# Folha de exercícios 7 - Soluções

1.

a) $\iint_D f \, dA = \int_0^{\sqrt{2}} \int_0^{2\pi} f(r\cos\theta, r\,\mathrm{sen}\,\theta) \cdot r \, d\theta \, dr$

b) $\iint_D f \, dA = \int_0^{0.5} \int_0^{\pi/2} f(r\cos\theta, r\,\mathrm{sen}\,\theta) \cdot r \, d\theta \, dr$

c) $\iint_D f \, dA = \int_1^{2} \int_{\pi/2}^{3/2\,\pi} f(r\cos\theta, r\,\mathrm{sen}\,\theta) \cdot r \, d\theta \, dr$

d) $\iint_D f \, dA = \int_0^{2} \int_{\pi/4}^{3/4\,\pi} f(r\cos\theta, r\,\mathrm{sen}\,\theta) \cdot r \, d\theta \, dr$

2.

$$\iint_D e^{x^2+y^2} \, dx\, dy = \int_0^1 \int_0^{2\pi} e^{r^2} \cdot r \, d\theta \, dr = \int_0^1 \left[\theta \cdot e^{r^2} \cdot r\right]_{\theta=0}^{2\pi} dr$$

$$= \int_0^1 2\pi \cdot e^{r^2} \cdot r \, dr = \pi \int_0^1 2r\, e^{r^2} dr$$

$$= \pi \left[ e^{r^2} \right]_{r=0}^{1} = \pi (e-1)$$

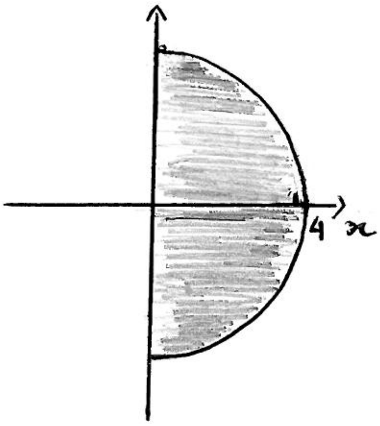
Note-se que, sendo $x = r\cos\theta$ e $y = r\,\mathrm{sen}\,\theta$, vem

$$e^{x^2+y^2} = e^{r^2\cos^2\theta + r^2\mathrm{sen}^2\theta}$$

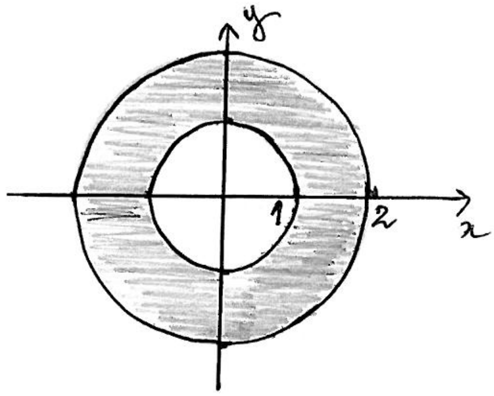
$$= e^{r^2(\cos^2\theta + \mathrm{sen}^2\theta)}$$

$$= e^{r^2}$$

3. a)




e)




4.

$$\iint_D f(x,y)\,dx\,dy = \int_1^2\int_0^{\pi/4} \frac{1}{(r^2)^{3/2}}\cdot r\,d\theta\,dr$$

$$= \int_1^2\int_0^{\pi/4} \frac{1}{r^2}\,d\theta\,dr = \int_1^2\int_0^{\pi/4} r^{-2}\,d\theta\,dr$$

$$= \int_1^2 \left[\theta \cdot r^{-2}\right]_{\theta=0}^{\pi/4} dr = \int_1^2 4.r^{-2}\,dr$$

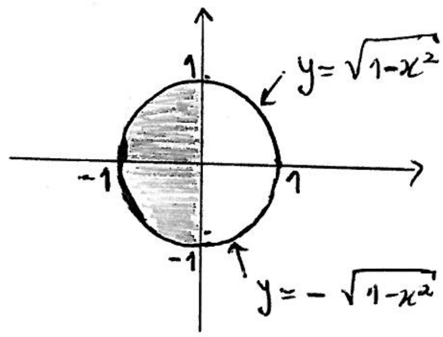
$$= \left[\frac{\pi}{4}\cdot\frac{r^{-1}}{-1}\right]_{r=1}^{2} = \left[-\frac{\pi}{4r}\right]_{r=1}^{2} = -\frac{\pi}{8}+\frac{\pi}{4} = \frac{\pi}{8}$$

