

Folha de exercícios 6 - soluções

1.

$$\begin{aligned} \iint_R (x^2 y^2 + x) dA &= \int_0^2 \int_{-1}^0 (x^2 y^2 + x) dy dx = \int_0^2 \left[x^2 \cdot \frac{y^3}{3} + xy \right]_{y=-1}^0 dx \\ &= \int_0^2 \left[0 - \left(-\frac{x^2}{3} - x \right) \right] dx = \int_0^2 \left(\frac{x^2}{3} + x \right) dx = \left[\frac{x^3}{9} + \frac{x^2}{2} \right]_{x=0}^2 = \frac{8}{9} + \frac{4}{2} = \frac{26}{9} \end{aligned}$$

2.

$$\begin{aligned} \text{a)} \quad \int_0^3 \int_0^4 (4x+y) dx dy &= \int_0^3 \left[2x^2 + yx \right]_{x=0}^4 dy = \int_0^3 (32 + 4y) dy = \\ &= \left[32y + 2y^2 \right]_{y=0}^3 = 114 \end{aligned}$$

$$\text{b)} \quad \int_0^3 \int_0^2 6xy dy dx = 54$$

3.

$$\text{a)} \quad \iint_D f dA = \int_1^4 \int_1^2 f(x,y) dy dx = \int_1^2 \int_1^4 f(x,y) dx dy$$

$$\text{b)} \quad \iint_D f dA = \int_0^4 \int_{\frac{1}{3}x - \frac{1}{3}}^2 f(x,y) dy dx = \int_0^1 \int_1^{3y+1} f(x,y) dx dy + \int_1^2 \int_1^4 f(x,y) dx dy$$

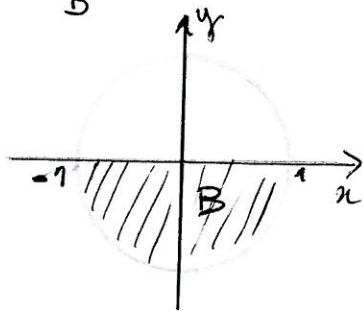
$$\text{c)} \quad \iint_D f dA = \int_{-1}^3 \int_{-2}^{-\frac{5}{4}x + \frac{1}{4}} f(x,y) dy dx = \int_{-2}^1 \int_{-1}^{\frac{1}{3} - \frac{4}{3}y} f(x,y) dx dy$$

$$\text{d)} \quad \iint_D f dA = \int_0^3 \int_0^{2x} f(x,y) dy dx + \int_3^5 \int_0^{-3x+15} f(x,y) dy dx = \int_0^6 \int_{\frac{y}{2}}^{5-\frac{y}{3}} f(x,y) dx dy$$

4.

a) $\iint_R dA > 0$, pois representa a área do semi-círculo R .

b) $\iint_B 5x \, dA = 0$, pois o domínio de integração B é simétrico relativamente ao eixo dos yy e a função integranda, $f(x,y) = 5x$, é ímpar com respeito à variável x , isto é, $f(x,y) = -f(-x,y)$.

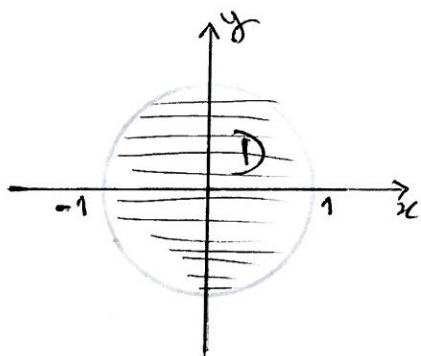


De facto,

$$\begin{aligned} \iint_B 5x \, dA &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^0 5x \, dy \, dx \\ &= \int_{-1}^1 \underbrace{5x \sqrt{1-x^2}}_{\text{função ímpar}} \, dx = 0 \end{aligned}$$

c) $\iint_D 5x \, dA = 0$, pela mesma razão que em b).

