Tema 13. Oscilatii armonice libere

Oscilații $\begin{cases} \text{libere sau proprii} \\ \text{forțate} \end{cases}$

Relația dintre frecvență și perioadă

frecvență liniară $\nu = \frac{N}{t} = \frac{N}{NT} = \frac{1}{T}$ frecvența ciclică $\omega = 2\pi\nu = \frac{2\pi}{T}$

Pentru orice mărime fizică ξ care variază periodic

$$\xi(t+T) = \xi(t)$$

Dacă variația are loc conform unei legi armonice, atunci oscilațiile se numesc armonice și au loc în conformitate cu legea

$$\boxed{\xi(t) = A\sin(\omega_0 t + \varphi_0)}$$

$\xi_{\max} = A = \text{const} > 0$ – amplitudinea oscilației;

$\Phi(t) = \omega_0 t + \varphi_0$ – faza oscilației;

$\Phi(0) = \varphi_0$ – faza inițială a oscilației;

Tema 13. Oscilatii armonice libere

Viteza și accelerația oscilațiilor

$$\left.\begin{aligned}\dot{\xi} &= A\omega_0\cos(\omega_0 t+\varphi_0) = A\omega_0\sin(\omega_0 t+\varphi_0+\pi/2)\\ \ddot{\xi} &= -A\omega_0^2\sin(\omega_0 t+\varphi_0) = A\omega_0^2\sin(\omega_0 t+\varphi_0+\pi)\end{aligned}\right\} \Rightarrow \quad \ddot{\xi} = -\omega_0^2\xi$$

Ecuația diferențială a oscilațiilor armonice

$$\boxed{\ddot{\xi} + \omega_0^2\xi = 0}$$

Considerăm oscilațiile mecanice ale unui punct material de-a lungul axei x. $\xi(t) = x(t)$. Atunci

$$\upsilon = \dot{x} = A\omega_0\cos(\omega_0 t+\varphi_0) = \upsilon_0\cos(\omega_0 t+\varphi_0),$$

$$a = \ddot{x} = -A\omega_0^2\sin(\omega_0 t+\varphi_0) = -a_0\sin(\omega_0 t+\varphi_0) = -\omega_0^2 x,$$

$$F = ma \;\Rightarrow\; F = -m\omega_0^2 x \;\Rightarrow\; \vec{F} = -m\omega_0^2 x\cdot\vec{i}$$

Forțele proporționale cu abaterea de la poziția de echilibru (liniare) se numesc **quasielastice**

Tema 13. Oscilatii armonice libere

Energia cinetică a unui punct material care oscilează

$$E_c = \frac{m\upsilon^2}{2} = \frac{m\upsilon_0^2 \cos^2\left(\omega_0 t + \varphi_0\right)}{2} = \frac{1}{2} m A^2 \omega_0^2 \cos^2\left(\omega_0 t + \varphi_0\right) \Rightarrow$$

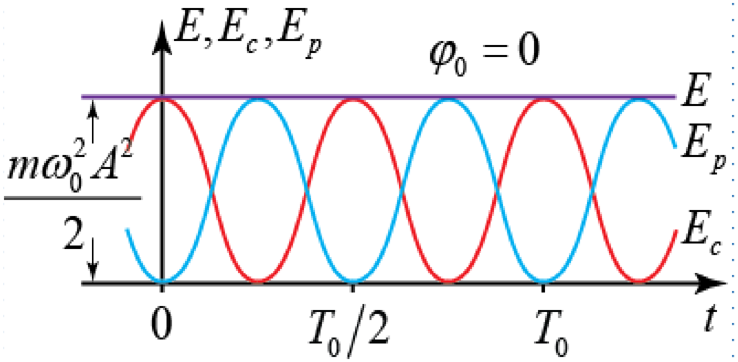
$$\Rightarrow E_c = \frac{1}{4} m A^2 \omega_0^2 \left[1 + \cos\left(2\omega_0 t + 2\varphi_0\right)\right]$$

