ASE 381P.1 Linear Systems Analysis
Unique Number: 13360, Fall 2019

Final Project

Date Given: November 10, 2019
Due: **9.30 am, December 5, 2019**

This project has three parts. The first and second parts are mandatory for everyone. The third part has two choices - students are required to pick one from these two choices. You are expected to submit a typeset report (use your favorite journal style/format) that is nominally about 10-15 pages long (single column, 12pt font, single-spaced). A hard-copy submission is required.

The final report is due at 9.30am December 5, 2019. This is a hard deadline.

Part (A):

The purpose and scope of this task is a rigorous review and analysis of zero-input uniform exponential stability results for linear time-varying systems with regards to the pointwise-in-time eigenvalues of matrix $A(t)$. Specifically, recall from the Yamabe-Marcus example we discussed in class that having the eigenvalues of $A(t)$ restricted to within the open left-half of the complex plane insufficient for ensuring exponential stability. On the other hand, Ref. [1] below provides a different example (very counter-intuitive) for an exponentially stable system having its $A(t)$ matrix such that it has a pointwise-in-time eigenvalue with positive real part for all time $t \geq 0$, but is "slowly" time-varying.

A survey of results on uniform exponential stability under the hypothesis that pointwise eigenvalues of slowly varying $A(t)$ have negative real parts is provided in Ref. [2]. Additional results along these same lines can be seen in Ref. [3]. Finally, a relatively recent result in Ref. [4] provides stability conditions for slowly-varying linear state equations where eigenvalues can have positive real-parts, so long as they have negative real-parts "average."

Your project task is to perform a critical review and summary of all the aforementioned results (and any important references therein). Be sure to provide a critical review of the stability proofs and accompanying discussions.

Part (B):

Consider the *Mathieu equation*

$$\ddot{y}(t) = -\xi \dot{y}(t) + (a - 2q\cos(2t))y(t)$$

wherein a, $\xi > 0$, and q are real-valued constants. Note that when $\xi = 0$ (undamped Mathieu equation), this is a second-order periodic system, i.e., it can be written in the form: $\dot{x}(t) =$

$A(t)x(t)$ with $A(t) = A(t+T)$ for some finite period $T > 0$. The undamped Mathieu equation plays a central role in modeling many important physical models and accordingly, has a rich history in literature (seminal work by R.E. Langer, 1932 paper, attached). There is also a relatively recent result (in 1971) providing an improved solution estimate for this problem by S. Ramarajan and S. Rao (attached).

Your task is to characterize range of values for parameters a and q such that the undamped Mathieu equation admits bounded solutions. To assist with this task, please refer to the excellent lecture notes by Prof. Rand from Cornell University (attached). You will read Prof. Rand's lecture notes and summarize in your own words all the important boundedness results governing the Mathieu equation. Finally, you are required to address whether introducing damping (i.e., "small" enough values of $\xi > 0$) would still ensure bounded solutions for the Mathieu equation.

Part (C):

This part of the project has two choices C.1 and C.2 - you are required to select and perform one out of these two tasks.

Option C.1:

Consider the linearized dynamics of planar motion of a mass particle inside an inverse-square gravitational field:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega_0^2 & 0 & 0 & 2\omega_0 \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega_0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u(t)$$

Suppose, the sensor measures two of the states according to $y = [x_1, x_3]^T$.

Your tasks are as follows:

(a) For this LTI system, analytically derive the state-transition matrix. You may use a computer and/or symbolic manipulation software. Derive the impulse response and also the transfer function matrix for this system.

(b) Assuming a sampling time t_s, derive a discrete-time representation for the open-loop system. In particular, compute the corresponding A_d and B_d matrices for the discrete-time system. Establish the discrete-time state-transition matrix and use $t_s = 0.01$ for your calculations. Comment on the locations of the eigenvalues of A_d on the complex plane.

(c) Analyze the zero-input (US, UES) and zero-state (BIBO) stability properties of the open-loop system.

(d) Investigate the system controllability and observability properties. Using your results from part (a) above, compute the controllability gramian matrix.

(d) Starting at initial conditions $x(0) = [0, 0, \theta, 0]^T$, using results from our class, derive a controller that drives the state to $x(t_f) = [0, 0, (\omega_0 t_f - \theta), 0]^T$ during a finite time period t_f. Note here that θ is typically some small value (has the interpretation of a phase angle) in the range $\theta \in [-\pi/8, \pi/8]$. Using $\theta = \pi/10$, and $\omega_0 = 1/(2\pi)$, simulate your closed-loop system on the computer and document the results of your numerical studies for total energy consumption ($J = \int_0^{t_f} \|u(\sigma)\|_2^2 \, d\sigma$) as a function of the transfer time $t_f \in [0.1, 2]$. Do these results change if $\theta = -\pi/10$ (i.e., sign reversal)? If so, how?

(e) Design a full-order observer using the measured signals such that the eigenvalues of the observer are placed at $[-0.5,\ -2.5,\ -0.5 \pm j1]$.

(f) Implement a finite-time observer based on the results of Engel and Kreisselmeier [5] (attached). Use finite-time $D = 2$ seconds for this study.

Option C.2:

The simplified linearized longitudinal equations of a helicopter near *hover* condition can be modeled by the following normalized third-order LTI system

$$\begin{bmatrix} \dot{q}(t) \\ \dot{\theta}(t) \\ \dot{u}(t) \end{bmatrix} = \begin{bmatrix} -0.4 & 0 & \alpha \\ 1 & 0 & 0 \\ -1.4 & 9.8 & -0.02 \end{bmatrix} \begin{bmatrix} q(t) \\ \theta(t) \\ u(t) \end{bmatrix} + \begin{bmatrix} \beta \\ 0 \\ 9.8 \end{bmatrix} \delta(t)$$

where $q(t)$ is the pitch-rate, $\theta(t)$ is the fuselage pitch angle, and $u(t)$ is the forward speed. The rotor tilt angle $\delta(t)$ is the control input. The initial conditions are $q(0) = 0.02$, $\theta(0) = \pi/6$, and $u(0) = 3$.

Suppose, the single sensor measures a composite signal comprising horizontal velocity and the pitch angle, i.e., $y(t) = u(t) + \gamma\theta(t)$. The constant parameters α, β and γ nominally assume values of -0.01, 6.3, and 0.0 respectively.

Your tasks are as follows:

(a) Evaluate e^{At} and the impulse-response function for this system.

(b) Taking nominal values for all the parameters, determine the zero-input stability properties of the system, i.e., if US or UES holds. Also, determine if this system is uniformly BIBO stable.

(c) Investigate the system controllability and observability properties for the open-loop system. Determine if the parameters α, β, and γ can be perturbed such that controllability and/or observability are destroyed.

(d) Consider a full-state feedback controller of the form $\delta(t) = -k_1 q(t) - k_2\theta(t) - k_3 u(t) + pr(t)$ with $r(t)$ being any auxiliary (reference) command signal. Determine the (constant) control gains k_1, k_2, and k_3 such that the closed-loop system is UES with eigenvalues $[-2, -3 \pm j2]$. Select suitable value for the constant feed-forward gain parameter p such that the output $y(t)$ perfectly tracks step-reference commands, i.e., $r(t) = r_0$ for all $t \geq 0$. Simulate the closed-loop system resulting from your design on the computer for a time duration of 10 s and document the results of your numerical studies.

(e) Design an observer using the measured signal such that the eigenvalues of the observer are placed at $[-5, -2 \pm j3]$. Set the initial conditions for your estimated states to be zero. Use this observer along with the same k_1, k_2, k_3, and p values from the feedback controller in part (d) to simulate the output-feedback controller response for reference step- commands on a time-duration of 10s. Compare your results with the full-state feedback results of part (d). You may adopt the following cost metric for comparison purposes,

$$J(t) = \int_0^t [y(\sigma) - r_0]^2 d\sigma$$

(f) Implement a finite-time observer based on the recent results of Engel and Kreisselmeier [5] (attached). Use finite-time $D = 2$ seconds for this study.

References:

1. R.A. Skoog and G.Y. Lau. "Instability of Slowly-Varying Systems," *IEEE Transactions on Automatic Control*, Vol. 17, No. 1, pp. 86-92, 1972.

2. A. Ilchmann, D.H. Owens, and D. Pratzel-Wolters. "Sufficient Conditions for Stability of Linear Time-Varying Systems," *Systems & Control Letters*, Vol. 9, pp. 157-163, 1987.

3. C.A. Desoer. "Slowly-Varying $\dot{x} = A(t)x$," *IEEE Transactions on Automatic Control*, Vol. 14, pp. 780-781, 1969.

4. V. Solo. "On the Stability of Slowly-Varying Systems," *Mathematics of Control, Signals, and Systems*, Vol. 7, pp. 331-350, 1994.

5. R. Engel and G. Kreisselmeier. "A Continuous-time Observer which Converges in Finite-time," *IEEE Transactions on Automatic Control*, Vol. 47, No. 7, July 2002.

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Mathematics of Control, Signals, and Systems

On the Stability of Slowly Time-Varying Linear Systems*

Victor Solo†

Abstract. New conditions are given in both deterministic and stochastic settings for the stability of the system $\dot{x} = A(t)x$ when $A(t)$ is slowly varying. Roughly speaking, the eigenvalues of $A(t)$ are allowed to "wander" into the right half-plane as long as "on average" they are strictly in the left half-plane.

Key words. Time-varying systems, Linear systems, Stability, Stochastic stability.

1. Introduction

Recent work in adaptive control [CG], [GS], [K2], [MG], [MGHM], [N], [S1], [TI], [ZW] and robust control [SB] has drawn renewed attention to conditions ensuring (exponential) stability of the time-varying linear system

$$\dot{x}(t) = A(t)x(t), \qquad t \geq 0, \tag{1}$$

where $x(t)$ is a p-vector and $A(t)$ varies slowly. A number of results are now available. The earliest is due to [R2], while the first general result is due to Desoer [D]. His assumptions have become more or less standard; they are:

(I) $\|A(t)\| \leq A < \infty$ ($\|\cdot\|$ is the induced Euclidean norm).
(II) $\|\dot{A}(t)\| \leq \dot{A} < \infty$.
(III) All eigenvalues of $A(t)$ have real part $\leq -\sigma < 0$.

Desoer uses (I) and (III) to generate a bound of the form

$$\|e^{A(\tau)t}\| \leq Me^{-\alpha t}, \qquad \alpha > 0, \tag{2}$$

and then uses a Lyapunov argument to show (1) is exponentially stable if $\dot{A}$ of (II) is small enough. Perhaps the best results are due to Coppel [C] who also starts with a bound of the form (2) (the bound in [C] is the best known) and then uses a Bellman–Gronwall-based argument to prove exponential stability of (1) if

$$\dot{A} < \sigma^2/(4M \ln M)^2. \tag{3}$$

For given M and α in (2), the bound (3) seems to be the best available.

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Our aim is to relax (III) by allowing the eigenvalues to "wander" into the right half-plane while "on average" remaining in the left half-plane. Our proof technique takes the following route: we modify the technique of [D] to get a bound on $\|e^{A(\tau)t}\|$ and then modify the Bellman–Gronwall argument of [C] to get exponential stability. There are some further nontrivial subtle points that arise however.

If $A(t)$ is periodic, then Floquet theory [R1] provides necessary and sufficient conditions for stability. However, in fact the Floquet approach requires calculation of the transition matrix of (1), which is generally intractable [R1, p. 57] except in the second-order case. Even then only special cases have been solved [R1, p. 57]. Thus our results are of relevance even in the periodic case.

Turning to the stochastic case, there is little to report. We introduce a stationarity assumption:

(1′) $A(t)$ is strictly stationary ergodic; $E\|A(t)\| = A < \infty$.

If we denote the system transition matrix by $\Phi(t, 0)$, then the Furstenburg–Kesten Theorem [K3] tells us that, as $t \to \infty$,

$$t^{-1} \log\|\Phi(t, 0)\| \to \theta < \infty \quad \text{a.s.}$$

If we knew $\theta < 0$ a.s., then we would have a form of stochastic stability. The trouble is that there seems to be no known general way to calculate θ. Khasminski [K1] uses a transformation method on the problem but his resulting procedure does not seem to lead anywhere either: see [FLJC] for some recent material.

Aside from the recently developed discrete-time result in [S1] there do not appear to be any stochastic stability results for slowly time-varying linear systems. Below we develop a stochastic stability result similar to the deterministic one in which the eigenvalues of $A(t)$ need only be in the left half-plane "on average." After this work was completed the author became aware of some related work, [I] and [B2], and this is discussed later.

To forestall a confusion that often arises with regard to be stability of (1) it is worth pointing out where averaging methods fit in. We explicitly show the slow variation by rewriting (1) as

$$\dot{x}(t) = A(\varepsilon t)x(t),$$

where ε is small and $\dot{A}(t) = O(1)$. Now change the time scale to $\tau = \varepsilon t$, then we get

$$dy/d\tau = \varepsilon^{-1}A(\tau)y(\tau), \qquad y(\tau) = x(\tau\varepsilon^{-1}).$$

Averaging methods apply to a system

$$dy/d\tau = \mu A(\tau)y(\tau),$$

where μ cannot be arbitrarily large. Our results deal with the harder problem where μ can be arbitrarily large.

The remainder of this article is organized as follows. Section 2 contains an exponential bound used in both deterministic and stochastic cases. Section 3 covers deterministic stability, while Section 4 is devoted to the stochastic case. Some final remarks are given in Section 5.

2. An Exponential Bound

Our results are made possible by the following bound:

Lemma 1. *Consider the eigenvalue of $A(t)$ whose real part (call it $\alpha(t)$) is greatest. Then, for any $\varepsilon > 0$,*

$$\|e^{A(\tau)t}\| \le M'_\varepsilon e^{[\alpha(\tau)+\varepsilon\|A(\tau)\|]t}, \qquad \textit{for all} \quad \tau, t \ge 0, \tag{4}$$
$$M'_\varepsilon = 3(1 + 2/\varepsilon)^{p-1}/2.$$

Proof. We can assume $\|A(\tau)\| > 0$, since otherwise the result is trivially true. If $\lambda(\tau)$ is an eigenvalue of $A(\tau)$, then we have the elementary bound

$$|\lambda(\tau)| \le \|A(\tau)\|. \tag{5}$$

Now consider the following Cauchy integral:

$$e^{A(\tau)t} = \int_\Gamma (sI - A(\tau))^{-1} e^{st}\, ds/2\pi j, \tag{6}$$

where Γ is a contour consisting of an arc of the circle centered at the origin, of radius

$$R = \|A(\tau)\| + \varepsilon\|A(\tau)\|, \tag{7}$$

and a vertical chord whose intercept on the real axis is at the point

$$\alpha = \alpha(\tau) + \varepsilon\|A(\tau)\|. \tag{8}$$

Note that the distance from any point on this contour to any eigenvalue of $A(\tau)$ is $\ge \varepsilon\|A(\tau)\|$. Now we use the following elementary bound: if N is a $p \times p$ matrix of full rank, then

$$\|N^{-1}\| \le \|N\|^{p-1}/|\det N|.$$

Putting this in (6) gives, via (8),

$$\|e^{A(\tau)t}\| \le \int_\Gamma \|sI - A(\tau)\|^{p-1}|\det|sI - A(\tau)|^{-1}|\,|ds| e^{\alpha(\tau)+\varepsilon\|A(\tau)\|}/2\pi. \tag{9}$$

For s on Γ we have, from (7),

$$\|sI - A(\tau)\| \le \|A(\tau)\|(2 + \varepsilon),$$

so that (9) becomes

$$\|e^{A(\tau)t}\| \le e^{\alpha(\tau)+\varepsilon\|A(\tau)\|}[\|A(\tau)\|(2 + \varepsilon)]^{p-1}D, \tag{10}$$

$$D = \int_\Gamma |\det|sI - A(\tau)|^{-1}|\,|ds|/2\pi. \tag{11}$$

Next we have

$$|\det[sI - A(\tau)]| = \prod_1^p |s - \sigma_r|,$$

where σ_r, $1 \le r \le p$, are the eigenvalues of $A(\tau)$. On Γ, $|s - \sigma_r| \ge \varepsilon\|A(\tau)\|$ so that

$$|\det[sI - A(\tau)]| \ge (\varepsilon\|A(\tau)\|)^{p-2}|s - \sigma_1||s - \sigma_2|.$$

Thus, from (11),

$$D \le [\varepsilon\|A(\tau)\|]^{(p-2)}J, \tag{12}$$

where

$$J = \int_\Gamma |s - \sigma_1|^{-1}|s - \sigma_1|^{-2}|ds|/2\pi$$

(we could have said $D \le (\varepsilon\|A(\tau)\|)^{-p}$ but (16) below improves this bound). Next,

$$J \le (J_1 J_2)^{1/2}, \tag{13}$$

where

$$J_r = \int_\Gamma |s - \sigma_r|^{-2}|ds|/2\pi, \qquad r = 1, 2.$$

We now evaluate J_r in two steps. On the circular part C of Γ put $s = Re^{j\theta}$ so

$$\int_C |s - \sigma_r|^{-2}|ds|/2\pi \le \int_{-\pi}^{\pi} |Re^{j\theta} - \sigma_r|^{-2}R\, d\theta/2\pi$$
$$= R^{-1}\int_{-\pi}^{\pi} |e^{j\theta} - \sigma_r/R|^{-2}R\, d\theta/2\pi.$$