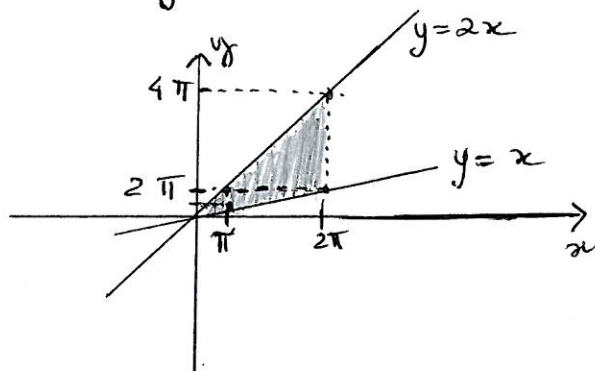
d) $\iint_D xy \, dA = 0$, por razão análoga à descrita em



b) mas considerando agora a simetria do domínio de integração D em relação ao eixo dos xx e o facto da função integranda ser uma função ímpar com respeito à variável y .

5. a) $\pi \leq x \leq 2\pi$

$x \leq y \leq 2x$



$$\int_{\pi}^{2\pi} \int_x^{2x} \sin x \, dy \, dx = \int_{\pi}^{2\pi} \sin x \left[y \right]_{y=x}^{2x} dx = \int_{\pi}^{2\pi} x \sin x \, dx = \quad (*)$$

$$= x \cdot (-\cos x) \Big|_{\pi}^{2\pi} - \int_{\pi}^{2\pi} -\cos x \, dx = (\pi \cos \pi - 2\pi \cos 2\pi) + [\sin x]_{\pi}^{2\pi}$$

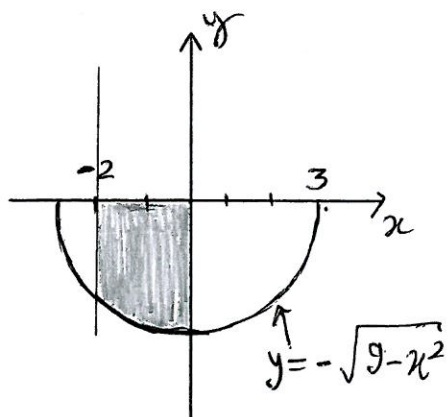
$$= -\pi - 2\pi + 0 = -3\pi$$

Obs.: em (*) foi usada a técnica de primitivação por partes:

$$\int f \cdot g' = f \cdot g - \int f' \cdot g$$

b) $-2 \leq x \leq 0$

$-\sqrt{9-x^2} \leq y \leq 0$



$$\int_{-2}^0 \int_{-\sqrt{9-x^2}}^0 2xy \, dy \, dx = \int_{-2}^0 x \left[y^2 \right]_{y=-\sqrt{9-x^2}}^0 dx =$$

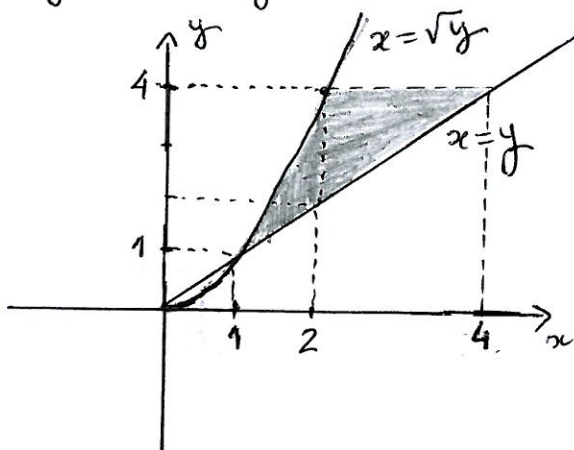
$$= \int_{-2}^0 -x(-\sqrt{9-x^2})^2 dx = \int_{-2}^0 -x(9-x^2) dx =$$

$$= \int_{-2}^0 (x^3 - 9x) dx = \left[\frac{x^4}{4} - \frac{9}{2}x^2 \right]_{x=-2}^0 =$$

