Determinarea unei baze primal admisibile

Considerăm problema de programare liniară în forma standard:

$$\inf\left\{ c^\top \cdot x \mid A\cdot x = b,\ x \geq \mathbf{0} \right\} \qquad \text{(P)}$$

unde $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $\boxed{b \geq \mathbf{0},}$ $c \in \mathbb{R}^n$.

Acestei probleme îi asociem **problema artificială**:

$$\min\left\{ \mathbf{e}^\top \cdot x^a \mid A\cdot x + \mathbf{I}_m \cdot x^a = b,\ x \geq \mathbf{0},\ x^a \geq \mathbf{0} \right\} \qquad (\mathrm{P_a})$$

unde $\mathbf{e} = (1,\dots,1)^\top \in \mathbb{R}^m$, $x^a = \left(x_{n+1}, x_{n+2}, \dots, x_{n+m}\right)^\top \in \mathbb{R}^m$,

iar $\mathbf{I}_m$ este matricea unitate de ordinul m .

Proprietăţi ale problemei ($\mathrm{P_a}$) :

- matricea restricţiilor: $\left(A \vdots \mathbf{I}_m\right) \in \mathbb{R}^{m\times(n+m)}$, $rang\left(A \vdots \mathbf{I}_m\right) = m < n+m$;
- $\mathbf{I}_m$ este o bază primal admisibilă: $\mathbf{I}_m^{-1} \cdot b = b \geq \mathbf{0}$;
- are o soluţie optimă finită: $x^a \geq \mathbf{0} \;\; \Rightarrow \;\; \bar{z}_a = \mathbf{e}^\top \cdot x^a = \sum_{i=1}^{m} x_{n+i} \geq 0.$

Concluzie: (P$_a$) se poate rezolva cu algoritmul simplex.

Fie B baza optimă a problemei (P$_a$) iar $\mathcal{B}$ mulţimea indicilor de bază.

Teoremă. *Dacă valoarea minimă a problemei* (P$_a$), $\overline{z}_a > 0$, *atunci problema iniţială* (P) *nu are soluţie.*

Demonstraţie. Prin absurd, dacă (P) are o soluţie admisibilă, conform TFPL are şi o soluţie admisibilă de bază.

Fie B_* baza corespunzătoare. Ea este formată doar din coloane ale matricei A !

Avem: $B_*^{-1} \cdot b \geq \mathbf{0}$, deci B_* este bază primal admisibilă **şi** pentru (P$_a$), iar variabilele x^a sunt secundare!

Deci, (P$_a$) are o soluţie admisibilă (de bază), pentru care,

$$x^a = \mathbf{0} \quad \Rightarrow \quad \mathbf{e}^\top \cdot x^a = 0 < \overline{z}_a \rightarrow$$ valoarea optimă. Contradicţie. (q.e.d.)

Teoremă. *Dacă* $\mathcal{B} \cap \{n+1, \ldots, n+m\} = \emptyset$, *atunci* $\overline{z}_a = 0$ *şi* B *este o bază primal admisibilă a problemei iniţiale* (P).

Demonstraţie. Evident, B conţine numai coloane a matricei A. (q.e.d.)

Teoremă. *Dacă valoarea minimă a lui* (P$_a$) *este* $\overline{z}_a = 0$ *şi există* $n + i_0 \in \mathcal{B}$, *pentru care* $y_{i_0 j} = 0, \ \forall\ j = \overline{1, n}, \quad i_0 = loc_{\mathcal{B}}(n + i_0)$, *atunci,* $rang(A) \le m - 1$ *şi restricţia* i_0 *din* (P) *este o combinaţie liniară de celelalte restricţii.*

Demonstraţie. Notăm: $B^{-1} = \left(\beta_{ij}\right)_{\substack{1 \le i \le m \\ 1 \le j \le m}}$ şi $Y = \left(y_{i\,j}\right) = B^{-1} \cdot A.$

Din ipoteză, $0 = y_{i_0 j} = \sum_{k=1}^{m} \beta_{i_0 k} a_{kj} = \sum_{k=1,\ k \ne i_0}^{m} \beta_{i_0 k} a_{kj} + \beta_{i_0 i_0} a_{i_0 j}\,, \ \forall\ j = \overline{1, n}.$

