Programare liniară în numere întregi

Considerăm problema:

$$\inf \left\{ c^{\top} \cdot x \,\middle|\, A \cdot x = b,\ x \in \mathbb{R}^n_+, \qquad \right\} \qquad \text{(P*)}$$

unde: $A \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m, \quad c \in \mathbb{R}^n, \quad rang(A) = m < n.$

$\varnothing \neq \mathcal{K} \subsetneq \{1, 2, \ldots, n\}$ $\Longrightarrow$ Problemă de programare liniară mixtă

$\mathcal{K} = \{1, 2, \ldots, n\}$ $\Longrightarrow$ <u>Problemă de programare liniară în numere întregi</u>

Notaţii:

$$\begin{cases} \mathcal{P}_0 = \left\{ x \in \mathbb{R}^n \mid A \cdot x = b,\ x \geq \mathbf{0} \right\} \\ \mathcal{P}^* = \left\{ x \in \mathcal{P}_0 \mid x_k \in \mathbb{Z},\ k \in \mathcal{K} \right\} \end{cases}$$

Evident, $\mathcal{P}_0 \supset \mathcal{P}^*$

Putem considera problemele:

(P_0) $\inf \left\{ c^{\top} \cdot x \,\middle|\, x \in \mathcal{P}_0 \right\}$ $\Longrightarrow$ Se rezolvă cu algoritmul simplex.

(P*) $\inf \left\{ c^{\top} \cdot x \,\middle|\, x \in \mathcal{P}^* \right\}$ $\Longrightarrow$ Trebuie elaborată o metodă!

Propoziţie. *Fie* $\bar{x} \in \mathcal{P}_0$ *soluţia optimă a lui* (P_0). Dacă $\bar{x} \in \mathcal{P}^*$, atunci $\bar{x}$ este soluţie optimă şi pentru problema (P^*).

Demonstraţie. $\mathcal{P}_0 \supset \mathcal{P}^*$

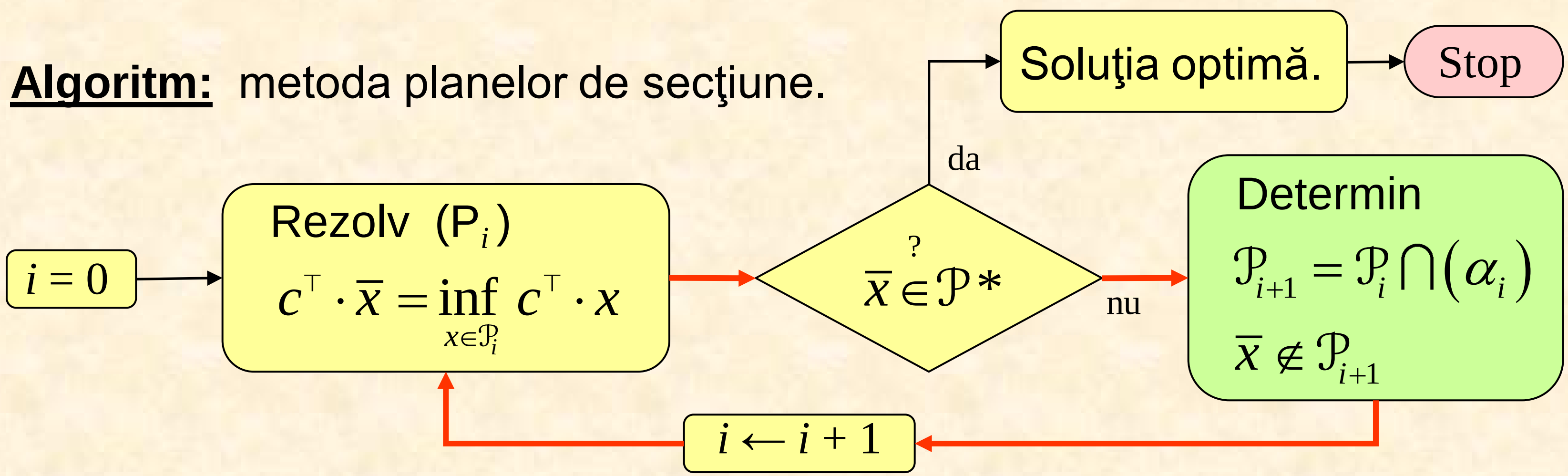
Avem:

$$c^\top \cdot \bar{x} = \inf\left\{c^\top \cdot x \,\middle|\, x \in \mathcal{P}_0\right\} \le \inf\left\{c^\top \cdot x \,\middle|\, x \in \mathcal{P}^*\right\} \le c^\top \cdot \bar{x}$$