Energia potențială a unui punct material care oscilează

$$E_p = \frac{m\omega_0^2 x^2}{2} = \frac{1}{2} m A^2 \omega_0^2 \sin^2\left(\omega_0 t + \varphi_0\right) \Rightarrow$$

$$\Rightarrow E_p = \frac{1}{4} m A^2 \omega_0^2 \left[1 - \cos\left(2\omega_0 t + 2\varphi_0\right)\right] =$$

$$= \frac{1}{4} m A^2 \omega_0^2 \left[1 + \cos\left(2\omega_0 t + 2\varphi_0 + \pi\right)\right]$$



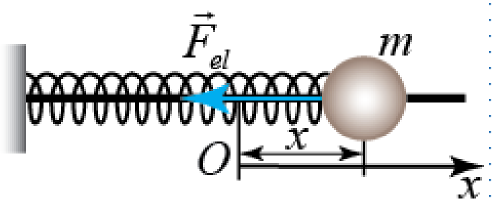
$$E = E_c + E_p = \frac{m A^2 \omega_0^2}{2} = \text{const}$$

Tema 13. Oscilatii armonice libere

Pendulul elastic

$$ma = F_{el.} \Rightarrow m\ddot{x} = -kx \Rightarrow \ddot{x} + \frac{k}{m}x = 0$$

$$\omega_0^2 = \frac{k}{m} \Rightarrow \omega_0 = \sqrt{\frac{k}{m}}$$



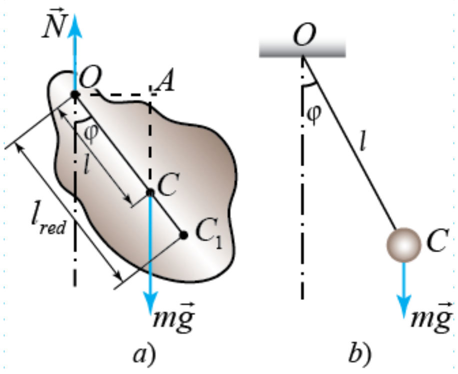
Perioada pendulului elastic

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}$$

Punctul material care efectuează oscilații armonice de-a lungul unei drepte sub acțiunea forței elastice sau cvasielastice se numește **oscilator liniar armonic**.

Tema 13. Oscilatii armonice libere

Pendulul fizic



$$\left.\begin{aligned} M &= I\varepsilon \\ M &= -mgl\sin\varphi \\ \varepsilon &= \ddot{\varphi} \end{aligned}\right\} \Rightarrow I\ddot{\varphi} = -mgl\sin\varphi \Rightarrow \ddot{\varphi} + \frac{mgl}{I}\sin\varphi = 0$$

Pentru unghiuri mici $\sin\varphi \approx \varphi \Rightarrow \ddot{\varphi} + \omega_0^2\varphi = 0$

$$\omega_0 = \sqrt{\frac{mgl}{I}} \qquad T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{I}{mgl}}$$

Dacă masa pendulului fizic este concentrată în centrul său de greutate, atunci pendulul fizic se transformă într-un pendul matematic

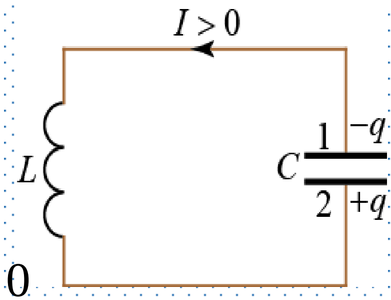
$$T_0 = 2\pi\sqrt{\frac{I}{mgl}} = 2\pi\sqrt{\frac{l}{g}}$$

