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ALGEBRĂ
CURS 7

K corp comutativ $f \in K[x]$, $\underbrace{\text{grad } f}_{n} \geq 1$.
Atunci $\exists$ un corp L, $K \leq L$ astfel încât f are n rădăcini în L.

(ținând cont și de multiplicități)

$f \in \mathbb{C}[x]$ grad $f = n$
$\Rightarrow f$ are n răd în $\mathbb{C}$

~~p prim~~ K corp finit $\Rightarrow |K| = p^{\alpha}$ p. prim.
p prim $\alpha \in \mathbb{N}^*$ $\Rightarrow \exists$ K corp a.î. $|K| = p^{\alpha}$.
K corp finit $\Rightarrow K$ comutativ.

Scop:

Th: $f \in K[x]$, K corp com. grad $f \geq 1$ $\Rightarrow$ f se scrie unic (mai puțin ordinea factorilor) ca prod. de polin monice, ireductibile

$f \in K[x]$ s.n. monic $\overset{def}{\Longleftrightarrow} f(x) = x^n$

$f, g \in K[x]$ nu ambele zero.
$\Rightarrow f\,K[x] + g\,K[x] = h\,K[x]$
$\exists\, h \in K[x]$.

$f\,K[x] + g\,K[x] = \{f \cdot q + g \cdot r \,/\, q, r \in K[x]\}$

$h\,K[x] = \{h \cdot s \,/\, s \in K[x]\}$

Prop $a, b \in \mathbb{Z}$ nu ambele $0 \Rightarrow \exists\, c, d \in \mathbb{Z}$ a.î.
$a \cdot c + b \cdot d = (a, b)$

$$\boxed{a\mathbb{Z} + b\mathbb{Z} = (a, b)\mathbb{Z}}$$

$d = \min \{ k = au + bw \in \mathbb{N}^* \mid u, w \in \mathbb{Z} \}$

$\downarrow$ nevidă (îl cont. pe a sau pe $-a$)

O să arăt că $d = (a, b)$

$d = au_0 + bw_0$ ($\exists\, u_0, w_0 \in \mathbb{Z}$)

$d = (a, b)$

1) $\begin{cases} d \mid a \\ d \mid b \end{cases}$

2) Dacă $\left.\begin{array}{l} e \mid a \\ e \mid b \end{array}\right\} \overset{?}{\Rightarrow} e \mid d$

Presup. că $r \neq 0 \Rightarrow r \geq 1$. $r \in \mathbb{N}^*$

$r = a - dq = a - q(au_0 + bw_0) = a(\underbrace{1 - qu_0}_{\in \mathbb{Z}}) + b(\underbrace{-qw_0}_{\in \mathbb{Z}})$

$\Rightarrow r \geq d$
(minimalitatea d)

$\Rightarrow a = d \cdot q + r$.

Deci $r = 0 \Rightarrow d \mid a$

$r \in \{0, 1, 2, \ldots, d-1\}$

$\left.\begin{array}{l} e \mid a \\ e \mid a \end{array}\right| \; e \mid \overline{au_0 + bw_0} = d$

$a\mathbb{Z} + b\mathbb{Z} = (a, b)\mathbb{Z}$ ex.

Dem (Prop)

Aleg $h \in K[x]$ a.î. grad h = cel mai mic grad t — $t = f \cdot g + g \cdot r$, $g, r \in K[x]$

h monic

$h = f \cdot g_0 + g \cdot r_0$.

t monic

Arătăm că $f = h \cdot f_1$

$g = h \cdot g_1$

$f_1, g_1 \in K[x]$

$f = h \cdot f_1 + r$ $r \in K[x]$; grad $r <$ grad h.

Presupem că $r \neq 0$

$r = f - h f_1 = f - (f g_0 + g r_0)$

$= f(1 - g_0) + g(-r_0) = a_n x^n + \dots + a_0$

$a_n \neq 0$ db.

$= \frac{1}{a_n} r = \frac{1 - g_0}{a_n} f + g \cdot \left(\frac{-r_0}{a_n}\right)$

$r_1 = \frac{1}{a_n} \cdot r \in K[x]$

monic.

$r_1 = \frac{1}{a_n} \cdot r \in K[x]$.

monic

$r_1 = f \cdot f_1 + g \cdot g_1$.

$h = f g_0 + g r_0$

$f = h \cdot f_1$

$g = h \cdot g_1$

$f_1, g_1, g_0, r_0 \in K(x]$

$h \cdot s = (f g_0 + g r_0) s = f \cdot (g_0 s) + g (r_0 s) \in f K[x] + g K[x].$

„$\subseteq$" $f g$ — $= h f_1 q + h g_1 r = h (f_1 q + g_1 r) \in h K[x].$

$\cancel{f_1 g}$ $r \in K[x]$

Dem th

I scrierea f · ired ✓

Dacă f nu e ired. $\Rightarrow f(x) = f_1(x) \cdot g_1(x)$

f_1, f_2 monice

grad $f_1 <$ grad f.

grad $f_2 <$ grad f.

