)] = A \sin(\omega_0 t + \varphi_0)$$

$$A^2 = A_1^2 + A_2^2 + 2A_1A_2 \cos(\varphi_{02} - \varphi_{01})$$

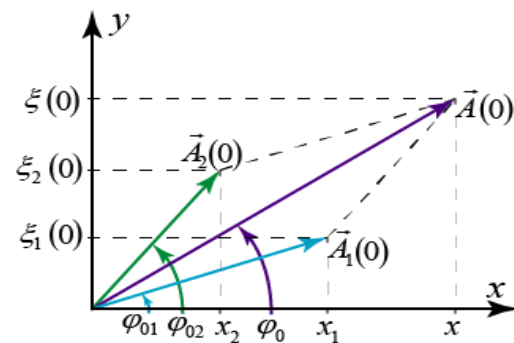
$$\operatorname{tg} \varphi_0 = \frac{\xi(0)}{x} = \frac{\xi_1(0) + \xi_2(0)}{x_1 + x_2} = \frac{A_1 \sin \varphi_{01} + A_2 \sin \varphi_{02}}{A_1 \cos \varphi_{01} + A_2 \cos \varphi_{02}}$$

$$\Delta\varphi = \pm 2m\pi \quad \longrightarrow \quad A_{\max} = A_1 + A_2$$

$$\Delta\varphi = \pm(2m+1)\pi \quad \longrightarrow \quad A_{\min} = |A_1 - A_2|$$



a)



b)

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## Cazul frecvențelor de oscilație care se deosebesc foarte puțin. Bătăi

$$\xi_1(t) = A_0 \sin(\omega_1 t)$$

$$\xi_2(t) = A_0 \sin(\omega_1 + \Delta\omega)t$$

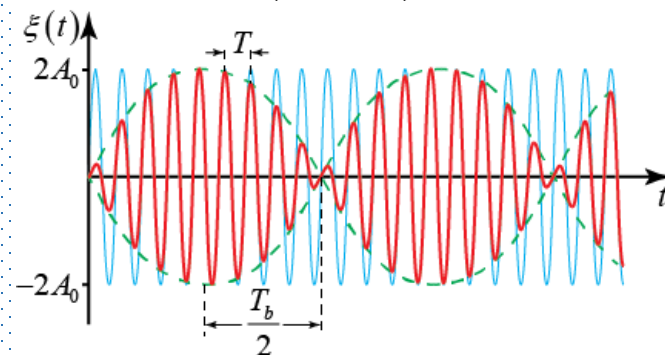
$$\Delta\omega = \omega_2 - \omega_1 \ll \omega_1$$

$$\sin \alpha + \sin \beta = 2 \sin\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right)$$

$$\xi(t) = \xi_1(t) + \xi_2(t) = 2A_0 \cos\left(\frac{\Delta\omega}{2}t\right) \sin \omega_1 t$$

$$T_b = \frac{2\pi}{\Delta\omega} = \frac{4\pi}{\omega_2 - \omega_1} \quad \longrightarrow \quad \nu_b = \frac{1}{T_b} = \frac{\omega_2 - \omega_1}{4\pi}$$

$$T = \frac{2\pi}{\omega_1} \ll T_b$$



Orice oscilație nearmonică, dar periodică poate fi prezentată ca o superpoziție a oscilațiilor armonice

$$\xi(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n \sin(n\omega_0 t + \varphi_n)$$

**Analiza armonică permite determinarea spectrului de frecvențe.**



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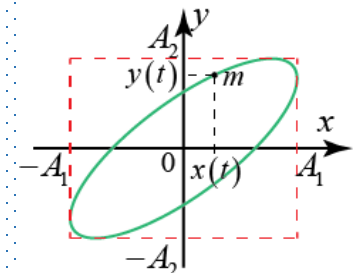
## Compunerea oscilațiilor armonice reciproc perpendiculare