5.

a) Região de integração:

$$-1 \leq x \leq 0$$

$$-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$$




$$y = \pm\sqrt{1-x^2} \Rightarrow y^2 = 1-x^2$$

$$\Rightarrow x^2+y^2 = 1$$

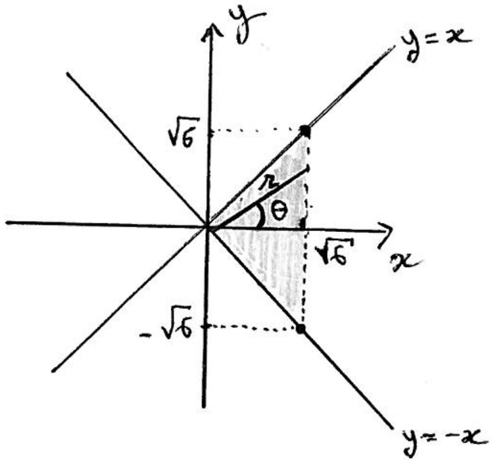
Mudança para coordenadas polares:

$$\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x\,dy\,dx = \int_0^1\int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} r\cos\theta \cdot r\,d\theta\,dr = \int_0^1 \left[r^2 \operatorname{sen}\theta\right]_{\theta=\frac{\pi}{2}}^{\frac{3}{2}\pi} dr$$

$$= \int_0^1 -2r^2\,dr = -2\,\frac{r^3}{3}\Big]_{r=0}^{1} = -\frac{2}{3}$$

b) Região de integração:

$$0 \leq x \leq \sqrt{6}$$
$$-x \leq y \leq x$$




Mudança para coordenadas polares:

$$\begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \end{cases}$$

- $x = \sqrt{6}, \quad -\sqrt{6} \leq y \leq \sqrt{6}$

$$-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$$

$$x = r\cos\theta \implies \sqrt{6} = r\cos\theta \implies r = \frac{\sqrt{6}}{\cos\theta} \quad \left(\cos\theta \neq 0, \ -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}\right)$$

Região de integração em coordenadas polares:

$$-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$$
$$0 \leq r \leq \sqrt{6}/\cos\theta$$

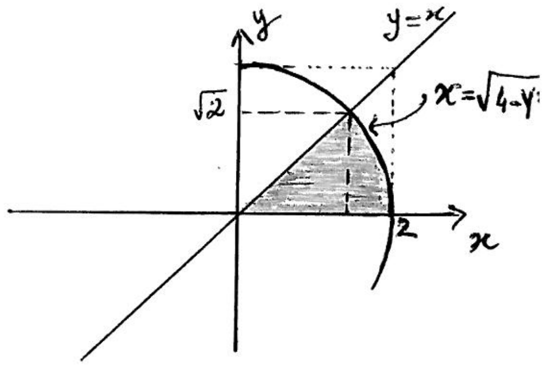
$$\int_0^{\sqrt{6}}\int_{-x}^{x} dy\,dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_0^{\sqrt{6}/\cos\theta} r\,dr\,d\theta = \int_{-\frac{\pi}{4}}^{\pi/4}\left[\frac{r^2}{2}\right]_{r=0}^{\sqrt{6}/\cos\theta} d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{3}{\cos^2\theta}\,d\theta = \left[3\,\text{tg}\,\theta\right]_{\theta=-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= 3\,\text{tg}\,\frac{\pi}{4} - 3\,\text{tg}\left(-\frac{\pi}{4}\right) = 3 - (-3) = 6$$

c) Regiao de integração:

$$0 \leq y \leq \sqrt{2}$$
$$y \leq x \leq \sqrt{4-y^2}$$




- $y = x$
- $x = \sqrt{4-y^2} \Rightarrow x^2+y^2 = 4$

Mudança para coordenadas polares:

$$\int_0^{\sqrt{2}}\int_y^{\sqrt{4-y^2}} dx\,dy = \int_0^{\pi/4}\int_0^{2} r\,dr\,d\theta = \int_0^{\pi/4} \left.\frac{r^2}{2}\right]_{r=0}^{2} d\theta$$

$$= \int_0^{\pi/4} 2\,d\theta = \left[2\theta\right]_{\theta=0}^{\pi/4} = \frac{\pi}{2}$$

6.