Deoarece B conţine vectorul e^{i_0}, în B^{-1} vom avea $\beta_{i_0 i_0} = 1.$

Deci, $a_{i_0 j} = -\sum_{k=1,\ k \ne i_0}^{m} \beta_{i_0 k} a_{kj}\,, \ \forall\ j = \overline{1, n}\,, \ \Leftrightarrow \ A_{i_0} = -\sum_{k=1,\ k \ne i_0}^{m} \beta_{i_0 k} A_k\,,$

adică, linia A_{i_0} este combinaţie liniară de celelalte linii. Deci, $rang(A) \le m - 1.$

Sistemul fiind compatibil, rezultă şi $b_{i_0} = -\sum_{k=1,\ k \ne i_0}^{m} \beta_{i_0 k} b_k\,.$

(q.e.d.)

Teoremă. *Dacă valoarea minimă a lui* (P$_a$) *este* $\overline{z}_a = 0$ *şi există* $n + i_0 \in \mathcal{B}$, *pentru care* $i_0 = loc_{\mathcal{B}}(n + i_0)$, $\exists\, k \in \{1, \dots, n\}$, $y_{i_0 k} \neq 0$, *atunci, se poate efectua o schimbare de bază prin care vectorul unitar* e^{i_0} *din* B *să fie înlocuit de coloana* A^k*.*

Demonstraţie. Din Lema substituţiei, $y_{i_0 k} \neq 0 \;\Rightarrow\; \det\left(\tilde{B}\right) \neq 0.$

În plus, din formulele de schimbare a bazei, deoarece

$$\overline{x}_{i_0} = 0 \;\Rightarrow\; \begin{cases} \tilde{x}_i = \overline{x}_i\,,\ \forall\, i = \overline{1, m} \quad \Leftrightarrow \quad \tilde{B}^{-1} \cdot b = B^{-1} \cdot b \geq \mathbf{0}, \\ \tilde{z} = \overline{z}\,. \end{cases}$$

(q.e.d.)

Observaţie. Dacă $\overline{z}_a = 0$, toate variabilele artificiale au valoarea zero ! inclusiv cele care au mai rămas în bază.

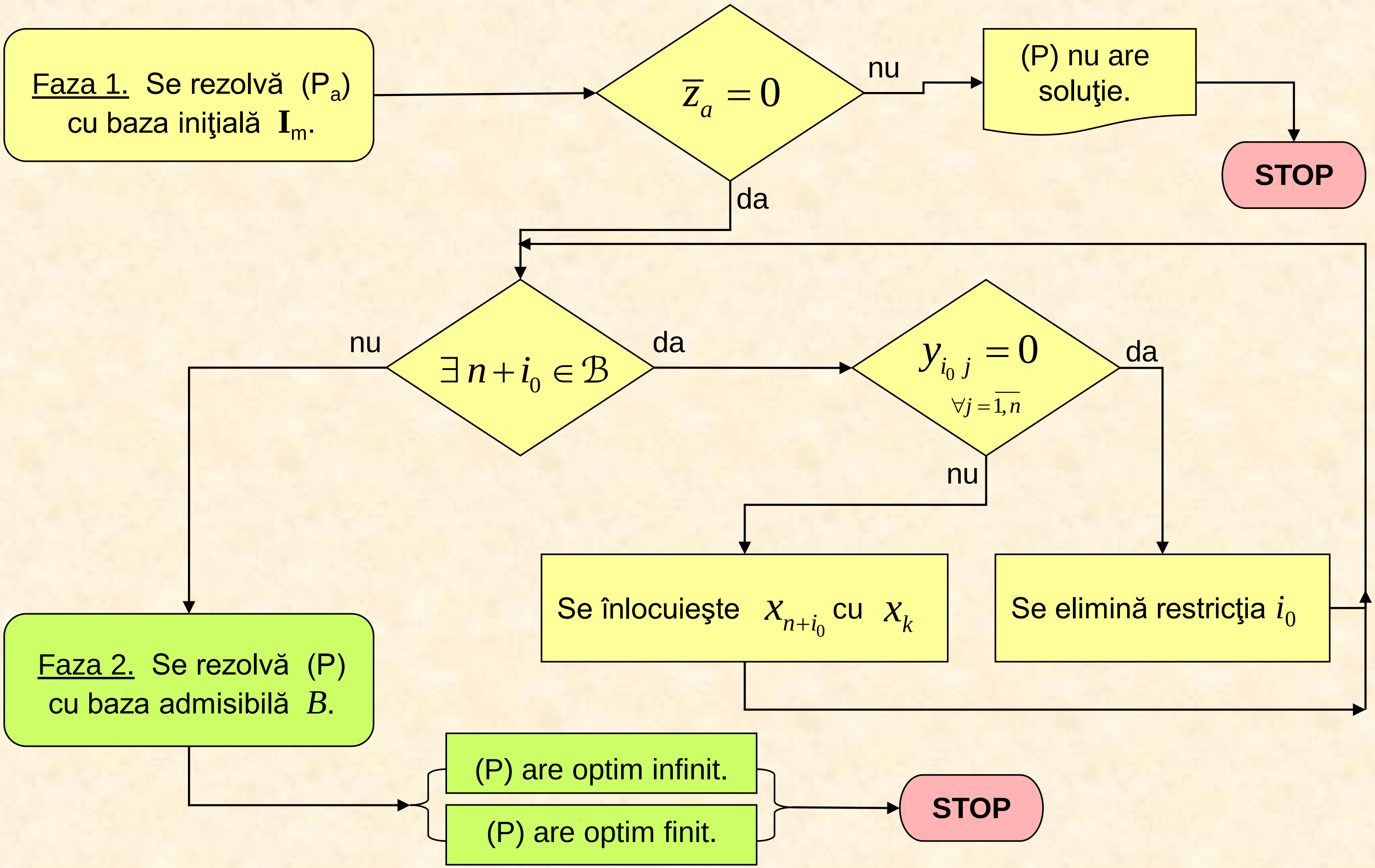
Variabilele artificiale din bază care au valoarea zero, pot fi:

eliminate împreună cu restricţia asociată.

sau

înlocuite cu o variabilă a problemei date.

Metoda celor două faze



Exemple ………

Problemă cu optim finit.

Considerăm problema: $\inf\{-2x_1 + 6x_3 - 2x_4 - 3x_5\}$

$$\begin{cases} 2x_1 - 2x_2 + x_3 - x_4 \le -6 \\ 5x_1 + 4x_2 - 3x_3 - 3x_4 = 20 \\ 2x_1 - x_2 + 2x_3 + x_4 + x_5 = 0 \end{cases}$$

$$x_i \ge 0, \quad \forall i = \overline{1,5}.$$

În prima restricţie introducem variabila ecart x_6 (forma standard).

Înmulţim apoi prima restricţie cu -1 pentru ca termenul liber să conţină doar valori nenegative (condiţia pentru Faza I).

Problema devine: $\inf\{-2x_1 + 6x_3 - 2x_4 - 3x_5 + 0x_6\}$

$$\begin{cases} -2x_1 + 2x_2 - x_3 + x_4 - x_6 = 6 \\ 5x_1 + 4x_2 - 3x_3 - 3x_4 = 20 \\ 2x_1 - x_2 + 2x_3 + x_4 + x_5 = 0 \end{cases}$$

$$x_i \ge 0, \quad \forall i = \overline{1,6}.$$

Faza I

	$\bar{x}$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
x_7	6	–2	2	–1	1	0	–1	1	0	0
x_8	20	5	4	–3	–3	0	0	0	1	0
x_9	0	2	–1	2	1	1	0	0	0	1
	26	5	5	–2	–1	1	–1	0	0	0

	$\bar{x}$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
x_7	6	0	1	1	2	1	–1	1	0
x_8	20	0	$\frac{13}{2}$	–8	$\frac{-11}{2}$	$\frac{-5}{2}$	0	0	1
x_1	0	1	$\frac{-1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0
	26	0	$\frac{15}{2}$	–7	$\frac{-7}{2}$	$\frac{-3}{2}$	–1	0	0