By Parseval's theorem

$$= R^{-1}\sum_0^\infty |\sigma_r/R|^2 = R^{-1}[1 - |\sigma_r/R|^2]^{-1}$$
$$\le R^{-1}[1 - |\sigma_r|/R]^{-1} = (R - |\sigma_r|)^{-1}$$
$$\le (\varepsilon\|A(\tau)\|)^{-1}. \tag{14}$$

On the vertical part V of Γ, $s = \alpha + j\omega$ (see (8)), so

$$\int_V |s - \sigma_r|^{-2}|ds|/2\pi = \int_{-\omega_0}^{\omega_0} |\alpha + j\omega - \sigma_r|^{-2}\, d\omega/2\pi,$$

where ω_0 is the vertical coordinate where C intersects V. If $\sigma_r = a_r + jb_r$, then this is

$$\int_{-\omega_0}^{\omega_0} [(\alpha - a_r)^2 + (\omega - b_r)^2]^{-1}\, d\omega/2\pi.$$

However, $|\alpha - a_r| \ge \varepsilon\|A(\tau)\|$, so this is

$$\le \int_{-\omega_0}^{\omega_0} [(\varepsilon\|A(\tau)\|)^2 + (\omega - b_r)^2]^{-1}\, d\omega/2\pi$$
$$= \int_{-(\omega_0 - b_r)}^{\omega_0 - b_r} [(\varepsilon\|A(\tau)\|)^2 + \rho^2]^{-1}\, d\rho/2\pi$$

$$\leq \int_{-\infty}^{\infty} [(\varepsilon\|A(\tau)\|)^2 + \rho^2]^{-1}\, d\rho/2\pi$$

$$= (\varepsilon\|A(\tau)\|)^{-1} \int_{-\infty}^{\infty} (1 + y^2)^{-1}\, dy/2\pi$$

$$= (\varepsilon\|A(\tau)\|)^{-1}/2. \tag{15}$$

Putting (14) and (15) together gives

$$J_r \leq 3(\varepsilon\|A(\tau)\|)^{-1}/2, \qquad r = 1, 2.$$

Thus from (12) and (13)

$$D \leq 3(\varepsilon\|A(\tau)\|)^{-(p-1)}/2. \tag{16}$$

Using this in (10) gives

$$\|e^{A(\tau)t}\| \leq 3e^{\alpha(\tau)+\varepsilon\|A(\tau)\|}\{[\|A(\tau)\|(2 + \varepsilon)]/[\varepsilon\|A(\tau)\|\}^{p-1}/2,$$

which gives the result. ■

3. Deterministic Stability

It adds little extra work to treat a perturbed form of (1) so we consider the system

$$\dot{x}(t) = [A(t) + P(t)]x(t), \tag{17}$$

where $P(t)$ is a small amplitude perturbation. We develop our main deterministic result (Theorem 2) in a series of theorems and lemmas. We introduce, then, the following assumptions:

(A1) $\overline{\lim}_{L\to\infty} L^{-1} \int_{t_0}^{t_0+L} \|A(s)\|\, ds \leq A < \infty$, for all t_0. Equivalently, $a < \infty$ exists such that

$$\int_{t_0}^{t_0+L} \|A(s)\|\, ds \leq a + LA < \infty, \qquad \text{for all } t_0.$$

(A2) For any given $T > 0$, there is γ, $0 < \gamma \leq 1$, with

$$\overline{\lim_{n\to\infty}}\, n^{-1} \sum_{n_0}^{n_0+n} \|A(t_2 + (t-1)T) - A(t_1 + (t-1)T)\| \leq T^\gamma \beta$$

for all t_1, t_2 with $|t_1 - t_2| \leq T$. Equivalently, $b > 0$ exists such that

$$\sum_{n_0}^{n_0+n} \|A(t_2 + (t-1)T) - A(t_1 + (t-1)T)\| \leq Tb + T^\gamma(n+1)\beta$$

whenever $|t_1 - t_2| \leq T$.

(A3) Let $\alpha(t)$ be the real part of the eigenvalue of $A(t)$ whose real part is greatest. For any given $T > 0$,

$$\overline{\lim_{N\to\infty}}\, N^{-1} \sum_{n_0}^{n_0+N} \alpha(s + nT) \leq \bar{\alpha} < 0, \qquad \text{for all } s, n_0.$$

Equivalently, $c > 0$ exists such that

$$\sum_{n_0}^{n_0+N} \alpha(s + nT) \le c + N\bar{\alpha} < 0, \qquad \text{for all } s, n_0.$$

(A4) $\overline{\lim}_{L->\infty} L^{-1} \int_{t_0}^{t_0+L} \|P(s)\| \, ds \le \delta < \infty$, for all t_0. Equivalently, $d > 0$ exists such that $\int_{t_0}^{t_0+L} \|P(s)\| \, ds \le d + L\delta$, for all t_0.

Note that, from condition (A3), $\alpha(\sigma)$ is bounded above by $c + \bar{\alpha}$ for any σ, but c is allowed to be arbitrarily large.

The above conditions are not straightforward to understand. To deal with this let us note immediately that if $A(t)$ is differentiable, then (A2) is implied by the more interpretable condition:

(A2*) $\overline{\lim}_{L->\infty} L^{-1} \int_{t_0}^{t_0+L} \|\dot{A}(s)\| \, ds \le \beta < \infty$, for all t_0.

This follows because, on taking $t_2 \ge t_1$, $|t_2 - t_1| \le T$,

$$\begin{aligned}
&\overline{\lim}\, n^{-1} \sum_1^n \|A(t_2 + (t-1)T) - A(t_1 + (t-1)T)\| \\
&\le \overline{\lim}\, n^{-1} \sum_1^n \int_{t_1+(t-1)T}^{t_2+(t-1)T} \|\dot{A}(s)\| \, ds \\
&\le \overline{\lim}\, n^{-1} \sum_1^n \int_{t_1+(t-1)T}^{t_1+tT} \|\dot{A}(s)\| \, ds, \qquad \text{since} \quad |t_2 - t_1| \le T, \\
&\le \overline{\lim}\, T(nT)^{-1} \sum_1^n \int_{t_1+(t-1)T}^{t_1+tT} \|\dot{A}(s)\| \, ds \\
&\le T\beta.
\end{aligned}$$

We do not however use (A2*) in what follows because we wish to avoid a differentiability assumption on $A(s)$.

Condition (A3) is discussed further below. In the meantime and before presenting the first stability result, it will assist discussion in several places to develop the following simple lemma.

Lemma 2. *Under* (A1) *and* (A2) *we have, for any given* $T > 0$,

$$\sum_{n_0}^{n_0+N} \|A(s + nT)\| \le b + a/T + N[A + T^\gamma \beta], \qquad N \ge 1,$$

$$\|A(s)\| \le b + T^\gamma \beta + a/T + A.$$

Proof. We can write, for $N \ge 1$,

$$\begin{aligned}
\sum_{n_0}^{n_0+N} A(s + nT) &= \sum_{n_0}^{n_0+N-1} T^{-1} \int_{nT}^{(n+1)T} A(s + nT) \, du \\
&= \omega_1 + \omega_2, \\
\omega_1 &= \sum_{n_0}^{n_0+N-1} \left[T^{-1} \int_{nT}^{(n+1)T} A(s + u) \, du \right],
\end{aligned}$$

$$\omega_2 = \sum_{n_0}^{n_0+N-1} \left[T^{-1} \int_{nT}^{(n+1)T} (A(s+nT) - A(s+u))\, du \right]$$
$$= T^{-1} \int_0^T \sum_{n_0}^{n_0+N-1} (A(s+nT) - A(s+\sigma+nT))\, d\sigma.$$

Now apply (A2) to see

$$|\omega_2| \leq T^{-1} \int_0^T \sum_{n_0}^{n_0+N-1} \|A(s+nT) - A(s+\sigma+nT)\|\, d\sigma$$
$$\leq T^{-1} \int_0^T (b + T^\delta N\beta)\, d\sigma$$
$$= b + T^\gamma N\beta. \tag{18}$$

On the other hand, from (A1)

$$|w_1| \leq \|T^{-1} \int_{n_0T+s}^{n_0T+s+NT} A(\sigma)\, d\sigma\|$$
$$\leq T^{-1}a + NA. \tag{19}$$

Putting (18) and (19) together gives the result for $N \geq 1$. By writing

$$A(s) = T^{-1} \int_0^T (A(s) - A(u+s))\, du + T^{-1} \int_0^T A(u+s)\, du$$

we find easily from (A1) and (A2) that

$$\|A(s)\| \leq b + T^\gamma\beta + a/T + A,$$

so the result is established. ■

We are now ready for the first stability result.

Theorem 1. *Under* (A1)–(A4), (17) *is exponentially stable provided we choose* $\varepsilon > 0$ *so small that*

$$\bar{\alpha} + \varepsilon < 0, \tag{20}$$

then have δ *so small that*

$$\bar{\alpha} + \varepsilon + M_\varepsilon\delta < 0, \tag{21}$$

where $M_\varepsilon = 3(2A_1/\varepsilon + 1)^{p-1}/2$, $A_1 = A + b$. *Then have* β *so small that*

$$\bar{\alpha} + \varepsilon + M_\varepsilon\delta + 2(\ln M_\varepsilon)^{\gamma/(\gamma+1)}[\beta(M_\varepsilon + \varepsilon/A_1)]^{1/(\gamma+1)} < 0. \tag{22}$$

Proof. Given any s we can write

$$\dot{x}(\sigma) = A(s)x(\sigma) + [A(\sigma) - A(s) + P(\sigma)]x(\sigma),$$

then solving this for $s \leq \sigma \leq t$ gives

$$x(t) = e^{A(s)(t-s)}x(s) + \int_s^t e^{A(s)(t-s)}[A(\sigma) - A(s) + P(\sigma)]x(\sigma)\, d\sigma.$$

In Lemma 1 replace ε by ε/A_1 and use the result to find

$$\begin{aligned}\|x(t)\| &= M_\varepsilon \exp\{[\alpha(s) + \varepsilon\|A(s)\|/A_1](t-s)\}\,\|x(s)\| \\ &\quad + M_\varepsilon \int_s^t \exp\{[\alpha(s) + \varepsilon\|A(s)\|/A_1](t-s)\} \\ &\quad \times \{\|A(\sigma) - A(s)\| + \|P(\sigma)\|\}\,\|x(\sigma)\|\, d\sigma,\end{aligned}$$

where M_ε is given in the theorem statement. From Gronwall's inequality we now get

$$\begin{aligned}\|x(t)\| \le \|x(s)\| M_\varepsilon \exp\Bigg\{&[\alpha(s) + \varepsilon\|A(s)\|/A_1](t-s) \\ &+ M_\varepsilon \int_s^t \|A(\sigma) - A(s)\|\, d\sigma + M_\varepsilon \int_s^t \|P(\sigma)\|\, d\sigma\Bigg\},\end{aligned}$$

which we can write as

$$\begin{aligned}\|x(t)\| \le \|x(s)\| M_\varepsilon \exp\Bigg\{&\rho(s)(t-s) + M_\varepsilon \int_0^{t-s} \|A(s+u) - A(s)\|\, du \\ &+ M_\varepsilon \int_s^t \|P(\sigma)\|\, d\sigma\Bigg\}, \qquad (23)\end{aligned}$$

$$\rho(s) = \alpha(s) + \varepsilon\|A(s)\|/A_1. \qquad (24)$$

Let $T > 0$, to be large, be chosen. First consider the case $|t - s| \le T$. Then using the elementary bound $|\alpha(s)| \le \|A(s)\|$ and applying (A2), (A4), and Lemma (2) we get

$$\begin{aligned}|\rho(s)| &= |(\alpha(s) + \varepsilon\|A(s)\|/A_1)| \\ &\le [1 + \varepsilon/A_1](b + T^\gamma\beta + a/T + A) M_\varepsilon \int_0^{t-s} \|A(u+s) - A(s)\|\, du \\ &\le M_\varepsilon T(b + T^\gamma \beta)\end{aligned}$$

and

$$M_\varepsilon \int_s^t \|P(\sigma)\|\, d\sigma \le M_\varepsilon(d + T\delta).$$

Thus, from (23),

$$\|x(t)\| \le b'_T \|x(s)\|, \qquad s \le t \le s + T,$$

$$b'_T = M_\varepsilon \exp\{(1 + \varepsilon/A_1)(b + T^\gamma\beta + a/T + A)T + M_\varepsilon[Tb + T^{\gamma+1}\beta + d + T\delta]\},$$

so we can write

$$\|x(t)\| \le b_T e^{-\theta(t-s)} \|x(s)\|, \qquad s \le t \le s + T, \qquad (25)$$

where $\theta > 0$ is to be chosen and

$$b_T = b'_T e^{\theta T}.$$

Now consider the case $|t-s| > T$ so that, for some integer n,

$$s + nT \le t \le s + (n+1)T$$

and denote

$$x_n(s) = x(s + nT).$$

Then, from (24),

$$\|x(t)\| \le b_T \|x_n(s)\| e^{-\theta(t-s-nT)}. \tag{26}$$

We return to this shortly. We next must bound $\|x_n(s)\|$. Using (23) again we find

$$\|x_n(s)\| \le \|x_{n-1}(s)\| M_\varepsilon \exp\Big[T\rho(s + (n-1)T) + M_\varepsilon \int_0^T \|A(s + (n-1)T + \sigma) - A(s + (n-1)T)\| \, d\sigma + M_\varepsilon \int_{s+(n-1)T}^{s+nT} \|P(\sigma)\| \, d\sigma \Big].$$

Iterate this to find

$$\|x_n(s)\| \le M_\varepsilon^n \|x(s)\| \exp\Big[T \sum_1^n \rho(s + (t-1)T) + M_\varepsilon \int_0^T \sum_1^n \|A(s + (t-1)T + \sigma) - A(s + (t-1)T)\| \, d\sigma + M_\varepsilon \int_s^{s+nT} \|P(\sigma)\| \, d\sigma \Big]. \tag{27}$$

Now apply (A2)–(A4) and Lemma 2 to find

$$T \sum_1^n \rho(s + (t-1)T) \le T(c + n\bar{\alpha}) + \varepsilon T A_1^{-1}(b + a/T + n(A + T^\gamma \beta)),$$

$$M_\varepsilon \int_0^T \sum_1^n \|A(s + (t-1)T + \sigma) - A(s + (t-1)T)\| \, d\sigma \le T M_\varepsilon (b + T^\gamma n\beta),$$

$$M_\varepsilon \int_{sT}^{s+nT} \|P(\sigma)\| \, d\sigma \le M_\varepsilon (d + nT\delta).$$

Then (27) becomes, for some constant $G_{T\varepsilon}$ (recall $A_1 = A + b$),

$$\|x_n(s)\| \le G_{T\varepsilon} M_\varepsilon^n \|x(s)\| \exp\{nT(\bar{\alpha} + \varepsilon + M_\varepsilon \delta)\} \exp\{nT^{\gamma+1}\beta(M_\varepsilon + \varepsilon/A_1)\}.$$

Now choose T so that

$$M_\varepsilon = \exp\{nT^{\gamma+1}\beta(M_\varepsilon + \varepsilon/A_1)\},$$

which is equivalent to

$$T = T_\beta = [\ln M_\varepsilon / [\beta(M_\varepsilon + \varepsilon/A_1)]]^{1/(\gamma+1)}, \tag{28}$$

so that now

$$\|x_n(s)\| \leq G_{T\varepsilon}\|x(s)\|(M_\varepsilon^{2/T}\exp\{\bar{\alpha}+\varepsilon+M_\varepsilon\delta\})^{nT}.$$

Now choose β so small (i.e., T_β so large) that

$$e^{-\theta} = M_\varepsilon^{2/T}\exp\{\bar{\alpha}+\varepsilon+M_\varepsilon\delta\} < 1. \tag{29}$$

Then

$$\|x_n(s)\| \leq G_{T\varepsilon}\|x(s)\|e^{-\theta nT}. \tag{30}$$

Using this in (26) gives

$$\|x(t)\| \leq b_T G_{T\varepsilon}\|x(s)\|e^{-\theta(t-s)}, \qquad |t-s| > T.$$

Coupling (26) and (30) gives

$$\|x(t)\| \leq B_{T\varepsilon}\|x(s)\|e^{-\theta(t-s)}, \qquad t \geq s,$$

$$B_{T\varepsilon} = G_{T\varepsilon}b_T,$$

and this is the desired result. Putting (28) into (29) gives (22). ■

Note that (22) can be rewritten as

$$\beta < |\bar{\alpha}+\varepsilon+M_\varepsilon\delta|^{\gamma+1}/[2^{\gamma+1}(\ln M_\varepsilon)^\gamma(M_\varepsilon+\varepsilon/A_1)]. \tag{31}$$

The reader will no doubt be concerned about condition (A3). Conditions (A1), (A2), and (A4) are straightforward to understand but (A3), while it comes naturally out of the proof technique, looks as though it may be hard to check. In view of Lemma 2 one is naturally led to try to replace (A3) by

(A3′) $\overline{\lim}_{L\to\infty} L^{-1}\int_{t_0}^{t_0+L}\alpha(u)\,du \leq \bar{\alpha}' < 0$

and then deduce (A3) by showing $|\dot{\alpha}(t)| = O(\beta)$ and repeating the argument of Lemma 2. Alas, it is not true in general that $|\dot{\alpha}|$ is uniformly bounded in t let alone $O(\beta)$. This is easily seen with a second-order example.