$$= -\frac{16}{4} + 18 = 14$$

$$y = -\sqrt{9-x^2} \Rightarrow y^2 + x^2 = 9$$

e) $1 \leq y \leq 4$
 $\sqrt{y} \leq x \leq y$



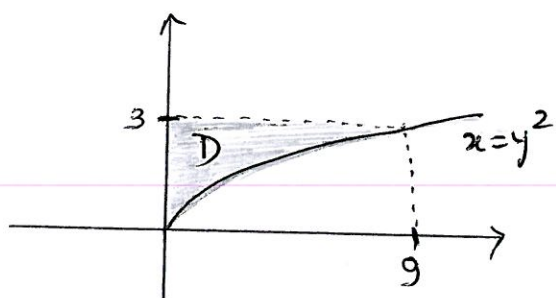
$$x = \sqrt{y} \Rightarrow y = x^2$$

$$\begin{aligned} \int_1^4 \int_{\sqrt{y}}^y x \, dx \, dy &= \int_1^4 \left[\frac{x^2}{2} \right]_{x=\sqrt{y}}^y dy = \\ &= \int_1^4 \frac{1}{2} (y^2 - y) \, dy = \\ &= \frac{1}{2} \left[\frac{y^3}{3} - \frac{y^2}{2} \right]_{y=1}^4 = \\ &= \frac{1}{2} \left(\frac{64}{3} - 8 - \frac{1}{3} + \frac{1}{2} \right) = \frac{27}{4} \end{aligned}$$

6.

$$0 \leq y \leq 3$$

$$0 \leq x \leq y^2$$



$$\text{Volume}(U) = \iint_D f(x,y) \, dA$$

U - região de \mathbb{R}^3 compreendida entre D e a superfície de equação $z = f(x,y)$.

$$\begin{aligned} \iint_D f(x,y) \, dA &= \int_0^3 \int_0^{y^2} (x+y) \, dx \, dy = \int_0^3 \left[\frac{x^2}{2} + yx \right]_{x=0}^{y^2} dy = \\ &= \int_0^3 \left(\frac{y^4}{2} + y^3 \right) dy = \left[\frac{y^5}{10} + \frac{y^4}{4} \right]_{y=0}^3 = \frac{3^5}{10} + \frac{3^4}{4} \end{aligned}$$

7.

5

a) Note-se que $0 < f(x,y) = \frac{2}{e^{(x-1)^2+y^2}} \leq 2, \forall (x,y) \in \mathbb{R}^2$.

Linhas (ou curvas) de nível $k \in]0, 2]$:

$$f(x,y) = k \Leftrightarrow 2e^{-(x-1)^2-y^2} = k$$

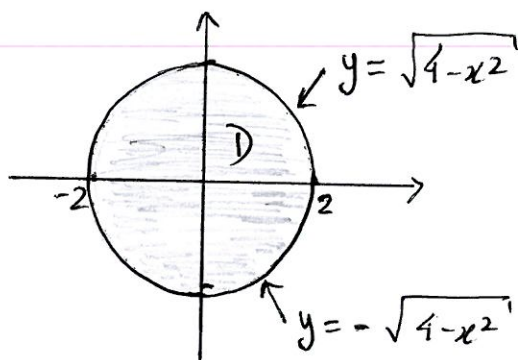
$$\Leftrightarrow e^{-(x-1)^2-y^2} = \frac{k}{2}$$

$$\Leftrightarrow -(x-1)^2 - y^2 = \ln \frac{k}{2}$$

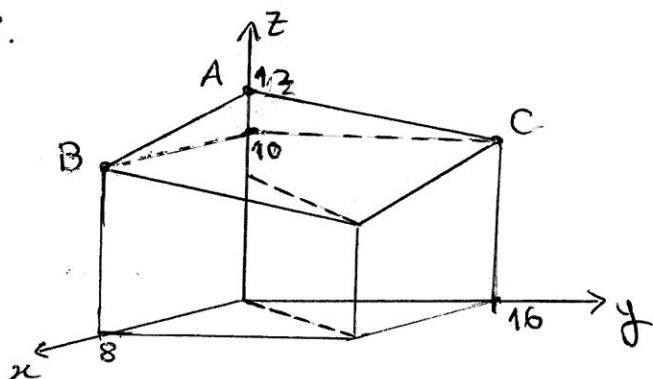
$$\Leftrightarrow (x-1)^2 + y^2 = -\ln \frac{k}{2}$$

As curvas de nível de f são circunferências de centro $(1,0)$ e raio $\sqrt{-\ln \frac{k}{2}}$, $0 < k \leq 2$. Observe-se que $\frac{k}{2} \leq 1$ e, portanto, $\ln \frac{k}{2} \leq \ln 1 = 0$.

