Algebră
seminar 6

$K = \mathbb{Z}_{11}$

$v_1 = \begin{pmatrix} \bar{1} \\ \bar{2} \\ \bar{3} \end{pmatrix} \quad v_2 = \begin{pmatrix} \bar{1} \\ \bar{1} \\ \bar{4} \end{pmatrix} \quad v_3 = \begin{pmatrix} \bar{1} \\ \bar{3} \\ \bar{7} \end{pmatrix}$

$V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{Z}_{11} \right\}$ $(V,+)$ este $\mathbb{Z}_{11}$ spațiu vectorial din 3

v_1, v_2, v_3 bază în $V \overset{\text{OA}}{\iff} \det \begin{vmatrix} \bar{1} & \bar{1} & \bar{1} \\ \bar{2} & \bar{1} & \bar{3} \\ \bar{3} & \bar{4} & \bar{7} \end{vmatrix} \neq \bar{0}$

$$\begin{vmatrix} \bar{1} & \bar{1} & \bar{1} \\ \bar{2} & \bar{1} & \bar{3} \\ \bar{3} & \bar{4} & \bar{7} \end{vmatrix} = \begin{vmatrix} \bar{1} & \bar{0} & \bar{0} \\ \bar{2} & -\bar{1} & \bar{1} \\ \bar{3} & \bar{1} & \bar{4} \end{vmatrix} = \bar{1} \cdot (-1)^{1+1} \begin{vmatrix} -\bar{1} & \bar{1} \\ \bar{1} & \bar{4} \end{vmatrix} = -\bar{5} = \bar{6} \neq \bar{0} \Rightarrow \text{bază}$$

$c_2 - c_1,\ c_3 - c_1$

$v = \begin{pmatrix} \bar{0} \\ \bar{8} \\ \bar{9} \end{pmatrix} \in V$

Găsiți $x, y, z \in \mathbb{Z}_{11}$ a.î. $v = x v_1 + y v_2 + z v_3$

$$\begin{cases} x + y + z = \bar{0} \\ \bar{2}x + \bar{y} + \bar{3}z = \bar{8} \\ \bar{3}x + \bar{4}y + \bar{7}z = \bar{9} \end{cases}$$

I / Cramer

$$x = \frac{\begin{vmatrix} 0 & 1 & 1 \\ 8 & 1 & 3 \\ 9 & 4 & 7 \end{vmatrix}}{\bar{6}} = 2 \cdot \begin{vmatrix} 0 & 0 & 1 \\ 8 & -2 & 3 \\ 9 & -3 & 7 \end{vmatrix} = 2(-24+18) = -\overline{12} = -\bar{1}$$

$c_2 - c_3$

$6x \equiv 1 \ (11)$

$x = 2 \ (11)$

$$y = \bar{2} \cdot \begin{vmatrix} \bar{1} & \bar{0} & \bar{1} \\ \bar{2} & \bar{8} & \bar{3} \\ \bar{3} & \bar{4} & \bar{7} \end{vmatrix} = \bar{2} \begin{vmatrix} \bar{1} & \bar{0} & \bar{0} \\ \bar{2} & \bar{8} & \bar{1} \\ \bar{3} & \bar{4} & \bar{4} \end{vmatrix} = \bar{2} \cdot \bar{28} = \bar{2}$$

$x = -\bar{1}$
$y = \bar{2}$
$z = -\bar{1}$

b.

$$z = \bar{2} \begin{vmatrix} 1 & 1 & 0 \\ 2 & 1 & 8 \\ 3 & 4 & 4 \end{vmatrix} = 2 \begin{vmatrix} 1 & 0 & 0 \\ 2 & -1 & 8 \\ 3 & 1 & 4 \end{vmatrix} = 2(-12) = \bar{10} = -\bar{1}$$

II $L_2 - L_1$

$$\begin{cases} x + \bar{2}z = 8 \\ -x + \bar{3}z = \bar{9} \end{cases}$$

$\bar{5}z = \bar{17} = -\bar{5}$

$z = -\bar{1}$

$x - 2 = 8$

$x = \bar{10} = -\bar{1}$

$-1 + y - 1 = 0 \Rightarrow y = \bar{2}$

$P(\lambda) = \det|A - \lambda I_n|$

$P_\lambda = \bar{v}_1 \lambda \ldots$

2)
$$D_n = \begin{vmatrix} 1+a_1 & & 1 \\ & 1+a_2 & \\ & & \ddots \\ 1 & & 1+a_n \end{vmatrix} = a_1 a_2 \ldots a_n \left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} + 1 \right)$$