	$\bar{x}$	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_7	$\frac{38}{13}$	0	0	$\frac{29}{13}$	$\frac{37}{13}$	$\frac{18}{13}$	–1	1
x_2	$\frac{40}{13}$	0	1	$\frac{-16}{13}$	$\frac{-11}{13}$	$\frac{-5}{13}$	0	0
x_1	$\frac{20}{13}$	1	0	$\frac{5}{13}$	$\frac{1}{13}$	$\frac{4}{13}$	0	0
	$\frac{38}{13}$	0	0	$\frac{29}{13}$	$\frac{37}{13}$	$\frac{18}{13}$	–1	0

	$\bar{x}$	x_1	x_2	x_3	x_4	x_5	x_6
x_4	$\frac{38}{37}$	0	0	$\frac{29}{37}$	1	$\frac{18}{37}$	$\frac{-13}{37}$
x_2	$\frac{146}{37}$	0	1	$\frac{-21}{37}$	0	$\frac{1}{37}$	$\frac{-11}{37}$
x_1	$\frac{54}{37}$	1	0	$\frac{12}{37}$	0	$\frac{10}{37}$	$\frac{1}{37}$
	0	0	0	0	0	0	0

Faza II

	$\bar{x}$	x_1	x_2	x_3	x_4	x_5	x_6
x_4	$\frac{38}{37}$	0	0	$\frac{29}{37}$	1	$\frac{18}{37}$	$\frac{-13}{37}$
x_2	$\frac{146}{37}$	0	1	$\frac{-21}{37}$	0	$\frac{1}{37}$	$\frac{-11}{37}$
x_1	$\frac{54}{37}$	1	0	$\frac{12}{37}$	0	$\frac{10}{37}$	$\frac{1}{37}$
	$\frac{-184}{37}$	0	0	$\frac{-304}{37}$	0	$\frac{55}{37}$	$\frac{24}{37}$

	$\bar{x}$	x_1	x_2	x_3	x_4	x_5	x_6
x_5	$\frac{19}{9}$	0	0	$\frac{29}{18}$	$\frac{37}{18}$	1	$\frac{-13}{18}$
x_2	$\frac{35}{9}$	0	1	$\frac{-11}{18}$	$\frac{-1}{18}$	0	$\frac{-5}{18}$
x_1	$\frac{8}{9}$	1	0	$\frac{-1}{9}$	$\frac{-5}{9}$	0	$\frac{2}{9}$
	$\frac{-37}{9}$	0	0	$\frac{-191}{18}$	$\frac{-55}{18}$	0	$\frac{31}{18}$

	$\bar{x}$	x_1	x_2	x_3	x_4	x_5	x_6
x_5	5	$\frac{13}{4}$	0	$\frac{5}{4}$	$\frac{1}{4}$	1	0
x_2	5	$\frac{5}{4}$	1	$\frac{-3}{4}$	$\frac{-3}{4}$	0	0
x_6	4	$\frac{9}{2}$	0	$\frac{-1}{2}$	$\frac{-5}{2}$	0	1
	–15	$\frac{-31}{4}$	0	$\frac{-39}{4}$	$\frac{5}{4}$	0	0

	$\bar{x}$	x_1	x_2	x_3	x_4	x_5	x_6
x_4	20	13	0	5	1	4	0
x_2	20	11	1	3	0	3	0
x_6	54	37	0	12	0	10	1
	–40	–24	0	–16	0	–5	0

Soluţie optimă !

Problemă fără soluție.

Considerăm problema:

$$\inf\left\{-2x_1+6x_3-2x_4-3x_5\right\}$$

$$\begin{cases} 2x_1-2x_2+x_3-x_4 & \le -6 \\ 5x_1+4x_2-3x_3-3x_4 & = 20 \\ 2x_1+5x_2+2x_3+x_4+x_5 & = 4 \end{cases}$$

$$x_i \ge 0, \quad \forall i=\overline{1,5}.$$

Faza I

	$\bar{x}$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
x_7	6	–2	2	–1	1	0	–1	1	0	0
x_8	20	5	4	–3	–3	0	0	0	1	0
x_9	4	2	5	2	1	1	0	0	0	1
	30	5	11	–2	–1	1	–1	0	0	0

