Soluções

- **Integral de linha** (campos escalares)

1. (a)

$$\int_{\mathcal{C}} (y+x)\,ds = \int_1^2 (2t+t)\,\|(t,2t)'\|\,dt = \int_1^2 3t\,\|(1,2)\|\,dt = \int_1^2 3\sqrt{5}t\,dt = \left[\frac{3\sqrt{5}}{2}t^2\right]_1^2 = \frac{9\sqrt{5}}{2}$$

(b)

$$\int_{\mathcal{C}} x\,ds = \int_0^1 t^3\,\|(t^3,t)'\|\,dt = \int_0^1 t^3\,\|(3t^2,1)\|\,dt = \int_0^1 t^3\,\sqrt{9t^4+1}\,dt = \frac{1}{36}\int_0^1 36t^3\,(9t^4+1)^{1/2}\,dt$$
$$= \left[\frac{(9t^4+1)^{3/2}}{3/2}\right]_0^1 = \frac{2}{3}\left(10^{3/2}-1\right)$$

2. Seja $\mathcal{C}$ a parte superior da circunferência unitária.

Parametrização de $\mathcal{C}$: $(x,y)=(\cos t, \operatorname{sen} t),\ t\in[0,\pi]$

$$\int_{\mathcal{C}} (2+x^2y)\,ds = \int_0^\pi \left(2+(\cos t)^2 \operatorname{sen} t\right)\,\|(\cos t,\operatorname{sen} t)'\|\,dt = \int_0^\pi \left(2+\cos^2 t \operatorname{sen} t\right)\,\|(-\operatorname{sen} t,\cos t)\|\,dt$$
$$= \int_0^\pi \left(2+\cos^2 t \operatorname{sen} t\right)\,\sqrt{(-\operatorname{sen} t)^2+(\cos t)^2}\,dt = \int_0^\pi \left(2+\cos^2 t \operatorname{sen} t\right)\cdot 1\,dt$$
$$= \left[2t - \frac{\cos^3 t}{3}\right]_0^\pi = 2\pi - \frac{-1}{3} - \left(-\frac{1}{3}\right) = 2\pi + \frac{2}{3}.$$

3. Parametrização de $\mathcal{C}_1$: $(x,y)=(t,t^2),\ t\in[0,1]$

Parametrização de $\mathcal{C}_2$: $(x,y)=(1,t),\ t\in[1,2]$

$$\int_{\mathcal{C}} 2x\,ds = \int_{\mathcal{C}_1} 2x\,ds + \int_{\mathcal{C}_2} 2x\,ds = \int_0^1 2t\,\|(1,2t)\|\,dt + \int_1^2 2\,\|(0,1)\|\,dt = \int_0^1 2t\,(1+4t^2)^{1/2}\,dt + \int_1^2 2\,dt$$
$$= \left[\frac{1}{4}\cdot\frac{2}{3}(1+4t^2)^{3/2}\right]_0^1 + \left[2t\right]_1^2 = \frac{1}{6}\left(5^{3/2}-1\right) + (4-2) = \frac{5\sqrt{5}+11}{6}$$

4. (a) Parametrização de $\mathcal{C}$: $(x,y)=(t,t^2),\ t\in[-2,1]$

$$\int_{\mathcal{C}} (x-2y^2)\,dy = \int_{-2}^1 \left(t-2(t^2)^2\right)\,(t^2)'\,dt = \int_{-2}^1 \left(2t^2-4t^5\right)\,dt = \left[\frac{2}{3}t^3 - \frac{2}{3}t^6\right]_{-2}^1 = -48$$

(b) Parametrização de $\mathcal{C}$: $(x,y)=(t^4,t),\ t\in[-1,1]$

$$\int_{\mathcal{C}} \operatorname{sen} x\,dx = \int_{-1}^1 \operatorname{sen} t^4\,(t^4)'\,dt = \int_{-1}^1 4t^3 \operatorname{sen} t^4\,dt = \left[-\cos t^4\right]_{-1}^1 = -\cos 1 + \cos 1 = 0$$

(c) Parametrização do segmento de reta de $(0,0)$ a $(2,0)$:
$(x,y)=(0,0)+t\big((2,0)-(0,0)\big)=(2t,0),\ t\in[0,1]$
Parametrização do segmento de reta de $(2,0)$ a $(3,2)$:
$(x,y)=(2,0)+t\big((3,2)-(2,0)\big)=(2+t,2t),\ t\in[0,1]$

$$\int_{\mathcal{C}} xy\,dx + (x-y)\,dy = \int_0^1 2t\cdot 0\cdot(2t)'\,dt + (2t-0)\cdot(0)'\,dt +$$
$$+\int_0^1 (2+t)(2t)(2+t)'\,dt + (2+t-2t)\cdot(2t)'\,dt$$
$$= 0 + \int_0^1 (2t^2+2t+4)\,dt = \left[\frac{2}{3}t^3+t^2+4t\right]_0^1 = \frac{17}{3}$$

