Construirea unei metode de ordin k numerice de aproximare a problemei Cauchy pentru ecuaţii diferenţiale

Curs Nr. 5

$$\begin{cases} x' = f(t,x) \\ x(t_0) = x_0 \end{cases} \tag{1}$$

$$x(t+st) = x(t) + \frac{x^{(1)}(t)}{1!}(st) + \frac{x^{(2)}(t)}{2!}(st)^2 + \cdots + \frac{x^{(k)}(t)}{k!}(st)^k + O\left((st)^{k+1}\right)$$

Notăm $st = h$

$$\frac{x(t+h) - x(t)}{h} = \frac{x^{(1)}(t)}{1!} + \frac{x^{(2)}(t)}{2!}h + \cdots + \frac{x^{(k)}(t)}{k!}h^{k-1} + O\left(h^k\right)$$

Metodă de aproximare pentru (1):

$$\begin{cases} x_0 \\ x_{j+1} = x_j + h\emptyset(t_j; x_j; h), \quad j = \overline{0, N-1} \end{cases} \tag{2}$$

unde $(t_0, t_0 + T),\ N \in \mathbb{N}*,\ h = \dfrac{T}{N} > 0$

Metoda (2) este de ordin k dacă pentru orice soluţie x a problemei Cauchy (1), avem:

$$\max_{j=\overline{0,N-1}} \frac{x(t_j+1) - x(t_j)}{h} - \emptyset(t_j, x_j, h) = O\left(h^k\right)$$

Luăm $\emptyset(t,x,h) = g_0(t,x) + g_1(t,x)h + \cdots + g_{k-1}(t,x)h^{k-1}$

$g_j(t,x) = \dfrac{x^{(j+1)}(t)}{(j+1)!},\ j = \overline{0, k-1}$

$g_0(t,x) = \dfrac{x^{(1)}(t)}{1!} = x'(t) = f(t,x)$

Exemplu :

$$\begin{cases} x' = 1 - \frac{x}{t},\ t \in [t_0, t_0 + T],\ t_0 > 0 \\ x(t_0) = t_0 \end{cases}$$

a) Se poate determina soluţia exactă (TEMĂ)
b) Determinaţi metoda numerică de ordin $k = 3$.

$$f(t,x) = 1 - \frac{x}{t}$$
$$g_0(t,x) = 1 - \frac{x}{t}$$

$$g_1(t,x) = \frac{x_{(2)}(t)}{2!} = \frac{1}{2}\frac{d}{dt}(x'(t)) = \frac{1}{2}\frac{d}{dt}(f(t,x)) = \frac{1}{2}\left[\frac{\partial f}{\partial t}(t,x) + \frac{\partial f}{\partial x}(t,x)\frac{dx}{dt}\right] =$$
$$= \frac{1}{2}\left[\frac{\partial f}{\partial t}(t,x) + \frac{\partial f}{\partial x}(t,x)\ f(t,x)\right]$$

Pentru exemplul considerat: $\frac{\partial f}{\partial t} = \frac{x}{t^2}$

$$\frac{\partial f}{\partial x} = \frac{-1}{t}$$

$$g_1(t,x) = \frac{1}{2}\left[\frac{x}{t^2} + \left(-\frac{1}{t}\right)\cdot\left(1 - \frac{x}{t}\right)\right] = \frac{1}{2}\left(\frac{x}{t^2} - \frac{1}{t} + \frac{x}{t^2}\right) \Rightarrow$$
$$\Rightarrow g_1(t,x) = \frac{1}{2}\left(\frac{2x}{t^2} - \frac{1}{t}\right)$$

Am construit metoda de ordin 2.