Lungimea unui pendul matematic care oscilează cu aceeași perioadă cu cea a pendulului fizic se numește **lungime redusă**

$$\frac{I}{mgl} = \frac{l_{red}}{g} \Rightarrow l_{red} = \frac{I}{ml}$$

Tema 13. Oscilatii armonice libere

Oscilatii armonice libere intr-un circuit oscilant electric



$$\left.\begin{aligned} & IR=\varphi_1-\varphi_2+\mathscr{E}_{ai} \\ & \mathscr{E}_{ai}=-L\frac{dI}{dt}=-L\dot{I} \\ & I=\frac{dq}{dt}=\dot{q},\quad \varphi_1-\varphi_2=-\frac{q}{C} \end{aligned}\right\} \Rightarrow IR=-\frac{q}{C}-L\dot{I} \Rightarrow \ddot{q}+\frac{R}{L}\dot{q}+\frac{1}{LC}q=0$$

$$\ddot{q}+\frac{1}{LC}q=0 \qquad \omega_0=\frac{1}{\sqrt{LC}} \qquad T_0=2\pi\sqrt{LC}$$

$$q=q_0\sin\left(\omega_0 t+\varphi_0\right)$$

$$I=\dot{q}=q_0\omega_0\cos\left(\omega_0 t+\varphi_0\right)=I_0\cos\left(\omega_0 t+\varphi_0\right)=I_0\sin\left(\omega_0 t+\varphi_0+\pi/2\right)$$

$$U=\varphi_2-\varphi_1=\frac{q}{C}=\frac{q_0}{C}\sin\left(\omega_0 t+\varphi_0\right)=U_0\sin\left(\omega_0 t+\varphi_0\right) \qquad I_0=\frac{q_0}{\sqrt{LC}}=\frac{U_0}{\sqrt{L/C}}$$

$$W_e=\frac{q^2}{2C}=\frac{q_0^2}{2C}\sin^2\left(\omega_0 t+\varphi_0\right)=\frac{q_0^2}{4C}\left[1-\cos\left(2\omega_0 t+2\varphi_0\right)\right]$$

$$W_m=\frac{LI^2}{2}=\frac{LI_0^2}{2}\cos^2\left(\omega_0 t+\varphi_0\right)=\frac{LI_0^2}{2}\left[1+\cos\left(2\omega_0 t+2\varphi_0\right)\right]=\frac{LI_0^2}{2}\left[1-\cos\left(2\omega_0 t+2\varphi_0+\pi\right)\right]$$

$$W_e+W_m=\frac{q_0^2}{2C}=\frac{LI_0^2}{2}=\text{const}$$

Tema 13. Oscilatii armonice libere

Compunerea oscilațiilor armonice colineare

$$\xi_1(t) = A_1 \sin\left[\Phi_1(t)\right] = A_1 \sin\left(\omega_0 t + \varphi_{01}\right)$$

$$\xi_2(t) = A_2 \sin\left[\Phi_2(t)\right] = A_2 \sin\left(\omega_0 t + \varphi_{02}\right)$$

$$\xi(t) = \xi_1(t) + \xi_2(t)$$

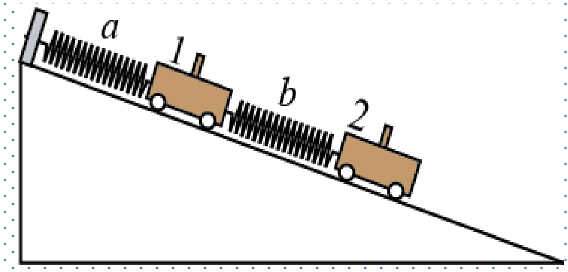
$$\xi(t) = A\sin\left[\Phi(t)\right] = A\sin\left(\omega_0 t + \varphi_0\right)$$

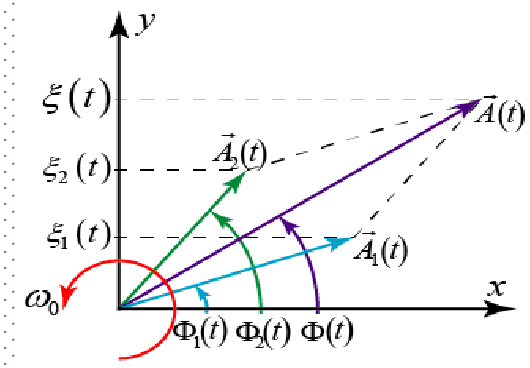
$$\boxed{A^2 = A_1^2 + A_2^2 + 2A_1A_2\cos\left(\varphi_{02} - \varphi_{01}\right)}$$

$$\boxed{\operatorname{tg}\varphi_0 = \frac{\xi(0)}{x} = \frac{\xi_1(0) + \xi_2(0)}{x_1 + x_2} = \frac{A_1\sin\varphi_{01} + A_2\sin\varphi_{02}}{A_1\cos\varphi_{01} + A_2\cos\varphi_{02}}}$$