$f_1 =$ ired.

dacă f_1 nu e ired. $f_1 = f_2(x) g_2(x)$ f_2, g_2 monice

grad $f_2 <$ grad f_1 — —

grad $f >$ grad $f_1 >$ grad f_2 ———

$f = h_1 \cdot h_2 = h_1 t_1$

h_1 ired, t_1 ired.

II unicitatea

$f = f_1^{\alpha_1} \cdots f_r^{\alpha_r} = g_1^{\beta_1} \cdots g_s^{\beta_s}$

$f_1, \cancel{f_2} \ldots f_r$ ired, monice

$f_i \neq f_j$ pt $i \neq j$.

$g_1 \ldots g_s$ ired monice $g_i \neq g_j$ pt. $i \neq j$

$\alpha_i, \beta_j \in \mathbb{N}^*$

Vreau să arăt că $f_1 = g_j$ $\exists j$ a.î.

$f \in K[x]$, f - ired, $g, h \in \cancel{K}[x]$.
$g \cdot h = f \cdot t \quad t \in K[x]$

$\Rightarrow$ Fie $g = f \cdot g_1 \quad g \in K[x]$.
Fie $h = f \cdot h_1$, $h_1 \in K[x]$

$\overset{prop}{\Rightarrow} fK[x] + gK[x] = t_1 K[x]$
$f K[x] + h K[x] = t_2 K[x]$
$f \cdot 1 + g \cdot 0 = t_1 \cdot t_3. \quad g = t_1 \cdot t_4$

$\boxed{f = t_1 \cdot t_3} \Rightarrow t_1 = 1$ Caz prost.
f ired sau $\boxed{t_1 = f}$ e perfect ✓

$$\begin{cases} f_i = g\, \bar{v}(i) \\ \alpha_i = \beta\, \bar{v}(i) \end{cases} \forall i = \overline{1,r}$$

$1 = f \cdot g_1 + g \cdot g_2$
$1 = f \cdot g_3 + h \cdot g_4$ Înmulțese

$1 = \underline{f^2 g_1 g_3} + \underline{f h g_1 g_4} + \underline{f g g_2 g_3} + g h g_2 g_4$

$g_1, g_2, g_3, g_4 \in K[x]$

$t \cdot f_1 = g_1^{\beta_1} \cdot g_2^{\beta_2} \cdots g_s^{\beta_s} = g_1 (g_1^{\beta_1 - 1} g_2^{\beta_2} \cdots)$

f_1 ired $\xrightarrow{OBS}$ $\boxed{g_1 = f_1 \cdot g_1}$ $\Rightarrow g_1 = 1. \quad g_1 \sim f_1$

$g_1^{\beta_1 - 1} \cdot g_2^{\beta_2} \cdots g^{\beta_s} = f_1 \cdot g_2 \cdot$

$f_1^{\alpha_1} \cdots f_2^{\alpha_2} = f_1^{\beta_1} g_2^{\beta_2} \cdots g_s^{\beta_s}$

$f_1^{\alpha_1 - 1} \cdot f_2^{\alpha_2} \cdots f_r^{\alpha_r} = f_1^{\beta_1 - 1} \cdot g_2^{\beta_2} \cdots g_s^{\beta_s}.$

$f_1 \cdot A = f_1 \cdot B \Rightarrow A = B.$

$f_1 (A - B) = 0.$

Dacă $A - B \neq 0 \rightarrow (x^m + \ldots)(at\, x^{k} \ldots)$

$\alpha_1 < \beta_1$

$f_1 \cdot g = f_2^{\alpha_2} \cdots$

Inel factor $\quad R$ inel comutativ

$\frac{R}{I} = \{\bar{r} \mid r \in R\}$

$\bar{r} = \bar{s} \overset{def}{\Leftrightarrow} r - s \in I$

$\bar{r} \oplus \bar{s} = \overline{r + s}$

$\bar{r} \odot \bar{s} = \overline{r \cdot s}$

$\frac{\mathbb{Z}}{n\mathbb{Z}}.$

$\mathbb{Z} \quad \bar{a} = \bar{b}$

$a - b \in \mathbb{Z}.$

Atunci $\dfrac{K[x]}{fK[x]}$ corp.

$$g \in gK[x] + fK[x] = K[x]$$

$$\bar{g} \cdot \bar{g}_1 = \bar{1}$$