SE 7. ECUAȚII DIFERENȚIALE CU DERIVATE PARȚIALE 17.11.2009

REZOLVARE EXERCIȚII

EX. 2

$x'' + \sin x = 0.$

Trebuie să arătăm că există o infinitate de soluții φ_0 cu $\lim_{t \to \infty} \varphi_0(t) = \pi$.

$x = y(x(t))$

$x' = y' x' \Rightarrow x'' = y' y$ Nu are soluții staționare $(\frac{1}{y} \neq 0)$

$y'y = -\sin x \Rightarrow y' = -\frac{\sin x}{y} \Rightarrow \frac{dy}{dx} = -\frac{\sin x}{y} \Rightarrow y\,dy = -\sin x\, dx$

$\Rightarrow \frac{y^2}{2} = \cos x + \frac{k}{2} \Rightarrow y^2 = 2\cos x + k \Rightarrow y = \pm\sqrt{2\cos x + k}$

$\Rightarrow y = \pm\sqrt{2(\cos x + 1)} \Rightarrow y = \pm\sqrt{2 \cdot 2\cos^2\left(\frac{x}{2}\right)} \Rightarrow y = \pm 2\cos\left(\frac{x}{2}\right) \Rightarrow$

$\Rightarrow x' = \pm 2\cos\left(\frac{x}{2}\right)$

$\frac{dx}{dt} = 2\cos\left(\frac{x}{2}\right) \Rightarrow$

Soluții staționare $\cos\left(\frac{x}{2}\right) = 0 \Rightarrow \frac{x}{2} = \frac{\pi}{2} + k\pi \Rightarrow x_k = \pi + 2k\pi,\ k \in \mathbb{Z}$

$\Rightarrow x_0 = \pi$ îndeplinește condițiile cerute

$\frac{dx}{2\cos\frac{x}{2}} = dt \Rightarrow \frac{dx}{2\left(\frac{1 - tg^2\frac{x}{4}}{1 + tg^2\frac{x}{4}}\right)} = dt \Rightarrow \frac{1}{2}\int \frac{1 + tg^2\frac{x}{4}}{1 - tg^2\frac{x}{4}}\, dx = t + k_2$

$\Rightarrow -\frac{1}{2}\int \frac{\frac{1}{\cos^2\frac{x}{4}}}{tg^2\frac{x}{4} - 1}\, dx = t + k_2 \Rightarrow -2\int \frac{\frac{1}{4}\frac{1}{\cos^2\frac{x}{4}}}{tg^2\frac{x}{4} - 1}\, dx = t + k_2 \Rightarrow$

$$\Rightarrow -2\int \frac{\left(tg\frac{x}{4}\right)}{\left(tg\frac{x}{4}\right)^2-1}\,dx = t+k_2 \Rightarrow$$

$$z = tg\frac{x}{4}$$

$$\Rightarrow \int \frac{1}{z^2-1}\,dz = t+k_2 \Rightarrow \cancel{-2}\cdot\frac{1}{\cancel{2}}\ln\left|\frac{z-1}{z+1}\right| = t+k_2 \Rightarrow$$

$$\Rightarrow \frac{z+1}{z-1} = e^{t+k_2} \Rightarrow z+1 = z e^{t+k_2} - e^{t+k_2} \Rightarrow$$

$$\Rightarrow z\left(e^{t+k_2}-1\right) = 1 + e^{t+k_2} \Rightarrow$$

$$\Rightarrow tg\frac{x}{4} = \frac{e^{t+k_2}+1}{e^{t+k_2}-1} \Rightarrow x = 4\,arctg\,\frac{e^{t+k_2}+1}{e^{t+k_2}-1} \quad (= \varphi_0(t))$$

$$\lim_{t\to\infty}\varphi_0(t) = 4\,arctg\,1 \Rightarrow \lim_{t\to\infty}\varphi_0(t) = \pi.$$

$\boxed{EX\ 3}$ $$x' = \frac{t^2+(1+2t^2)}{t}x$$

$$\Rightarrow x' = \frac{2t^2+1}{t}x + t \quad (\text{ec. afină})$$

$$\frac{d\bar{x}}{dt} = a(t)\bar{x}$$

$$\bar{x} = c\cdot e^{\int a(t)} \Rightarrow c\cdot e^{\int(2t+\frac{1}{t})dt} = c\cdot e^{t^2+\ln t} = c\cdot e^{t^2}\cdot e^{\ln t} =$$
$$= c\cdot t\cdot e^{t^2}$$

$$x(t) = c(t)\cdot t\cdot e^{t^2} \Rightarrow (c(t)\cdot t\cdot e^{t^2})' = \frac{2t^2+1}{\cancel{t}}(c(t)\cdot\cancel{t}\cdot e^{t^2}) + t$$

$$\Rightarrow c'(t)\cdot t\cdot e^{t^2} + c(t)(e^{t^2}\cancel{+te^{t^2}\cdot 2t}) = (2t^2\cancel{+1})c(t)\cdot e^{t^2} + t$$

$$\Rightarrow c'(t)\cdot t\cdot e^{t^2} = t \Rightarrow c'(t) = e^{-t^2}$$

$$\Rightarrow c(t) = \int e^{-t^2}dt + c_1$$

$$\Rightarrow c(t) = \int_0^t e^{-s^2}ds + c_1$$

$$x(t) = t\cdot e^{t^2}\cdot\left(\int_0^t e^{-s^2}ds + c_1\right) \Rightarrow$$

$$\boxed{(*)\quad \int_{-\infty}^{+\infty} e^{-s^2}ds = \sqrt{\pi} \quad (\text{amintire: repartiția Gauss})}$$