$9:70$

$n = 1 \quad 1 + a_1$

$$n = 2 \quad \begin{vmatrix} 1+a_1 & 1 \\ 1 & 1+a_2 \end{vmatrix} = a_1 + a_2 + a_1 a_2$$

$$n_3 \quad \begin{vmatrix} 1+a_1 & 1 & 0 \\ 1 & 1+a_2 & 1 \\ 1 & 1 & 1+a_3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ -a_1 & a_2 & 1 \\ -a_1 - a_3 - a_1 a_3 & a_3 & 1+a_3 \end{vmatrix}$$

$$= \begin{vmatrix} -a_1 & a_2 \\ -a_1 - a_3 - a_1 a_3 & 1+a_3 \end{vmatrix} = a_1 a_3 + a_1 a_2 + a_2 a_3 + a_1 a_2 a_3$$

$$D_4=\begin{vmatrix} 1+a_1 & 1 & 1 & 1 \\ 1 & 1+a_2 & 1 & 1 \\ 1 & 1 & 1+a_3 & 1 \\ 1 & 1 & 1 & 1+a_4 \end{vmatrix} = (1+a_4)D_3 - \begin{vmatrix} 1+a_1 & 1 & 1 \\ 1 & 1+a_2 & 1 \\ 1 & 1 & 1 \end{vmatrix}_{a_3=0} +$$

$$+\underbrace{\begin{vmatrix} 1+a_1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1+a_3 & 1 \end{vmatrix}}_{\text{inversam } l_2 \text{ cu } l_3} + \underbrace{\begin{vmatrix} 1 & 1 & 1 \\ 1+a_2 & 1 & 1 \\ 1 & 1+a_3 & 1 \end{vmatrix}}_{\text{inversam de 2 ori}}$$

$$= a_1a_2+a_1a_3+a_2a_3+a_1a_2a_3+a_1a_2a_4+a_1a_3a_4+a_2a_3a_4+a_1a_2a_3a_4$$
$$-\cancel{a_1a_2}+a_1a_3-a_2a_3$$

inducție după n

- verificare ($n=1,2,3,4$)

presupunem adevărat pt $nr \le n-1$ și demonstrăm pt n

$$(1+a_n)D_{n-1}-\begin{vmatrix} 1+a_1 & 1 & & 1 \\ \vdots & 1+a_2 & & 1 \\ \vdots & \vdots & 1+a_{n-2} & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1+a_1 & \cdots & & 1 \\ \vdots & & 1+a_{n-3} & \vdots \\ 1 & & 1 & \vdots \\ 1 & & 1 \quad a_{n-1} & 1 \end{vmatrix}$$

$$\begin{vmatrix} a_{11}+b_{11}, & a_{12}+b_{12}, & \dots, & a_{1n}+b_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & \vdots & & a_{nn} \end{vmatrix} = \det(a_{ij})+\begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & \cdots & & a_{nn} \end{vmatrix}$$

$$D_n=a_1a_2\dots a_{n-1}+\dots+a_{23}\dots a_n+\dots+a_1\dots a_n$$

$$D_n = a_1 \underset{a_2 \dots a_n}{D_{n-1}} + \begin{vmatrix} 1 & 1 & \dots & 1 \\ 1 & 1+a_2 & & 1 \\ \vdots & & \ddots & \\ 1 & & 1 & 1+a_n \end{vmatrix}$$

$$\begin{array}{c} L_2-L_1 \\ L_3-L_1 \\ \vdots \\ L_n-L_1 \end{array} \begin{vmatrix} 1 & 1 & \dots & 1 \\ 1 & 1+a_2 & \dots & 1 \\ \vdots & & \ddots & \vdots \\ \vdots & 1 & & \\ 1 & \dots & \dots & 1+a_n \end{vmatrix} = \begin{vmatrix} 1 & 1 & \dots & 1 \\ 0 & a_2 & \dots & 0 \\ & \dots & \dots & \\ & \cdot & & \cdot \\ 0 & \dots & \dots & a_n \end{vmatrix} = a_2 \cdot \ldots \cdot a_n$$

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \vdots & & & \\ x_1^{n-1} & \dots & \dots & x_n^{n-1} \end{vmatrix} = \prod_{1 \le i < j \le n} (x_j - x_i)$$

Ex inducție după n

$$D = \begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \vdots & & & \\ x_1^n & \dots & \dots & x_n^n \end{vmatrix} = ?$$

$(n+1) \times (n+1)$

$$\begin{vmatrix} 1 & 1 & \dots & 1 & 1 \\ x_1 & x_2 & \dots & x_n & x \\ x_1^2 & x_2^2 & \dots & x_n^2 & x^2 \\ \vdots & & & & \vdots \\ x_1^n & x_2^n & \dots & x_n^n & x^n \end{vmatrix} = \prod_{1 \le i < j \le n} (x_j - x_i)(x - x_1)(x - x_2) \dots (x - x_n) \in K[x]$$

$x_j \in K$

↳ coef com

$$x^n \begin{vmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_n \\ \vdots & & \\ x_1^{n-1} & \dots & x_n^{n-1} \end{vmatrix} - x^{n-1} \begin{vmatrix} \dots \end{vmatrix} \dots\dots x\Delta(-1)^{2+(n+1)} + \dots$$