a) Coordenadas cartesianas

$$\int_1^3\int_{-1}^{2} f(x,y)\,dy\,dx$$

b) Coordenadas polares

$$\int_0^3\int_0^{2\pi} f(r\cos\theta, r\sin\theta)\,r\,d\theta\,dr$$

c) Coordenadas cartesianas

$$\int_0^2\int_{\frac{x}{2}-1}^{3} f(x,y)\,dy\,dx$$

d) Coordenadas polares

$$\int_1^2\int_{\frac{\pi}{2}}^{\pi} f(r\cos\theta, r\sin\theta)\cdot r\,d\theta\,dr$$

7. Descrição da região em coordenadas cilíndricas:

$$\begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \\ z = z \end{cases}$$

$$\left\{(r,\theta,z):\ 0 \leq r \leq 6,\quad 0 \leq \theta \leq \frac{\pi}{2} - \frac{\pi}{3}\ ,\ 0 \leq z \leq 4\right\}$$

8.

$$\iiint_U f\,dV = \int_0^4 \int_{\frac{\pi}{4}}^{3\pi/4} \int_{-1}^{1} \left[(r\cos\theta)^2 + (r\,\text{sen}\,\theta)^2 + z^2\right]\cdot r\ dz\,d\theta\,dr$$

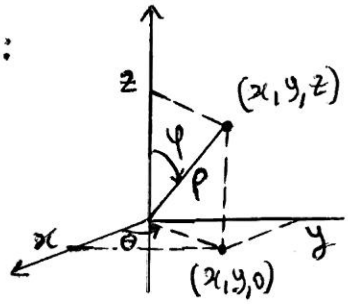
$$= \int_0^4 \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_{-1}^{1} (r^2 + z^2)\cdot r\ dz\,d\theta\,dr$$

$$= \int_0^4 \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_{-1}^{1} (r^3 + rz^2)\,dz\,d\theta\,dr = \int_0^4 \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \left[zr^3 + r\frac{z^3}{3}\right]_{z=-1}^{1} d\theta\,dr$$

$$= \int_0^4 \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 2\left(r^3 + \frac{r}{3}\right) d\theta\,dr = \int_0^4 \left[2\left(r^3 + \frac{r}{3}\right)\theta\right]_{\theta=\frac{\pi}{4}}^{\frac{3\pi}{4}} dr$$

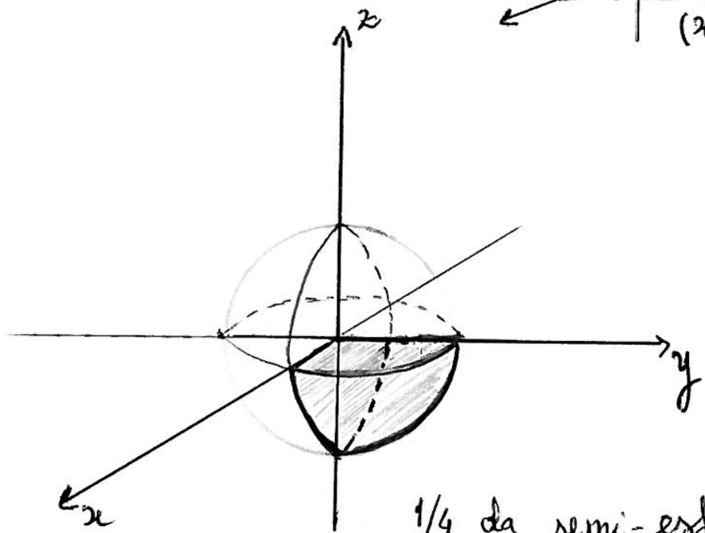
$$= \int_0^4 \pi\left(r^3 + \frac{r}{3}\right) dr = \pi\cdot\left[\frac{r^4}{4} + \frac{r^2}{6}\right]_{r=0}^{4} = \pi\left(64 + \frac{16}{6}\right) = \pi\left(64 + \frac{8}{3}\right)$$

9. Região de integração em coordenadas esféricas:




$$\begin{cases} x = \rho \,\text{sen}\varphi \cos\theta \\ y = \rho \cos\varphi \,\text{sen}\theta \\ z = \rho \cos\varphi \end{cases}$$

$$\left\{ (\rho, \theta, \varphi) : \quad 0 \leq \theta \leq \pi/2, \quad \frac{\pi}{2} \leq \varphi \leq \pi, \quad 0 \leq \rho \leq 1 \right\}$$