Consider the system (1) with $p = 2$ and

$$A(t) = \begin{pmatrix} -2a_1(t) & -a_2(t) \\ 1 & 0 \end{pmatrix},$$

then the characteristic polynomial is

$$s^2 + 2a_1(t)s + a_2(t) = 0,$$

which has roots (eigenvalues)

$$\lambda_1, \lambda_2 = -a_1(t) \pm \sqrt{\Delta(t)},$$

$$\Delta = a_1^2(t) - a_2(t).$$

Now suppose that Δ has a local minimum at some point t_1 and that $\Delta(t_1) = 0$. So, in the neighborhood of t_1, $\Delta(t) > 0$ and so

$$\alpha(t) = -a_1(t) + \sqrt{\Delta(t)}, \text{ near } t_1 \quad \Rightarrow \quad \dot{\alpha}(t) = -\dot{a}_1(t) + \dot{\Delta}(t)/2\sqrt{\Delta(t)}$$

$$\Rightarrow \quad \dot{\alpha}(t) \to \infty \qquad \text{as} \quad t \to t_1.$$

Having a local minimum at zero is hardly a pathological case, so we see that bounding $|\dot{\alpha}(t)|$ is a hopeless cause. In this second-order case it is easy to see then that a perturbation h in a coefficient leads to the generic result: perturbation in $\alpha(t) = O(h^{1/2})$. In the pth-order case it is conjectured that the perturbation is $O(h^{1/p})$.

Nevertheless, we can accomplish the replacement of (A3) by (A3′) by using a weaker condition than the boundedness of the derivatives; namely uniform continuity. To get uniform continuity it seems to be necessary to strengthen (A2) to a uniform Lipschitz condition:

(A2′) $\|A(t+h) - A(t)\| \leq \beta h^\gamma$ for all t and some γ, $0 < \gamma \leq 1$.

This allows $A(t)$ to be piecewise continuous rather than just allowing differentiability. We see immediately that it implies (A2). However, if $\gamma = 1$, it is stronger than (A2*). In any case Lemma 2 still holds and we can now generate our major deterministic result.

Theorem 2. *Under* (A1), (A2′), (A3′), *and* (A4), (17) *is exponentially stable provided we choose* $\varepsilon > 0$ *so small that*

$$\bar{\alpha}' + 2\varepsilon < 0, \tag{32}$$

then have δ *so small that* (*with* $M_\varepsilon = \frac{3}{2}(2A/\varepsilon + 1)$)

$$\bar{\alpha}' + 2\varepsilon + M_\varepsilon \delta < 0, \tag{33}$$

then have β *so small that*

$$\bar{\alpha}' + 2\varepsilon + M_\varepsilon \delta + 2(\ln M_\varepsilon)^{\gamma/(\gamma+1)}[\beta(M_\varepsilon + \varepsilon/A)]^{1/(\gamma+1)} < 0 \tag{34}$$

and also so small that

$$\sigma(\beta^{1/(\gamma+1)}(\ln M_\varepsilon/(M_\varepsilon + \varepsilon/A))^{1/(\gamma+1)}) < \varepsilon, \tag{35}$$

where $\sigma(\cdot)$ *is a certain nonincreasing function defined in the proof below.*

Proof. Note that in (A2) $b = 0$, hence the new value for M_ε. First we show $\alpha(t)$ is uniformly continuous. From (A1) and (A2′), Lemma 2 holds (with $b = 0$) so we have

$$\|A(s)\| \leq k\beta + a/k + A, \qquad \text{for all} \quad k > 0, \quad \text{any} \quad s \leq 0.$$

Set $k = \sqrt{a/\beta}$ to find

$$\|A(s)\| \leq 2\sqrt{a\beta} + A, \qquad \text{all} \quad s \geq 0.$$

Let us now insist that β be small enough that

$$\beta < (|\bar{\alpha}'|/2)^{\gamma+1}. \tag{36}$$

Since, in (34), $M_\varepsilon > 1$, we see (36) is already implied by (34). Then we have

$$\|A(s)\| \leq 2\sqrt{a}(|\alpha^{-1}|/2)^{\gamma+1} + A, \qquad \text{all} \quad s \geq 0. \tag{37}$$

However, now $\alpha(\mathbf{A})$ is a continuous function of $\mathbf{A}$ [S2, p. 328] on a bounded set (because of (37)) and so is uniformly continuous [A1, p. 329] in $\mathbf{A}$. Thus given $\delta_0 > 0$ there is $\eta(\delta_0)$ such that

$$\|\mathbf{A}_1 - \mathbf{A}_2\| < \eta(\delta_0) \quad \Rightarrow \quad |\alpha(\mathbf{A}_1) - \alpha(\mathbf{A}_2)| < \delta_0.$$

Now let t_1, t_2 be such that $|t_1 - t_2|^\gamma < \eta(\delta_0)/\beta$, then by (A2′)

$$\|\mathbf{A}(t_1) - \mathbf{A}(t_2)\| \le \beta |t_1 - t_2|^\gamma \le \eta(\delta_0).$$

So we conclude $\alpha(t)$ is uniformly continuous in t.

Inverting the uniform continuity property we can say there is a nonincreasing function $\sigma(\eta)$ with $\sigma(\eta) \to 0$ as $\eta \to 0$, such that

$$\|\mathbf{A}_1 - \mathbf{A}_2\| < \eta \quad \Rightarrow \quad |\alpha(\mathbf{A}_1) - \alpha(\mathbf{A}_2)| < \sigma(\eta).$$

Note that $\sigma(\eta)$ is not time dependent. Next if t_1, t_2 are such that $|t_1 - t_2| < h$, then by (A2′)

$$\|\mathbf{A}(t_1) - \mathbf{A}(t_2)\| \le h^\gamma \beta \quad \Rightarrow \quad |\alpha(t_1) - \alpha(t_2)| \le \sigma(h^\gamma \beta). \tag{38}$$

Now we show (A3′) ⇒ (A3). We proceed as in the proof of Lemma 2. We can write

$$N^{-1} \sum_{n_0}^{n_0+N} \alpha(s + nT) = \alpha_N + e_N,$$

$$\alpha_N = N^{-1} \sum_{n_0}^{n_0+N-1} T^{-1} \int_{nT}^{(n+1)T} \alpha(s + u)\, du$$

$$= (NT)^{-1} \int_{s+n_0T}^{s+n_0T+NT} \alpha(t)\, dt,$$

$$e_N = N^{-1} \sum_{n_0}^{n_0+N-1} T^{-1} \int_{nT}^{(n+1)T} [\alpha(s + nT) - \alpha(s + u)]\, du.$$

Then

$$|e_N| \le (NT)^{-1} \sum_{n_0}^{n_0+N-1} \int_0^T |\alpha(s + nT) - \alpha(s + nT + t)|\, dt.$$

In view of the uniform continuity of $\alpha(t)$ we have form (38) that

$$|\alpha(s + nT) - \alpha(s + nT + \sigma)| \le \sigma(\beta T^\gamma), \qquad 0 \le \sigma \le T,$$

$$\Rightarrow \quad |e_N| \le \sigma(\beta T^\gamma).$$

Thus via (D3′) we find

$$\overline{\lim_{N\to\infty}}\, N^{-1} \sum_{n_0}^{n_0+N} \alpha(s + nT) \le \bar{\alpha}' + \sigma(\beta T^\gamma).$$

Now choose $T = T_\beta$ of (28), then

$$\beta T_\beta^\gamma = \left[\frac{\ln M_\varepsilon}{M_\varepsilon + \varepsilon/A_1}\right]^{\gamma/(\gamma+1)} \beta^{1/(\gamma+1)}.$$

Now take β so small that

$$\sigma(\beta T_\beta^\gamma) < \varepsilon. \tag{39}$$

Then (A3) is established with $\bar{\alpha} = \bar{\alpha}' + \varepsilon$. Now we can apply Theorem 1 and the result is established. ∎

The above conjecture is that for the pth-order system (17), $\sigma(x) = \text{const. } x^{1/p}$.

If $A(t)$ is almost periodic we might expect that the upper limits in (A1) and (A3′) could be replaced by limits. This leads to:

Theorem 3. *Consider system* (17) *with assumption* (A2′). *Suppose* $A(t)$, $P(t)$ *are almost periodic, then the two limits below exist*:

$$\lim_{L->\infty} L^{-1} \int_{t_0}^{t_0+L} \|A(s)\| \, ds = A, \qquad \textit{for all } t_0, \tag{40}$$

$$\lim_{L->\infty} L^{-1} \int_{t_0}^{t_0+L} \|P(s)\| \, ds = \delta, \qquad \textit{for all } t_0. \tag{41}$$

We suppose $A < \infty$, $\delta < \infty$. *Further*, $\alpha(t)$ *is almost periodic so that the following limit exists*:

$$\lim_{L->\infty} L^{-1} \int_{t_0}^{t_0+L} \|\alpha(s)\| \, ds = \bar{\alpha}', \qquad \textit{for all } t_0. \tag{42}$$

Then provided (32)–(35) *hold*, (17) *is exponentially stable.*

Proof. A standard result on almost periodic functions [W, p. 189] ensures (40) and (41) exist. Next Lemma 2 ensures $\|A(t)\|$ is uniformly bounded and since $\alpha(A)$ is then a continuous function of A on a bounded set, a standard result [W, p. 189] ensures $\alpha(t)$ is almost periodic. Thus (42) exists. Thus (A1), (A2′), (A3′), and (A4) all hold and the result follows from Theorem 2. ∎

4. Stochastic Stability

Let us consider the forced system

$$\dot{x}(t) = G(t)x(t) + v(t). \tag{43}$$

In the deterministic case, if $v(t)$ is uniformly bounded, then $x(t)$ is uniformly bounded if the homogeneous system

$$\dot{x}(t) = G(t)x(t) \tag{44}$$

is exponentially stable. In the stochastic case the situation is not as simple as this.

Let $\Phi(t, s)$ be the transition matrix of (15) and introduce the following assumptions:

(s1) $G(t)$ is strictly stationary ergodic, $-\infty \le t \le \infty$. $E\|G(t)\| = G < \infty$.
(s2) $E\|v(t)\| \le c < \infty$, for all t.
(s3) $\overline{\lim}_{L->\infty} t^{-1} \log\|\Phi(t, 0)\| \le -\gamma < 0$ a.s.

As indicated in the Introduction, (s1) ensures that the upper limit in (s3) exists as a limit. So we are really just assuming $-\gamma < 0$. However, we have not indicated this in the assumption because we want to emphasize that our results in no way depend on the Furstenberg–Kesten theorem. In any case we now have:

Lemma 3. *For* (43) *with assumptions* (s1)–(s3), $x(t)$ *is stable in probability, i.e.,*

$$\lim_{B->\infty} \sup_{0\leq t\leq\infty} P(\|x(t)\| > B) = 0. \tag{45}$$

Proof. Solving (43) gives

$$x(t) = \Phi(t, 0)x(0) + \int_0^t \Phi(t, u)v(u)\, du. \tag{46}$$

Now, by (s3), $\|\Phi(t, 0)x(0)\| \to 0$ a.s., as $t \to \infty$, so we need only consider the second term in (46). Put $\theta = \gamma/2$ and consider that

$$\begin{aligned} T_{2t} = |\text{second term}| &\leq \int_0^t \|\Phi(t, u)\|\, \|v(u)\|\, du \\ &\leq \int_0^t \|\Phi(t, u)\| e^{(t-u)\theta} e^{-(t-u)\theta} \|v(u)\|\, du \\ &\leq \sup_{0\leq u\leq t} \|\Phi(t, u)\| e^{(t-u)\theta} \int_0^t e^{-(t-u)\theta}\|v(u)\|\, du \\ &= T_{at} T_{bt}, \quad \text{say.} \end{aligned}$$

However, by (s2), for $B > 0$,

$$P(T_{bt} > B) < c \int_0^t e^{-(t-u)\theta}\, du/B < c/\theta B,$$

so we need only treat T_{at} (because $P(T_{2t} > B) \leq P(T_{at} > B^{1/2}) + P(T_{bt} > B^{1/2})$). Let us first ensure that $T_{bt} < \infty$ a.s., for each t. For $t \geq u$, the transition matrix obeys

$$d\Phi(t, u)/dt = G(t)\Phi(t, u), \qquad \Phi(u, u) = I.$$

Integrating this gives

$$\Phi(t, u) = I + \int_u^t G(s)\Phi(s, u)\, ds \quad \Rightarrow \quad \|\Phi(t, u)\| \leq 1 + \int_u^t \|G(s)\|\, \|\Phi(s, u)\|\, ds.$$

Now apply the Bellman–Gronwall lemma

$$\|\Phi(t, u)\| \leq \exp\left\{\int_u^t \|G(s)\|\, ds\right\}.$$

Thus

$$T_{at} = \sup_{0\leq u\leq t} \|\Phi(t, u)\| e^{(t-u)\theta} \leq \exp\left\{\int_0^t \|G(s)\|\, ds + t\theta\right\}. \tag{47}$$

Thus, for any $B > 0$,

$$
\begin{aligned}
P(T_{at} > B) &= P(\log T_{at} > \log B) \\
&< E(\log T_{at})/\log B \\
&< (tA + t\theta)/\log B, \qquad \text{by (s1)},
\end{aligned}
$$

so the finiteness (a.s.) of T_{at} is established for each t. Next use the strict stationarity to see

$$
\begin{aligned}
P(T_{at} > B) &= P\left(\sup_{0 \le u \le t} \|\Phi(0, u - t)\| e^{(t-u)\theta} > B \right) \\
&= P\left(\sup_{0 \le u \le t} \|\Phi(0, \sigma)\| e^{\sigma\theta} > B \right).
\end{aligned}
$$

By (s3) a random time $t_0 < \infty$ (a.s.) and a random variable C exist such that, for all $t \ge t_0$,

$$
\|\Phi(0, -\sigma)\| e^{\sigma\theta} \le C e^{-2\theta\sigma + \theta\sigma} = C e^{-\theta\sigma} \le C.
$$

On the other hand, from (47)

$$
\sup_{0 \le \sigma \le t_0} \|\Phi(0, -\sigma)\| e^{\sigma\theta} \le \exp\left\{ \int_0^t \|G(s)\| \, ds + t_0 |\theta| \right\} = e^D.
$$

However, if $b > 0$,

$$
\begin{aligned}
P(D > b) &= P\left(\int_0^{t_0} (\theta + \|G(s)\|) \, ds > b \right) \\
&\le P\left(\int_0^{t_0} (\theta + \|G(s)\|) \, ds > b, t_0 \le b^{1/2} \right) + P(t_0 \ge b^{1/2}) \\
&\le P\left(\int_0^{b^{1/2}} (\theta + \|G(s)\|) \, ds > b \right) + P(t_0 \ge b^{1/2}) \\
&\le b^{1/2}(\theta + G)/b + P(t_0 \ge b^{1/2}) \qquad \text{by (s1)} \\
&\to 0, \qquad \text{as} \quad b \to \infty.
\end{aligned}
$$

Thus there is a random variable $W (= C + e^D) < \infty$ a.s. with (when $B > 1$)

$$
p_t = P\left(\sup_{0 \le \sigma \le t} \|\Phi(0, -\sigma)\| e^{\sigma\theta} > B \right) < P(W > B),
$$

so that

$$
\lim_{B \to \infty} \sup_{0 \le t < \infty} p_t \le \lim_{B \to \infty} P(W > B) = 0,
$$

and so (45) is established. ■

Stability in probability is a minimal stochastic stability or boundedness property. It ensures no mass escapes to infinity. It is elsewhere called tightness [B1], and it ensures the existence of an invariant measure [B1, p. 290].

Of course, satisfying as Lemma 3 is, it does not get us very far unless we can produce a means of checking (s3). This we now do. As before we allow a small amplitude perturbation and write

$$G(t) = A(t) + P(t) \tag{48}$$

and introduce the following assumption with regard to the time-varying system (17):

(S1) $A(t)$ is strictly stationary ergodic, $E\|A(t)\| = A < \infty$.

(S2) $E\|A(t+h) - A(t)\| \leq \beta |h|^{\gamma}$, $0 < \gamma \leq 1$, h small enough.

(S3) Let $\alpha(t)$ be the real part of the eigenvalue of $A(t)$ whose real part is greatest. Then $\alpha(t)$ is strictly stationary ergodic and by the elementary bound $|\alpha(t)| \leq \|A(t)\|$ we deduce from (S1) that $E|\alpha(t)| < \infty$. We then suppose

$$E[\alpha(t)] = \bar{\alpha} < 0.$$

(S4) $P(t)$ is strictly stationary ergodic, $E\|P(t)\| = \delta < \infty$.

