Formule pentru schimbarea bazei

Fiecare iteraţie a algoritmului simplex este caracterizată de inversa bazei primal admisibile B^{-1}.

$$\bar{x} = B^{-1} \cdot b; \quad \bar{u}^{\top} = c_{\mathcal{B}}^{\top} \cdot B^{-1};$$

$$\bar{z} = c_{\mathcal{B}}^{\top} \cdot \bar{x} = c_{\mathcal{B}}^{\top} \cdot B^{-1} \cdot b = \bar{u}^{\top} \cdot b;$$

$$Y = B^{-1} \cdot A;$$

$$z^{\top} - c^{\top} = c_{\mathcal{B}}^{\top} \cdot Y - c^{\top} = \bar{u}^{\top} \cdot A - c^{\top}$$

Componentele vectorului $\bar{u}$ se numesc *multiplicatori simplex*.

Componentele lui $z - c$ se numesc *costuri reduse*.

Recalcularea elementelor din algoritmul simplex în urma schimbării unei baze se face cu ajutorul Lemei substituţiei. (Sunt cunoscuţi indicii s_r şi k, precum şi vectorul Y^k.)

Valoarea nouă ← Formulă de calcul cu valori vechi

Valorile pentru noua inversă a matricei de bază:

Notăm: $B^{-1} = \left(\beta_{ij}\right)_{\substack{1\le i\le m \\ 1\le j\le m}}$ $\tilde{B}^{-1} = \left(\tilde{\beta}_{ij}\right)_{\substack{1\le i\le m \\ 1\le j\le m}}$

Avem:

$$\tilde{B}^{-1} = E_r\left(\eta\right)\cdot B^{-1},$$

de unde rezultă:

$$\boxed{\begin{aligned} &\tilde{\beta}_{ij} = \beta_{ij} - \frac{y_{ik}\beta_{rj}}{y_{rk}} \ \text{ pentru } i=\overline{1,m},\ i\ne r,\ \ j=\overline{1,m}; \\ &\tilde{\beta}_{rj} = \frac{\beta_{rj}}{y_{rk}} \ \text{ pentru } j=\overline{1,m}. \end{aligned}}$$

Valorile soluţiei de bază:

$$\tilde{x} = \tilde{B}^{-1} \cdot b = E_r(\eta) \cdot B^{-1} \cdot b = E_r(\eta) \cdot \bar{x} =$$

$$= \begin{pmatrix} \ddots & \vdots & & \vdots & \\ \cdots & 1 & \cdots & \dfrac{-y_{ik}}{y_{rk}} & \cdots \\ & \vdots & \ddots & \vdots & \\ \cdots & 0 & \cdots & \dfrac{1}{y_{rk}} & \cdots \\ & \vdots & & \vdots & \ddots \end{pmatrix} \cdot \begin{pmatrix} \vdots \\ \bar{x}_i \\ \vdots \\ \bar{x}_r \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \bar{x}_i - \dfrac{y_{ik}}{y_{rk}} \bar{x}_r \\ \vdots \\ \dfrac{\bar{x}_r}{y_{rk}} \\ \vdots \end{pmatrix}$$

$$\boxed{\begin{array}{l} \tilde{x}_i = \bar{x}_i - \dfrac{y_{ik}}{y_{rk}} \bar{x}_r \quad \text{pentru } i \neq r; \\ \tilde{x}_r = \dfrac{\bar{x}_r}{y_{rk}} \quad \text{unde } r = loc(k) \text{ pentru } k \in \tilde{\mathcal{B}}. \end{array}}$$

Valorile pentru multiplicatorii simplex:

$$\tilde{u}^\top = c_{\tilde{\mathcal{B}}}^\top \cdot \tilde{B}^{-1} = \left(\cdots, c_{s_i}, \cdots, c_k, \cdots\right) \cdot E_r(\eta) \cdot B^{-1}$$