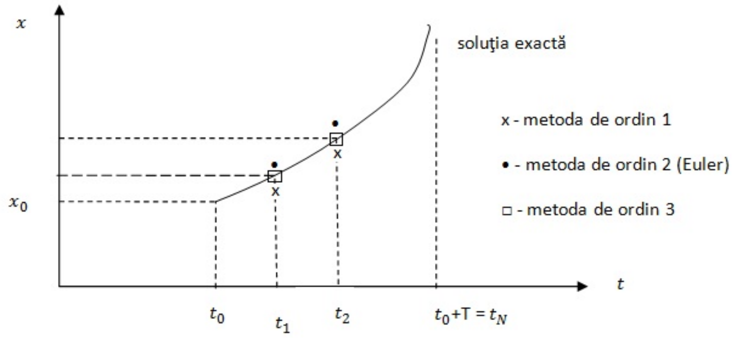
ordin 1 $\Rightarrow x_0$

$$x_{j+1} = x_j + hf(t_j, x_j) = x_j + h\left(1 - \frac{x_j}{t_j}\right),\ j = \overline{0, N-1}$$

ordin 2 $\Rightarrow x_0$

$$x_{j+1} = x_j + h\left(g_0(t_j,x_j) + g_1(t_j,x_j)h\right) =$$
$$= x_j + h\left[\left(1 - \frac{x_j}{t_j}\right) + h\frac{1}{2}\left(\frac{2x_j}{t_j^2} - \frac{1}{t_j}\right)\right],\ j = \overline{1, N-1}$$

$$t_0 = 1, ; x_0 = \frac{3}{2},\ [t_0, t_0 + T] = [1,2],\ N = 10$$



$$
\begin{aligned}
g_2(t,x) &= \frac{x^{(3)}}{3!} = \frac{1}{6}\frac{d}{dt}\left[x^{(2)}(t)\right] = \frac{1}{6}\frac{d}{dt}\left[\frac{\partial f}{\partial t}(t,x) + \frac{\partial f}{\partial x}(t,x)f(t,x)\right] = \\
&= \frac{1}{6}\left[\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}(t,x) + \frac{\partial f}{\partial x}(t,x)f(t,x)\right) + \right. \\
&\left. + \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial t}(t,x) + \frac{\partial f}{\partial x}(t,x)f(t,x)\right)\underbrace{\frac{dx}{dt}}_{f(t,x)}\right] = \\
&= \frac{1}{6}\left[\frac{\partial^2 f}{\partial t^2}(t,x) + \frac{\partial^2 f}{\partial x \partial t}(t,x)f(t,x) + \frac{\partial f}{\partial x}(t,x)\frac{\partial f}{\partial t}(t,x) + \right. \\
&\left. + \left(\frac{\partial^2 f}{\partial x \partial t}(t,x) + \frac{\partial^2 f}{\partial x^2}(t,x)f(t,x) + \left(\frac{\partial f}{\partial x}(t,x)\right)^2\right)f(t,x)\right] = \\
&= \frac{1}{6}\left[\frac{\partial^2 f}{\partial t^2}(t,x) + \frac{\partial f}{\partial x}(t,x)\frac{\partial f}{\partial t}(t,x) + \right. \\
&\left. + \left(2\frac{\partial^2 f}{\partial x \partial t}(t,x) + \frac{\partial^2 f}{\partial x^2}(t,x) + \left(\frac{\partial f}{\partial x}(t,x)\right)^2\right)f(t,x)\right]
\end{aligned}
$$

Pentru exemplul considerat:

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= 0 \\
\frac{\partial^2 f}{\partial t^2} &= \frac{-2x}{t^3} \\
\frac{\partial^2 f}{\partial x \partial t} &= \frac{1}{t^2}
\end{aligned}
$$

$$
\begin{aligned}
g_2(t,x) &= \frac{1}{6}\left[\frac{-2x}{t^3} + \left(-\frac{1}{t}\right)\cdot\frac{x}{t^2} + \left[2\cdot\frac{1}{t^2} + 0\cdot\left(1-\frac{x}{t}\right) + \left(-\frac{1}{t}\right)^2\right]\cdot\left(1-\frac{x}{t}\right)\right] = \\
&= \frac{1}{6}\left(\frac{-2x}{t^3} - \frac{x}{t^3} + \frac{2}{t^2} - \frac{2x}{t^3} + \frac{1}{t^2} - \frac{x}{t^3}\right) = \\
&= \frac{1}{6}\left(-\frac{6x}{t^3} + \frac{3}{t^2}\right) = \frac{1}{2}\left(\frac{1}{t^2} - \frac{2x}{t^3}\right)
\end{aligned}
$$