Dacă $c_1 \neq -\frac{\sqrt{\pi}}{2}$ atunci $\lim_{t\to\infty} x(t) = \infty\cdot\left(\frac{\sqrt{\pi}}{2}+c_1\right) =$

$$= \infty\cdot \operatorname{sgn}\left(\frac{\sqrt{\pi}}{2}+c_1\right)$$

Dacă $c_1 = -\frac{\sqrt{2}}{2}$ atunci $\lim_{t\to\infty} x(t) = \infty\cdot 0$ nedeterm. $\Rightarrow \lim_{t\to\infty}\dfrac{\int_0^t e^{-s^2}ds + \frac{\sqrt{\pi}}{2}}{\frac{1}{t\cdot e^{-t^2}}} =$

$$\overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{t\to\infty}\frac{\cancel{e^{-t^2}}\cdot 1}{\cancel{e^{-t^2}}\left(-\frac{1}{t^2}-2\right)} = \frac{1}{-2} = -\frac{1}{2} = l \in R.$$

$$\left(\frac{1}{t}e^{-t^2}\right)' = \frac{1}{t^2}e^{-t^2} + \frac{1}{t}\cdot(-2t)\cdot e^{-t^2}$$

EXERCIȚII CURS

$\boxed{EX}$ Se cere un sistem fundamental de soluții pt. sistemul

$$\begin{cases} x_1' = 2x_1 + x_2 \\ x_2' = 4x_1 + x_2 \end{cases}$$ - sistem liniar cu coef. constanți

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 2 & +1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Determinăm valorile proprii pt. matricea A

$$\det(A - \lambda I_2) = 0 \to \begin{vmatrix} 2-\lambda & +1 \\ 4 & 1-\lambda \end{vmatrix} = 0 \to 2 - 2\lambda - \lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - 3\lambda - 2 = 0$$

$$\Delta = 9 + 8 = 17$$

$$\lambda_{1,2} = \frac{3 \pm \sqrt{17}}{2}$$

$\{\lambda_1, \lambda_2\} \overset{not}{=} Spect(A) = \sigma(A)$

mulțimea valorilor proprii

$\lambda_1 = \frac{3+\sqrt{17}}{2} \Rightarrow$ det $\vec{u}$ a.î. $(A - \lambda_1 I_2)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\downarrow$

$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

ord. multip: $k_1 = 1, k_2 = 1$

$n = 2$.

$$\begin{pmatrix} \frac{1-\sqrt{17}}{2} & 1 \\ 4 & \frac{-1-\sqrt{17}}{2} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} \frac{1-\sqrt{17}}{2}u_1 + u_2 = 0 \\ u_1 + \frac{-1-\sqrt{17}}{2} = 0 \quad \Big| \cdot \frac{-1+\sqrt{17}}{8} \end{cases}$$

$$\Rightarrow \cancel{4}\cdot\frac{-1+\sqrt{17}}{\frac{8}{2}}u_1 + \frac{(-1)^2 - (\sqrt{17})^2}{16}u_2 = 0 \Rightarrow -\frac{(1-\sqrt{17})}{2}u_1 - u_2 = 0$$

ec. nu sunt indep.

$u_2 = \frac{-1+\sqrt{17}}{2} u_1$, $u_1 \in \mathbb{R}$; alegem $u_1 = 1 \Rightarrow u_2 = \frac{\sqrt{17}-1}{2} \Rightarrow$

$\Rightarrow$ avem $\varphi_1(t) = e^{\lambda_1 t} \cdot u$

$$\boxed{\varphi_1(t) = e^{\frac{3+\sqrt{17}}{2} \cdot t} \cdot \begin{pmatrix} 1 \\ \frac{\sqrt{17}-1}{2} \end{pmatrix}}$$

$\lambda_2 = \frac{3-\sqrt{17}}{2}$, $k_2 = 1 \Rightarrow$ det $u \in \mathbb{R}^2 \setminus \{0\}$ a.î. $(A - \lambda_2 I_2) u = 0 \Rightarrow$

$$\Rightarrow \begin{pmatrix} \frac{1+\sqrt{17}}{2} & 1 \\ 4 & \frac{-1+\sqrt{17}}{2} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} \frac{1+\sqrt{17}}{2} u_1 + u_2 = 0 \\ 4u_1 + \left(\frac{-1+\sqrt{17}}{2}\right) u_2 = 0 \end{cases} \Big/ \frac{\sqrt{17}+1}{8} \Rightarrow \text{prima ec}$$

