Ecuatii diferentiale si cu derivate partiale

Curs Nr. 1

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Ecuatie diferentiala de ordin k ($k \in \mathbb{N}^*$)

$$\boxed{F\left(t, x, x^{(1)}, \dots, x^{(k)}\right) = 0 \qquad (1)}$$

Se cunoaste *F* de *k+2* variabile, $F: D \subset \mathbb{R}^{k+2} \to \mathbb{R}$. Se cere sa se determine $x(\cdot): D \subset \mathbb{R} \to \mathbb{R}$ astfel incat sa verifice ecuatia (1), adica trebuie gasita $\varphi(\cdot): D_1 \subset \mathbb{R} \to \mathbb{R}$ astfel incat $F\left(t, \varphi(t), \varphi^{(1)}(t), \varphi^{(k)}(t)\right) = 0, \forall t \in D$.

Ecuatia (1) poate fi:

1. **Liniara:**
   $$\sum_{j=0}^{k} a_j x^{(j)} = f(t)$$
   $f, a_0, \dots, a_k$ sunt functii continue

2. **Liniara cu coeficienti constanti:**
   $$\sum_{j=0}^{k} a_j x^{(j)} = f(t)$$
   Daca $f(t) = 0$ atunci ecuatia este omogena.

3. **Cvasiliniara de ordin k:**
   $x^{(k)} = g\left(t, x, x^{(1)}, \dots, x^{(k-1)}\right)$

Ecuatie diferentiala de ordin 1

$F(t, x, x') = 0$ ***forma implicita***

$\boxed{(2) \quad x' = f(t, x)}$ ***forma explicita*** $\boxed{\frac{dx}{dt} = f(t, x) \quad (2')}$

$f(\cdot,\cdot): D \subset \mathbb{R}^2 \to \mathbb{R}$ defineste ecuatia (2).

<u>Cerinte:</u> a) Se da un $\varphi_0(\cdot)$ si verificam ca este solutie pentru (2).

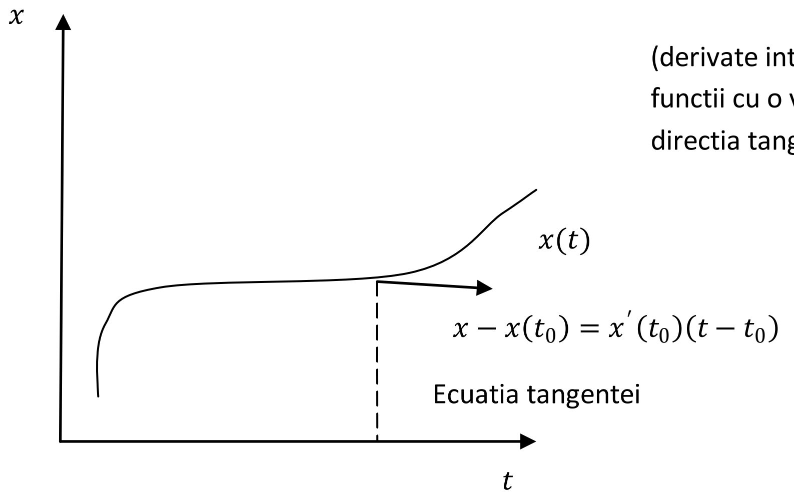
$\varphi_0: I_0 \subset \mathbb{R} \to \mathbb{R}$

$\varphi_0(t) = f\left(t, \varphi_0(t)\right), \forall t \in I_0$

b) Se cere determinarea solutiei generale sau a unei solutii particulare care indeplineste conditia $x(t_0) = x_0$ (problema Cauchy)

DEF.: Spunem ca s-a dat o **problema Cauchy** (f, t_0, x_0) pentru ecuatia (2) daca se cauta o functie $\varphi(\cdot): I \subset \mathbb{R} \to \mathbb{R}$ astfel incat $\begin{cases} \varphi'(t) = f(t, \varphi(t)), \forall t \in I \\ \varphi(t_0) = x_0 \end{cases}$

<u>Interpretare a solutiei ecuatiei (2)</u>: inseamna a se gasi o familie de functii $x(\cdot)$, pentru care se cunoaste din (2) directia tangentei.