Condition (S2) allows nondifferentiable stationary processes whose spectrum goes as ω^{-1-2r}, $0 < r < 1$, as frequency $\omega \to \infty$.

With these assumptions we can establish (s3).

Theorem 4. *Consider the system* (44), (48) *with assumptions* (S1)–(S4). *Choose* $\varepsilon > 0$ *so that* (20) *holds and require* δ, β *to be small such that* (21) *and* (22) *hold* (*but with* $M_\varepsilon = 3(2A(\varepsilon + 1))^{p-1}/2$), *then* (s3) *holds.*

Proof. Lemma 1 holds in the current setting. In that lemma replace ε by ε/A and repeat the calculations in the proof of Theorem 1 leading to (23) where now, in place of (24),

$$\rho(s) = \alpha(s) + \varepsilon\|A(s)\|/A. \tag{49}$$

These calculations are unchanged if we replace $x(t)$ by $\Phi(t, 0)$, so that in place of (23) we have, for $s \leq t \leq s + T$,

$$\|\Phi(t,0)\| \leq \|\Phi(s,0)\| M_\varepsilon \exp\left\{\rho(s)(t-s) + M_\varepsilon \int_s^t \|P(\sigma)\|\, d\sigma \right.$$
$$\left. + M_\varepsilon \int_0^{t-s} \|A(s+u) - A(s)\|\, du\right\}. \tag{50}$$

Now consider $t > T$ and suppose, for some integer n, $nT \leq t \leq (n+1)T$ and denote

$$\Phi_n = \Phi(nT, 0).$$

Then from (50)

$$\|\Phi(t,0)\| \leq \|\Phi_n\| M_\varepsilon \exp\left\{\rho(nT)(t - nT) + M_\varepsilon \int_{nT}^t \|P(\sigma)\|\, d\sigma \right.$$
$$\left. + M_\varepsilon \int_0^T \|A(nT+u) - A(nT)\|\, du\right\}$$

$$\leq \|\Phi_n\| M_\varepsilon \exp\left\{T|\rho(nT)|(t - nT) + M_\varepsilon \int_{nT}^{(n+1)T} \|P(\sigma)\| \, d\sigma \right.$$
$$\left. + M_\varepsilon \int_0^T \|A(nT + u) - A(nT)\| \, du\right\}.$$

Now apply the elementary bound, $|\alpha(\sigma)| \leq \|A(\sigma)\|$ to get from (49) that

$$\|\Phi(t, 0)\| \leq \|\Phi_n\| M_\varepsilon \exp\left\{T(1 + \varepsilon/A)\|A(nT)\| + M_\varepsilon \int_{nT}^{(n+1)T} \|P(\sigma)\| \, d\sigma \right.$$
$$\left. + M_\varepsilon \int_0^T \|A(nT + u) - A(nT)\| \, du\right\}. \tag{51}$$

We return to this expression below. Now we bound $\|\Phi_n(s)\|$. Using (50) again we get

$$\|\Phi_n\| \leq \|\Phi_{n-1}\| M_\varepsilon \exp\left\{T\rho((n - 1)T) + M_\varepsilon \int_{(n-1)T}^{n+T} \|P(\sigma)\| \, d\sigma \right.$$
$$\left. + M_\varepsilon \int_0^T \|A((n - 1)T + u) - A((n - 1)T)\| \, du\right\}.$$

Iterate this to find (since $\Phi(0, 0) = I$)

$$\|\Phi_n\| \leq M_\varepsilon^n \exp\left\{T \sum_1^n \rho((t - 1)T) + M_\varepsilon \int_0^{nT} \|P(\sigma)\| \, d\sigma \right.$$
$$\left. + M_\varepsilon \sum_1^n \int_0^T \|A((t - 1)T + u) - A((t - 1)T)\| \, du\right\}. \tag{52}$$

Coupling (52) with (51) we find

$$\|\Phi(t, 0)\|^{1/t} = b_n^{nT/t}[W]^{nT/t}, \tag{53}$$

$$b_n = M_\varepsilon^{1/nT} \exp\left\{n^{-1}\|A(nT)\|(1 + \varepsilon/A) + M_\varepsilon \int_{nT}^{(n+1)T} \|P(\sigma)\| \, d\sigma/nT \right.$$
$$\left. + (nT)^{-1} M_\varepsilon \int_0^T \|A(nT + u) - A(nT)\| \, du\right\},$$

$$W = M_\varepsilon^{1/nT} \exp\left\{n^{-1} \sum_1^n \rho((t - 1)T) + M_\varepsilon \int_0^{nT} \|P(\sigma)\| \, d\sigma/nT \right.$$
$$\left. + (nT)^{-1} M_\varepsilon \sum_1^n \int_0^T \|A((t - 1)T + u) - A((t - 1)T)\| \, du\right\}. \tag{54}$$

We now let $t \to \infty$ so that $n \to \infty$ while $nT/t \to 1$. Below we show

$$b_n \to 1 \quad \text{a.s.} \tag{55}$$

so that $b_n^{nT/t} \to 1$ a.s. In (54) apply the ergodic theorem to see that (via (49))

$$n^{-1} \sum_1^n \rho((t-1)T) \to \bar{\alpha} + \varepsilon \quad \text{a.s.},$$

$$(nT)^{-1} M_\varepsilon \sum_1^n \int_0^T \|A((t-1)T + u) - A((t-1)T)\| \, du$$

$$\to M_\varepsilon T^{-1} E\left(\int_0^T \|A(u) - A(0)\| \, du \right) \quad \text{a.s.}, \tag{56}$$

$$M_\varepsilon (nT)^{-1} \int_0^{nT} \|P(\sigma)\| \, d\sigma \to M_\varepsilon \delta \quad \text{a.s.}$$

Note that the summand in (56) is strictly stationary ergodic and has finite expectation because of (S2). In any case from (53)–(55)

$$\overline{\lim_{t \to \infty}} \, \|\Phi(t, 0)\|^{1/t} \leq M_\varepsilon^{1/T} \exp\{\bar{\alpha} + \varepsilon + M_\varepsilon \delta\} \exp\{M_\varepsilon T^\gamma \beta\} \quad \text{a.s.}, \tag{57}$$

where we have used (S2) in (56). Now choose T so that

$$M_\varepsilon^{1/T} = \exp\{T^\gamma M_\varepsilon \beta\},$$

which is equivalent to

$$T = T_\beta = [\ln M_\varepsilon / (\beta M_\varepsilon)]^{1/(\gamma+1)}.$$

Then (57) becomes

$$\begin{aligned} \overline{\lim} \, t^{-1} \|\Phi(t, 0)\| &\leq (2/T_\beta) \ln M_\varepsilon + \bar{\alpha} + \varepsilon + M_\varepsilon \delta \\ &= \bar{\alpha} + \varepsilon + M_\varepsilon \delta + 2(\ln M_\varepsilon)^{\gamma/(\gamma+1)} (\beta M_\varepsilon)^{1/(\gamma+1)}. \end{aligned}$$

From which the first conclusion of the theorem follows. The second conclusion follows from Lemma 3. It remains to show (55). We must show, as $n \to \infty$,

$$\|A(nT)\|/n \to 0 \quad \text{a.s.}, \tag{58}$$

$$(nT)^{-1} \int_{nT}^{(n+1)T} \|P(\sigma)\| \, d\sigma \to 0 \quad \text{a.s.}, \tag{59}$$

$$(nT)^{-1} \int_0^T \|A(nT + u) - A(nT)\| \, du \to 0 \quad \text{a.s.} \tag{60}$$

Denote $a_n = \|A(nT)\|$ and consider that

$$a_n/n = \sum_1^n a_s/n - \left[\sum_1^{n-1} a_s/(n-1) \right] (1 - 1/n).$$

By the ergodic theorem the right-hand side converges a.s. to $A - A = 0$. Thus (58) is established: (59) and (60) follow similarly. ∎

We now obtain our main result.

Theorem 5. *For the system* (43), *with assumptions* (S1)–(S4), (s2) *and* ε, δ, β *satisfying* (20)–(22), *then* $x(t)$ *is stable in probability.*

Proof. Follows from Lemma 3 and Theorem 4. ■

5. Concluding Remarks

After this work was completed the author became aware of [I] (see also [B2]) which presents a condition that can be regarded as a means of checking (s3) (although it only deals with $v = 0$ in (43)). He assumes there is a Lyapunov matrix P such that

$$E[\lambda_{\max}(A^T(t) + PA(t)P^{-1})] < 0. \tag{61}$$

By similarity $A^T(t) + PA(t)P^{-1}$ has the same eigenvalues as $P^{-1/2}A^T(t)P^{1/2} + P^{1/2}A(t)P^{-1/2}$, so the condition is equivalent to

$$E[\lambda_{\max}(P^{-1/2}A^T(t)P^{1/2} + P^{1/2}A(t)P^{-1/2})] < 0. \tag{62}$$

If $A(t)$ has unit eigenvector $\xi(t)$ with corresponding eigenvalue $\lambda_A(t)$, then

$$\xi^H P^{1/2}(P^{-1/2}A^T(t)P^{1/2} + P^{1/2}A(t)P^{-1/2})P^{1/2}\xi = (2\,\mathrm{Re}\,\lambda_A(t))\xi^H P\xi, \tag{63}$$

where superscript H denotes complex conjugate transpose. Thus

$$2\,\mathrm{Re}(\lambda_A(t)) \le \lambda_{\max}(P^{-1/2}A^T(t)P^{1/2} + P^{1/2}A(t)P^{-1/2}). \tag{64}$$

So, if (62) holds, then so does (S3). That is, (S3) is a weaker requirement than (62).

In any case, only for second-order systems (e.g., [A2]) has any way of finding such a P been demonstrated. Our result, on the other hand, gives quite explicit conditions for (s3) and hence Theorem 5 to hold for systems of any order.

References

[A1] T. Apostol, *Mathematical Analysis*, Addison-Wesley, Reading, MA, 1977.

[A2] S. T. Ariaratnam, Almost-sure stability of some linear stochastic systems, *J. Appl. Mech.*, **56** (1989), 175–178.

[B1] P. Billingsley, *Probability and Measure*, Wiley, New York, 1979.

[B2] G. Blankenship, Stability of differential equations with random coefficients, *IEEE Trans. Automat. Control*, **22** (1977), 834–838.

[CG] H. F. Chen and L. Guo, *Identification and Stochastic Adaptive Control*, Birkhauser, Boston, 1991.

[C] W. A. Coppel, *Dichotomies in Stability Theory*, Lecture Notes in Mathematics, vol. 629, Springer-Verlag, Berlin, 1978.

[D] C. A. Desoer, Slowly varying system $\dot{x} = Ax$, *IEEE Trans. Automat. Control*, **14** (1970), 339–340.

[FLJC] X. Feng, K. A. Loparo, Y. Ji, and H. J. Chizeck, Stochastic stability properties of jump linear systems, *IEEE Trans. Automat. Control*, **37** (1992), 38–53.

[GS] G. C. Goodwin and K. S. Sin, *Adaptive Filtering Prediction and Control*, Prentice-Hall, Englewood Cliffs, NJ, 1984.

[I] E. F. Infante, On the stability of some linear nonautonomous random systems, *J. Appl. Mech.*, **35** (1968), 7–12.

[K1] R. Z. Khasminski, *Stochastic Stability of Differential Equations*, Sijthoff and Nordhoff, Baltimore, MD, 1980.

[K2] G. Kreisselmeier, Adaptive control of a class of slowly time varying plants, *Systems Control. Lett.*, **8** (1986), 97–103.

[K3] U. Krengel, *Ergodic Theorems*, de Gruyter, New York, 1985.

[MG] R. H. Middleton and G. C. Goodwin, Adaptive control of the linear time varying plants, *IEEE Trans. Automat. Control*, **33** (1988), 150–155.

[MGHM] R. H. Middleton, G. C. Goodwin, D. J. Hill, and D. Q. Mayne, Design issues in adaptive control, *IEEE Trans. Automat. Control*, **33** (1988), 50–57.

[N] K. S. Narendra and A. M. Annaswamy, *Stable Adaptive Systems*, Prentice-Hall, Englewood Cliffs, NJ, 1989.

[R1] J. A. Richards, *Analysis of Periodically Time-Varying Systems*, Springer-Verlag, Berlin, 1983.

[R2] H. H. Rosenbrook, The stability of linear time-dependent control systems, *J. Electron and Control*, **15** (1963), 73–80.

[SB] S. Sastry and M. Bodson, *Adaptive Control: Stability, Convergence and Robustness*, Prentice-Hall, Englewood Cliffs, NJ, 1989.

[S1] V. Solo, A One Step Ahead Adaptive Controller With Slowly Time Varying Parameters, Technical Report, Dept. of EECS, Johns Hopkins University, Baltimore, MD, 1991.

[S2] E. D. Sontag, *Mathematical Control Theory*, Springer-Verlag, Berlin, 1990.

[TI] K. S. Tsakalis, and P. A. Ioannou, Adaptive control of linear time varying plants: a new model reference controller structure, *IEEE Trans. Automat. Control*, **34** (1989), 1038–1046.

[W] N. Wiener, *The Fourier Integral and Certain of Its Applications*, Dover, New York, 1958.

[ZW] G. Zames and L. Y. Wang, Local–global double algebras for slow H^∞ adaptation: Part I —inversion and stability, *IEEE Trans. Automat. Control*, **36** (1991), 130–142.

time-invariant linear optimal regulators," in *Proc. 3rd Annu. Southeast. Symp. System Theory*, 1971.
[12] R. E. Kalman, "When is a linear control system optimal?" *Trans. ASME*, J. Basic Eng., ser. D, vol. 86, pp. 51–60, 1964.
[13] R. E. Kalman and R. S. Bucy, "New results in linear filtering and prediction theory," *Trans. ASME*, J. Basic Eng., ser. D, vol. 83, pp. 95–100, 1961.
[14] A. E. Bryson, Jr., and D. E. Johansen, "Linear filtering for time-varying systems using measurements containing colored noise," *IEEE Trans. Automat. Contr.*, vol. AC-10, pp. 4–10, Jan. 1965.
[15] S. Butman, "A method for optimizing control-free costs in systems with linear controllers," *IEEE Trans. Automat. Contr.* (Short Papers), vol. AC-13, pp. 554–556, Oct. 1968.
[16] E. G. Gilbert, "The decoupling of multivariable systems by state feedback," *SIAM J. Contr.*, vol. 7, Feb. 1969.
[17] B. Moore and L. Silverman, private communication, Aug. 1971.

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Instability of Slowly Varying Systems

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***Abstract*—Instability criteria are obtained for systems described by $\dot{x} = A(t)x$ when the parameters are slowly varying. In particular it is shown that, when $A(t)$ has eigenvalues in the right-half plane and all eigenvalues are bounded away from the imaginary axis, then if $\sup_{t\geq 0} \|\dot{A}(t)\|$ is sufficiently small, the system has unbounded solutions. Results are also given for systems of the form $\dot{x} = A(t)x + f(x, t)$, and the dichotomy of solutions is studied in both the linear and nonlinear cases.**

I. Introduction

In this paper the question of instability is considered for systems described by $\dot{x} = A(t)x$ in which the parameters are "slowly varying." In particular, it is our aim to obtain conditions under which the stability properties of the time-varying system can be predicted from the stability properties of the frozen-time systems [i.e., from the eigenvalues of $A(t)$]. Regarding stability, it is known that, if the eigenvalues of $A(t)$ lie in $\operatorname{Re} \lambda < \sigma_0 < 0$ for all t and $\sup_{t\geq 0} \|\dot{A}(t)\|$ is sufficiently small, then all solutions of $\dot{x} = A(t)x$ go to zero as $t \to \infty$ (c.f., [1]–[3]). One would intuitively expect also that, if $A(t)$ had eigenvalues in the right-half plane, then the system would have unbounded solutions if $\sup_{t\geq 0} \|\dot{A}(t)\|$ was sufficiently small. It is shown here that this is indeed the case provided that no eigenvalues cross the imaginary axis. It is also shown by an example (Section III) that, if the eigenvalues are allowed to cross the imaginary axis, then even though there is always an eigenvalue with positive real part, the system can be asymptotically stable for arbitrarily small $\sup_{t\geq 0} \|\dot{A}(t)\|$. Thus, this additional restriction is unavoidable for the preceding type of result to hold. These results are also extended in a straightforward manner to nonlinear systems of the form $\dot{x} = A(t)x + f(x, t)$ where $\|f(x, t)\|/\|x\| \to 0$ as $\|x\| \to 0$.

The main result is proved along lines similar to the proof of the stability criteria of [1]–[3] in which Lyapunov methods were used. However, the method of constructing a Lyapunov function used in [1]–[3] cannot be used in the

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case when $A(t)$ has eigenvalues in both the left- and right-half planes since the equation $A'(t)Q(t) + Q(t)A(t) = -I$ need not have a solution for all t. This difficulty is circumvented by the use of the decomposition theorem of Section II. This theorem is then used in Section III to prove the instability criteria, and in Section IV it is used to study the dichotomy of solutions.