Metoda de ordin 3:

$$
\begin{aligned}
& x_0 \\
& x_{j+1} = x_j + h\left[g_0(t_j,x_j) + g_1(t_j,x_j)h + g_2(t_j,x_j)h^2\right] = \\
& \quad = x_j + h\left[\left(1-\frac{x_j}{t_j}\right) + \frac{h}{2}\left(\frac{2x_j}{t_j^2} - \frac{1}{t_j}\right) + \frac{h^2}{6}\left(\frac{-6x_j}{t_j^3} + \frac{3}{t_j^2}\right)\right],\ j=\overline{0,N-1}
\end{aligned}
$$

Temă : Determinaţi metoda de ordin 4.

Dependenţa continuă a problemei Cauchy de datele iniţiale şi de parametrii

Fie problema Cauchy:

$$
\begin{cases} x' = f(t,x;\lambda) \\ x(t_0) = x_0 \end{cases} \tag{1}
$$

unde $f : D\times\Lambda \to \mathbb{R},\ D\subset\mathbb{R}^2,\ D$ mulţime deschisă, $\Lambda = [\lambda_0,\lambda_1]\subset R$

Din teorema Cauchy-Picard ştim că pentru $\lambda\in\Lambda$ fixat, dacă $f_1(t,x) = f(t,x,\lambda)$ este continuă şi $\dfrac{\partial f}{\partial x}$ este continuă, atunci există şi este unică soluţia problemei Cauchy (1). Ne intereseazaă în ce condiţii există o unică soluţie $\phi(t;t_0,x_0,\lambda)$ soluţie a problemei Cauchy (1).

$$
\begin{aligned}
& \phi : I_1\times I_0\times D_1\times\Lambda \to \mathbb{R} \\
& I_1\subset I_0,\ (t_0,x_0)\in I_1\subset\mathbb{R} \\
& D_1\subset\mathbb{R} \text{ astfel încât } I_0\times D_1\subset D
\end{aligned}
$$

$$
\text{Ipoteze : }\begin{cases} f(\cdot;\cdot,\cdot) \text{ este continuă în toate argumentele} \\ \exists\dfrac{\partial f}{\partial x} \text{ este continuă pe } D\times\Lambda \\ \text{Fie } (\overline{t_0},\overline{x_0})\in D \end{cases} \tag{2}
$$

Teorema de dependenţă continuă de date :

Din ipotezele (2) rezultă că :

$$\exists \overline{\alpha} > 0 \text{ astfel încât } \overbrace{[\overline{t_0} - \overline{\alpha}, \overline{t_0} + \overline{\alpha}] \times [\overline{x_0} - \overline{\alpha}, \overline{x_0} + \overline{\alpha}]}^{D_1} \subset D$$

a) $\forall (t_0, x_0) \in [\overline{t_0} - \overline{\alpha}, \overline{t_0} + \overline{\alpha}] \times [\overline{x_0} - \overline{\alpha}, \overline{x_0} + \overline{\alpha}]$ şi $\forall \lambda \in [\lambda_0, \lambda_1]$,
există şi este unică soluţia problemei (1) notată $x = \phi(t; t_0, x_0, \lambda)$,
$\phi(\cdot; \cdot, \cdot, \cdot) : [\overline{t_0} - \overline{\alpha}, \overline{t_0} + \overline{\alpha}] \times [\overline{t_0} - \overline{\alpha}, \overline{t_0} + \overline{\alpha}] \times [\overline{x_0} - \overline{\alpha}, \overline{x_0} + \overline{\alpha}] \times D \to \mathbb{R}$
b) $\phi(\cdot; \cdot, \cdot, \cdot)$ de la punctul a este funţie uniform continuă pe domeniul de definiţie.