(derivate intr-un punct al unei functii cu o variabila da directia tangentei)

Ecuatie diferentiala de ordin 1 integrabila prin cuadraturi

1. **Ecuatie cu variabile separabile**

$$\boxed{\frac{dx}{dt} = a(t)b(t) \quad (3)}$$

$cu\ a(\cdot), b(\cdot)$ functii continue

Rezolvare: Se determina solutiile stationare obtinute din $b(x) = 0$, apoi se separa variabilele $(b(x) \neq 0)$: $\frac{dx}{b(x)} = a(t)dt \implies \int \frac{dx}{b(x)} = \int a(t)dt$

2. **Ecuatie liniara**

$$\boxed{\frac{dx}{dt} = a(t)x \quad (4)}$$

Tema: Sa se arate folosind ecuatia (3) ca solutia generala a ecuatiei (4) este de forma

$$x(t) = ce^{\int a(t)dt}, cu\ c\ constanta\ din\ \mathbb{R}$$

3. **Ecuatie afina**

$$\boxed{\frac{dx}{dt} = a(t)x + b(t) \quad (5)}$$

$cu\ a(\cdot), b(\cdot)$ functii continue

Rezolvare: Se rezolva ecuatia omogena atasata $\frac{d\bar{x}}{dt} = a(t)\bar{x}$ si se aplica metoda variatiei constantelor. (Tema)

4. Ecuatie omogena (functia *f* nu depinde arbitrar de *t* si *x*)

$$\boxed{\frac{dx}{dt} = g\left(\frac{x}{t}\right) \qquad (6)}$$

Adica $f(\cdot,\cdot)$ din ecuatia (2) are proprietatea de a fi functie omogena

$\Leftrightarrow f(\lambda t, \lambda x) = f(t,x) \forall t \in \mathbb{R}, (\lambda t, \lambda x) \in D_f$

Rezolvare: Schimbare de variabila: $\frac{x}{t} = z, \frac{x(t)}{t} = z(t)$, se obtine o noua ecuatie doar in z.

$x(t) = tz(t) \Longrightarrow x'(t) = tz'(t) + z(t)$

Ecuatia (6) devine $z + tz' = g(z)$

si se reduce la $z' = \frac{1}{t}(g(z) - z)$ ecuatie cu variabile separabile.

5. Ecuatie Bernoulli

$$\boxed{\frac{dx}{dt} = a(t)x + b(t)x^{\alpha} \quad (7)}$$

$cu\ a(\cdot), b(\cdot)$ functii continue, $\alpha \epsilon \mathbb{R} \setminus \{0,1\}$

Observatie: $\alpha = 0 \Longrightarrow ec.afina;\ \alpha = 1, x\ se\ ia\ ca\ factor\ \Longrightarrow ec.liniara$

Rezolvare: Varianta 1

a. Se identifica $a(\cdot), b(\cdot)$

b. Se rezolva ecuatia liniara atasata $\frac{d\bar{x}}{dt} = a(t)\bar{x} \overset{2.}{\Rightarrow} \bar{x}(t) = ce^{\int a(t)dt}, c \in \mathbb{R}$

c. Utilizand metoda variatiei constantelor determinam functia $c(\cdot)$ astfel incat $x(t) = c(t)\, e^{\int a(t)dt}$ sa verifice (7)

$\left(c(t)e^{\int a(t)dt}\right)' = a(t)xc(t)e^{\int a(t)dt} + b(x)\left(c(t)e^{\int a(t)dt}\right)^{\alpha}$

$c'(t)e^{\int a(t)dt} + c(t)\left(e^{\int a(t)dt}\right)' a(t) = a(t)xc(t)e^{\int a(t)dt} + b(t)(c(t))^{\alpha} e^{\alpha \int a(t)dt}$

$c'(t) = \left(c(t)\right)^{\alpha} b(t)e^{(\alpha-1)\int a(t)dt}$ ecuatie cu variabile separabile pentru determinarea lui c(·).

Solutii stationare exista numai daca $\alpha > 0 :\ c^{\alpha} = 0 \Longrightarrow c = 0 \Longrightarrow x = 0$ este solutie stationara pt (7).

$\frac{dc}{c^{\alpha}} = b(t)e^{(\alpha-1)\int a(t)dt}dt$

Varianta 2

Se face schimbarea de variabile $x = z^{1-\alpha}, x(t) = (z(t))^{1-\alpha}$

$x(t) = (z(t))^{\frac{1}{1-\alpha}} \Longrightarrow \frac{dx}{dt} = \frac{1}{1-\alpha}\left(z(t)\right)^{\frac{1}{1-\alpha}-1} z'(t) = \frac{1}{1-\alpha}\left(z(t)\right)^{\frac{\alpha}{1-\alpha}} z'(t).$

Ecuatia (7) devine: $\frac{1}{1-\alpha}\left(z(t)\right)^{\frac{\alpha}{1-\alpha}} z'(t) = a(t)\left(z(t)\right)^{\frac{1}{1-\alpha}} + b(t)\left(z(t)\right)^{\frac{\alpha}{1-\alpha}}$

$z' = (1-\alpha)a(t)z^{\frac{1}{1-\alpha}-\frac{\alpha}{1-\alpha}} + b(t)$

$z' = (1-\alpha)a(t)z + b(t)$ ecuatie afina in z, se aplica algoritmul pentru ecuatia afina.

