Momentos, transformadas, convergências, teoremas limite, processo de Poisson, ...

60.

c) t.L. de $X \frown U[a,b]$

$L_X(t) = E(e^{-tX})$, $|t| < t_0$

$\therefore L_X(t) = \sum_i e^{-t x_i} p_i$, caso discreto

$L_X(t) = \int_{-\infty}^{+\infty} e^{-tx} f(x)\,dx$, caso contínuo

$$f(x) = \frac{1}{b-a}, \quad a < x < b$$

$$L_X(t) = E(e^{-tX}) = \int_{-\infty}^{+\infty} e^{-tx} f(x)\,dx$$

$$= \int_a^b e^{-tx} \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\left(-\frac{1}{t} e^{-tx}\right)\Big|_a^b$$

$$= \frac{1}{b-a}\left(\frac{1}{t} e^{-at} - \frac{1}{t} e^{-bt}\right) = \frac{e^{-at} - e^{-bt}}{t(b-a)}, \quad t \in \mathbb{R}$$

f) t.L. de $X \frown Geom(p)$

$$p_j = P(X=j) = (1-p)^{j-1} p, \quad j = 1, 2, \ldots$$

$$L_X(t) = E(e^{-tX}) = \sum_{j \geq 1} e^{-tj} (1-p)^{j-1} p$$

$$= \sum_{j \geq 1} e^{-t(j-1)} e^{-t} (1-p)^{j-1} p$$

$$= e^{-t} p \underbrace{\sum_{j \geq 1} (e^{-t}(1-p))^{j-1}}_{\sum_{i \geq 0} x^i = \frac{1}{1-x},\ |x| < 1}$$

$$= p e^{-t} \frac{1}{1 - e^{-t}(1-p)}, \quad \text{se } |\underbrace{e^{-t}(1-p)}_{>0}| < 1 \xrightarrow{\text{logo}} \text{se } e^{-t}(1-p) < 1$$

i.e., se $e^{-t} < \frac{1}{1-p}$ ou seja $-t < -\log(1-p)$, ou ainda $t > \underbrace{\log(1-p)}_{<0}$

$$\therefore L_X(t) = \frac{p e^{-t}}{1 - e^{-t}(1-p)}, \quad t > \log(1-p)$$

61.

c) Calc. $\mu'_n = E(X^n)$ com $X \sim Exp(\lambda)$; $\lambda > 0$

$L(t) = \dfrac{\lambda}{\lambda + t}$, $t > -\lambda$

$L'(t) = -\lambda(\lambda+t)^{-2}$

$L''(t) = +2\lambda(\lambda+t)^{-3}$

$E(X^n) = (-1)^n L^{(n)}(0)$

$L'''(t) = -2 \times 3 \lambda (\lambda+t)^{-4}$

$\vdots$

$L^{(n)}(t) = (-1)^n \dfrac{n!}{(\lambda+t)^{n+1}}$

$\therefore L^{(n)}(0) = (-1)^n \dfrac{n!}{\lambda^{n+1}}$ logo $E(X^n) = \dfrac{n!}{\lambda^{n+1}}$

b) $L(t) = e^{-\lambda(1-e^{-t})}$, $t \in \mathbb{R}$

$X_i \sim$ Poisson (λ_i) indep. $i = 1, 2, \dots, n$

A t.L identifica a distribuição

$\Downarrow$

$L_{\sum_{i=1}^n X_i}(t) = L_{X_1}(t) \dots L_{X_n}(t)$

$= \prod_{i=1}^n e^{-\lambda_i(1-e^{-t})} = \underbrace{e^{-(\sum \lambda_i)(1-e^{-t})}}_{\text{t.L. Poisson}(\sum \lambda_i)}$

e como a transformada de Laplace identifica a distribuição, conclui-se que $\sum_{i=1}^n X_i \sim$ Poisson $(\sum_{i=1}^n \lambda_i)$

62. $Z \frown N(0,1)$; $L(t) = e^{\frac{t^2}{2}}$, $t \in \mathbb{R}$

$E(Z)^n = (-1)^n L^{(n)}(0)$ $L(0) = 1$

$L'(t) = t\,L(t)$ - - - - - $\to L'(0)$ $\therefore E(Z) = 0$

$L''(t) = L(t) + t\,L'(t) = L(t) + t^2 L(t)$
$= (1+t^2)\,L(t)$ $\to$ $L''(0) = 1$ $\therefore E(Z^2) = 1$

$L'''(t) = 2t\,L(t) + (1+t^2)\underbrace{L'(t)}_{t\,L(t)} = (3t+t^3)\,L(t)$ $\to$ $L'''(0) = 0$ $\therefore E(Z^3) = 0$ secções

$\beta_1 = 0$ pq $\beta_1 = E\left(\left(\frac{Z-\mu}{\sigma}\right)^3\right)$
$= E(Z^3)$

$L^{(4)}(t) = (3+3t^2)\,L(t) + (3t+t^3)\underbrace{L'(t)}_{tL(t)} = (3+3t^2+3t^2+t^4)\,L(t)$ $\to$ $L^{(4)}(0) = 3$

$\therefore E(Z^4) = 3$

e $\beta = 3$

Donde $\beta_1 = 0$ e $\beta_2 = 3$ para v.a. $X \frown N(\mu, \sigma)$ porque

$X = \sigma Z + \mu$ e os coef. β_1 e β_2 são invariantes para transformações lineares e $\sigma > 0$

✓

X é transformação linear de Z.