By way of preliminaries it is noted that we will be primarily concerned with vector differential equations in which the vector space is $\mathcal{R}^n$ (n-tuples of real numbers). The usual Euclidian norm of $x \in \mathcal{R}^n$ will be denoted by $\|x\|$ and the corresponding induced norm on the space of linear mappings of $\mathcal{R}^n$ into $\mathcal{R}^n$ will be denoted by $\|\cdot\|$.

II. A Decomposition Theorem

The key result to be used in proving the instability criteria is the following result on the decomposition of R^n into two time-varying subspaces invariant under $A(t)$.

Theorem 1: Let the matrix-valued function $t \rightarrow A(t)$ be continuously differentiable and satisfy the following conditions.

Condition 1: $\alpha = \sup_{t\geq 0} \|A(t)\| < \infty$ and $\dot{\alpha} = \sup_{t\geq 0} \|\dot{A}(t)\| < \infty$.

Condition 2: The eigenvalues of $A(t)$, say, $\lambda_1(t), \cdots, \lambda_n(t)$, are bounded away from a closed Jordan curve Γ in the complex plane for all $t \geq 0$, the set $\Lambda_1(t) = \{\lambda_1(t), \cdots, \lambda_k(t)\}$ lies inside Γ, and the set $\Lambda_2(t) = \{\lambda_{k+1}(t), \cdots, \lambda_n(t)\}$ lies outside Γ.

Then there exists a continuously differentiable matrix valued function $t \rightarrow T(t)$ such that: 1)

$$\sigma = \sup_{t\geq 0} \|T(t)\| < \infty$$

$$\tilde{\sigma} = \sup_{t\geq 0} \|T^{-1}(t)\| < \infty$$

$$\dot{\sigma} = \sup_{t\geq 0} \|\dot{T}(t)\| \leq K\dot{\alpha}$$

for some $0 < K < \infty$; and 2)

$$T^{-1}(t)A(t)T(t) = \begin{bmatrix} A_1(t) & 0 \\ 0 & A_2(t) \end{bmatrix}$$

where $A_1(t)$ is $k \times k$ and its eigenvalues are $\lambda(t), \cdots, \lambda_k(t)$; $A_2(t)$ is $(n-k) \times (n-k)$ and its eigenvalues are $\lambda_{k+1}(t), \cdots, \lambda_n(t)$; $\sup_{t\geq 0} \|\dot{A}_1(t)\| \leq K_1\dot{\alpha}$, and $\sup_{t\geq 0} \|\dot{A}_2(t)\| \leq K_2\,\dot{\alpha}$ for some $0 < K_1, K_2 < \infty$.

The geometrical interpretation of this theorem is simply that if $A(\cdot)$ is slowly varying one can find a slowly varying basis $\{x_1(t), \cdots, x_n(t)\}$ of R^n such that for every $t \geq 0$ $x_1(t), \cdots, x_k(t)$ spans the direct sum, say, $\Sigma_1(t)$, of the generalized eigenspaces corresponding to $\lambda_1(t), \cdots, \lambda_k(t)$, and $x_{k+1}(t), \cdots, x_n(t)$ span the direct sum, say, $\Sigma_2(t)$, of the generalized eigenspaces corresponding to $\lambda_{k+1}(t), \cdots, \lambda_n(t)$. Since $\Sigma_1(t)$ and $\Sigma_2(t)$ are invariant under $A(t)$, it follows that by taking the first k columns of $T(t)$ to be $x_1(t), \cdots, x_k(t)$ and the remaining columns to be $x_{k+1}(t), \cdots, x_n(t)$, the decomposition of 2 results.

In the special case when $A(t)$ is constant, this result is of course well known. The important fact to note here is that, when $A(\cdot)$ is not constant, not only can one find a $T(\cdot)$ which transforms $A(\cdot)$ into block diagonal form, but $T(\cdot)$ can also be chosen such that $\|\dot{T}(t)\|$ is bounded by $\sup_{t\geq 0} \|\dot{A}(t)\|$.

Proof of Theorem 1: Let $\Sigma_1(t)$ denote the direct sum of the generalized eigenspaces[1] corresponding to $\lambda_1(t), \cdots, \lambda_k(t)$, and $\Sigma_2(t)$ the direct sum of the generalized eigenspaces of $\lambda_{k+1}(t), \cdots, \lambda_n(t)$. The projection $P_1(t)$ onto $\Sigma_1(t)$ along $\Sigma_2(t)$ is given by the Dunford–Taylor integral [4]

$$P_1(t) = \frac{1}{2\pi j} \int_\Gamma R(\xi, t)\, d\xi \tag{1}$$

where Γ is the positively oriented closed Jordan curve of Condition 2, and the resolvent $R(\xi, t)$ of $A(t)$ is given by (I = identity)

$$R(\xi, t) = [\xi I - A(t)]^{-1}. \tag{2}$$

Note that the projection $P_2(t)$ onto $\Sigma_2(t)$ along $\Sigma_1(t)$ is simply $P_2(t) = I - P_1(t)$.

Since the eigenvalues of $A(t)$ are bounded away from Γ for all t, it follows that there exists some ρ, $0 < \rho < \infty$, such that $\|R(\xi, t)\| < \rho$ for all $\xi \in \Gamma$ and all $t \geq 0$. Thus, $\|P_1(t)\| \leq \rho\gamma$ for all $t \geq 0$ where $2\pi\gamma$ = arc length of Γ. Since $A(\cdot)$ is continuously differentiable, it follows from (1) that $P_1(\cdot)$ is continuously differentiable and, in fact,

$$\begin{aligned} \dot{P}(t) &= \frac{1}{2\pi j} \int_\Gamma \frac{\partial R(\xi, t)}{\partial t}\, d\xi \\ &= \frac{1}{2\pi j} \int_\Gamma R(\xi, t)\dot{A}(t)R(\xi, t)\, d\xi. \end{aligned} \tag{3}$$

Thus

$$\|\dot{P}_1(t)\| \leq \rho^2\gamma\dot{\alpha}. \tag{4}$$

Also, since $P_2(t) = I - P_1(t)$, it follows that $\|P_2(t)\| \leq 1 + \rho\gamma$ and $\|\dot{P}_2(t)\| = \|\dot{P}_1(t)\| \leq \rho^2\gamma\dot{\alpha}$.

Note that our objective is to obtain bases for $\Sigma_1(t)$ and $\Sigma_2(t)$ which are continuously differentiable and have derivatives whose norms are bounded by $\dot{\alpha}$. It is clear from the above that for any time t_1 there exists an interval J about t_1 such that one can select k columns from $P_1(t)$ and $(n-k)$ columns from $P_2(t)$ which meet these requirements for $t \in J$. The difficulty is that these columns of $P_1(\cdot)$ and $P_2(\cdot)$ need not remain linearly independent for all $t \geq 0$. In order to obtain continuously differentiable bases for all $t \geq 0$ for $\Sigma_1(t)$ and $\Sigma_2(t)$ we invoke Dolezal's theorem [6], which asserts in this case that there exist continuously differentiable matrix functions of t, $M_1(\cdot)$, and $M_2(\cdot)$ such that

$$P_1(t)M_1(t) = [\hat{P}_1(t)|0] \tag{5}$$

$$P_2(t)M_2(t) = [\hat{P}_2(t)|0] \tag{6}$$

where $\hat{P}_1(t)$ and $\hat{P}_2(t)$ are, respectively, $n \times k$ and $n \times$

[1] By "generalized eigenspace corresponding to $\lambda_i(t)$" is meant the subspace $\{x | [A(t) - \lambda_i(t)I]^k x = \theta$ for some integer $k > 0\}$.

$(n - k)$ of full rank and continuously differentiable. Also, it is clear that $M_1(\cdot)$ and $M_2(\cdot)$ can be chosen such that $\hat{P}_1'(t)\hat{P}_1(t) \equiv I_{k\times k}$ and $\hat{P}_2'(t)\hat{P}_2(t) \equiv I_{n-k\times n-k}$.

Now, the columns of $\hat{P}_1(t)$ form a basis of $\Sigma_1(t)$, and those of $\hat{P}_2(t)$ form a basis of $\Sigma_2(t)$. Note also that, since $P_1(t)$ and $P_2(t)$ are projections, $P_1(t)\hat{P}_1(t) \equiv \hat{P}_1(t)$ and $P_2(t)\hat{P}_2(t) \equiv \hat{P}_2(t)$. The final step in the proof is to use $\hat{P}_1(\cdot)$ and $\hat{P}_2(\cdot)$ to obtain bases for $\Sigma_1(t)$ and $\Sigma_2(t)$ such that the norms of their derivatives are bounded by $\dot{\alpha}$. This is accomplished by obtaining basis vectors for $\Sigma_1(t)$ such that their derivatives are orthogonal to $\Sigma_1(t)$ and basis vectors for $\Sigma_2(t)$ such that their derivatives are orthogonal to $\Sigma_2(t)$. In particular, define $\tilde{P}_1(t)$ by

$$\tilde{P}_1(t) = \hat{P}_1(t)\Phi_1(t) \tag{7}$$

where $\Phi_1(\cdot)$ is the $k \times k$ matrix solution of

$$\dot{\Phi}_1(t) = -\hat{P}_1'(t)\dot{\hat{P}}_1(t)\Phi_1(t), \qquad \Phi_1(0) = I. \tag{8}$$

Then

$$\dot{\tilde{P}}_1(t) = [\dot{\hat{P}}_1(t) - \hat{P}_1(t)\hat{P}_1'(t)\dot{\hat{P}}_1(t)]\Phi_1(t) \tag{9}$$

and thus, since $\hat{P}_1'(t)\hat{P}_1(t) = I$, it follows that $\hat{P}_1'(t)\dot{\tilde{P}}_1(t) = 0$. Therefore, the columns of $\dot{\tilde{P}}_1(t)$ are orthogonal to $\Sigma_1(t)$. Note also that

$$0 = \frac{d}{dt}\hat{P}_1'(t)\hat{P}_1(t) = \dot{\hat{P}}_1'(t)\hat{P}_1(t) + \hat{P}_1'(t)\dot{\hat{P}}_1(t) \tag{10}$$

and thus

$$\frac{d}{dt}\Phi_1'(t)\Phi_1(t) = -\Phi_1'(t)[\dot{\hat{P}}_1'(t)\hat{P}_1(t) + \hat{P}_1'(t)\dot{\hat{P}}_1(t)]\Phi_1(t) = 0. \tag{11}$$

As a result, since $\Phi_1(0) = I$, it follows that $\Phi_1'(t)\Phi_1(t) = I$. Thus, $\tilde{P}_1'(t)\tilde{P}_1(t) = \Phi_1'(t)\hat{P}_1'(t)\hat{P}_1(t)\Phi_1(t) = I$, and therefore the norms of the columns of $\tilde{P}_1(t)$ are all unity.

The columns of $\tilde{P}_1(t)$ yield the desired basis of $\Sigma_1(t)$ for

$$P_1(t)\tilde{P}_1(t) = \tilde{P}_1(t), \tag{12}$$

and thus differentiating (12) gives

$$\dot{P}_1(t)\tilde{P}_1(t) = (I - P_1(t))\dot{\tilde{P}}_1(t). \tag{13}$$

Let

$$\delta = \inf_{t\geq 0} \inf_{x\in\Sigma_1^{\perp}(t)} \frac{\|(I - P_1(t))x\|}{\|x\|}.$$

[Note that $P_1(t)$ bounded implies that $\delta > 0$.] Then from (13) it follows that if $p_j^1(t)$ is the jth column of $\tilde{P}_1(t)$, then (note $\|p_j^1(t)\| = 1$)

$$\|\dot{p}_j^1(t)\| \leq \frac{\rho^2\gamma\dot{\alpha}}{\delta}. \tag{14}$$

In a similar manner, defining $\tilde{P}_2(t)$ by

$$\tilde{P}_2(t) = \hat{P}_2(t)\Phi_2(t) \tag{15}$$

where $\Phi_2(\cdot)$ is the $(n - k) \times (n - k)$ matrix solution of

$$\dot{\Phi}_2(t) = -\hat{P}_2'(t)\dot{\hat{P}}_2(t)\Phi_2(t), \qquad \Phi_2(0) = I, \tag{16}$$

it follows that, if $p_j^2(t)$ is the jth column of $\tilde{P}_2(t)$, then

$$\|\dot{p}_j^2(t)\| \leq \frac{\rho^2\gamma\dot{\alpha}}{\delta}. \tag{17}$$

The matrix $T(t)$ is now taken to be the matrix with its first k-columns the vectors $p_j^1(t)$, $j = 1, 2, \cdots, k$, and the remaining columns to be $p_j^2(t)$, $j = 1, 2, \cdots, n - k$. From (14), (17), and the equivalence of norms on R^n, the bounds on the norm of $T^{-1}(\cdot)$ and the norms of $T(\cdot)$, $A_1(\cdot)$, and $A_2(\cdot)$ and their derivatives follow easily.

Q.E.D.

III. Instability for Slowly Varying Systems

Consider systems which are described by

$$\dot{x}(t) = A(t)x(t). \tag{18}$$

One would intuitively expect that, if for each t, $A(t)$ had at least one eigenvalue in the half-plane $\operatorname{Re} s > \sigma_0 > 0$ and if $\|\dot{A}(t)\|$ were sufficiently small, then the system of (18) would have unbounded solutions. To see that $A(\cdot)$ must be slowly varying and some restrictions on A and $\dot{A}$ are necessary for such a result to hold, consider the following examples.

Example 1: Let $A(t)$ be given by

$$A(t) = \begin{bmatrix} -1 + \alpha\cos\omega t\sin\omega t & \alpha\cos^2\omega t + \omega \\ -\alpha\sin^2\omega t - \omega & -1 - \alpha\cos\omega t\sin\omega t \end{bmatrix}.$$

Then the corresponding transition matrix $\Phi(t, t_0)$ is given by

$$\Phi(t, t_0) = e^{-(t-t_0)}\begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix}\begin{bmatrix} 1 & \alpha(t-t_0) \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\omega t_0 & -\sin\omega t_0 \\ \sin\omega t_0 & \cos\omega t_0 \end{bmatrix}.$$

Thus, with this A matrix all solutions of (18) are exponentially bounded. The eigenvalues are independent of t and given by

$$\lambda = -1 \pm \sqrt{-\alpha\omega - \omega^2}.$$

With $\omega = 1$ and $\alpha = -5$ the eigenvalues of $A(t)$ are at $+1$ and -3 for all t. Note, however, that for any $\alpha < -2$, if

$$0 < \omega < -\frac{\alpha}{2} - \frac{1}{2}\sqrt{\alpha^2 - 4}$$

or

$$\omega > -\frac{\alpha}{2} + \frac{1}{2}\sqrt{\alpha^2 - 4},$$

then the eigenvalues of $A(t)$ have negative real parts.

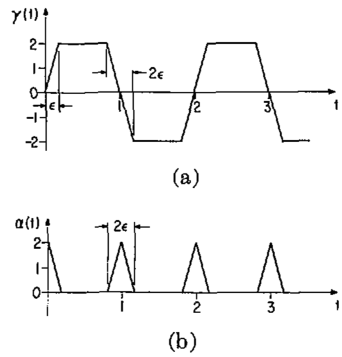

Fig. 1.

Thus, for slowly varying $A(\cdot)$ and rapidly varying $A(\cdot)$, the eigenvalues of $A(\cdot)$ correctly predict the stability properties of the system. They do not, however, when ω lies between the two previously mentioned bounds.

Example 2: Let $A(t)$ be given by

$$A(t) = \begin{bmatrix} -1 + \alpha(t) & \gamma(t) \\ \gamma(t) & -1 + \alpha(t) \end{bmatrix}.$$

Then the transition matrix $\Phi(t, t_0)$ is given by

$$\Phi(t, t_0) = \exp\left[-(t - t_0) + \int_{t_0}^{t} \alpha(\tau)\, d\tau\right] \cdot \begin{bmatrix} \cosh \Gamma(t, t_0) & \sinh \Gamma(t, t_0) \\ \sinh \Gamma(t, t_0) & \cosh \Gamma(t, t_0) \end{bmatrix}$$

where $\Gamma(t, t_0) = \int_{t_0}^{t} \gamma(\tau)\, d\tau$. The eigenvalues of $A(t)$ are seen to be given by

$$\lambda(t) = -1 + \alpha(t) \pm \gamma(t).$$

Now take $\alpha(t)$ and $\gamma(t)$ to be as in Fig. 1. Since $\alpha(t) + |\gamma(t)| \equiv 2$, it follows that one of the eigenvalues of $A(t)$ is fixed at $\lambda = +1$. The other eigenvalue is at $\lambda = -3$ when $\alpha(t) = 0$, and moves over to $+1$ and back to -3 when $\alpha(t) \neq 0$.

It is easily seen, however, that $[-(t - t_0) + \int_{t_0}^{t} \alpha(\tau)\, d\tau] \to -\infty$ as $t \to \infty$ for all $0 < \epsilon < 1/2$ and also that $\Gamma(t, t_0)$ is bounded. Hence the solutions of (18) with this $A(t)$ are exponentially bounded when $0 < \epsilon < 1/2$.