5. Parametrização do segmento de reta $\mathcal{C}$:

$(x,y,z) = (1,0,1) + t\big((0,3,6) - (1,0,1)\big) = (1,0,1) + t(-1,3,5) = (1-t, 3t, 1+5t),\ t \in [0,1]$

$$\int_{\mathcal{C}} xy^2 z\, ds = \int_0^1 (1-t)(3t)^2(1+5t)\, \|(1-t,3t,1+5t)'\|\, dt = \int_0^1 9(t^2+4t^3-5t^4)\, \|(-1,3,5)\|\, dt$$
$$= 9\sqrt{35}\int_0^1 (t^2+4t^3-5t^4)\, dt = 9\sqrt{35}\left[\frac{t^3}{3} + t^4 - t^5\right]_0^1 = 3\sqrt{35}.$$

6.

$$\int_{\mathcal{C}} y \operatorname{sen} z\, ds = \int_0^{2\pi} \operatorname{sen} t \operatorname{sen} t\, \|(\cos t, \operatorname{sen} t, t)'\|\, dt = \int_0^{2\pi} \operatorname{sen}^2 t\, \|(-\operatorname{sen} t, \cos t, 1)\|\, dt$$
$$= \int_0^{2\pi} \operatorname{sen}^2 t\, \sqrt{(-\operatorname{sen} t)^2 + (\cos t)^2 + 1)}\, dt = \sqrt{2}\int_0^{2\pi} \operatorname{sen}^2 t\, dt$$
$$= \sqrt{2}\int_0^{2\pi} \frac{1-\cos 2t}{2}\, dt = \frac{\sqrt{2}}{2}\left[t - \frac{\operatorname{sen} 2t}{2}\right]_0^{2\pi} = \sqrt{2}\pi$$

7. Parametrização de $\mathcal{C}_1$:

$(x,y,z) = (2,0,0) + t\big((3,4,5) - (2,0,0)\big) = (2,0,0) + t(1,4,5) = (2+t, 4t, 5t),\ t \in [0,1]$

Parametrização de $\mathcal{C}_2$:

$(x,y,z) = (3,4,5) + t\big((3,4,0) - (3,4,5)\big) = (3,4,5) + t(0,0,-5) = (3,4,5-5t),\ t \in [0,1]$

$$\int_{\mathcal{C}_1} y\, dx + z\, dy + x\, dz = \int_0^1 4t\,(2+t)'\, dt + 5t\,(4t)'\, dt + (2+t)\,(5t)'\, dt = \int_0^1 (4t + 20t + 10 + 5t)\, dt$$
$$= \int_0^1 (29t+10)\, dt = \left[\frac{29}{2}t^2 + 10t\right]_0^1 = \frac{49}{2} = 24.5$$

$$\int_{\mathcal{C}_2} y\, dx + z\, dy + x\, dz = \int_0^1 4\cdot\ (3)'\, dt + (5-5t)\,(4)'\, dt + 3\,(5-5t)'\, dt$$
$$= \int_0^1 -15\, dt = \big[-15t\big]_0^1 = -15$$

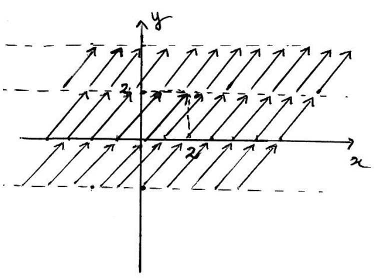
Assim, como $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, vem

$$\int_{\mathcal{C}} y\, dx + z\, dy + x\, dz = \int_{\mathcal{C}_1} y\, dx + z\, dy + x\, dz + \int_{\mathcal{C}_2} y\, dx + z\, dy + x\, dz = 24.5 - 15 = 9.5$$

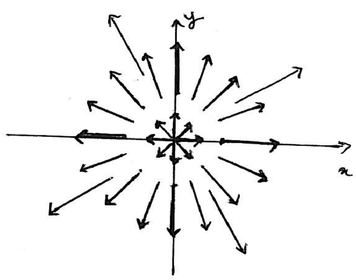
- **Integral de linha (campos vetoriais)**

8.

a) $\mathbf{F}(x,y) = (2,2)$



b) $\mathbf{F}(x,y) = (x,y)$



9.

$$\int_{\mathbf{r}} \mathbf{F}\cdot d\mathbf{r} = \int_0^2 \mathbf{F}\big(\mathbf{r}(t)\big)\cdot \mathbf{r}'(t)\, dt = \int_0^2 \mathbf{F}(1,t,e^t)\cdot(1,t,e^t)'\, dt = \int_0^2 (\cos \mathrm{e}^t, \mathrm{e}^1, \mathrm{e}^t)\cdot(0,1,\mathrm{e}^t)\, dt$$
$$= \int_0^1 (\mathrm{e} + \mathrm{e}^{2t})\, dt = \left[\mathrm{e}\, t + \frac{\mathrm{e}^{2t}}{2}\right]_0^2 = 2\,\mathrm{e} + \frac{1}{2}\,(\mathrm{e}^4 - 1)$$