Coeficientul lui x este $\Delta(-1)^{n+3} = \prod_{1 \le i < j \le n} (x_j - x_i) (-1)^{n+1} (x_2 x_3 \dots x_n + x_1 x_3 \dots x_n + \dots$
$x_1 \dots x_{n-1})$

$$D_n = \begin{vmatrix} \alpha+\beta, & \alpha\beta & & & 0 \\ 1, & \alpha+\beta, & \alpha\beta & & \\ & 1, & \alpha+\beta, & \alpha\beta & \\ & & & \ddots & \\ 0 & & & & \alpha\beta \\ & & & 1 & \alpha+\beta \end{vmatrix}$$

dezvoltăm după prima linie

$$= (\alpha+\beta) D_{n-1} - \alpha\beta \begin{vmatrix} 1 & \alpha\beta & \dots & 0 \\ 0 & \alpha+\beta & \alpha\beta & 0 \dots 0 \\ \vdots & 0 & & \\ 0 & 0 & 1 & \alpha+\beta \end{vmatrix}$$

$D_n = ?$
$\alpha, \beta \in \mathbb{C}$

$$D_n = (\alpha+\beta) D_{n-1} - \alpha\beta D_{n-2}$$

0

$a_{n+2} = a\, a_{n+1} + b\, a_n$

$t^2 = at + b$

x_1, x_2 rădăcini

Dacă $x_1 \neq x_2 \Rightarrow a_n = A x_1^n + B x_2^n$

$\begin{cases} F_0 = 0, F_1 = 1 \\ F_{n+2} = F_{n+1} + F_n \end{cases}$

$x^2 = x + 1$

$x^2 - x - 1 = 0$

$x_{1,2} = \frac{1 \pm \sqrt{5}}{2}$

$F_n = A \left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$

$n = 0$
$n = 1$

$0 = F_0 = A + B$

$B = -A$

$1 = F_1 = A \frac{1+\sqrt{5}}{2} + B \frac{1-\sqrt{5}}{2} = A(\sqrt{5}) \Rightarrow A = \frac{1}{\sqrt{5}}$ $B = -\frac{1}{\sqrt{5}}$

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

$$F_6 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^6 - \left(\frac{1-\sqrt{5}}{2}\right)^6\right] = \frac{1}{2^6\sqrt{5}}\left[(6+2\sqrt{5})^3 - (6-2\sqrt{5})^3\right] =$$

$$= \frac{1}{2^6\sqrt{5}}\left[6^3 + 3\cdot 6^2\cdot 2\sqrt{5} + 3\cdot 6\cdot 20 + 2^3\cdot 5\sqrt{5}\right) - 6^3 + 3\cdot 6^2\cdot 2\sqrt{5} - 3\cdot 6\cdot 20 + 2^3\, 5\sqrt{5}\Big)$$

$$= \frac{2}{2^6}\left[216 + 40\right] = \frac{2\cdot 256}{2^6} = 2^3 = 8$$

$$D_n = (\alpha+\beta) D_{n-1} - \alpha\beta D_{n-2}$$

$$x^2 = (\alpha+\beta)x - \beta\alpha$$

$$x^2 - (\alpha+\beta)x + \alpha\beta = 0$$

$$x_1 = \alpha \quad x_2 = \beta$$

$$D_n = A\alpha^n + B\beta^n$$

$$D_0 = 1$$

$$D_1 = \alpha+\beta$$

$$D_2 = \begin{vmatrix} \alpha+\beta & \alpha\beta \\ 1 & \alpha+\beta \end{vmatrix} = \alpha^2 + \beta^2 + \alpha\beta$$

$$\begin{vmatrix} 1 & 1 \\ \alpha & \beta \end{vmatrix} = \beta - \alpha$$

$$1 = D_0 = A + B$$

$$\alpha+\beta = D_1 = A\alpha + B\beta$$

$$A = \frac{\begin{vmatrix} 1 & 1 \\ \alpha+\beta & \beta \end{vmatrix}}{\beta-\alpha} = \frac{-\alpha}{\beta-\alpha}$$

$$B = \frac{\begin{vmatrix} 1 & 1 \\ \alpha & \alpha+\beta \end{vmatrix}}{\beta-\alpha} = \frac{\beta}{\beta-\alpha}$$

Dacă $\beta \neq \alpha$

$$D_n = \frac{-\alpha}{\beta-\alpha}\alpha^n + \frac{\beta}{\beta-\alpha}\beta^n = \frac{\beta^{n+1} - \alpha^{n+1}}{\beta-\alpha}$$

$\beta = \alpha$

$$D_n = \begin{vmatrix} \alpha + \beta_n & \alpha\beta_n & & 0 \\ 1 & \alpha + \beta_n & \alpha\beta_n & \\ & 1 & \cancel{\alpha} + \beta_n & \alpha\beta_n \\ 0 & & & \ddots \end{vmatrix}$$

$\beta_n \xrightarrow{n} \alpha$

$\beta_n \neq \alpha$

$$\lim_{x \to \alpha} \frac{x^{n+1} - \alpha^{n+1}}{x - \alpha} = (n+1)\cancel{x}^{n} \quad \alpha = \beta$$