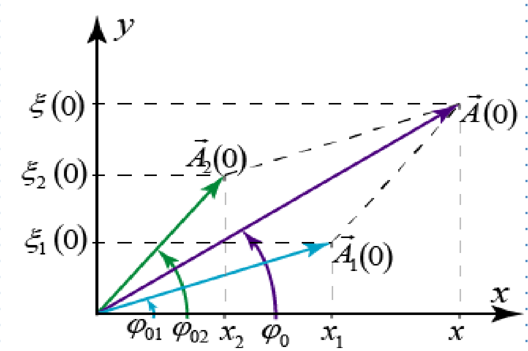
$$\Delta\varphi = \pm 2m\pi \quad \Longrightarrow \quad A_{\max} = A_1 + A_2$$

$$\Delta\varphi = \pm(2m+1)\pi \quad \Longrightarrow \quad A_{\min} = \left|A_1 - A_2\right|$$





a)



b)

Tema 13. Oscilatii armonice libere

Cazul frecvențelor de oscilație care se deosebesc foarte puțin. Bătăi

$$\xi_1(t) = A_0 \sin(\omega_1 t)$$

$$\xi_2(t) = A_0 \sin(\omega_1 + \Delta\omega)t$$

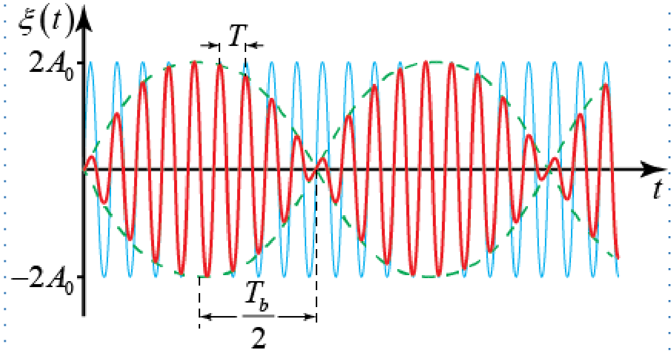
$$\Delta\omega = \omega_2 - \omega_1 \ll \omega_1$$

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\xi(t) = \xi_1(t) + \xi_2(t) = 2A_0 \cos\left(\frac{\Delta\omega}{2}t\right)\sin\omega_1 t$$

$$T_b = \frac{2\pi}{\Delta\omega} = \frac{4\pi}{\omega_2 - \omega_1} \Rightarrow \nu_b = \frac{1}{T_b} = \frac{\omega_2 - \omega_1}{4\pi}$$

$$T = \frac{2\pi}{\omega_1} << T_b$$



Orice oscilație nearmonică, dar periodică poate fi prezentată ca o superpoziție a oscilațiilor armonice

$$\xi(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n \sin\left(n\omega_0 t + \varphi_n\right)$$

Analiza armonică permite determinarea spectrului de frecvențe.

Tema 13. Oscilatii armonice libere

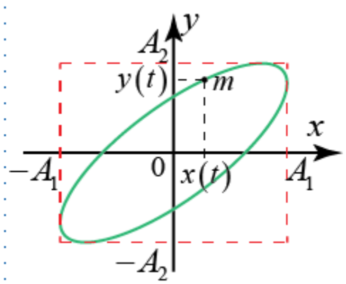
Compunerea oscilațiilor armonice reciproc perpendiculare