The most interesting point about this example is that the same situation prevails when $A(t)$ is replaced by $A(\beta t)$ for all β. Thus, even though β is small so that the system is slowly varying, the eigenvalues of $A(t)$ do not predict the stability behavior of (18). The essential thing to note here is that one of the eigenvalues of $A(t)$ crosses the imaginary axis.

If we make $A(t)$

$$A(t) = \begin{pmatrix} -1 + \alpha(t) + \epsilon(t) & \gamma(t) \\ \gamma(t) & -1 + \alpha(t) \end{pmatrix},$$

then the eigenvalues are given by

$$\lambda = -1 + \alpha(t) + \frac{\epsilon(t)}{2} \pm \sqrt{\frac{\epsilon^2(t)}{4} + \gamma^2(t)},$$

and if $\epsilon(t)$ is such that $\epsilon(t) \neq 0$ when $\gamma(t) = 0$, then the eigenvalues of $A(t)$ will never be equal, and if $\epsilon(t)$ is sufficiently small, the system will be asymptotically stable since with $\epsilon = 0$ the system is exponentially stable. Hence the main phenomenon occurring in these examples is an eigenvalue crossing the imaginary axis, not two eigenvalues becoming equal.

The main result of this section is based primarily on the decomposition theorem of the previous section and the following well-known theorem from Lyapunov theory [5].

Theorem 2: For the system

$$\dot{x}(t) = f(x(t), t), \qquad f(0, t) \equiv 0, \tag{19}$$

assume $f(\cdot, t)$ is continuous and (19) has a solution for all $t \geq t_0$ and $x(t_0) \in R^n$. Let $v(x, t)$ be a real valued function satisfying the following conditions.

Condition 1: v is continuously differentiable in x and t, and $v(0, t) \equiv 0$.

Condition 2: There exist at each time t points arbitrarily close to $x = 0$ at which $v(x, t) < 0$.

Condition 3: $\dot{v}(\phi(t, t_0, x_0), t) \leq -\delta\|\phi(t, t_0, x_0)\|^2$ for some $\delta > 0$ and all solutions $\phi(t, t_0, x_0)$ of (19) with $x(t_0) = x_0$.

Then the null solution of (19) is unstable in the sense of Lyapunov.

Concerning the instability of slowly varying systems in the form of (18), we have the following.

Theorem 3: Let $A(\cdot)$ satisfy the conditions of Theorem 1, and in addition, assume that the eigenvalues $\lambda_1(t), \cdots, \lambda_k(t)$ lie in $\mathrm{Re}\,\lambda < -\sigma_1 < 0$ for all $t \geq 0$ and that $\lambda_{k+1}(t), \cdots, \lambda_n(t)$ lie in $\mathrm{Re}\,\lambda > \sigma_2 > 0$ for all $t \geq 0$. Then if $\dot{\alpha}$ is sufficiently small, the null solution of (18) is unstable in the sense of Lyapunov.

Proof: From Theorem 1 it is known that there exists a matrix function $T(\cdot)$ such that conclusion 2) of the theorem holds. Let the matrices $Q_1(t)$ and $Q_2(t)$ be solutions of

$$A_1'(t)Q_1(t) + Q_1(t)A_1(t) = -I_{k\times k} \tag{20}$$

and

$$A_2'(t)Q_2(t) + Q_2(t)A_2(t) = -I_{n-k\times n-k}. \tag{21}$$

Since the eigenvalues of $A_1(t)$ lie in $\mathrm{Re}\,\lambda < 0$ for all $t \geq 0$ and the eigenvalues of $A_2(t)$ lie in $\mathrm{Re}\,\lambda > 0$ for all $t \geq 0$, it is known that (20) and (21) have a unique solution for all t. In fact, $Q_1(t)$ and $Q_2(t)$ are given explicitly by

$$Q_1(t) = \int_0^{\infty} e^{A_1'(t)\tau} e^{A_1(t)\tau}\, d\tau \tag{22}$$

$$Q_2(t) = -\int_{-\infty}^{0} e^{A_2'(t)\tau} e^{A_2(t)\tau}\, d\tau. \tag{23}$$

From (22) and (23) it is seen that $Q_1(t)$ is positive definite and $Q_2(t)$ is negative definite for all $t \geq 0$.

Now make the change of variables $T(t)z(t) = x(t)$, then

$$\dot{z}(t) = [T^{-1}(t)A(t)T(t) - T^{-1}(t)\dot{T}(t)]z(t). \quad (24)$$

For the system (24), choose the Lyapunov function $v(z, t) = z'Q(t)z$ where

$$Q(t) = \begin{bmatrix} Q_1(t) & 0 \\ 0 & Q_2(t) \end{bmatrix}. \quad (25)$$

Then, along solutions of (24), $\dot{v}(t)$ is seen to be

$$\dot{v}(t) = -z'(t)z(t) + z'(t)[\dot{Q}(t) - \dot{T}'(t)T^{-1'}(t)Q(t) - Q(t)T^{-1}(t)\dot{T}(t)]z(t). \quad (26)$$

Now, from Theorem 1 it is known that $\|\dot{T}(t)\| \leq K\dot{\alpha}$ and that $\|T^{-1}(t)\| \leq \zeta$ for some $\zeta < \infty$ and all $t \geq 0$. Also, it follows easily from (22) and (23) that $\|Q(t)\| \leq q_1$ and $\|\dot{Q}(t)\| \leq q_2\dot{\alpha}$ for some $0 < q_1, q_2 < \infty$ and all $t \geq 0$ (c.f., [2] or [3] for details of these types of estimates). Thus, there exists some positive constant $c < \infty$ such that

$$\sup_{t \geq 0} \|\dot{Q}(t) - \dot{T}'(t)T^{-1'}(t)Q(t) - Q(t)T^{-1}(t)\dot{T}(t)\| \leq c\dot{\alpha}. \quad (27)$$

As a result, from (26) it is seen that, if $\dot{\alpha} < (1/c)$, then $\dot{v}(t) \leq -\gamma z'(t)z(t)$ for some $\gamma > 0$. Since $Q_2(t)$ is negative definite for all $t \geq 0$, it follows that $v(z, t)$ is for each t negative for some z arbitrarily close to zero. Hence by Theorem 2, the null solution of (24), and thus the null solution of (18), is unstable if $\sup_{t \geq 0} \|\dot{A}(t)\| \triangleq \dot{\alpha}$ is sufficiently small. Q.E.D.

By contrast to the above result, we can state the following regarding rapidly varying systems.

~~*Theorem 4*~~: Let $A(t)$ by periodic of period T, and let

$$\bar{A} = \frac{1}{T}\int_0^T A(\sigma)\, d\sigma \quad (28)$$

have at least one eigenvalue in Re $\lambda > \sigma_0 > 0$. Then the system

$$\dot{x}(t) = A(t/\epsilon)x(t) \quad (29)$$

will be unstable for ϵ sufficiently small.

The analogous result to this in proving stability when all eigenvalues of $\bar{A}$ have negative real parts has been given by Brockett [3]. There it is shown that, if $\Phi(t, t_0)$ is the transition matrix for $A(t/\epsilon)$, then

$$\Phi\left(t_0 + \frac{T}{\epsilon}, t_0\right) = I + \epsilon\bar{A}(t - t_0) + 0(\epsilon). \quad (30)$$

Thus, if $\bar{A}$ has an eigenvalue in Re, $\lambda > 0$, then for ϵ sufficiently small $\Phi(t_0 + T/\epsilon, t_0)$ will have an eigenvalue lying outside the unit disk, which implies (29) is unstable.

Regarding nonlinear time-varying systems, we have the following result which follows easily from the proof of Theorem 3 and well-known arguments.

Theorem 5: For the system

$$\dot{x}(t) = A(t)x(t) + f(x, t), \quad (31)$$

let $A(\cdot)$ satisfy the assumptions of Theorem 3, and in addition, assume $(\|f(x, t)\|/\|x\|) \to 0$ uniformly in t as $\|x\| \to 0$. Then if $\sup_{t \geq 0} \|\dot{A}(t)\|$ is sufficiently small, the null solution of (31) is unstable.

IV. Dichotomies of Solutions and Conditional Stability

In the time-invariant case it is well known that, if the matrix A has k eigenvalues with negative real parts, then for the system

$$\dot{x}(t) = Ax(t) \quad (32)$$

there exists a k-dimensional subspace of R^n such that, if $\phi(t)$ is a solution of (32) with $\phi(t_0)$ in this subspace, then $\|\phi(t)\| \leq \alpha e^{-\sigma(t-t_0)}$ for some constants $\alpha, \sigma > 0$. In the nonlinear system

$$\dot{x}(t) = Ax(t) + f(x(t)) \quad (33)$$

with $(\|f(x)\|/\|x\|) \to 0$ as $\|x\| \to 0$, there is a k-dimensional manifold containing the origin such that any solution $\phi(t)$ of (33) with $\phi(t_0)$ on the manifold satisfies $\phi(t) \to 0$ as $t \to \infty$ [7]. Also, if $\phi(t_0)$ is not on the manifold, then $\phi(t)$ is bounded away from the origin for all $t \geq t_0$.

The purpose of this section is to prove similar types of results when A is a function of t but slowly varying. In particular, for linear systems we have the following.

Theorem 6: Consider the system (18) and let $A(\cdot)$ satisfy the assumptions of Theorem 3. Then for $\dot{\alpha}$ sufficiently small there exists for each $t \geq 0$ a k-dimensional subspace $S(t)$ such that, if $\phi(t)$ is a solution of (18) with $\phi(t_0) \in S(t_0)$, then $\|\phi(t)\| \leq \beta e^{-\sigma(t-t_0)}$ for some positive constants β and σ. Moreover, if $\phi(t_0) \notin S(t_0)$, then $\|\phi(t)\| \to \infty$ as $t \to \infty$.

Proof: Let $\Phi_1(t, t_0)$ and $\Phi_2(t, t_0)$ be the transition matrices for $A_1(\cdot)$ and $A_2(\cdot)$, respectively. Since $\|\dot{A}_1(t)\| \leq K_1\dot{\alpha}$ and $A_1(t)$ has all of its eigenvalues in Re $\lambda < -\sigma_1 < 0$ for all t, it is known from the stability results for slowly varying systems [2], [3] that, if $\dot{\alpha}$ is sufficiently small, then for some positive constants γ_1 and c_1,

$$\|\Phi_1(t, t_0)\| \leq c_1 e^{-\gamma_1(t-t_0)}, \qquad \text{for } t \geq t_0; \quad (34)$$

also, since the eigenvalues of $A_2(t)$ are in Re $\lambda > \sigma_2 > 0$ for all t and $\|\dot{A}_2(t)\| \leq K_2\dot{\alpha}$, it follows that, for some positive constants γ_2 and c_2,

$$\|\Phi_2(t, t_0)\| \leq c_2 e^{\gamma_2(t-t_0)}, \qquad \text{for } t \leq t_0. \quad (35)$$

Let

$$\hat{\Phi}_1(t, t_0) = \begin{bmatrix} \Phi_1(t, t_0) & 0 \\ 0 & 0 \end{bmatrix} \quad (36)$$

and

$$\hat{\Phi}_2(t, t_0) = \begin{bmatrix} 0 & 0 \\ 0 & \Phi_2(t, t_0) \end{bmatrix}. \quad (37)$$

Now consider the system obtained from (18) after the change of variables $T(t)z(t) = x(t)$ where $T(\cdot)$ is the matrix function of Theorem 1. Then

$$\dot{z}(t) = [T^{-1}(t)A(t)T(t) - T^{-1}(t)\dot{T}(t)]z(t). \quad (38)$$

For each vector $y \in R^n$, define the mapping H_y by

$$(H_y\phi)(t) = \hat{\Phi}_1(t, t_0)y - \int_{t_0}^{t} \hat{\Phi}_1(t, \tau)T^{-1}(\tau)\dot{T}(\tau)\phi(\tau)\, d\tau + \int_{t}^{\infty} \hat{\Phi}_2(t, \tau)T^{-1}(\tau)\dot{T}(\tau)\phi(\tau)\, d\tau. \quad (39)$$

From the bounds given in (34) and (35) and the fact that $\|T^{-1}(\tau)\| \le \zeta$ and $\|\dot{T}(\tau)\| \le K\dot{\alpha}$ for all τ, it follows easily that H_y maps[2] $L_\infty{}^n(t_0, \infty)$ into itself. It will now be shown that for $\dot{\alpha}$ sufficiently small H_y is a contraction on $L_\infty{}^n(t_0, \infty)$ and thus it has a unique fixed point $\phi_y{}^*$. It is easily seen that this fixed point $\phi_y{}^*$ is a solution of (38).

To show that H_y is a contraction, first note that

$$(H\phi_1)(t) - (H\phi_2)(t) = -\int_{t_0}^{t} \hat{\Phi}_1(t, \tau)T^{-1}(\tau)\dot{T}(\tau)(\phi_1(\tau) - \phi_2(\tau))\, d\tau + \int_{t}^{\infty} \hat{\Phi}_2(t, \tau)T^{-1} \cdot \dot{T}(\tau)(\phi_1(\tau) - \phi_2(\tau))\, d\tau. \quad (40)$$

Thus, [$\|\cdot\|_\infty$ denotes the norm on $L_\infty{}^n(t_0, \infty)$]

$$\|H_y\phi_1 - H_y\phi_2\|_\infty \le K\zeta\dot{\alpha}\|\phi_1 - \phi_2\|_\infty \left[c_1 \int_{t_0}^{t} e^{-\gamma_1(t-\tau)}\, d\tau + c_2 \int_{t}^{\infty} e^{\gamma_2(t-\tau)}\, d\tau\right] \le K\zeta\gamma\dot{\alpha}\|\phi_1 - \phi_2\|_\infty \quad (41)$$

where

$$\gamma = \left[\frac{c_1}{\gamma_1} + \frac{c_2}{\gamma_2}\right].$$

Thus, if $\dot{\alpha} < (1/K\zeta\gamma)$, then H_y is a contraction and has a unique fixed point ϕ_y. Also, by using successive approximations to obtain $\phi_y{}^*(t)$ by starting with the zero function, it can be shown that

$$\|\phi_y{}^*(t)\| \le \frac{1}{1-r}\|\hat{\Phi}_1(t, t_0)y\| \le \frac{c_1\|y\|}{1-r} e^{-\gamma_1(t-t_0)} \quad (42)$$

where $r = K\zeta\gamma\dot{\alpha} < 1$.

Since in the definition of H_y only the first k-components of y play a role, the last $n - k$-components can always be chosen to be zero. From (39) and the fact that $H_y\phi_y{}^* = \phi_y{}^*$, it is seen that the first k-components of $\phi_y{}^*(t_0)$ are simply the first k-components of y and the remaining components are given by

$$[\phi_y{}^*(t_0)]_j = \left(\int_{t_0}^{\infty} \hat{\Phi}_2(t_0, \tau)T^{-1}(\tau)\dot{T}(\tau)\phi_y{}^*(\tau)\, d\tau\right)_j, \quad j = k+1, \cdots, n. \quad (43)$$

Thus it is seen that there exists a k-parameter family of solutions of (38) which are exponentially bounded [they are exponentially bounded by virtue of (42)]. Since the system is linear, it follows that $\phi_{y_1}{}^* + \phi_{y_2}{}^* = \phi_{y_1+y_2}{}^*$, and thus for each t_0 there is a k-dimensional subspace $\tilde{S}(t_0)$ of R^n such that if $\phi(t)$ is a solution of (38) with $\phi(t_0) \in \tilde{S}(t_0)$ then $\phi(t)$ is exponentially bounded. Finally, transforming back to the system (18), it is seen that the theorem follows with $S(t) = T(t)\tilde{S}(t)$.

It remains to be shown that, if $\phi(t_0) \notin \tilde{S}(t_0)$ and ϕ is a solution of (38), then ϕ is unbounded. Assume $\|\phi(t)\|$ is bounded. Then from (38) it follows that

$$\begin{aligned}\phi(t) &= [\hat{\Phi}_1(t, t_0) + \hat{\Phi}_2(t, t_0)]\phi(t_0) - \int_{t_0}^{t} [\hat{\Phi}_1(t, \tau) + \hat{\Phi}_2(t, \tau)]T^{-1}(\tau)\dot{T}(\tau)\phi(\tau)\, d\tau \\ &= \hat{\Phi}_1(t, t_0)\phi(t_0) + \hat{\Phi}_2(t, t_0)\, a - \int_{t_0}^{t} \hat{\Phi}_1(t, \tau)\dot{T}^{-1}(\tau)T(\tau)\phi(\tau)\, d\tau + \int_{t}^{\infty} \hat{\Phi}_2(t, \tau)T^{-1}(\tau)\dot{T}(\tau)\phi(\tau)\, d\tau \quad (44)\end{aligned}$$

where

$$a = \phi(t_0) - \int_{t_0}^{\infty} \hat{\Phi}_2(t_0, \tau)T^{-1}(\tau)\dot{T}(\tau)\phi(\tau)\, d\tau. \quad (45)$$

The integrals in (44) and (45) exist under the assumption that $\|\phi(\tau)\|$ is bounded by virtue of (34) and (35). Now, if $\phi(t_0) \notin \tilde{S}(t_0)$, then from (43) and (45) it is seen that the last $n - k$-components of a are not all zero. Thus, $\hat{\Phi}_2(t, t_0)a$ will be unbounded[3] contradicting the assumption that ϕ is bounded. Q.E.D.