	$\bar{x}$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
x_7	$\frac{22}{5}$	$-\frac{14}{5}$	0	$-\frac{9}{5}$	$\frac{3}{5}$	$-\frac{2}{5}$	-1	1	0	$-\frac{2}{5}$
x_8	$\frac{84}{5}$	$\frac{17}{5}$	0	$-\frac{23}{5}$	$-\frac{19}{5}$	$-\frac{4}{5}$	0	0	1	$-\frac{4}{5}$
x_2	$\frac{4}{5}$	$\frac{2}{5}$	1	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	0	0	0	$\frac{1}{5}$
	$\frac{106}{5}$	$\frac{3}{5}$	0	$-\frac{32}{5}$	$-\frac{16}{5}$	$-\frac{6}{5}$	-1	0	0	$-\frac{11}{5}$

	$\bar{x}$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
x_7	10	0	7	1	2	1	-1	1	0	1
x_8	10	0	$-\frac{17}{2}$	-8	$-\frac{11}{2}$	$-\frac{5}{2}$	0	0	1	$-\frac{5}{2}$
x_1	2	1	$\frac{5}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$
	$\boxed{20}$	0	$-\frac{3}{2}$	-7	$-\frac{7}{2}$	$-\frac{3}{2}$	-1	0	0	$-\frac{5}{2}$

Problema artificială are soluţie optimă: $20 > 0$ $\Rightarrow$ Problema iniţială nu are soluţii !

Problemă cu optim infinit.

Considerăm problema:

$$\inf\left\{-2x_1+6x_3-2x_4-3x_5\right\}$$

$$\begin{cases} 2x_1-2x_2+x_3-x_4 & \le -6 \\ 5x_1+4x_2-3x_3-3x_4 & =20 \\ 2x_1-x_2+2x_3+\frac{1}{2}x_4+x_5 & =0 \end{cases}$$

$$x_i\ge 0,\quad \forall i=\overline{1,5}.$$

Faza I

	$\bar{x}$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
x_7	6	–2	2	–1	1	0	–1	1	0	0
x_8	20	5	4	–3	–3	0	0	0	1	0
x_9	0	2	–1	2	1	1	0	0	0	1
	26	5	5	–2	$-\frac{3}{2}$	1	–1	0	0	0

$$
\begin{array}{c|c||c|c|c|c|c|c|c|c|c|}
 & \bar{x} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\
\hline
x_7 & 6 & 0 & 1 & 1 & \frac{3}{2} & 1 & -1 & 1 & 0 & 1 \\
\hline
x_8 & 20 & 0 & \frac{13}{2} & -8 & -\frac{17}{4} & -\frac{5}{2} & 0 & 0 & 1 & -\frac{5}{2} \\
\hline
x_1 & 0 & 1 & -\frac{1}{2} & 1 & \frac{1}{4} & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\
\hline
 & 26 & 0 & \frac{15}{2} & -7 & -\frac{11}{4} & -\frac{3}{2} & -1 & 0 & 0 & -\frac{5}{2} \\
\hline
\end{array}
$$

$$
\begin{array}{c|c||c|c|c|c|c|c|c|c|c|}
 & \bar{x} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\
\hline
x_7 & \frac{38}{13} & 0 & 0 & \frac{29}{13} & \frac{28}{13} & \frac{18}{13} & -1 & 1 & -\frac{2}{13} & \frac{18}{13} \\
\hline
x_2 & \frac{40}{13} & 0 & 1 & -\frac{16}{13} & -\frac{17}{26} & -\frac{5}{13} & 0 & 0 & \frac{2}{13} & -\frac{5}{13} \\
\hline
x_1 & \frac{20}{13} & 1 & 0 & \frac{5}{13} & -\frac{1}{13} & \frac{4}{13} & 0 & 0 & \frac{1}{13} & \frac{4}{13} \\
\hline
 & \frac{38}{13} & 0 & 0 & \frac{29}{13} & \frac{28}{13} & \frac{18}{13} & -1 & 0 & -\frac{15}{13} & \frac{5}{13} \\
\hline
\end{array}
$$