$$b) \text{ Volume}(R) = \iint_D f(x,y) dA = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 2e^{-(x-1)^2-y^2} dy dx$$



8.



$$A = (0, 0, 12)$$

$$B = (8, 0, 10)$$

$$C = (9, 16, 10)$$

$$\vec{AB} \times \vec{AC} = (32, 16, 128)$$

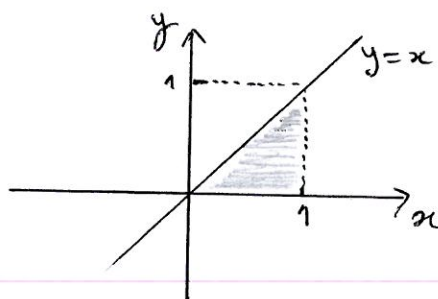
Equações do plano que contém os pontos A, B e C :

$$(2, 1, 8) \cdot (x, y, z-12) = 0 \quad \Leftrightarrow \quad z = 12 - \frac{x}{4} - \frac{y}{8}$$

Volume do edifício :

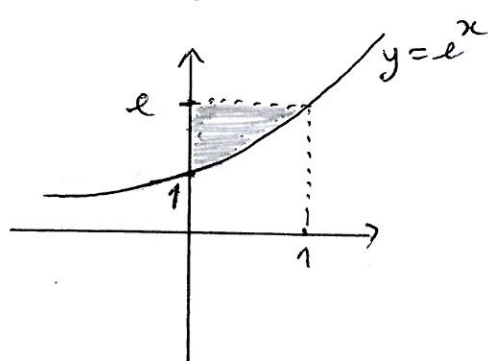
$$\begin{aligned} \int_0^8 \int_0^{16} \left(12 - \frac{x}{4} - \frac{y}{8} \right) dy dx &= \int_0^8 \left[12y - \frac{x}{4}y - \frac{y^2}{16} \right]_{y=0}^{16} dx = \\ &= \int_0^8 (176 - 4x) dx = \left[176x - 2x^2 \right]_{x=0}^8 = 1280 \end{aligned}$$

9. a) $0 \leq y \leq 1$
 $y \leq x \leq 1$



$$\begin{aligned} \int_0^1 \int_y^1 e^{x^2} dx dy &= \int_0^1 \int_0^x e^{x^2} dy dx = \int_0^1 \left[e^{x^2} y \right]_{y=0}^x dx = \\ &= \int_0^1 e^{x^2} \cdot x dx = \left[\frac{e^{x^2}}{2} \right]_{x=0}^1 = \\ &= \frac{1}{2}(e-1) \end{aligned}$$

b) $0 \leq x \leq 1$
 $e^x \leq y \leq e$



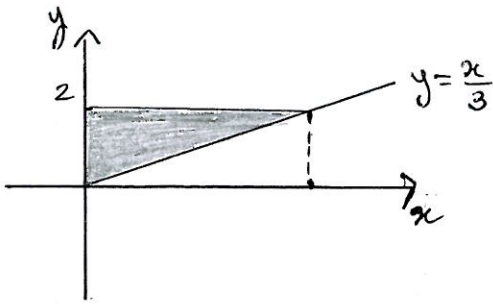
$1 \leq y \leq e$
 $0 \leq x \leq \ln y$

$$\begin{aligned} \int_0^1 \int_{e^x}^e \frac{y}{\ln y} dy dx &= \int_1^e \int_0^{\ln y} \frac{y}{\ln y} dx dy = \\ &= \int_1^e \left[\frac{y}{\ln y} \cdot x \right]_{x=0}^{\ln y} dy = \int_1^e y dy = \\ &= \left[\frac{y^2}{2} \right]_{y=1}^e = \frac{1}{2}(e^2 - 1) \end{aligned}$$

10.