Demonstraţie: Fără să se piardă din generalitate, se poate considera $\overline{t_0} = 0$, $\overline{x_0} = 0$.
Se face schimbarea de variabilă:

$$\begin{cases} s = t - t_0 \\ y = x - x_0 \end{cases}$$

(t_0, x_0) din condiţia Cauchy

Fie $\delta > 0$ astfel încât $\underbrace{[-2\delta, 2\delta] \times [-2\delta, 2\delta]}_{D_2} \subset D$ şi $[\overline{t_0}, \overline{x_0}] = (0,0) \in D_1$.

$$y(s(t)) = x(t) - x(0),\ s(t) = t - t_0 \Rightarrow s(t_0) = 0,\ y(s(t_0)) = 0$$

$$(1), (3) \Rightarrow \begin{cases} y' = f(s + t_0, y + x_0, t_0, x_0, \lambda) \\ y(0) = 0 \end{cases}$$

Notăm $g(s; y, t_0, x_0, \lambda) = f(s + t_0, y + x_0, t_0, x_0, \lambda)$

Dacă f continuă $\Rightarrow g(\cdot; \cdot, \cdot, \cdot)$ este continuă pe $[-\delta, \delta] \times [-\delta, \delta] \times [-\delta, \delta] \times [-\delta, \delta] \times \Lambda$.
$\exists \dfrac{\partial f}{\partial x}$ continuă $\Rightarrow \dfrac{\partial g}{\partial y}$ continuă.
Aplicând teorema Cauchy-Picard pentru (3) : pentru orice $(t_0, x_0, \lambda) = \mu \in [-\delta, \delta]^2 \times \Lambda$ problema (3) are soluţii:
$y = \psi(s; \mu)$ astfel încât $\psi : [-\beta, \beta] \to [-\delta, \delta]$, unde $\beta = \min \left\{ \delta, \dfrac{\delta}{M} \right\}$, $M = \sup_{[-\delta,\delta] \times \Lambda} |g(s, y; \mu)|$.

Considerăm $\phi : [-\alpha, \alpha]^3 \times \Lambda \to [-\delta, \delta]$ unde $\alpha = \dfrac{\beta}{2}$, $\phi(t, \mu) = \psi(t - t_0; \mu) + x_0$ şi arătăm că:

- ϕ derivabilă şi verifică (1)

  $\phi'(t; \mu) = \psi'(t - t_0, \mu)(t - t_0)' = \psi'(t - t_0, \mu) = f(t; \phi(t, \mu); \mu)$
  ϕ verifică ecuaţia diferenţială (1)
  $\phi'(t_0; \mu) = \phi'(0, \mu) + x_0 = 0 + x_0 = x_0$

- graficul lui ϕ este $\Gamma_\phi = (t, \phi(t;\mu))|t \in [-\alpha, \alpha] \subset D$

  Avem $\Gamma_\psi = (s, \psi(s;\mu)|s \in [-\beta, \beta] \subset [-\delta, \delta]$
  $[-2\delta, 2\delta]^2 \subset D$
  Dacă $|\phi(t;\mu)| < 2\delta \Rightarrow \phi(t;\mu) \in [-2\delta, 2\delta]$
  $\phi(t;\mu) = |\psi(t-t_0);\mu) + x_0| \leq \underbrace{|\psi(t-t_0;\mu)|}_{\leq 2\delta} + \underbrace{x_0}_{\leq \delta} \leq 2\delta$
  Deci ϕ este bine definită. $\phi : [-\alpha, \alpha]^3 \times \Lambda \to [-\delta, \delta]$
  $\alpha = \dfrac{\beta}{2} = \dfrac{1}{2}\min\left\{\delta, \dfrac{\delta}{M}\right\} \leq \delta$
  Avem pentru a) : $\overline{\alpha} = \dfrac{\beta}{2}$.