Componenta *j*:

$$\tilde{u}_j = \left(\cdots, c_{s_i}, \cdots, \sum_{i \neq r} \frac{-c_{s_i} y_{ik}}{y_{rk}} + \frac{c_k}{y_{rk}}, \cdots\right) \cdot \begin{pmatrix} \vdots \\ \beta_{ij} \\ \vdots \\ \beta_{rj} \\ \vdots \end{pmatrix} =$$

$$= \sum_{i \neq r} c_{s_i} \beta_{ij} - \left(\sum_{i \neq r} c_{s_i} y_{ik} - c_k\right) \frac{\beta_{rj}}{y_{rk}} + c_{s_r} \beta_{rj} - c_{s_r} \beta_{rj} \frac{y_{rk}}{y_{rk}}$$

$$\boxed{\tilde{u}_j = \bar{u}_j - (z_k - c_k) \frac{\beta_{rj}}{y_{rk}}, \quad 1 \leq j \leq m.}$$

Pentru matricea $\tilde{Y} = \tilde{B}^{-1} \cdot A,$ coloana $\tilde{Y}^{j},\ j = \overline{1,n}$ este:

$$\tilde{Y}^{j} = \tilde{B}^{-1} \cdot A^{j} = E_r(\eta) \cdot B^{-1} \cdot A^{j} = E_r(\eta) \cdot Y^{j} = \begin{pmatrix} \vdots \\ y_{ij} - \dfrac{y_{ik}}{y_{rk}} y_{rj} \\ \vdots \\ \dfrac{y_{rj}}{y_{rk}} \\ \vdots \end{pmatrix}$$

$$\boxed{\begin{aligned} &\tilde{y}_{ij} = y_{ij} - \frac{y_{ik}}{y_{rk}} y_{rj} \quad \text{pentru } i = \overline{1,m},\ i \neq r; \\ &\tilde{y}_{rj} = \frac{y_{rj}}{y_{rk}} \end{aligned}}$$

Valoarea funcţiei obiectiv:

$$\tilde{z} = c_{\tilde{\mathcal{B}}}^{\top} \cdot \tilde{B}^{-1} \cdot b = \tilde{u}^{\top} \cdot b = \sum_{j=1}^{m} \tilde{u}_j b_j$$

$$\tilde{z} = \sum_{j=1}^{m} \left(\overline{u}_j - (z_k - c_k) \frac{\beta_{rj}}{y_{rk}} \right) b_j = \sum_{j=1}^{m} \overline{u}_j b_j - \frac{(z_k - c_k)}{y_{rk}} \sum_{j=1}^{m} \beta_{rj} b_j$$

$$\boxed{\tilde{z} = \overline{z} - \frac{(z_k - c_k)}{y_{rk}} \overline{x}_r}$$

Valoarea costurilor reduse:

$$\tilde{z}_j - c_j = c_{\tilde{\mathcal{B}}}^{\top} \cdot \tilde{B}^{-1} \cdot A^j - c_j = \tilde{u}^{\top} \cdot A^j - c_j =$$

$$= \sum_{i=1}^{m} \tilde{u}_i a_{ij} - c_j = \sum_{i=1}^{m} \left(\bar{u}_i - \left(z_k - c_k \right) \frac{\beta_{ri}}{y_{rk}} \right) a_{ij} - c_j =$$

$$= \left(\sum_{i=1}^{m} \bar{u}_i a_{ij} - c_j \right) - \frac{\left(z_k - c_k \right)}{y_{rk}} \sum_{i=1}^{m} \beta_{ri} a_{ij}$$

$$\boxed{\tilde{z}_j - c_j = \left(z_j - c_j \right) - \frac{\left(z_k - c_k \right) y_{rj}}{y_{rk}}, \;\; 1 \le j \le n.}$$

Organizarea calculelor

Tabloul simplex standard

$x_{\mathcal{B}}$	$\bar{x}$	$Y = B^{-1} \cdot A$
	$\bar{z}$	$z^\top - c^\top$

$$
\begin{array}{ccc|c|ccccccc|}
 & & & & \cdots & c_j & \cdots & c_k & \cdots \\
\underline{c_{\mathcal{B}}} & \underline{x_{\mathcal{B}}} & & \bar{x} & \cdots & x_j & \cdots & x_k & \cdots \\
\hline
\vdots & \vdots & & \vdots & & \vdots & & \vdots & \\
c_{s_i} & x_{s_i} & & \bar{x}_i & \cdots & y_{ij} & \cdots & y_{ik} & \cdots \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots & \\
c_{s_r} & x_{s_r} & & \bar{x}_r & \cdots & y_{rj} & \cdots & \boxed{y_{rk}} & \cdots \\
\vdots & \vdots & & \vdots & & \vdots & & \vdots & \\
\hline
 & & & \bar{z} & \cdots & z_j - c_j & \cdots & z_k - c_k & \cdots \\
\hline
\end{array}
$$