$$a) \quad 0 \leq x \leq 6$$

$$\frac{x}{3} \leq y \leq 2$$



$$0 \leq y \leq 2$$

$$0 \leq x \leq 3y$$

b) Invertendo a ordem de integração, temos

$$I = \int_0^2 \int_0^{3y} x \sqrt{y^3 + 1} \, dx \, dy = \int_0^2 \sqrt{y^3 + 1} \cdot \left[\frac{x^2}{2} \right]_{x=0}^{3y} dy$$

$$= \int_0^2 \frac{9}{2} y^2 \sqrt{y^3 + 1} \, dy = \frac{3}{2} \int_0^2 3y^2 (y^3 + 1)^{1/2} dy$$

$$= \frac{3}{2} \left[(y^3 + 1)^{3/2} \right]_{y=0}^2 = \frac{3}{2} (9^{3/2} - 1) = \frac{78}{2}$$

11.

$$\iiint_P xyz \, dv = \int_0^1 \int_1^2 \int_2^3 xyz \, dz \, dy \, dx = \int_0^1 \int_1^2 xy \left[\frac{z^2}{2} \right]_2^3 dy \, dx$$

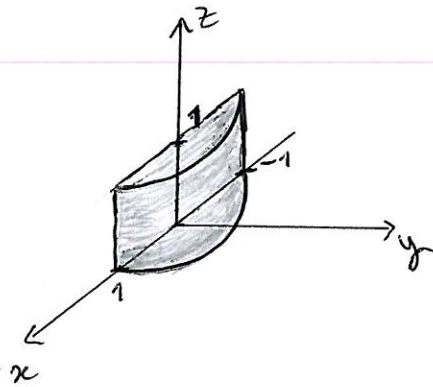
$$= \frac{5}{2} \int_0^1 \int_1^2 xy \, dy \, dx = \frac{5}{2} \int_0^1 x \left[\frac{y^2}{2} \right]_{y=1}^2 dx = \frac{15}{2} \int_0^1 x \, dx = \frac{15}{8}$$

12.

$$\begin{aligned}
 \iiint_U f \, dv &= \int_0^2 \int_{-1}^1 \int_2^3 (x^2 + 5y^2 - z) \, dz \, dy \, dx \\
 &= \int_0^2 \int_{-1}^1 \left[x^2 z + 5y^2 z - \frac{z^2}{2} \right]_{z=2}^3 \, dy \, dx = \\
 &= \int_0^2 \int_{-1}^1 \left(x^2 + 5y^2 - \frac{5}{2} \right) \, dy \, dx = \int_0^2 \left[x^2 y + \frac{5}{3} y^3 - \frac{5}{2} y \right]_{y=-1}^1 \, dx = \\
 &= \int_0^2 \left(2x^2 - \frac{5}{3} \right) \, dx = \left[\frac{2}{3} x^3 - \frac{5}{3} x \right]_{x=0}^2 = 2
 \end{aligned}$$

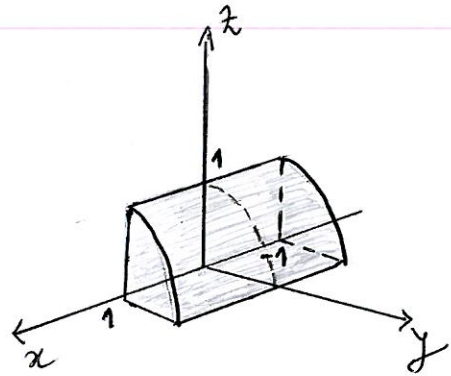
13.

a) $0 \leq z \leq 1$
 $-1 \leq x \leq 1$
 $0 \leq y \leq \sqrt{1-x^2}$



$-1 \leq x \leq 1$
 $0 \leq y \leq \sqrt{1-x^2}$
 $0 \leq z \leq 1$

b) $0 \leq z \leq 1$
 $-1 \leq x \leq 1$
 $0 \leq y \leq \sqrt{1-z^2}$



$0 \leq z \leq 1$
 $0 \leq y \leq \sqrt{1-z^2}$
 $-1 \leq x \leq 1$

14.