b) Cum $\phi(t;\mu) = \psi(t-t_0;\mu)+x_0$ avem că ϕ este uniform continuă dacă ψ este uniform continuă.
Din demonstraţia teoremei Cauchy-Picard ştim că :

$$\psi_i \xrightarrow[i\to\infty]{unif} \psi, \text{ cu}$$

$$\begin{cases} \psi_0(s,\mu) = 0 \\ \psi_{i+1}(s;\mu) = 0 + \int\limits_0^s g(r, \psi_i(r;\mu);\mu)\ \mathrm{d}r\ ,\ \forall i \geq 0,\ s \in [-\beta, \beta] \end{cases}$$

$(\phi_i)_{i\geq 0}$ este uniform continuă dacă :

$$\begin{aligned} &\forall \epsilon > 0,\ \exists \eta(\epsilon) > 0, \forall (s',\mu'), (s'',\mu'') \text{ astfel încât } |s' - s''| < \eta(\epsilon) \\ &\text{şi } \left\|\mu' - \mu''\right\| < \eta(\epsilon) = |t_0| + |x_0| + |\lambda| \\ &\text{Avem } |\psi_i(s',\mu') - \psi_i(s'',\mu'')| < \epsilon \end{aligned} \qquad (*)$$

Notăm $h_i(r;\mu) = g(r, \psi_i(r;\mu);\mu)$. g este continuă pe intervalul compact $[-\delta, \delta]^4 \times \Lambda$.
Deci g şi h_i sunt uniform continue.

$$\Rightarrow \begin{cases} \forall \epsilon,\ \exists \eta(\epsilon)\ \forall (r',\mu'),\ (r'',\mu'') \text{ cu } |r' - r''| < \eta(\epsilon), |\mu' - \mu''| < \eta(\epsilon) \\ |h_i(r',\mu') - h_i(r'',\mu'')| < \epsilon \end{cases}$$

Demonstrăm (*) prin inducţie:

$\psi_0(s;\mu) = 0$
Presupunem adevărat pentru ψ_i şi demonstrăm pentru ψ_{i+1}:

$$
\begin{aligned}
|\psi_{i+1}(s',\mu') - \psi_{i+1}(s'',\mu'')| &= \left| \int_0^{s'} h_i(r;\mu')\,\mathrm{d}r - \int_0^{s''} h_i(r;\mu'')\,\mathrm{d}r \right| = \\
&= \left| \int_0^{s'} \left| h_i(r;\mu') - h_i(r,\mu'') \right| \,\mathrm{d}r - \int_{s'}^{s''} h_i(r,\mu'')\,\mathrm{d}r \right| \leq \\
&\leq \left| \int_0^{s'} \underbrace{(h_i(r,\mu') - h_i(r,\mu''))}_{<\epsilon}\,\mathrm{d}r \right| - \left| \int_{s'}^{s''} \underbrace{h_i(r,\mu'')}_{\leq M_i}\,\mathrm{d}r \right| \leq \\
&\leq \left| \int_0^{s'} \epsilon\,\mathrm{d}r \right| + \left| \int_{s'}^{s''} M_i\,\mathrm{d}r \right| = \\
&= \epsilon \underbrace{|s'|}_{\delta} + M_i \underbrace{|s''-s'|}_{<\delta} \leq \delta(\epsilon + M_i) < \overline{\epsilon} \\
&\text{unde } M_i = \sup_{[-\delta,\delta]^4 \times \Lambda} (h_i(r,\mu))
\end{aligned}
$$

De unde rezultă că (ψ_i) este şir uniform continuu.