$$\bar{z} = \sum_{i=1}^{m} c_{s_i} \bar{x}_i$$

$$z_j - c_j = \sum_{i=1}^{m} c_{s_i} y_{ij} - c_j$$

Tabloul simplex revizuit

$x_{\mathcal{B}}$	$\bar{x}$	B^{-1}	
	$\bar{z}$	$\bar{u}^{\top}$	

$\underline{c_{\mathcal{B}}}$	$\underline{x_{\mathcal{B}}}$	$\bar{x}$				x_k
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
c_{s_i}	x_{s_i}	$\bar{x}_i$	$\cdots$	β_{ij}	$\cdots$	y_{ik}
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
c_{s_r}	x_{s_r}	$\bar{x}_r$	$\cdots$	β_{rj}	$\cdots$	$\boxed{y_{rk}}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
		$\bar{z}$	$\cdots$	$\bar{u}_j$	$\cdots$	$z_k - c_k$

$$\bar{z} = \sum_{i=1}^{m} c_{s_i} \bar{x}_i$$

$$\bar{u}_j = \sum_{i=1}^{m} c_{s_i} \beta_{ij}$$

$$z_k - c_k = \bar{u}^{\top} \cdot A^k - c_k > 0$$

$$Y^k = B^{-1} \cdot A^k$$

Regula dreptunghiului

Elementul $y_{rk} \neq 0,$ se numeşte pivot. Restul elementelor le redenumim t_{ij} .

- Linia pivotului se împarte la pivot:

$$\tilde{t}_{rj} = \frac{t_{rj}}{y_{rk}}, \quad \forall\ j = \overline{0, n}.$$

- Coloana pivotului devine un vector unitar:

$$\tilde{t}_{rk} = 1 \text{ şi } \tilde{t}_{ik} = 0, \quad \forall\ i = \overline{1, m+1},\ i \neq r.$$

- Restul elementelor din tablou, se calculează după *regula dreptunghiului*:

$$\tilde{t}_{ij} = t_{ij} - \frac{t_{rj}\, t_{ik}}{y_{rk}}, \quad \begin{cases} \forall\ i = \overline{1, m+1},\ i \neq r, \\ \forall\ j = \overline{0, n},\ \ j \neq k. \end{cases}$$

Exemplu.

$$\inf\left\{2x_1 - 3x_2 + x_3 - x_4 + 4x_5\right\}$$

$$\begin{cases} x_1 - x_2 + 2x_3 - 2x_4 + x_5 = 1 \\ -x_1 + 2x_2 - x_3 + x_4 + 2x_5 = 3 \\ x_1 - 2x_2 - x_3 + 2x_4 + x_5 = -5 \end{cases}$$

$$x_i \geq 0,\ i = \overline{1,5}$$

$$B = \left(A^1 A^2 A^3\right) = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -2 & -1 \end{pmatrix} \qquad B^{-1} = \begin{pmatrix} 2 & \frac{5}{2} & \frac{3}{2} \\ 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$$\overline{x} = B^{-1}b = \begin{pmatrix} 2 & \frac{5}{2} & \frac{3}{2} \\ 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \geq 0$$

$$Y = B^{-1}A = \begin{pmatrix} 2 & \frac{5}{2} & \frac{3}{2} \\ 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 & -2 & 1 \\ -1 & 2 & -1 & 1 & 2 \\ 1 & -2 & -1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \frac{3}{2} & \frac{17}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{9}{2} \\ 0 & 0 & 1 & -\frac{3}{2} & -\frac{3}{2} \end{pmatrix}$$

$$\overline{z} = c_{\mathcal{B}}^{\top} B^{-1} b = c_{\mathcal{B}}^{\top} \overline{x} = \begin{pmatrix} 2 & -3 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = -4$$

$$c_{\mathcal{B}}^{\top} Y - c^{\top} = \begin{pmatrix} 2 & -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \frac{3}{2} & \frac{17}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{9}{2} \\ 0 & 0 & 1 & -\frac{3}{2} & -\frac{3}{2} \end{pmatrix} - \begin{pmatrix} 2 & -3 & 1 & -1 & 4 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

