Algebră
curs 13

Teorema Kronecker - Capelli

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{cases}$$

$a_{ij}, b_i \in K$ K comutativ

$$c_j := \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix} \qquad V \simeq K^n$$

$$V = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \middle| x_j \in K \ \forall j = \overline{1,n} \right\}$$

Sistemul are soluție (compatibil) $\Leftrightarrow \dim_K \langle c_1, \dots, c_n \rangle = \dim_K \langle c_1, \dots, c_n, b \rangle$

V - K sp. vectorial

$v_1, v_2, \dots, v_k \in V$

$$\langle v_1, v_2, \dots, v_k \rangle = \left\{ \sum_{i=1}^{k} x_i v_i \,\middle|\, x_j \in K \ \forall j = \overline{1,k} \right\}$$

$\Leftrightarrow$ rang A = rang A^e

$$A^e \underset{not}{=} \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ \vdots & & & & \\ a_{n1} & \dots & \dots & a_{nn} & b_n \end{pmatrix}$$

Presupunem că rang A = rang A^e = $k \le n$

$\dim_K \langle c_1, \dots, c_k \rangle = \dim_K \langle l_1, \dots, l_n \rangle$

Cum se calculează soluția

Presupun că $c_1, \dots, c_k$ / $l_1, \dots, l_k$ sunt liniar independenți / K

$x_{k+1}, x_{k+2}, \dots, x_n \in K$

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1k}x_k = -a_{1\,k+1}x_{k+1} \dots - a_{1n}x_n + b_1$$

$$a_{21}x_1 + \dots + a_{2k}x_k = -a_{2\,k+1}x_{k+1} \dots - a_{2n}x_n + b_2$$

$\vdots$

$$a_{k1}x_1 + \dots + a_{kk}x_k = -a_{k\,k+1}x_{k+1} \dots - a_{kn}x_n + b_k$$

$x_1, \ldots, x_k$ sunt unic det de $x_{k+1}, \ldots, x_n$ (formulele Cramer)
Soluția este unică $\Leftrightarrow k = n$

(*) $a_1, \ldots, a_{23} \in \mathbb{R}$ $\qquad 23 \Leftrightarrow 2n+1$

$\forall i \in \{1, \ldots, 23\} \Rightarrow \{1, \ldots, 23\} \setminus \{i\} = G_1 \cup G_2;$ $\quad G_1 \cap G_2 = \emptyset$
$|G_1| = |G_2| = 11$

și $\sum_{j \in G_1} a_j = \sum_{j \in G_2} a_j$

Să se arate că $a_1 = a_2 = \ldots = a_{23}$

Obs 1) Dacă $a_1, \ldots, a_{23}$ au prop (*) $\left.\right\}$ $\Rightarrow a_1 + a, a_2 + a, \ldots, a_{23} + a$ au prop (*)
$a \in \mathbb{R}$

$a_1 + \ldots + a_{11} = a_{12} + \ldots + a_{22} \quad / + 11a$

$(a_1 + a) + \ldots + (a_{11} + a) = (a_{12} + a) + \ldots + (a_{22} + a)$

2) $a \in \mathbb{R} \Rightarrow a a_1 + \ldots + a a_{23}$ au prop (*)

I) Demonstrez enunțul în cazul în care toate numerele ~~sunt~~ $\in \mathbb{Q}$

$a_1\ a_2 \ldots, a_{23} \in \mathbb{Q}$ au prop (*)
Pot presupune că sunt toate întregi

$a a_1 = a a_2 = \ldots = a a_{23}$ $\qquad a \in \mathbb{N}^*$
$\Rightarrow a_1 = a_2 = \ldots = a_{23}$

Pot presupune că $a_1 = 0$

O: $a_1 - a_1, a_2 - a_1, \ldots, a_{23} - a_1 \in \mathbb{Z}$ au prop (*)

O: $0, a_2, \ldots, a_{23} \in \mathbb{Z}$ au prop (*)

$S = a_1 + \ldots + a_{23} =$ par

$S = S - a_1 =$ par $\Rightarrow$
$S - a_i =$ par $\forall i \Rightarrow a_i =$ par $\forall i = \overline{1,23}$

(2)

$0 = \frac{a_1}{2}, \frac{a_2}{2}, \dots, \frac{a_{23}}{2}$ au prop. (*)

nr întregi

Aplică același argument $\Rightarrow \frac{a_i}{2}$ par $\forall i$

Analog $2^k \mid a_i \ \forall k$

$\Rightarrow a_i = 0 \ \forall i$ Dacă $a_i \neq 0 \Rightarrow a_i = \pm 2^m \cdot$ (impar)

$\Rightarrow a_1 = \dots = a_{23} = 0$

II $a_1, a_2, \dots, a_{23} \in \mathbb{R}$

$$\begin{cases} 0a_1 \pm a_2 \pm a_3 \pm \dots \pm a_{23} = 0 \\ \pm a_1 + 0a_2 \pm \dots \pm a_{23} = 0 \\ \quad \vdots \\ \pm a_1 \pm a_2 \pm \dots \pm a_{23} \cdot 0 = 0 \end{cases}$$