63. X, $\mu = E(X) = 70$ kg
$\sigma = \sqrt{var(X)} = 10$kg

$X_1, X_2, \dots, X_n$, $(n=49)$ iid. c/ X
$S_n = X_1 + \dots + X_n$

a) $\mu_S = E(S) = \sum_{i=1}^{n} E(X_i) = n\mu$

$\sigma_S^2 = var(S) \underset{\{X_i\}_{i=1,\dots,n} \text{ indep}}{=} \sum_{i=1}^{n} var(X_i) = n\sigma^2$

$\therefore \sigma_S = \sigma\sqrt{n}$

TLC

Se $X_1, X_2, \dots$ c/ $E(X) = \mu$ e $var(X) = \sigma^2 < +\infty$, então

$\frac{S_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{d} Z \frown N(0,1)$

$\frac{\bar{X} - \mu}{\sigma}\sqrt{n} \xrightarrow{d} Z \frown N(0,1)$

b) $P(S > 3500) \underset{TLC}{\simeq}$?? (pressupondo que n=49 é grande o suficiente para aplicar o problema)

$P(S > 3500) \underset{\substack{TLC \\ n=49}}{\simeq} P(Y > 3500) \Rightarrow$ pnorm(3500, 49×70, 70, lower = F)

$= 0.1587$

$\downarrow$
aprox. $N(\underbrace{n\mu}_{49\times70}, \underbrace{\sigma\sqrt{n}}_{10\sqrt{49}})$
(v.a. Y)

64. $Y: \begin{cases} -1 & 1 \\ 1/2 & 1/2 \end{cases}$ $\therefore Y \sim U\{-1, 1\}$

independentes identicamente distribuídas

$Y_1, \dots, Y_n$ iid c/ Y ; $n = 100$

$P(0 \leq S_n \leq 50) = ?$

$S_n = \sum_{i=1}^{n} Y_i$

$Y = 2X - 1$, c/ $X \sim bi(1, 1/2)$

(i) Calc. prob. exata usando

$S_n = \sum_{i=1}^{n} Y_i = \sum_{i=1}^{n} (2X_i - 1) = 2 \underbrace{\sum_{i=1}^{n} X_i}_{\sim bi(n, 1/2)} - n$

$X_1, \dots, X_i$ iid c/ X

Então $P(0 \leq S_{100} \leq 50) = P(0 \leq 2 \sum_{i=1}^{100} X_i - 100 \leq 50)$

$= P(50 \leq \underbrace{\sum_{i=1}^{100} X_i}_{W} \leq 75)$

$P(a < T \leq b) = F_T(b) - F_T(a)$

$= F_W(75) - F_W(49)$

=> pbinom(75, 100, 0.5) - pbinom(49, 100, 0.5)

0.5397945

ou

> sum(dbinom(50:75, 100, 0.5))

0.5397945

(ii) Calc. valor apox. usando TLC e $Y = 2X - 1$:

$P(0 \leq S_n \leq 50) \overset{\text{TLC c/ corr. de continuidade}}{\simeq} P(-0.5 < T < 50.5)$

$T \sim N(100 \times 0.5, \sqrt{25})$

=> pnorm(50.5, 50, 5) - pnorm(-0.5, 50, 5)

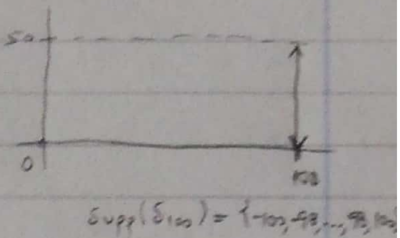
0.5398278

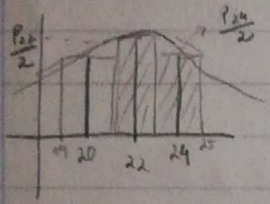
(iii) Calc. valor aprox. usando TLC e Y:

$P(0 \le S_n \le 50) \underset{TLC}{\simeq}$

$S_n = \sum_{i=1}^{n} Y_i$; $\mu = E(Y) = 0$, $\sigma^2 = E(Y^2) - 0 = 1$

c/ $\mu = 0, \sigma = 1$, $n = 100$, c/ corr., usando $U \sim N(\underbrace{0}_{n\mu}, \underbrace{\sqrt{100}}_{\sigma\sqrt{n}})$





$\simeq P(-1 < U < 51) =$

S_n tem supp $\{-100, -98, \ldots, 98, 100\}$

se a variável varia de 2 em 2 tenho de ir buscar 1 ponto a trás e um ponto à frente

=> pnorm (51, 0, 10) - pnorm (-1, 0, 10)

0.5398277

66. T = tp. atendimento ... (v.a. contínua)

$\mu = E(T) = 15$ min.

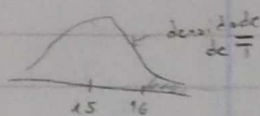
$\sigma = \sqrt{Var(T)} = 4.5$ min.

$\bar{T} \,\dot\sim\, N(15, 4.5/\sqrt{n})$ v.a. W

$T_1, T_2, \ldots, T_{50}$ iid c/ T

i) $P(\bar{T} \ge 16) \simeq P(W > 16)$ => pnorm (16, 15, 4.5 / sqrt(50), lower=F)

= 0.05805087



ii) $P(14.5 \le \bar{T} \le 15.5) \simeq P(14.5 < W < 15.5)$

=> pnorm (15.5, 15, 4.5/sqrt(50)) − pnorm (14.5, 15, 4.5/sqrt(50))

0.5679416