10. (a)

$$\begin{aligned}\int_{\mathbf{r}} \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 \mathbf{F}\big(\mathbf{r}(t)\big)\cdot \mathbf{r}'(t)\,dt = \int_0^1 \mathbf{F}(t,t,t)\cdot(t,t,t)'\,dt = \int_0^1 (t,t,t)\cdot(1,1,1)\,dt \\ &= \int_0^1 3t\,dt = \left[3\frac{t^2}{2}\right]_0^1 = \frac{3}{2}\end{aligned}$$

(b)

$$\begin{aligned}\int_{\mathbf{r}} \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}\big(\mathbf{r}(t)\big)\cdot \mathbf{r}'(t)\,dt = \int_0^{2\pi} \mathbf{F}(\operatorname{sen} t, 0, \cos t)\cdot(\operatorname{sen} t, 0, \cos t)'\,dt \\ &= \int_0^{2\pi} (\operatorname{sen} t, 0, \cos t)\cdot(\cos t, 0, -\operatorname{sen} t)\,dt = \int_0^{2\pi} 0\,dt = 0\end{aligned}$$

11. (a)

$$\begin{aligned}\int_{\mathbf{r}} x\,dx + y\,dy &= \int_0^{2\pi} \Big[\cos t\,\big(\operatorname{sen} t\big)' - \operatorname{sen} t\,\big(\cos t\big)'\Big]\,dt \\ &= \int_0^{2\pi} \big(\cos^2 t + \operatorname{sen}^2 t\big)\,dt = \int_0^{2\pi} 1\,dt = 2\pi\end{aligned}$$

(b)

$$\begin{aligned}\int_{\mathbf{r}} x\,dx + y\,dy &= \int_0^2 \Big[\cos(\pi t)\,\big(\cos(\pi t)\big)' + \operatorname{sen}(\pi t)\,\big(\operatorname{sen}(\pi t)\big)'\Big]\,dt \\ &= \int_0^2 [-\pi\cos(\pi t)\,\operatorname{sen}(\pi t) + \pi\operatorname{sen}(\pi t)\,\cos(\pi t)]\,dt = \int_0^{2\pi} 0\,dt = 0\end{aligned}$$

(c) Temos $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, com $\mathcal{C}_1$ sendo o segmento de reta que une o ponto $(1,0,0)$ ao ponto $(0,1,0)$ e $\mathcal{C}_2$ o segmento de reta que une o ponto $(0,1,0)$ ao ponto $(0,0,1)$.

Parametrização de $\mathcal{C}_1$:
$(x,y,z) = \mathbf{r}_1(t) = (1,0,0) + t\big((0,1,0)-(1,0,0)\big) = (1-t,t,0),\ t\in[0,1]$
Parametrização de $\mathcal{C}_2$:
$(x,y,z) = \mathbf{r}_2(t) = (0,1,0) + t\big((0,0,1)-(0,1,0)\big) = (0,1-t,t),\ t\in[0,1]$

Fazendo $\mathbf{F}(x,y,z) = (yz, xz, xy)$, podemos escrever

$$\begin{aligned}\int_{\mathcal{C}} yz\,dx + xz\,dy + xy\,dz &= \int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = \int_{\mathcal{C}_1} \mathbf{F}\big(\mathbf{r}_1(t)\big)\cdot\mathbf{r}_1'(t)\,dt + \int_{\mathcal{C}_2} \mathbf{F}\big(\mathbf{r}_2(t)\big)\cdot\mathbf{r}_2'(t)\,dt \\ &= \int_0^1 \mathbf{F}(1-t,t,0)\cdot(1-t,t,0)'\,dt + \int_0^1 \mathbf{F}(0,1-t,t)\cdot(0,1-t,t)'\,dt \\ &= \int_0^1 (0,0,t-t^2)\cdot(-1,1,0)\,dt + \int_0^1 (t-t^2,0,0)\cdot(0,-1,1)\,dt \\ &= \int_0^1 0\,dt + \int_0^1 0\,dt = 0\end{aligned}$$

12. Parametrização da curva, $\mathcal{C}$:
$(x,y,z) = \mathbf{r}(t) = (t,t^2,0),\ t\in[-1,2]$
Trabalho realizado por $\mathbf{F}$:

$$\begin{aligned}W &= \int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = \int_{\mathcal{C}} \mathbf{F}\big(\mathbf{r}(t)\big)\cdot\mathbf{r}'(t)\,dt = \int_{-1}^2 \mathbf{F}(t,t^2,0)\cdot(t,t^2,0)'\,dt \\ &= \int_{-1}^2 (t,t^2,0)\cdot(1,2t,0)\,dt = \int_{-1}^2 (t+2t^3)\,dt = \left[\frac{t^2}{2}+\frac{t^4}{2}\right]_{-1}^2 = 9\end{aligned}$$