$$\begin{cases} x(t) = A_1 \sin(\omega_0 t + \varphi_1), \\ y(t) = A_2 \sin(\omega_0 t + \varphi_2). \end{cases} \Rightarrow \begin{cases} \dfrac{x(t)}{A_1} = \sin(\omega_0 t + \varphi_1) = \sin\omega_0 t \cos\varphi_1 + \cos\omega_0 t \sin\varphi_1, & \Big| \times \sin\varphi_2 \\ \dfrac{y(t)}{A_2} = \sin(\omega_0 t + \varphi_2) = \sin\omega_0 t \cos\varphi_2 + \cos\omega_0 t \sin\varphi_2. & \Big| \times \sin\varphi_1 \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{x}{A_1}\sin\varphi_2 - \dfrac{y}{A_2}\sin\varphi_1 = \sin\omega_0 t\left(\sin\varphi_2\cos\varphi_1 - \cos\varphi_2\sin\varphi_1\right) = \sin\omega_0 t\sin(\varphi_2 - \varphi_1), \\ \dfrac{x}{A_1}\cos\varphi_2 - \dfrac{y}{A_2}\cos\varphi_1 = \cos\omega_0 t\left(\sin\varphi_1\cos\varphi_2 - \cos\varphi_1\sin\varphi_2\right) = -\cos\omega_0 t\sin(\varphi_2 - \varphi_1). \end{cases} \Rightarrow$$

$$\Rightarrow \frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} - 2\frac{x}{A_1}\frac{y}{A_2}\left(\sin\varphi_2\sin\varphi_1 + \cos\varphi_2\cos\varphi_1\right) = \sin^2(\varphi_2 - \varphi_1)\left(\sin^2\omega_0 t + \cos^2\omega_0 t\right) \Rightarrow$$

$$\Rightarrow \frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} - 2\frac{x}{A_1}\frac{y}{A_2}\cos(\varphi_2 - \varphi_1) = \sin^2(\varphi_2 - \varphi_1)$$

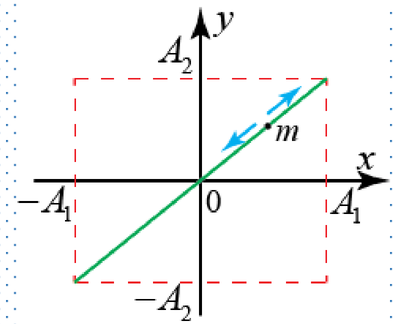


Tema 13. Oscilatii armonice libere

1) $\varphi_2 - \varphi_1 = 2n\pi; (2n+1)\pi$

$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} - 2\frac{x}{A_1}\frac{y}{A_2}\cos(\varphi_2 - \varphi_1) = \sin^2(\varphi_2 - \varphi_1)$$

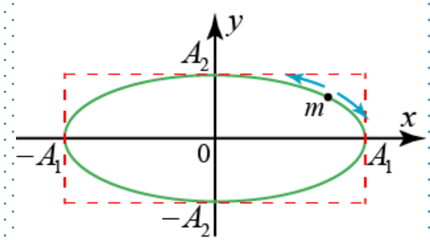
$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} \mp 2\frac{x}{A_1}\frac{y}{A_2} = 0 \Rightarrow \left(\frac{x}{A_1} \mp \frac{y}{A_2}\right)^2 = 0 \Rightarrow y = \pm\frac{A_2}{A_1}x$$



Oscilații polarizate liniar

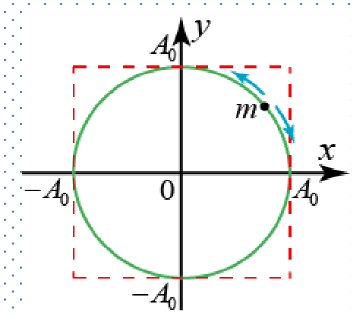
2) $\varphi_2 - \varphi_1 = (2n+1)\pi/2$

$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} = 1$$



$A = \sqrt{A_1^2 + A_2^2}$

Oscilații polarizate eliptic și circular



$A_1 = A_2 = A_0$

Figurile Lissajou

$$\begin{cases} x(t) = A_1\sin(m\omega_0 t + \varphi_1), \\ y(t) = A_2\sin(n\omega_0 t + \varphi_2), \end{cases}$$