(11 semne de +, 11 semne de −)

necunoscută a_1

Sistemul are soluții

Matricea sistemului $A = \begin{pmatrix} 0 & & \pm 1 \\ & \ddots & \\ \pm 1 & & 0 \end{pmatrix} \in \mathcal{M}_{23}(\mathbb{R})$ Pe fiecare linie avem un 0, 11 de +1, 11 de −1

$\det A = 0$ (adun toate coloanele la prima)

rang $A = 22$

$$\underbrace{\begin{vmatrix} 0 & & \pm 1 \\ & \ddots & \\ \pm 1 & & 0 \end{vmatrix}}_{\text{impar}} = \overset{\mathbb{Z}_2}{\begin{vmatrix} \hat{0} & & \hat{1} \\ & \ddots & \\ \hat{1} & & \hat{0} \end{vmatrix}} \overset{\text{adun } C_2, \dots, C_{22} \text{ la } C_1}{=} \begin{vmatrix} \hat{1} & \hat{0} & & \hat{1} \\ \hat{1} & \hat{0} & & \hat{1} \\ \vdots & & \ddots & \\ \hat{1} & & \hat{1} & \hat{0} \end{vmatrix} \underset{\substack{C_2 - C_1 \\ \vdots \\ C_{22} - C_1}}{=} \begin{vmatrix} \hat{1} & \hat{0} & \hat{0} & \cdots & \hat{0} \\ \hat{1} & \hat{1} & \hat{0} & \cdots & \hat{0} \\ \hat{1} & \hat{0} & \hat{1} & \cdots & \hat{0} \\ \vdots & & & \ddots & \vdots \\ \hat{1} & \hat{0} & \hat{0} & \cdots & \hat{1} \end{vmatrix} = \hat{1}$$

$a_{23} = \lambda$

$$\begin{cases} 0a_1 \pm a_2 \dots, \pm a_{22} = \mp\lambda \\ \pm a_1 + 0a_2 \pm \dots \pm a_{22} = \mp\lambda \\ \vdots \\ \pm a_1 \dots \dots \pm a_{22} = \mp\lambda \end{cases}$$ rezolvă cu Cramer

$$a_j = \frac{\begin{vmatrix} 0_0 & \mp\lambda & \pm 1 \\ \pm 1 & \mp\lambda \vdots & 0 \\ & \mp\lambda & \end{vmatrix}}{\begin{vmatrix} 0 & \ddots & \pm 1 \\ \pm 1 & & 0 \end{vmatrix}} = \lambda r_j \quad \boxed{r_j \in \mathbb{Q}} \quad \forall j = \overline{1,22}$$

$$\underset{a_1}{\lambda r_1}, \lambda r_2, \dots, \underset{a_{22}}{\lambda r_{22}}, \underset{a_{23}}{\lambda}$$

Dacă $\lambda = 1$

$r_1 r_2 \dots, r_{22}, 1$

au propr. (*)

Ef. pas I (cât toate nr. ...)

$r_1 = r_2 = \dots = r_{22} = 1$

$f_{ij} : \mathbb{R} \to \mathbb{R}$ derivabile $\forall i, j = \overline{1,n}$

$$f(x) = \begin{vmatrix} f_{11}(x), & f_{12}(x), & \dots & f_{1n}(x) \\ \vdots & & & \\ f_{n1}(x) & \dots & \dots & f_{nn}(x) \end{vmatrix}$$

<u>Teoremă</u>

f derivabilă și $$f'(x) = \begin{vmatrix} f_{11}'(x), & f_{12}'(x) \dots, & f_{1n}'(x) \\ f_{21}(x), & f_{22}(x) \dots & f_{2n}(x) \\ \vdots & & \\ f_{n1}(x) & \dots & \dots \end{vmatrix} + \begin{vmatrix} f_{11}(x) & \dots, & f_{1n}(x) \\ f_{21}'(x) & - & f_{2n}'(x) \\ \vdots & & \\ f_{n1}(x) & - \dots & f_{nn}(x) \end{vmatrix} + \dots$$

$$\dots + \begin{vmatrix} f_{11}(x) & \dots & f_{1n}(x) \\ \vdots & & \\ f_{n1}'(x) & \dots & f_{nn}'(x) \end{vmatrix}$$

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$$f(x) = \begin{vmatrix} x+a_1 & & & x \\ & x+a_2 & & \\ & & \ddots & \\ x & & & x+a_n \end{vmatrix} =$$

$$f'(x) = \begin{vmatrix} 1 & 1 & \dots & 1 \\ x & x+a_2 & x \dots & x \\ x & \dots & & x+a_n \end{vmatrix} + \dots + \begin{vmatrix} x+a_1 & \dots & x \\ \vdots & & \\ 1 & 1 \dots & 1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ x & x+a_2 & \dots & x \\ x & \dots & & x+a_n \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ x & a_2 & 0 & & 0 \\ x & 0 & a_3 & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ x & 0 & 0 & & a_n \end{vmatrix} = a_2 a_3 \dots a_n$$

$$f'(x) = a_2 a_3 \dots a_n + a_1 a_3 \dots a_n + \dots + a_1 a_2 \dots a_{n-1}$$

$$f(x) = x(a_2 a_3 \dots a_n + a_1 a_3 \dots a_n + \dots + a_1 a_2 \dots a_{n-1}) + \underset{\text{const}}{c}$$

$$c = f(0) = a_1 \dots a_n$$