$$a) \int_{-x}^x \int_0^{\sqrt{x^2-x^2}} \int_{-\sqrt{x^2-x^2-y^2}}^{\sqrt{x^2-x^2-y^2}} f(x,y,z) dz dy dx$$

$$b) \int_0^x \int_0^{\sqrt{x^2-x^2}} \int_0^{\sqrt{x^2-x^2-y^2}} f(x,y,z) dz dy dx$$

$$c) \int_0^x \int_0^{\sqrt{x^2-y^2}} \int_0^1 f(x,y,z) dx dz dy$$

$$d) \int_0^x \int_0^{\sqrt{x^2-x^2}} \int_0^1 f(x,y,z) dy dz dx$$

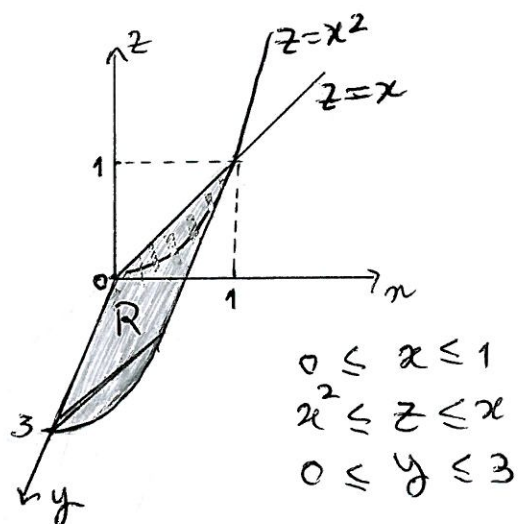
15.

$$\text{Volume} = \int_0^8 \int_0^{16} \int_0^{12 - \frac{x}{4} - \frac{y}{8}} 1 dz dy dx =$$

$$= \int_0^8 \int_0^{16} \left[z \right]_{z=0}^{12 - \frac{x}{4} - \frac{y}{8}} dy dx =$$

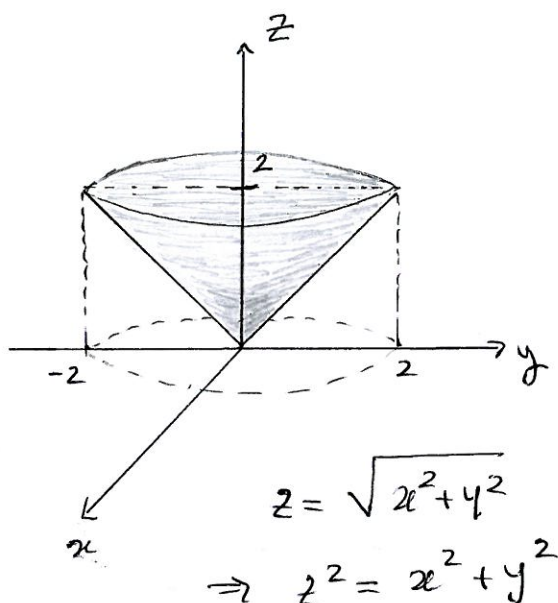
$$= \int_0^8 \int_0^{16} \left(12 - \frac{x}{4} - \frac{y}{8} \right) dy dx = 1280$$

16.



$$\begin{aligned} \text{Volume}(R) &= \int_0^1 \int_{x^2}^x \int_0^3 1 dy dz dx \\ &= \int_0^1 \int_{x^2}^x 3 dz dx \\ &= \int_0^1 3(x - x^2) dx = \frac{1}{2} \end{aligned}$$

$$U = \{ (x, y, z) \in \mathbb{R}^3 : z \geq \sqrt{x^2 + y^2} \text{ e } z \leq 2 \}$$



$$-2 \leq x \leq 2$$

$$-\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}$$

$$\sqrt{x^2 + y^2} \leq z \leq 2$$

$$z = 2 \Rightarrow x^2 + y^2 = 4$$

a) $\iiint_U \sqrt{x^2 + y^2} \, dV > 0$, pois a função integranda, $f(x, y, z) = \sqrt{x^2 + y^2}$, assume valores positivos para todo o $(x, y, z) \in U$.

b) $\iiint_U x \, dV = 0$, pois a função integranda, $f(x, y, z) = x$, é uma função ímpar com respeito a x e o domínio de integração é simétrico relativamente à origem.

c) $\iiint_U z - \sqrt{x^2 + y^2} \, dV > 0$, pois a função integranda, $f(x, y, z) = z - \sqrt{x^2 + y^2}$, assume valores positivos para todo o $(x, y, z) \in U$.