6. Ecuatie Riccati

$$\boxed{\frac{dx}{dt} = a(t)x^2 + b(t)x + c(t) \quad (8)}$$

$cu\ a(\cdot), b(\cdot), c(\cdot): J \subset \mathbb{R} \to \mathbb{R}$ functii continue

Observatie: $\left[\begin{array}{l} 1)\ a(t) \not\equiv 0 \text{ (diferit de functia identic 0), altfel este ecuatie afina} \\ 2)\ c(t) \not\equiv 0, \text{ altfel este ecuatie Bernoulli pentru } \alpha=2 \end{array}\right.$

Rezolvare: (se presupune cunoscuta o solutie $\varphi_0(\cdot)$ a ecuatiei (8))

Se face schimbarea de variabila $x = y + \varphi_0, x(t) = y(t) + \varphi_0(t)$

$\varphi_0'(t) = a(t)\varphi_0{}^2(t) + b(t)\varphi_0(t) + c(t) \qquad (9)$

$x'(t) = y'(t) + \varphi_0'(t)$

Ecuatia (8) devine: $y' + \varphi_0'(t) = a(t)\big(y + \varphi_0(t)\big)^2 + b(t)\big(y + \varphi_0(t)\big) + c(t)$

$y'(\varphi_0{}'(t)) = a(t)y^2 + 2a(t)y\varphi_0(t) + a(t)\varphi_0{}^2(t) + b(t)y + b(t)\varphi_0(t) + c(t)$

$y' = \underbrace{[2a(t)\varphi_0(t) + b(t)]}_{a_1(t)}y + \underbrace{[a(t)]}_{b_1(t)}y^2 \Longrightarrow$ pentru y o ecuatie Bernoulli pt $\alpha = 2$.

Observatie: Ecuatia rasturnata a ecuatiei: $\frac{dx}{dt} = f(t,x)$ este ecuatia: : $\frac{dt}{dx} = \frac{1}{f(t,x)}$, care, uneori, poate fi incadrata intr-unul din tipurile de mai sus. Se determina $t = t(x)$ care constituie solutie implicita si pentru ecuatia $\frac{dx}{dt} = f(t,x)$.

Exemplu (pentru ecuatie rasturnata): $x' = \frac{x}{3t - x^2}$

$$f(t,x) = \frac{x}{3t - x^2} \qquad f: D \subset \{(t,x) | x^2 \neq 3t\} \subset \mathbb{R}^2 \to \mathbb{R}$$

Ecuatia rasturnata: $\frac{dt}{dx} = \frac{3t - x^2}{x} = \underbrace{\frac{3}{x}}_{a(x)} t + \underbrace{-x}_{b(x)}$ $\qquad \frac{d\bar{t}}{dx} = \frac{3}{x}\bar{t}$

$$\bar{t}(x) = ce^{\int \frac{3}{x} dx} = ce^{3\ln|x|} = ce^{\ln|x|^3} = c\ln|x|^3 = cx^3, \quad c\epsilon\mathbb{R}$$

Variatia constantelor: cautam $c(x)$ astfel incat $t(x) = c(x)x^3$ sa fie solutie a ecuatiei rasturnate: $\frac{dt}{dx} = \frac{3t}{x} - x$

$$c'(x)x^3 + c(x)3x^2 = \frac{3}{x}c(x)x^3 - x$$

$c'(x) = -\frac{1}{x^2}$ (ecuatie de tip primitiva)

$$c(x) = \frac{1}{x} + k, k\epsilon\mathbb{R}$$

Solutia ecuatiei rasturnate: $t(x) = \left(\frac{1}{x} + k\right)x^3 \qquad t = x^2 + kx^3, k\epsilon\mathbb{R}$

Tema (ecuatie rasturnata) : $x' = \frac{1}{t^2 e^x - 2t} \qquad \frac{dt}{dx} = -2t + e^x t^2$ (Bernoulli cu $\alpha = 2$)