Regarding the nonlinear system

$$\dot{x}(t) = A(t)x(t) + f(x(t), t), \quad (46)$$

the following result can be stated.

Theorem 7: Consider the system (46) and let $A(\cdot)$ satisfy the condition of Theorem 3. In addition, it is assumed that $\|f(x, t)\|/\|x\| \to 0$ uniformly in t as $\|x\| \to 0$. Then for $\dot{\alpha}$ sufficiently small there exists at each time t a k-dimensional manifold $S(t)$ such that, if $\phi(t)$ is a solution of (46) with $\phi(t_0) \in S(t_0)$, then $\|\phi(t)\| \to 0$ as $t \to \infty$. Moreover, if $\phi(t_0) \notin S(t_0)$ and is near the origin, then $\phi(t)$ is bounded away from the origin for all $t \ge t_0$.

This theorem is proved along the same lines as Theorem 6, the only difference being that $\|\phi(t_0)\|$ must be assumed small so that the effects of $f(x(t), t)$ are small compared to the linear part. The details are omitted.

V. Conclusions

It has been shown that for systems described by $\dot{x}(t) = A(t)x(t)$, if the eigenvalues of $A(t)$ are bounded away from the imaginary axis for all $t \ge 0$ and if $\sup_{t\ge 0}\|\dot{A}(t)\|$ is

[2] $L_\infty{}^n(t_0, \infty)$ is the normed linear space of R^n-valued functions of t such that, if $\phi \in L_\infty{}^n(t_0, \infty)$, then $\sup_{t\ge t_0}\|\phi(t)\| < \infty$.

[3] The fact that all nonzero solutions of $\dot{q}(t) = A_2(t)q(t)$ are unbounded for $\dot{\alpha}$ sufficiently small follows from the Lyapunov function $v(q, t) = q'_2Q(t)q$ where $Q_2(t)$ is that obtained in Theorem 3. For $\dot{v}(t) = -q'q + q'\dot{Q}_2(t)q \le -\delta\|q\|^2$ for some $\delta > 0$ when $\dot{\alpha} < (1/q_2)$ (recall $\|\dot{Q}_2(t)\| \le q_2\dot{\alpha}$). Thus, by Theorem 2, solutions are unbounded.

sufficiently small, then the stability properties of the time-varying system are the same as those of the frozen-time systems. Also, it has been shown by examples that, if either of these two conditions is violated, then the result will not necessarily follow. Thus, the type of results presented here seems to be the best one can hope for. One direction for improving these criteria would be in obtaining better bounds on the derivative of $A(\cdot)$ such that the results still hold.

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References

[1] H. H. Rosenbrock, "The stability of linear time-dependent control systems," *J. Electron. Contr.*, vol. 15, pp. 73–80, July 1963.
[2] C. A. Desoer, "Slowly varying system $\dot{x} = A(t)x$," *IEEE Trans. Automat. Contr.* (Corresp.), vol. AC-14, pp. 780–781, Dec. 1969.
[3] R. W. Brockett, *Finite Dimensional Linear Systems.* New York: Wiley, 1970, sec. 32.
[4] T. Kato, *Perturbation Theory for Linear Operators.* New York: Springer, 1966, p. 67.
[5] W. Hahn, *Theory and Applications of Liapunov's Direct Method.* Englewood Cliffs, N.J.: Prentice-Hall, 1963, p. 18.
[6] V. Dolezal, *Dynamics of Linear Systems.* Prague, Czechoslovakia: Academia; Groningen, Netherlands: Noordhoff, 1967, p. 189.
[7] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations.* New York: McGraw-Hill, 1955.

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Short Papers

Minimum-Energy Control of a Traction Motor

PETAR KOKOTOVIĆ AND GURDIAL SINGH, MEMBER, IEEE

***Abstract*—Minimum-energy control is obtained for a nonlinear second-order model of a ground transportation vehicle with a dc traction motor for which regenerative braking is possible.**

Introduction

Optimal control theory has been extensively applied to aircraft guidance, while only several published works have treated optimum control of ground transportation vehicles [1]–[3]. In this paper the maximum principle is applied to a minimum-energy problem for a vehicle with a standard dc traction motor and, when the ohmic loss in the armature circuit is negligible, a simple feedback solution is obtained by Green's theorem.

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The description of vehicular horizontal motion most frequently used in traction calculations [4],[5] is

$$\dot{x} = v, \qquad \dot{v} = -k(v) + T \tag{1}$$

where x is distance; v is vehicle speed; $k(v)$ is vehicle resistance force per unit mass, an odd function of v, with the properties $k(0) = 0$, $dk(v)/dv = l(v) > 0$; and T is the tractive effort per unit mass.

For a dc motor whose armature circuit is described by $E = b\phi v + ir$, where E is the applied voltage, b is the back emf constant, ϕ is the flux in the air gap, i is the armature current, and r is the armature circuit resistance, the tractive effort T is $T = \alpha\phi i$ where α is a constant. While x and v are the state variables, either E, r, or ϕ can be used as the control variable. For acceleration it is common to use voltage or resistance control, while for the regenerative braking the flux control is used [4],[5].

Problem Statement

A solution to the minimum energy problem with voltage and resistance control is given in [6]. In this paper we consider the flux control problem when E = const, r = const, and the state equation (1) is rewritten as

$$
W_2 = \begin{array}{c} \begin{array}{ccccccccccc} & & & & l\text{th column} & & & & & (l+r)\text{th column} & \end{array} \\
\left[\begin{array}{ccccccccccc}
1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 1\cdots 1 \\
\lambda\big|_{\lambda_1} & \frac{\partial}{\partial\lambda}(\lambda)\big|_{\lambda_1} & 0 & \cdots & 0 & \lambda\big|_{\lambda_{l+1}} & \frac{\partial}{\partial\lambda}(\lambda)\big|_{\lambda_{l+1}} & 0 & \cdots & 0 & \lambda_{l+r+1}\cdots\lambda_n \\
\lambda^2\big|_{\lambda_1} & \frac{\partial}{\partial\lambda}(\lambda^2)\big|_{\lambda_1} & \frac{1}{2!}\frac{\partial^2}{\partial\lambda^2}(\lambda^2)\big|_{\lambda_1} & \cdots & 0 & \lambda^2\big|_{\lambda_{l+1}} & \frac{\partial}{\partial\lambda}(\lambda^2)\big|_{\lambda_{l+1}} & \frac{1}{2!}\frac{\partial^2}{\partial\lambda^2}(\lambda^2)\big|_{\lambda_{l+1}} & \cdots & 0 & \lambda_{l+r+1}^2\cdots\lambda_n^2 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
\lambda^{n-2}\big|_{\lambda_1} & \frac{\partial}{\partial\lambda}(\lambda^{n-2})\big|_{\lambda_1} & \frac{1}{2!}\frac{\partial^2}{\partial\lambda^2}(\lambda^{n-2})\big|_{\lambda_1} & \cdots & \frac{1}{(l-1)!}\frac{\partial^{l-1}}{\partial\lambda^{l-1}}(\lambda^{n-2})\big|_{\lambda_1} & \lambda^{n-2}\big|_{\lambda_{l+1}} & \frac{\partial}{\partial\lambda}(\lambda^{n-2})\big|_{\lambda_{l+1}} & \frac{1}{2!}\frac{\partial^2}{\partial\lambda^2}(\lambda^{n-2})\big|_{\lambda_{l+1}} & \cdots & \frac{1}{(r-1)!}\frac{\partial^{r-1}}{\partial\lambda^{r-1}}(\lambda^{n-2})\big|_{\lambda_{l+1}} & \lambda_{l+r+1}^{n-2}\cdots\lambda_n^{n-2} \\
\lambda^{n-1}\big|_{\lambda_1} & \frac{\partial}{\partial\lambda}(\lambda^{n-1})\big|_{\lambda_1} & \frac{1}{2!}\frac{\partial^2}{\partial\lambda^2}(\lambda^{n-1})\big|_{\lambda_1} & \cdots & \frac{1}{(l-1)!}\frac{\partial^{l-1}}{\partial\lambda^{l-1}}(\lambda^{n-1})\big|_{\lambda_1} & \lambda^{n-1}\big|_{\lambda_{l+1}} & \frac{\partial}{\partial\lambda}(\lambda^{n-1})\big|_{\lambda_{l+1}} & \frac{1}{2!}\frac{\partial^2}{\partial\lambda^2}(\lambda^{n-1})\big|_{\lambda_{l+1}} & \cdots & \frac{1}{(r-1)!}\frac{\partial^{r-1}}{\partial\lambda^{r-1}}(\lambda^{n-1})\big|_{\lambda_{l+1}} & \lambda_{l+r+1}^{n-1}\cdots\lambda_n^{n-1}
\end{array}\right] \end{array} \quad (11)
$$

$$f = W_2 C$$

$$C = W_2^{-1} f.$$

Conclusions

The method presented here involves determination of the modified Vandermonde matrix and its inverse. The initial vector f corresponding to each element of the transition matrix can be easily evaluated in view of the recursive relations. The coefficient matrix C is given by $V^{-1}f$ or $W^{-1}f$. These matrix techniques are general and exact and can be easily programmed on a digital computer.[1]

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References

[1] M. L. Liou, "A novel method of evaluating transient response," *Proc. IEEE*, vol. 54, pp. 20–23, January 1966.

[2] W. Everling and M. L. Liou, "On the evaluation of e^{AT} by power series," *Proc. IEEE* (Letters), vol. 55, p. 413, March 1967.

[3] M. L. Liou, "Evaluation of the transition matrix," *Proc. IEEE* (Letters), vol. 55, pp. 228–229, February 1967.

[4] K. R. Rao and N. Ahmed, "Heaviside's expansion of transition matrices," *Proc. IEEE* (Letters), vol. 56, pp. 884–886, May 1968.

[5] A. K. Choudhury, D. R. Choudhury, B. Roy, and A. K. Mandal, "On the evaluation of $e^{A\tau}$," *Proc. IEEE* (Letters), vol. 56, pp. 1110–1111, June 1968.

[6] M. L. Liou and G. E. Alderson, "Evaluation of state transition matrix and related topics," *Proc. 1968 Asilomar Conf. Circuits and Systems.*

[7] B. Roy, A. K. Mandal, D. R. Choudhury, and A. K. Choudhury, "On the evaluation of the state transition matrix," *Proc. IEEE* (Letters), vol. 57, pp. 234–235, February 1969.

[8] N. Ahmed and K. R. Rao, "A determinant formulation for the initial value theorem," *IEEE Trans. Education* (Correspondence), vol. E-10, pp. 255–256, December 1967.

[9] C. F. Chen and R. F. Yates, "A new matrix for the inverse Laplace transformation," *Trans. ASME, J. Basic Engrg.*, vol. 89, pp. 269–272, June 1967.

[10] G. C. Reis, "A matrix formulation for the inverse Vandermonde matrix," *IEEE Trans. Automatic Control* (Correspondence), vol. AC-12, p. 793, December 1967.

[1] A computer program in FORTRAN IV can be obtained, on request, from the authors.

Slowly Varying System $\dot{x} = A(t)x$

Abstract—A limiting case of great importance in engineering is that of slowly varying parameters. For systems described

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by $\dot{x} = A(t)x$, one would intuitively expect that if, for each t, the frozen system is stable, then the time-varying system should also be stable. Provided $\dot{A}(t)$ is small enough, Rosenbrock has shown that this is the case [1]. Rosenbrock used a continuity argument [1, p. 75]. In this correspondence explicit bounds and slightly sharper results are obtained. Finally, it is pointed out that these results are useful in the study of the exact behavior of nonlinear lumped systems with slowly varying operating points.

We first establish the following:

1) If the function $t \to A(t)$ is a matrix-valued piecewise continuous function bounded on $\mathbf{R}_+$, i.e.,

$$a_M \triangleq \sup_{t\geq 0} \| A(t) \| < \infty \quad (1)$$

2) if there is a $\sigma_0 > 0$ such that

$$\operatorname{Re} \lambda_i[A(t)] \leq -2\sigma_0 < 0, \quad \forall i, \quad \forall t \geq 0 \quad (2)$$

then there is a constant m (which depends only on σ_0 and a_M) such that

$$\| \exp \tau A(t) \| \leq m \exp(-\sigma_0 \tau), \quad \forall \tau \geq 0. \quad (3)$$

The inequality (3) is very useful in the study of the stability of slowly varying systems and in some problems of nonlinear circuits. In fact it allows one to give explicit bounds to the important result of Rosenbrock.

First observe that the boundedness of $A(\cdot)$ implies that the functions $t \to \lambda_i(t)$, $i = 1,2,\cdots,n$ are bounded on $\mathbf{R}_+$; indeed det $[\lambda(t)I - A(t)] = 0$ leads to an inequality of the form

$$| \lambda(t) | \leq \sum_{k=1}^{n} | a_k(t) | \, | \lambda(t) |^{-k}$$

where the $a_k(\cdot)$ are bounded since they are sums of products of elements of $A(\cdot)$. Let R be such that

$$| \lambda_i(t) | \leq R/2, \quad \forall i, \quad \forall t \geq 0. \quad (4)$$

The Laplace transform inversion theorem gives

$$\exp [A(t)\tau] = \frac{1}{2\pi j} \int_C [sI - A(t)]^{-1} \exp(s\tau)\, ds \quad (5)$$

where C is the left-half plane closed contour consisting of a vertical segment of abscissa $-\sigma_0$ and an arc of the circle centered on the origin and of radius R. Now for any $n \times n$ matrix M, any norm in $\mathbf{R}^n$, if $\| M \|$ denotes the induced norm then

$$\| M \|^{-1} \leq \beta \| M \|^{n-1} / | \det (M) |. \quad (6)$$

This inequality follows from Carmer's formula; β depends on n and the chosen norm on $\mathbf{R}^n$. Taking norms of both sides of (5), replacing the norm of the integral by the integral of the norm, replacing $| \exp s\tau |$ by $\exp(-\sigma_0\tau)$, and using (6), we obtain

$$\| \exp [A(t)\tau] \| \leq \beta \frac{\exp(-\sigma_0\tau)}{2\pi} \int_C \frac{\| sI - A(t) \|^{n-1}}{| \det [sI - A(t)] |} ds. \quad (7)$$

Noting that 1) the length of the contour C is $\leq 2\pi R$, 2) for all s on C, $| \det [sI - A(t)] | = \Pi | [s - \lambda_i(t)] | \geq \sigma_0^n$, and 3) $A(t)$ is bounded, we conclude that, for all $t \in \mathbf{R}_+$, the integral in (7) is bounded by

$$2\pi R k^{n-1} / \sigma_0^n \quad (8)$$

where k is a constant depending on R and a_M. From (7) and (8) the assertion (3) follows.

Application

Under assumptions 1) and 2), Rosenbrock has shown that if, in addition

$$\sup_{t\geq 0} \| \dot{A}(t) \| \triangleq \dot{a}_M \quad (9)$$

is sufficiently small, then the system $\dot{x} = A(t)x$ is asymptotically stable. We shall follow his derivation, show that the system is exponentially stable, and obtain explicit bounds.

Let $\epsilon_1 > 0$, and pick

$$V(x,t) = x^T(\epsilon_1 I + P(t))x \quad (10)$$

as a possible Lyapunov function. Then

$$\dot{V} = x^T(A^T P + PA)x + \epsilon_1 x^T(A^T + A)x + x^T \dot{P} x \quad (11)$$

(where the explicit dependence on t is understood). Pick $P(t)$ so that

$$A^T(t)P(t) + P(t)A(t) = -3I \quad (12)$$

or

$$P(t) = 3 \int_0^\infty \exp [\tau A^T(t)] \exp [\tau A(t)]\, d\tau. \quad (13)$$

Differentiating (12) and solving for $\dot{P}$, we obtain

$$\dot{P}(t) = \int_0^\infty \exp [\tau A^T(t)] \cdot [\dot{A}^T(\tau)P(\tau) + P(\tau)\dot{A}(\tau)] \cdot \exp [\tau A(t)]\, d\tau. \quad (14)$$

Hence from (3) and (9)

$$\| \dot{P}(t) \| \leq \dot{a}_M 3m^4 (2\sigma_0)^{-2}, \quad \forall t \geq 0. \quad (15)$$

Using (1), (12), and (15) in (11), we get

$$\dot{V} \leq x^T x[-3 + 2\epsilon_1 a_M + \dot{a}_M 3m^4 (2\sigma_0)^{-2}]. \quad (16)$$

Thus if, for example

$$0 < \epsilon_1 \leq 1/(2a_M)$$

and

$$\dot{a}_M \leq 4\sigma_0^2/(3m^4)$$

then

$$\dot{V} \leq -x^T x, \quad \forall x \in R^n, \quad \forall t \geq 0. \quad (17)$$

The conditions of the Lyapunov theorem hold [2]: V is positive definite since $\epsilon_1 > 0$ and $P(t)$ is positive definite; it is decrescent, since by (3), $\sup \| P(t) \| \leq 3m^2(2\sigma_0)^{-1}$. Hence, uniform asymptotic stability follows. Furthermore, from (10) and (17)

$$\dot{V}/V \leq -1/2b.$$

where $2b = \epsilon_1 + 3m^2(2\sigma_0)^{-1}$. Hence,

$$\| x(t,t_0,x_0) \| \leq \| x_0 \| [\epsilon_1 + 3m^2(2\sigma_0)^{-1}]^{1/2} \cdot \exp [-(t - t_0)/b]$$

i.e., any solution is uniformly exponentially stable. Q.E.D.