Tabloul simplex standard

$c_{\mathcal{B}}$	$x_{\mathcal{B}}$	c / $\overline{x}$	2 / x_1	-3 / x_2	1 / x_3	-1 / x_4	4 / x_5
2	x_1	2	1	0	0	3/2	17/2
-3	x_2	3	0	1	0	1/2	9/2
1	x_3	1	0	0	1	-3/2	-3/2
		-4	0	0	0	1	-2

Tabloul simplex standard

		c	2	-3	1	-1	4
$c_{\mathcal{B}}$	$x_{\mathcal{B}}$	$\overline{x}$	x_1	x_2	x_3	x_4	x_5
2	x_1	2	1	0	0	**3/2**	17/2
-3	x_2	3	0	1	0	1/2	9/2
1	x_3	1	0	0	1	-3/2	-3/2
		-4	0	0	0	1	-2

$$\min\left\{\frac{2}{\frac{3}{2}},\ \frac{3}{\frac{1}{2}}\right\} = \min\left\{\frac{4}{3},\ 6\right\} = \frac{4}{3}$$

Tabloul simplex standard

$c_{\mathcal{B}}$	$x_{\mathcal{B}}$	c / $\bar{x}$	2 / x_1	-3 / x_2	1 / x_3	-1 / x_4	4 / x_5
2	x_1	2	1	0	0	3/2	17/2
-3	x_2	3	0	1	0	1/2	9/2
1	x_3	1	0	0	1	-3/2	-3/2
		-4	0	0	0	1	-2

$$3-\frac{2\cdot\frac{1}{2}}{\frac{3}{2}}=3-\frac{2}{3}=\frac{7}{3}$$

$c_{\mathcal{B}}$	$x_{\mathcal{B}}$	$\bar{x}$	x_1	x_2	x_3	x_4	x_5
-1	x_4	4/3	2/3	0	0	1	17/3
-3	x_2	7/3		1	0	0	
1	x_3			0	1	0	
				0	0	0	

Tabloul simplex standard

	c		2	-3	1	-1	4
$c_{\mathcal{B}}$	$x_{\mathcal{B}}$	$\bar{x}$	x_1	x_2	x_3	x_4	x_5
2	x_1	2	1	0	0	3/2	17/2
-3	x_2	3	0	1	0	1/2	9/2
1	x_3	1	0	0	1	-3/2	-3/2
		-4	0	0	0	1	-2

$$1-\frac{2\cdot\left(-\frac{3}{2}\right)}{\frac{3}{2}}=1+2=3$$

$c_{\mathcal{B}}$	$x_{\mathcal{B}}$	$\bar{x}$	x_1	x_2	x_3	x_4	x_5
-1	x_4	4/3	2/3	0	0	1	17/3
-3	x_2	7/3		1	0	0	
1	x_3	3		0	1	0	
				0	0	0	

Tabloul simplex standard

		c	2	-3	1	-1	4
$c_{\mathcal{B}}$	$x_{\mathcal{B}}$	$\bar{x}$	x_1	x_2	x_3	x_4	x_5
2	x_1	2	1	0	0	3/2	17/2
-3	x_2	3	0	1	0	1/2	9/2
1	x_3	1	0	0	1	-3/2	-3/2
		-4	0	0	0	1	-2

$$-4-\frac{2\cdot 1}{\frac{3}{2}}=-4-\frac{4}{3}=-\frac{16}{3}$$

$c_{\mathcal{B}}$	$x_{\mathcal{B}}$	$\bar{x}$	x_1	x_2	x_3	x_4	x_5
-1	x_4	4/3	2/3	0	0	1	17/3
-3	x_2	7/3		1	0	0	
1	x_3	3		0	1	0	
		-16/3		0	0	0	

Tabloul simplex standard

$c_{\mathcal{B}}$	$x_{\mathcal{B}}$	$\bar{x}$	x_1	x_2	x_3	x_4	x_5
		c	2	-3	1	-1	4
2	x_1	2	1	0	0	3/2	17/2
-3	x_2	3	0	1	0	1/2	9/2
1	x_3	1	0	0	1	-3/2	-3/2
		-4	0	0	0	1	-2

$c_{\mathcal{B}}$	$x_{\mathcal{B}}$	$\bar{x}$	x_1	x_2	x_3	x_4	x_5
-1	x_4	4/3	2/3	0	0	1	17/3
-3	x_2	7/3	-1/3	1	0	0	5/3
1	x_3	3	1	0	1	0	7
		-16/3	-2/3	0	0	0	-23/3

Soluţie optimă !