	$\bar{x}$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
x_3	$\frac{38}{29}$	0	0	1	$\frac{28}{29}$	$\frac{18}{29}$	$-\frac{13}{29}$	$\frac{13}{29}$	$-\frac{2}{29}$	$\frac{18}{29}$
x_2	$\frac{136}{29}$	0	1	0	$\frac{31}{58}$	$\frac{11}{29}$	$-\frac{16}{29}$	$\frac{16}{29}$	$\frac{2}{29}$	$\frac{11}{29}$
x_1	$\frac{30}{29}$	1	0	0	$-\frac{13}{29}$	$\frac{2}{29}$	$\frac{5}{29}$	$-\frac{5}{29}$	$\frac{3}{29}$	$\frac{2}{29}$
	0	0	0	0	0	0	0	–1	–1	–1

Inversa bazei curente

Faza II

	$\bar{x}$	x_1	x_2	x_3	x_4	x_5	x_6
x_3	$\frac{38}{29}$	0	0	1	$\frac{28}{29}$	$\frac{18}{29}$	$-\frac{13}{29}$
x_2	$\frac{136}{29}$	0	1	0	$\frac{31}{58}$	$\frac{11}{29}$	$-\frac{16}{29}$
x_1	$\frac{30}{29}$	1	0	0	$-\frac{13}{29}$	$\frac{2}{29}$	$\frac{5}{29}$
	$\frac{168}{29}$	0	0	0	$\frac{252}{29}$	$\frac{191}{29}$	$-\frac{88}{29}$

	$\bar{x}$	x_1	x_2	x_3	x_4	x_5	x_6
x_4	$\frac{19}{14}$	0	0	$\frac{29}{28}$	1	$\frac{9}{14}$	$-\frac{13}{28}$
x_2	$\frac{111}{28}$	0	1	$-\frac{31}{56}$	0	$\frac{1}{28}$	$-\frac{17}{56}$
x_1	$\frac{23}{14}$	1	0	$\frac{13}{28}$	0	$\frac{5}{14}$	$-\frac{1}{28}$
	–6	0	0	–9	0	1	1

Problema are optim infinit !

Problemă cu restricții redundante.

Considerăm problema:

$$\inf\left\{-3x_1 + x_2 - x_3 + 2x_4 + 4x_5\right\}$$

$$\begin{cases} 2x_1 - x_2 + 3x_3 + x_4 & = 5 \\ x_1 - 3x_2 + 2x_3 \quad - x_5 & = 3 \\ 3x_1 - 4x_2 + 5x_3 + x_4 - x_5 & = 8 \end{cases}$$

$$x_i \geq 0, \quad \forall i = \overline{1,5}.$$

Faza I

	$\bar{x}$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
x_6	5	2	–1	3	1	0	1	0	0
x_7	3	1	–3	2	0	–1	0	1	0
x_8	8	3	–4	5	1	–1	0	0	1
	16	6	–8	10	2	–2	0	0	0

	$\bar{x}$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
x_6	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{7}{2}$	0	1	$\frac{3}{2}$	1	$-\frac{3}{2}$	0
x_3	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{3}{2}$	1	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0
x_8	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{7}{2}$	0	1	$\frac{3}{2}$	0	$-\frac{5}{2}$	1
	1	1	7	0	2	3	0	–5	0

	$\bar{x}$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
x_2	$\frac{1}{7}$	$\frac{1}{7}$	1	0	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{2}{7}$	$-\frac{3}{7}$	0
x_3	$\frac{12}{7}$	$\frac{5}{7}$	0	1	$\frac{3}{7}$	$\frac{1}{7}$	$\frac{3}{7}$	$-\frac{1}{7}$	0
x_8	0	0	0	0	0	0	–1	–1	1
	0	0	0	0	0	0	–2	–2	0

Faza II

	$\bar{x}$	x_1	x_2	x_3	x_4	x_5
x_2	$\frac{1}{7}$	$\frac{1}{7}$	1	0	$\frac{2}{7}$	$\frac{3}{7}$
x_3	$\frac{12}{7}$	$\frac{5}{7}$	0	1	$\frac{3}{7}$	$\frac{1}{7}$
	$-\frac{11}{7}$	$\frac{17}{7}$	0	0	$-\frac{15}{7}$	$-\frac{26}{7}$

	$\bar{x}$	x_1	x_2	x_3	x_4	x_5
x_1	1	1	7	0	2	3
x_3	1	0	–5	1	–1	–2
	–4	0	–17	0	–7	–11

Soluţia optimă !