$$\begin{cases} x(t) = A_1 \sin(\omega_0 t + \varphi_1), \\ y(t) = A_2 \sin(\omega_0 t + \varphi_2). \end{cases} \rightarrow \begin{cases} \frac{x(t)}{A_1} = \sin(\omega_0 t + \varphi_1) = \sin \omega_0 t \cos \varphi_1 + \cos \omega_0 t \sin \varphi_1, \\ \frac{y(t)}{A_2} = \sin(\omega_0 t + \varphi_2) = \sin \omega_0 t \cos \varphi_2 + \cos \omega_0 t \sin \varphi_2. \end{cases}$$

$$\rightarrow \begin{cases} \frac{x}{A_1} \sin \varphi_2 - \frac{y}{A_2} \sin \varphi_1 = \sin \omega_0 t (\sin \varphi_2 \cos \varphi_1 - \cos \varphi_2 \sin \varphi_1) = \sin \omega_0 t \sin(\varphi_2 - \varphi_1), \\ \frac{x}{A_1} \cos \varphi_2 - \frac{y}{A_2} \cos \varphi_1 = \cos \omega_0 t (\sin \varphi_1 \cos \varphi_2 - \cos \varphi_1 \sin \varphi_2) = -\cos \omega_0 t \sin(\varphi_2 - \varphi_1). \end{cases}$$

$$\rightarrow \frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} - 2 \frac{x}{A_1} \frac{y}{A_2} (\sin \varphi_2 \sin \varphi_1 + \cos \varphi_2 \cos \varphi_1) = \sin^2(\varphi_2 - \varphi_1) (\sin^2 \omega_0 t + \cos^2 \omega_0 t) \rightarrow$$

$$\rightarrow \frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} - 2 \frac{x}{A_1} \frac{y}{A_2} \cos(\varphi_2 - \varphi_1) = \sin^2(\varphi_2 - \varphi_1)$$

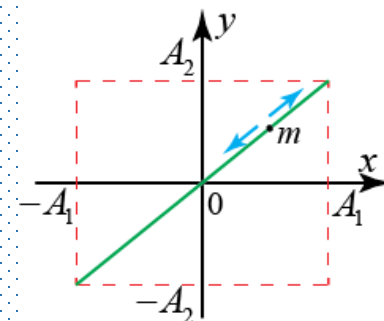


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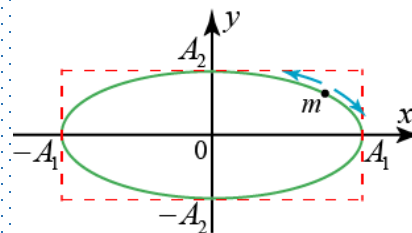
1)  $\varphi_2 - \varphi_1 = 2n\pi; (2n+1)\pi$   $\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} - 2 \frac{x}{A_1} \frac{y}{A_2} \cos(\varphi_2 - \varphi_1) = \sin^2(\varphi_2 - \varphi_1)$

$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} \mp 2 \frac{x}{A_1} \frac{y}{A_2} = 0 \quad \longrightarrow \quad \left( \frac{x}{A_1} \mp \frac{y}{A_2} \right)^2 = 0 \quad \longrightarrow \quad y = \pm \frac{A_2}{A_1} x$$

Oscilații polarizate liniar

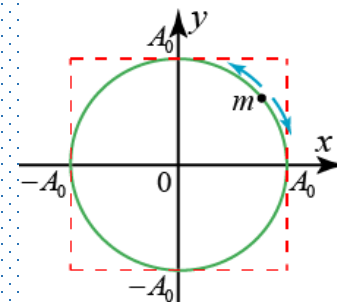


2)  $\varphi_2 - \varphi_1 = (2n+1)\pi/2$   $\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} = 1$



$$A = \sqrt{A_1^2 + A_2^2}$$

Oscilații polarizate eliptic și circular



$$A_1 = A_2 = A_0$$

Figurile Lissajou

$$\begin{cases} x(t) = A_1 \sin(m\omega_0 t + \varphi_1), \\ y(t) = A_2 \sin(n\omega_0 t + \varphi_2), \end{cases}$$