Tabloul simplex

revizuit

$c_{\mathcal{B}}$	$x_{\mathcal{B}}$	$\overline{x}$				
2	x_1	2				
-3	x_2	3				
1	x_3	1				
		-4				

$$B^{-1} = \begin{pmatrix} 2 & \frac{5}{2} & \frac{3}{2} \\ 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

Tabloul simplex

revizuit

$c_{\mathcal{B}}$	$x_{\mathcal{B}}$	$\bar{x}$				
2	x_1	2	2	5/2	3/2	
-3	x_2	3	1	3/2	1/2	
1	x_3	1	0	-1/2	-1/2	
		-4				

$$\bar{u}^{\top} = c_{\mathcal{B}}^{\top} B^{-1} = \begin{pmatrix} 2 & -3 & 1 \end{pmatrix} \begin{pmatrix} 2 & \frac{5}{2} & \frac{3}{2} \\ 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}$$

Tabloul simplex

revizuit

$c_{\mathcal{B}}$	$x_{\mathcal{B}}$	$\overline{x}$				
2	x_1	2	2	5/2	3/2	
-3	x_2	3	1	3/2	1/2	
1	x_3	1	0	-1/2	-1/2	
		-4	1	0	1	

$$z_4 - c_4 = \overline{u}^\top A^4 - c_4 = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} - (-1) = 1 > 0$$

Tabloul simplex revizuit

$c_{\mathcal{B}}$	$x_{\mathcal{B}}$	$\overline{x}$				
2	x_1	2	2	5/2	3/2	
-3	x_2	3	1	3/2	1/2	
1	x_3	1	0	-1/2	-1/2	
		-4	1	0	1	1

$$Y^4 = B^{-1}A^4 = \begin{pmatrix} 2 & \frac{5}{2} & \frac{3}{2} \\ 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix}$$

Tabloul simplex

revizuit

$c_{\mathcal{B}}$	$x_{\mathcal{B}}$	$\overline{x}$				x_4
2	x_1	2	2	5/2	3/2	3/2
-3	x_2	3	1	3/2	1/2	1/2
1	x_3	1	0	-1/2	-1/2	-3/2
		-4	1	0	1	1

$$\min\left\{ \frac{2}{\frac{3}{2}}, \frac{3}{\frac{1}{2}} \right\} = \min\left\{ \frac{4}{3}, 6 \right\} = \frac{4}{3}$$

Tabloul simplex

revizuit

$c_{\mathcal{B}}$	$x_{\mathcal{B}}$	$\overline{x}$				x_4
2	x_1	2	2	5/2	3/2	3/2
-3	x_2	3	1	3/2	1/2	1/2
1	x_3	1	0	-1/2	-1/2	-3/2
		-4	1	0	1	1

$$-\frac{1}{2}-\frac{\frac{5}{2}\left(\frac{-3}{2}\right)}{\frac{3}{2}}=2$$

$c_{\mathcal{B}}$	$x_{\mathcal{B}}$	$\overline{x}$				
-1	x_4	4/3	4/3	5/3	1	
-3	x_2	7/3	1/3	2/3	0	
1	x_3	3	2	2	1	
		-16/3	-1/3	-5/3	0	

Tabloul simplex revizuit

$c_{\mathcal{B}}$	$x_{\mathcal{B}}$	$\overline{x}$				
-1	x_4	4/3	4/3	5/3	1	
-3	x_2	7/3	1/3	2/3	0	
1	x_3	3	2	2	1	
		-16/3	-1/3	-5/3	0	

$$\left.\begin{aligned} z_1 - c_1 &= \overline{u}^\top A^1 - c_1 = \begin{pmatrix} -\tfrac{1}{3} & -\tfrac{5}{3} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - 2 = -\frac{2}{3} \leq 0 \\ z_5 - c_5 &= \overline{u}^\top A^5 - c_5 = \begin{pmatrix} -\tfrac{1}{3} & -\tfrac{5}{3} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} - 4 = -\frac{23}{3} \leq 0 \end{aligned}\right\}$$

Soluţie optimă !