$\Rightarrow u_2 = -\frac{1+\sqrt{17}}{2} u_1$, alegem $u_1 = 1$

$u_2 = \frac{-1-\sqrt{17}}{2} \Rightarrow \boxed{\varphi_2(t) = e^{\left(\frac{3-\sqrt{17}}{2}\right)t} \cdot \begin{pmatrix} 1 \\ \frac{-1-\sqrt{17}}{2} \end{pmatrix}}$

Soluția generală

$$\varphi(t) = c_1 \varphi_1(t) + c_2 \varphi_2(t) \quad , c_1, c_2 \in \mathbb{R}.$$

$\boxed{\text{SISTEM FUNDAMENTAL DE SOLUȚII } \{\varphi_1(t), \varphi_2(t)\}.}$

$x_1(t) = c_1 e^{\lambda_1 t} \cdot 1 + c_2 \cdot e^{\lambda_2 t} \cdot 1$

$x_2(t) = c_1 \cdot e^{\lambda_1 t} \cdot \frac{\sqrt{17}-1}{2} + c_2 \cdot e^{\lambda_2 t} \cdot \frac{-1-\sqrt{17}}{2}$, $c_1, c_2 \in \mathbb{R}$.

Acesta a fost un caz simplu cu sol. reale și ord. de multiplic. reale.

$\boxed{EX}$ Sistem fundamental de soluții pt.

$$\begin{cases} x_1' = -2x_1 - 3x_2 \\ x_2' = 3x_1 + 4x_2. \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \underbrace{\begin{pmatrix} -2 & -3 \\ 3 & 4 \end{pmatrix}}_{A.} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \text{det val proprii} \Rightarrow$$

$$\Rightarrow \begin{vmatrix} -2-\lambda & -3 \\ 3 & -4-\lambda \end{vmatrix} = 0 \Rightarrow -8 + 2\lambda - 4\lambda + \lambda^2 + 9 = 0$$

$$\lambda^2 - 2\lambda + 1 = 0$$

$$(\lambda - 1)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = 1.$$

$Spec(A) = \{1\}$

$m = 1$ (nr. val. prop)

$k = 2$

$k = 2 \Rightarrow$ det $p_0, p_1 \in \mathbb{R}^2$ a.î. $\varphi(t) = e^{\lambda_1 t}\left(\sum_{t=0}^{1} p_t e^{t}\right)$

nu toți nuli

$$\left.\begin{array}{l} \varphi(t) = e^t(p_0 + p_1 t) \\ x' = Ax \end{array}\right| \Rightarrow \cancel{e^t}(p_0 + p_1 t) + \cancel{e^t} p_1 =$$

$$= A\cancel{e^t}(p_0 + p_1 t) \Rightarrow \begin{cases} p_0 + p_1 = A p_0. \\ p_1 = A p_1 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} p_1 = (A - I_2) p_0. \\ 0 = (A - I_2) p_1 \end{cases} \Rightarrow \begin{cases} (A - I_2) p_1 = (A - I_2)^2 p_0. \\ \end{cases} \Rightarrow$$

$$\Rightarrow (A - I_2)^2 p_0 = 0 \Rightarrow p_0 \in \ker\{(A - I_2)^2\}$$

$$A - I_2 = \begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix}$$

$$(A - I_2)^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O_2 \Rightarrow p_0 \in \ker(O_2) = \mathbb{R}^2 \Rightarrow$$

$$\Rightarrow p_0 \in \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

(pentru p_0 se aleg valorile dintr-o bază a lui $\ker(O_2)$)

$$p_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow p_1 = \begin{pmatrix} -3 \\ 3 \end{pmatrix} \Rightarrow \text{o soluție } \varphi_1(t) = e^t\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3 \\ 3 \end{pmatrix}\right)$$

$$\boxed{\varphi_1(t) = e^t \begin{pmatrix} 1-3t \\ 3t \end{pmatrix}}$$

$$p_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow p_1 = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$$

$$\boxed{\varphi_2(t) = e^t \begin{pmatrix} -3t \\ 1+3t \end{pmatrix}}$$

sol. gen: $\varphi(t) = c_1 \varphi_1(t) + c_2 \varphi_2(t)$, $c_1, c_2 \in \mathbb{R}$.

TEMA / Să se determ. sol gen. a următoarelor sisteme:

① $\begin{cases} x_1' = 3x_1 - x_2 \\ x_2' = 4x_1 - x_2 \end{cases}$

② $\begin{cases} x_1' = x_1 - x_2 - x_3 \\ x_2' = x_1 + x_2 \\ x_3' = 3x_1 + x_3 \end{cases}$

③ $\begin{cases} x_1' = x_2 - x_3 \\ x_2' = 2x_1 + x_2 + x_3 \\ x_3' = x_3 - x_2 \end{cases}$