1/4 da semi-esfera inferior de centro em (0,0,0) e raio 1

10. Região de integração em coordenadas esféricas:

$$0 \leq \theta \leq 2\pi$$
$$\frac{\pi}{2} \leq \varphi \leq \pi$$
$$0 \leq \rho \leq 5$$

Temos

$$f(x,y,z) = f(\rho\,\text{sen}\varphi\cos\theta, \rho\,\text{sen}\varphi\,\text{sen}\theta, \rho\cos\varphi)$$

$$= \frac{1}{\left(\rho^2\text{sen}^2\varphi\cos^2\theta + \rho^2\text{sen}^2\varphi\,\text{sen}^2\theta + \rho^2\cos^2\varphi\right)^{1/2}}$$

$$= \frac{1}{\left(\rho^2\text{sen}^2\varphi\,(\cos^2\theta + \text{sen}^2\theta) + \rho^2\cos^2\varphi\right)^{1/2}}$$

$$= \frac{1}{\left(\rho^2\text{sen}^2\varphi + \rho^2\cos^2\varphi\right)^{1/2}}$$

$$= \frac{1}{(\rho^2)^{1/2}} = \frac{1}{\rho}$$

e

$$\iiint_U f(x,y,z)\,dx\,dy\,dz = \int_0^5 \int_{\frac{\pi}{2}}^{\pi} \int_0^{2\pi} \frac{1}{\rho} \cdot \rho^2 \operatorname{sen}\varphi \; d\theta\, d\varphi\, d\rho$$

$$= \int_0^5 \int_{\frac{\pi}{2}}^{\pi} \left[\rho \operatorname{sen}\varphi \cdot \theta\right]_{\theta=0}^{2\pi} d\varphi\, d\rho = \int_0^5 \int_{\frac{\pi}{2}}^{\pi} 2\pi\rho \operatorname{sen}\varphi \; d\varphi\, d\rho$$

$$= \int_0^5 2\pi\rho\left[-\cos\varphi\right]_{\varphi=\frac{\pi}{2}}^{\pi} d\rho = \int_0^5 2\pi\rho\, d\rho = \left[\pi\rho^2\right]_{\rho=0}^{5} = 25\pi$$

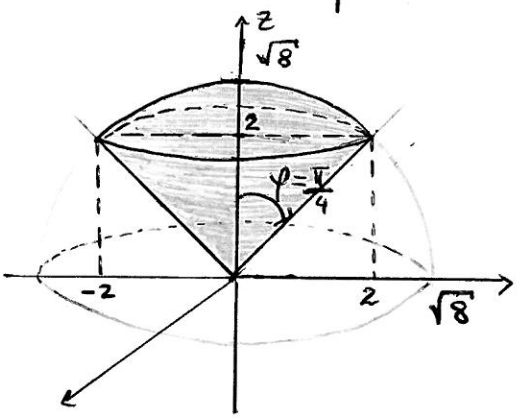
11.

- $z = \sqrt{8-x^2-y^2} \implies z^2+x^2+y^2 = 8$

  semi-superfície esférica superior de centro em $(0,0,0)$ e raio $\sqrt{8}$

- $z = \sqrt{x^2+y^2} \implies z^2 = x^2+y^2$

  Cone "superior" de vértice em $(0,0,0)$




$$\begin{cases} z = \sqrt{8-x^2-y^2} \\ z = \sqrt{x^2+y^2} \end{cases} \implies \begin{cases} z^2+x^2+y^2 = 8 \\ z^2 = x^2+y^2 \end{cases}$$

$$\implies \begin{cases} x^2+y^2 = 4 \\ z^2 = x^2+y^2 \end{cases} \implies \begin{cases} x^2+y^2 = 4 \\ z = \pm 2 \end{cases}$$

Região de integração em coordenadas esféricas:

$$0 \le \rho \le \sqrt{8}$$
$$0 \le \theta \le 2\pi$$
$$0 \le \varphi \le \pi/4$$