Problemă (eliminare variabilă artificială).

Considerăm problema:

$$\inf\left\{-3x_1 + x_2 - x_3 + 2x_4 + 4x_5\right\}$$

$$\begin{cases} 2x_1 - x_2 + 3x_3 + x_4 & = 5 \\ x_1 - 3x_2 + 2x_3 \quad - x_5 & = 3 \\ 3x_1 - 6x_2 + 5x_3 \quad - x_5 & = 8 \end{cases}$$

$$x_i \geq 0, \quad \forall i = \overline{1,5}.$$

Faza I

	$\bar{x}$	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_4	5	2	-1	3	1	0	0	0
x_6	3	1	-3	2	0	-1	1	0
x_7	8	3	-6	5	0	-1	0	1
	11	4	-9	7	0	-2	0	0

	$\bar{x}$	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_4	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{7}{2}$	0	1	$\frac{3}{2}$	$-\frac{3}{2}$	0
x_3	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{3}{2}$	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	0
x_7	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	0	0	$\frac{3}{2}$	$-\frac{5}{2}$	1
	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	0	0	$\frac{3}{2}$	$-\frac{7}{2}$	0

	$\bar{x}$	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_2	$\frac{1}{7}$	$\frac{1}{7}$	1	0	$\frac{2}{7}$	$\frac{3}{7}$	$-\frac{3}{7}$	0
x_3	$\frac{12}{7}$	$\frac{5}{7}$	0	1	$\frac{3}{7}$	$\frac{1}{7}$	$-\frac{1}{7}$	0
x_7	$\frac{2}{7}$	$\frac{2}{7}$	0	0	$-\frac{3}{7}$	$\frac{6}{7}$	$-\frac{13}{7}$	1
	$\frac{2}{7}$	$\frac{2}{7}$	0	0	$-\frac{3}{7}$	$\frac{6}{7}$	$-\frac{20}{7}$	0

	$\bar{x}$	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_5	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{7}{3}$	0	$\frac{2}{3}$	1	-1	0
x_3	$\frac{5}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	1	$\frac{1}{3}$	0	0	0
x_7	0	0	-2	0	-1	0	-1	1
	0	0	-2	0	-1	0	-2	0

Soluţie optimă
cu valoarea = 0,
şi cu variabila artificială
x_7 în bază !

	$\bar{x}$	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_5	$\frac{1}{3}$	$\frac{1}{3}$	1	0	0	1	$-\frac{5}{3}$	$\frac{2}{3}$
x_3	$\frac{5}{3}$	$\frac{2}{3}$	-1	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$
x_4	0	0	2	0	1	0	1	-1
	0	0	0	0	0	0	-1	-1

Faza II

	$\bar{x}$	x_1	x_2	x_3	x_4	x_5
x_5	$\frac{1}{3}$	$\frac{1}{3}$	1	0	0	1
x_3	$\frac{5}{3}$	$\frac{2}{3}$	–1	1	0	0
x_4	0	0	2	0	1	0
	$-\frac{1}{3}$	$\frac{11}{3}$	8	0	0	0

	$\bar{x}$	x_1	x_2	x_3	x_4	x_5
x_5	$\frac{1}{3}$	$\frac{1}{3}$	0	0	$-\frac{1}{2}$	1
x_3	$\frac{5}{3}$	$\frac{2}{3}$	0	1	$\frac{1}{2}$	0
x_2	0	0	1	0	$\frac{1}{2}$	0
	$-\frac{1}{3}$	$\frac{11}{3}$	0	0	–4	0

	$\bar{x}$	x_1	x_2	x_3	x_4	x_5
x_1	1	1	0	0	$-\frac{3}{2}$	3
x_3	1	0	0	1	$\frac{3}{2}$	–2
x_2	0	0	1	0	$\frac{1}{2}$	0
	–4	0	0	0	$\frac{3}{2}$	–11

	$\bar{x}$	x_1	x_2	x_3	x_4	x_5
x_1	1	1	3	0	0	3
x_3	1	0	–3	1	0	–2
x_4	0	0	2	0	1	0
	–4	0	–3	0	0	–11

Soluţia optimă !