Pe de altă parte, $\bar{x} \in \mathcal{P}^*$

Rezultă, $c^\top \cdot \bar{x} = \inf\left\{ c^\top \cdot x \,\middle|\, x \in \mathcal{P}^* \right\}$. (q.e.d.)

Algoritm: metoda planelor de secţiune.



Observaţie: $\mathcal{P}_0 \supset \mathcal{P}_1 \supset \cdots \supset \mathcal{P}_i \supset \mathcal{P}_{i+1} \supset \cdots \supset \mathcal{P}^*$.

Determinarea planelor de secţiune (Gomory).

Considerăm problema (P*) în care $\mathcal{K}=\{1,2,\ldots,n\}$.

Fie B o bază optimă pentru (P_i) şi presupunem că $\exists\, s_k \in \mathcal{B},\quad x_{s_k} \notin \mathbb{Z}$.

Avem: $$x_{s_k} = \bar{x}_k - \sum_{j\in\mathcal{R}} y_{kj} x_j$$

Introducem notaţiile: $$\bar{x}_k = \left[\bar{x}_k\right] + f_{k0} \quad \text{unde } 0 < f_{k0} < 1,$$
$$y_{kj} = \left[y_{kj}\right] + f_{kj} \quad \text{unde } 0 \le f_{kj} < 1.$$

Rezultă, $$\underbrace{x_{s_k} - \left[\bar{x}_k\right] + \sum_{j\in\mathcal{R}} \left[y_{kj}\right] x_j}_{\in\mathbb{Z}} = \underbrace{f_{k0} - \sum_{j\in\mathcal{R}} f_{kj} x_j}_{\in\mathbb{Z}} \le 0 < 1.$$

Dar, $\forall\, x \in \mathcal{P}^* \Longrightarrow \in \mathbb{Z} \Longrightarrow \in \mathbb{Z}$

În plus, avem: $$\sum_{j\in\mathcal{R}} f_{kj} x_j \ge 0 \quad \& \quad f_{k0} \in (0,1)$$

Se adaugă restricţia: $$\sum_{j\in\mathcal{R}} \left(-f_{kj}\right) x_j \le -f_{k0}\,.$$

Observaţie. Soluţia de bază corespunzătoare lui $\overline{x}$ **nu** verifică restricţia adăugată. Într-adevăr,

$$x = \begin{pmatrix} x_{\mathcal{B}} \\ x_{\mathcal{R}} \end{pmatrix} = \begin{pmatrix} \overline{x} \\ \mathbf{0} \end{pmatrix} \Longrightarrow \quad 0 = \sum_{j \in \mathcal{R}} \left(-f_{kj}\right) x_j \leq -f_{k0} < 0 .$$

Contradicţie!

Prin urmare, putem defini:

$$\mathcal{P}_{i+1} = \mathcal{P}_i \bigcap \left(\alpha_i\right) = \left\{ x \in \mathcal{P}_i \,\middle|\, \sum_{j \in \mathcal{R}} \left(-f_{kj}\right) x_j + y_{(i+1)} = -f_{k0}, \;\; y_{(i+1)} \geq 0 \right\}$$

Algoritmul se termină într-un număr finit de paşi:

- Există o soluţie optimă $\in \mathbb{Z}$
- Nu există soluţii în $\in \mathbb{Z}$

Implementare.