Volume: $\displaystyle\int_0^{\sqrt{8}} \int_0^{2\pi} \int_0^{\pi/4} 1 \cdot \rho^2 \operatorname{sen}\varphi \; d\varphi\, d\theta\, d\rho =$

$$= \int_0^{\sqrt{8}} \int_0^{2\pi} \rho^2 \left[-\cos\varphi\right]_{\varphi=0}^{\pi/4} d\theta\, d\rho = \int_0^{\sqrt{8}} \int_0^{2\pi} \rho^2 \left(1 - \frac{\sqrt{2}}{2}\right) d\theta\, d\rho$$

$$= \int_0^{\sqrt{8}} \cancel{2}\pi \rho^2 \frac{2-\sqrt{2}}{\cancel{2}}\, d\rho = \int_0^{\sqrt{8}} \pi\rho^2 (2-\sqrt{2})\, d\rho$$

$$= \left[\pi \frac{\rho^3}{3} (2-\sqrt{2})\right]_{\rho=0}^{\sqrt{8}} = \pi \frac{(\sqrt{8})^3}{3} (2-\sqrt{2}) = \frac{\pi \times 16 \times \sqrt{2}}{3} (2-\sqrt{2})$$

$$= \pi . \frac{32\sqrt{2} - 32}{3}$$

15

a) Coordenadas cartesianas

$$\iiint_U f\, dV = \int_0^1 \int_0^3 \int_0^5 f(x,y,z)\, dz\, dy\, dx$$

b) Coordenadas cilíndricas

$$\iiint_U f\, dV = \int_0^4 \int_0^{2\pi} \int_0^1 r . f(r\cos\theta, r\,\mathrm{sen}\,\theta, z)\, dz\, d\theta\, dr$$

c) Coordenadas cilíndricas

$$\iiint_U f\, dV = \int_0^2 \int_0^{\pi/2} \int_0^4 r\, f(r\cos\theta, r\,\mathrm{sen}\,\theta, z)\, dz\, d\theta\, dr$$

d) Coordenadas esféricas

$$\int_2^3 \int_0^{\pi} \int_0^{\pi} \rho^2 \,\mathrm{sen}\,\varphi\, f(\rho\,\mathrm{sen}\,\varphi\cos\theta, \rho\,\mathrm{sen}\,\varphi\,\mathrm{sen}\,\theta, \rho\cos\varphi)\, d\varphi\, d\theta\, d\rho$$

13.

a) Usando coordenadas polares, tem-se

$$\iint_D (x^2+y^2)\,dx\,dy = \int_0^2\int_0^{2\pi} r^2 \cdot r\, d\theta\, dr = \int_0 r^3 \cdot 2\pi\, dr = \left[2\pi \cdot \frac{r^4}{4}\right]_0^2 = 8\pi$$

b) Usando coordenadas cilíndricas, vem

$$\iiint_B z\, e^{x^2+y^2}\,dx\,dy\,dz = \int_0^2\int_0^{2\pi}\int_2^3 z\, e^{r^2} \cdot r\, dz\, d\theta\, dr$$

$$= \int_0^2\int_0^{2\pi} \left.\frac{z^2}{2} \cdot e^{r^2} \cdot r\right]_{r=2}^3 d\theta\, dr = \int_0^2\int_0^{2\pi} \frac{5}{2} e^{r^2} \cdot r\, d\theta\, dr$$

$$= \int_0^2 \left.\frac{5}{2} e^{r^2} r\, \theta\right]_0^{2\pi} dr = \int_0^2 5\pi e^{r^2} r\, dr = 5\pi \left.\frac{e^{r^2}}{2}\right]_{r=0}^{2}$$

$$= 5\pi \cdot \frac{e^4-1}{2}$$

c) Usando coordenadas esféricas, tem-se

$$\iiint_E \frac{dx\,dy\,dz}{\sqrt{2+(x^2+y^2+z^2)^{3/2}}} = \int_0^1\int_0^{\pi}\int_0^{2\pi} \frac{1}{\sqrt{2+(\rho^2)^{3/2}}}\, \rho^2 \operatorname{sen}\phi\, d\theta\, d\phi\, d\rho$$

$$= \int_0^1\int_0^{\pi}\int_0^{2\pi} (2+\rho^3)^{-1/2} \cdot \rho^2 \operatorname{sen}\phi\, d\theta\, d\phi\, d\rho =$$

$$= \int_0^1\int_0^{\pi} 2\pi (2+\rho^3)^{-1/2} \cdot \rho^2 \operatorname{sen}\phi\, d\phi\, d\rho = \int_0^1 2\pi (2+\rho^3)^{-1/2} \cdot \rho^2 \Big[-\cos\phi\Big]_{\phi=0}^{\pi} d\rho$$

$$= \int_0^1 4\pi (2+\rho^3)^{-1/2} \cdot \rho^2\, d\rho = \left[\frac{4\pi}{3}(2+\rho^3)^{1/2} \cdot 2\right]_{\rho=0}^1 = \frac{8\pi}{3}(\sqrt{3}-\sqrt{2})$$