Fie $B_{(i)}$ o bază optimă pentru (P_i). Pentru (P_{i+1}) considerăm matricea:

$$B_{(i+1)} = \begin{pmatrix} B_{(i)} & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{pmatrix} \Longrightarrow B_{(i+1)}^{-1} = \begin{pmatrix} B_{(i)}^{-1} & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{pmatrix}$$

Matricea $B_{(i+1)}$ este dual admisibilă pentru (P_{i+1}) :

$$\boxed{\overline{x}_{(i+1)}} = B_{(i+1)}^{-1} \cdot b_{(i+1)} = \begin{pmatrix} B_{(i)}^{-1} & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{pmatrix} \cdot \begin{pmatrix} b_{(i)} \\ -f_{k0} \end{pmatrix} = \begin{pmatrix} B_{(i)}^{-1} \cdot b_{(i)} \\ -f_{k0} \end{pmatrix} = \begin{pmatrix} \overline{x}_{(i)} \\ -f_{k0} \end{pmatrix} \boxed{\ngeq \mathbf{0}}$$

$$Y_{(i+1)}^j = B_{(i+1)}^{-1} A_{(i+1)}^j = \begin{pmatrix} B_{(i)}^{-1} & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{pmatrix} \begin{pmatrix} A_{(i)}^j \\ -f_{kj} \end{pmatrix} = \begin{pmatrix} Y_{(i)}^j \\ -f_{kj} \end{pmatrix}$$

$$\boxed{z_j^{(i+1)} - c_j} = c_{\mathcal{B}_{(i+1)}}^\top Y_{(i+1)}^j - c_j = \left(c_{\mathcal{B}_{(i)}}^\top, 0\right) \begin{pmatrix} Y_{(i)}^j \\ -f_{k0} \end{pmatrix} - c_j = z_j^{(i)} - c_j \boxed{\leq 0}$$

$$\overline{z}^{(i+1)} = c_{\mathcal{B}_{(i+1)}}^\top \overline{x}^{(i+1)} = \left(c_{\mathcal{B}_{(i)}}^\top, 0\right) \begin{pmatrix} \overline{x}^{(i)} \\ -f_{k0} \end{pmatrix} = \overline{z}^{(i)}$$

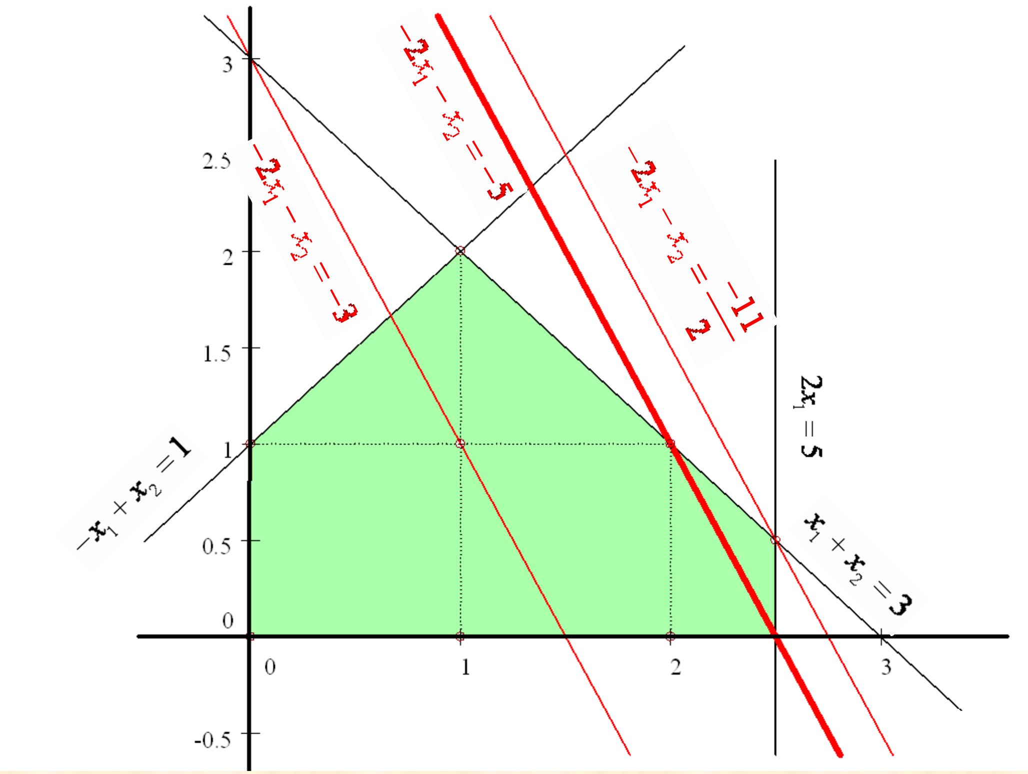
Tabloul simplex:

	$\bar{x}$		x_j		x_p		$y_{(i+1)}$
	$\vdots$		$\vdots$		$\vdots$		$\vdots$
x_{s_i}	$\bar{x}_i$	$\cdots$	y_{ij}	$\cdots$	y_{ip}	$\cdots$	0
	$\vdots$		$\vdots$		$\vdots$		$\vdots$
x_{s_k}	$[\bar{x}_k] + f_{k0}$	$\cdots$	$[y_{kj}] + f_{kj}$	$\cdots$	$[y_{kp}] + f_{kp}$	$\cdots$	0
	$\vdots$		$\vdots$		$\vdots$		$\vdots$
$y_{(i+1)}$	$-f_{k0}$	$\cdots$	$-f_{kj}$	$\cdots$	$\boxed{-f_{kp}}$	$\cdots$	1
	$\bar{z}$	$\cdots$	$z_j - c_j$	$\cdots$	$z_p - c_p$	$\cdots$	0

Exemplu. $$\min\{-2x_1 - x_2\}$$

$$\text{în raport cu} \begin{cases} -x_1 + x_2 \le 1 \\ x_1 + x_2 \le 3 \\ 2x_1 \le 5 \end{cases} \qquad x_1, x_2 \in \mathbb{Z}_+$$

Rezolvare grafică:



Rezolvarea cu algoritmul ciclic al lui Gomory:

	$\bar{x}$	x_1	x_2	x_3	x_4	x_5
x_3	1	–1	1	1	0	0
x_4	3	1	1	0	1	0
x_5	5	2	0	0	0	1
	0	2	1	0	0	0

	$\bar{x}$	x_1	x_2	x_3	x_4	x_5
x_3	$\frac{7}{2}$	0	1	1	0	$\frac{1}{2}$
x_4	$\frac{1}{2}$	0	1	0	1	$\frac{-1}{2}$
x_1	$\frac{5}{2}$	1	0	0	0	$\frac{1}{2}$
	–5	0	1	0	0	–1

	$\bar{x}$	x_1	x_2	x_3	x_4	x_5
x_3	3	0	0	1	–1	1
x_2	$\frac{1}{2}$	0	1	0	1	$\frac{-1}{2}$
x_1	$\frac{5}{2}$	1	0	0	0	$\frac{1}{2}$
	$\frac{-11}{2}$	0	0	0	–1	$\frac{-1}{2}$

	$\bar{x}$	x_1	x_2	x_3	x_4	x_5	y
x_3	3	0	0	1	–1	1	0
x_2	$\frac{1}{2}$	0	1	0	1	$\frac{-1}{2}$	0
x_1	$\frac{5}{2}$	1	0	0	0	$\frac{1}{2}$	0
y	$\frac{-1}{2}$	0	0	0	0	$\frac{-1}{2}$	1
	$\frac{-11}{2}$	0	0	0	–1	$\frac{-1}{2}$	0

	$\bar{x}$	x_1	x_2	x_3	x_4	x_5	y
x_3	3	0	0	1	–1	1	0
x_2	$\frac{1}{2}$	0	1	0	1	$\frac{-1}{2}$	0
x_1	$\frac{5}{2}$	1	0	0	0	$\frac{1}{2}$	0
y	$\frac{-1}{2}$	0	0	0	0	$\frac{-1}{2}$	1
	$\frac{-11}{2}$	0	0	0	–1	$\frac{-1}{2}$	0

	$\bar{x}$	x_1	x_2	x_3	x_4	x_5	y
x_3	2	0	0	1	–1	0	$\frac{1}{2}$
x_2	1	0	1	0	1	0	–1
x_1	2	1	0	0	0	0	1
x_5	1	0	0	0	0	1	–2
	–5	0	0	0	–1	0	–1