In particular, it implies that for some finite m'

$$\| \Phi(t,t_0) \| \leq m' \exp [-(t - t_0)/b], \quad \forall t_0, \forall t \geq t_0$$

The essential point is that m' is independent of t and t_0. This inequality implies that the conditions required by the basic theorem of the small signal theory of nonlinear lumped systems holds in the slowly varying case [3].

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References

[1] H. H. Rosenbrock, "The stability of linear time-dependent control systems," *J. Electronics and Control*, vol. 15, pp. 73–80, July 1963.
[2] W. Hahn, *Stability of Motion*. New York: Springer-Verlag, 1967, p. 199, Theorem 42.4.
[3] C. A. Desoer and K. K. Wong, "Small-signal behavior of nonlinear lumped networks," *Proc. IEEE*, vol. 56, pp. 14–22, January 1968.

Comments on "On Linear Sampled-Data Feedback Systems with Finite Pulse Width"

***Abstract*—Certain aspects of a recent correspondence by Jury and Kan[1] are discussed. It is pointed out that deterministic sampling schemes may be regarded**

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[1] E. I. Jury and E. P. F. Kan, *IEEE Trans. Automatic Control* (Correspondence), vol. AC-13, pp. 742–743, December 1968.

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Sufficient conditions for stability of linear time-varying systems

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Abstract: In this paper we consider sufficient conditions for the exponential stability of linear time-varying systems of the form $\dot{x}(t) = A(t)x(t)$, $t \geq 0$. Stability guaranteeing upper bounds for different measures of parameter variations are derived.

Keywords: Time-varying linear systems, Exponential stability.

1. Introduction

Stability analysis for time-varying linear systems is of increasing interest in control theory. One reason is the growing importance of adaptive controllers for which the underlying closed-loop adaptive system often is time-varying and linear.

In this paper we analyse exponential stability for systems of the form $\dot{x}(t) = A(t)x(t)$, $t \geq 0$, where $A(\cdot): \mathbb{R}_+ \to \mathbb{R}^{n \times n}$ is piecewise continuous and uniformly bounded. Furthermore for every $t \geq 0$ the eigenvalues of $A(t)$ are contained in a left half plane $\mathbb{C}^{-\varepsilon} = \{s \in \mathbb{C} \mid \operatorname{Re} s \leqslant -\varepsilon\}$ for some $\varepsilon > 0$. However this last condition is not strong enough to guarantee exponential stability. Additional restrictions on the parameter variations in $A(\cdot)$ have to be imposed.

In Section 2 we summarize different types of those sufficient parameter variation conditions including the well known criteria of Coppel [2] and Rosenbrock [5] and two new conditions due to Kreisselmeier [4] and Krause and Kumar [3]. We give a new *short* proof of the Krause and Kumar result which was recently published in this journal. However all mentioned conditions are qualitative results in the sense that if some measure of the parameter variation is 'sufficiently small' the exponential stability is ensured.

In Section 3 we derive *explicit formulas* for the parameter variations upper bound to guarantee exponential stability. These formulas involve some a priori knowledge of $\| A(t) \|$ and $\sigma(A(t))$.

2. Sufficient conditions for exponential stability

To derive stability results for linear time-varying systems of the form

$$\dot{x}(t) = A(t)x(t), \quad t \geq 0, \tag{2.1}$$

it is usually a priori required that $A(\cdot)$ belongs to the set $\mathcal{S}$ of all piecewise continuous matrix functions

$$A(\cdot): \mathbb{R}_+ \to \mathbb{R}^{n \times n} \tag{2.2}$$

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which satisfy:

there exists $M > 0$ such that $\| A(t) \| \le M$ for all $t \ge 0$, (2.3)

there exists $\alpha > 0$ such that $\sigma(A(t)) \subset \mathbb{C}^{-\alpha} := \{ s \in \mathbb{C} \mid \operatorname{Re} s < -\alpha \}$ for all $t \ge 0$. (2.4)

The following definition of exponential stability is standard:

2.1. Definition. A system (2.1) is called *exponentially stable* if there exist L, $\lambda > 0$ such that

$$\| \phi(t, t_0) \| \le L\, e^{-\lambda(t-t_0)} \quad \text{for all } t \ge t_0 \ge 0,$$

where $\phi(\cdot, \cdot)$ denotes the transition matrix of (2.1).

In general $A(\cdot) \in \mathcal{S}$ is neither necessary nor sufficient for exponential stability.

2.2. Examples. (i) Coppel [2,p.3]. Let

$$A(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} -1 & -5 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}.$$

Then $\sigma(A(t)) = \{-1\}$ for all $t \ge 0$ and a calculation of a fundamental matrix shows that $A(\cdot)$ is unstable.

(ii) Wu [6]. Let

$$A(t) = \begin{bmatrix} -\frac{11}{2} + \frac{15}{2} \sin 12t & \frac{15}{2} \cos 12t \\ \frac{15}{2} \cos 12t & -\frac{11}{2} - \frac{15}{2} \sin 12t \end{bmatrix}.$$

Then $\sigma(A(t)) = \{2, -13\}$ for all $t \ge 0$; however the associated system $\dot{x}(t) = A(t)x(t)$ is exponentially stable.

In order obtain sufficient conditions for exponential stability additional restrictions for the variation of the elements of $A(\cdot) \in \mathcal{S}$ have to be imposed.

It has been shown that if $\delta > 0$ is sufficiently small, then any of the following conditions guarantees exponential stability of (2.1):

$\| \dot{A}(t) \| \le \delta$ for all $t \ge 0$ [5]. (2.5)

$\| A(t_2) - A(t_1) \| \le \delta \| t_2 - t_1 \|$ for all $t_1, t_2 \ge 0$ [2,p.5]. (2.6)

$\sup_{0 \le \tau \le h} \| A(t+\tau) - A(t) \| \le \delta$ for some $h > 0$. (2.7)

$\dot{A}(\cdot)$ is continuous, $\| \dot{A}(\cdot) \|$ is uniformly bounded
and there exists $T > 0$ such that $\int_{t_0}^{t_0+T} \| \dot{A}(t) \| \, dt \le \delta T$ for all $t_0 \ge 0$. (2.8)

Condition (2.7) is a consequence of Theorem 3.2 (iii) in Section 3 of this paper and is less restrictive than the similar condition in Lemma 3 of [4]:

$$\lim_{t \to \infty} \sup_{0 \le \tau \le h} \| A(t+\tau) - A(t) \| = 0 \quad \text{for all } h > 0.$$

Furthermore (2.8) is less restrictive than the criterion in [3] which requires the integral inequality of (2.8) for all $T \ge T_0$ and some $T_0 > 0$. In fact condition (2.8) can be proved much shorter following the ideas of Rosenbrock's proof for condition (2.5).

'Simpler' proof of the Krause and Kumar condition. Let $A(\cdot) \in \mathcal{S}$, $\dot{A}(\cdot)$ continuous and uniformly bounded. Suppose the integral condition in (2.8) is satisfied for some $T > 0$, $\delta > 0$. Then the set

$$\tau_A(\varepsilon) := \{ t \in R_+ \mid \| \dot{A}(\cdot) \| > \varepsilon \}$$

is a union of open intervals since $\dot{A}(\cdot)$ is continuous. If $\mathscr{L}(I)$ denotes the Lebesgue measure of measurable sets $I \subset \mathbb{R}$, then by (2.8),

$$\mathscr{L}((t_0, t_0+T) \cap \tau_A(\varepsilon)) \cdot \varepsilon \leq \delta T \quad \text{for all } t_0 \geq 0.$$

Thus $\mathscr{L}((t_0, t_0+T) \cap \tau_A(\varepsilon)) \to 0$ as $\delta \to 0$. Let now

$$V(x, t) := x^{\mathrm{T}} R(t) x \tag{2.9}$$

where

$$R(t) := \int_0^\infty \mathrm{e}^{A^{\mathrm{T}}(t)s}\, \mathrm{e}^{A(t)s}\, \mathrm{d}s. \tag{2.10}$$

Similar to Rosenbrock's [5] proof we show that if δ is sufficiently small, then $V(x, t)$ is a Liapunov function for (2.1). Since $A(\cdot)$ is bounded there exist (cf. Brockett [1,p.203]) c_1, $c_2 > 0$ such that $c_1 I_n \leq R(t) \leq c_2 I_n$. Furthermore because $R(t)A(t) + A^{\mathrm{T}}(t)R(t) = -I_n$ we obtain

$$\dot{R}(t) = \int_0^\infty \mathrm{e}^{A^{\mathrm{T}}(t)s} \left[R(t)\dot{A}(t) + \dot{A}^{\mathrm{T}}(t)R(t) \right] \mathrm{e}^{A(t)s}\, \mathrm{d}s \tag{2.11}$$

and for ε sufficiently small,

$$\begin{aligned} \dot{V}(x(t), t) &= \langle x(t), \left[\dot{R}(t) - I_n\right] x(t) \rangle \\ &\leq -\tfrac{1}{2} \| x(t) \|^2 \leq -\frac{1}{2c_2} V(x(t), t) \quad \text{for } t \notin \tau_A(\varepsilon). \end{aligned}$$

Since $\| \dot{A}(t) \| \leq K^*$ for some $K^* > 0$,

$$\dot{V}(x(t), t) \leq K \| x(t) \|^2 \leq \frac{K}{c_1} V(x(t), t) \quad \text{for } t \in \tau_A(\varepsilon)$$

for some $K \geqslant K^*$. Since $\mathscr{L}(\tau_A(\varepsilon)) \to 0$ as $\delta \to 0$, there exists for δ sufficiently small $\omega > 0$ such that

$$\int_{t_0}^t \frac{\dot{V}(x(s), s)}{V(x(s), s)}\, \mathrm{d}s \leq -\omega(t - t_0) \quad \text{for all } t \geq t_0 + T,$$

and this proves exponential decaying of the solutions of (2.1). □

As an existence condition (2.8) is not really an improvement of (2.5). The above proof shows that if $\| \dot{A}(\cdot) \|$ is assumed to be uniformly bounded there always exists sufficiently small $\delta > 0$ such that the *average* parameter variation condition (2.8) implies that $A(\cdot)$ satisfies Rosenbrock's criterion. The converse direction is obvious. Hence, viewed as existence conditions (2.8) and (2.5) are *equivalent*. In order to show that (2.8) is *less conservative* than (2.5), upper bounds for the integral in (2.8) should be derived which guarantee exponential stability of (2.1). In the next section we determine such quantitative bounds for the δ's involved in (2.5)–(2.7).

3. Upper bounds for parameter variations

In order to prove the main result of this paper we need a lemma and some preparatory formulas:

3.1. Lemma [2]. *Suppose $A(\cdot) \in \mathscr{S}$, in particular $\| A(t) \| \leq M$. Then for every $\varepsilon \in (0, 2M)$,*

$$\| \mathrm{e}^{A(t)\sigma} \| \leq (2M/\varepsilon)^{n-1}\, \mathrm{e}^{(-\alpha+\varepsilon)\sigma} \quad \textit{for all } \sigma,\ t \geq 0.$$

For fixed $t_0 \in \mathbb{R}_+$, (2.1) can be rewritten in the form

$$\dot{x}(t) = A(t_0)x(t) + [A(t) - A(t_0)]x(t), \quad t \geq 0, \tag{3.1}$$

and for $x(t_0) = x_0 \in \mathbb{R}^n$ its solution is given by

$$x(t) = e^{A(t_0)(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t_0)(t-s)}[A(s) - A(t_0)]x(s)\, ds. \tag{3.2}$$

Thus for $A(\cdot) \in \mathcal{S}$ Coppel's Lemma yields

$$\begin{aligned} \| x(t) \| &\leq \kappa_\varepsilon\, e^{(-\alpha+\varepsilon)(t-t_0)} \| x_0 \| \\ &\quad + \kappa_\varepsilon \int_{t_0}^{t} e^{(-\alpha+\varepsilon)(t-s)} \| A(s) - A(t_0) \| \| x(s) \|\, ds \quad \text{for } t \geq t_0, \end{aligned} \tag{3.3}$$

where

$$\kappa_\varepsilon := (2M/\varepsilon)^{n-1}. \tag{3.4}$$

Applying Gronwalls's Lemma to (3.3) gives

$$\| x(t) \| \leq \kappa_\varepsilon \exp\left[(-\alpha+\varepsilon)(t-t_0) + \kappa_\varepsilon \int_{t_0}^{t} \| A(s) - A(t_0) \|\, ds\right] \| x_0 \| \quad \text{for all } t \geq t_0. \tag{3.5}$$

3.2. Theorem. *Suppose $A(\cdot) \in \mathcal{S}$. Then*

$$\dot{x}(t) = A(t)x(t), \quad t \geq 0, \tag{3.6}$$

is exponentially stable if one of the following conditions holds for all $t \geq 0$:

(i) $\alpha > 4M$.

(ii) *$A(\cdot)$ is piecewise differentiable and*

$$\| \dot{A}(t) \| \leq \delta < \frac{2}{2n-1} \frac{\alpha^{4n-2}}{2M^{4n-4}}.$$

(iii) *For some $k \geq 0$, $\eta \in (0, 1)$, $\alpha > 2M\eta + ((n-1)/k)\log \eta$ and*

$$\sup_{0 \leq \tau \leq k} \| A(t+\tau) - A(t) \| \leq \delta < \eta^{n-1}\left(\alpha - 2M\eta + \frac{n-1}{k} \log \eta\right).$$

(iv) *$\alpha > n-1$ and for some $\eta \in (0, 1)$,*

$$\sup_{h>0} \left\| \frac{A(t+h) - A(t)}{h} \right\| \leq \delta < 2\eta^{n-1}(\alpha - 2M\eta + (n-1) \log \eta).$$

Proof. (i) Since $\| A(s) - A(t_0) \| \leq 2M$ $\forall s$, $t_0 \geq 0$, inequality (3.5) implies for some $h > 0$ and $\varepsilon \in (0, 2M)$,

$$\| x(t) \| \leq \kappa_\varepsilon\, e^{[-\alpha+\varepsilon+\kappa_\varepsilon 2M](t-t_0)} \| x(t_0) \| \leq \kappa_\varepsilon\, e^{[\varepsilon+\kappa_\varepsilon 2M-4M-h]} \| x(t_0) \|.$$

The function

$$f: (0, 2M] \to R, \quad \varepsilon \mapsto \varepsilon + \kappa_\varepsilon 2M - 4M - h,$$

is continuous and $f(2M) = -h$. Thus there exists $\varepsilon \in (0, 2M]$ such that $f(\varepsilon) < 0$.

(ii) We prove that $V(x, t)$ as defined in (2.9) is a Liapunov function of (3.6). Its time derivative along solutions of (3.6) is

$$\frac{d}{dt} V(x(t), t) = x^{T}(t)[-I_n + \dot{R}(t)]x(t).$$

We have to show that

$$\dot{R}(t) < I_n \quad \text{for all } t \geq 0. \tag{3.7}$$

Applying Coppel's Lemma to (2.11) and (2.10) we obtain

$$\begin{aligned} \| \dot{R}(t) \| &\leq \int_0^\infty \left(\frac{2M}{\varepsilon}\right)^{2(n-1)} \mathrm{e}^{2(-\alpha+\varepsilon)s} \,\mathrm{d}s \cdot 2 \| R(t) \| \| \dot{A}(t) \| \\ &\leq 2\left[\left(\frac{2M}{\varepsilon}\right)^{2(n-1)} \int_0^\infty \mathrm{e}^{2(-\alpha+\varepsilon)s} \,\mathrm{d}s\right]^2 \delta = 2\left(\frac{2M}{\varepsilon}\right)^{4(n-1)} \left(\frac{1}{2(-\alpha+\varepsilon)}\right)^2 \delta \end{aligned}$$

and thus (3.7) holds if for some $\varepsilon \in (0, \alpha)$,

$$\delta < 2\left(\frac{\varepsilon}{2M}\right)^{4(n-1)} (\alpha - \varepsilon)^2 =: g(\varepsilon).$$

It is easily verified that $g(\cdot)$ achieves its maximum at $\varepsilon_0 = \alpha k/(k+1)$ on $(0, \alpha)$ and

$$g(\varepsilon_0) = \frac{2\alpha^{4n-2}}{(2M)^{4(n-1)}(n-1)}.$$

(iii) By (3.5) we have for every $l \in \mathbb{N}$, $k > 0$ and $t \in [t_0 + lk,\ t_0 + (l+1)k]$,

$$\begin{aligned} \| x(t) \| &= \kappa_\varepsilon \,\mathrm{e}^{\gamma(t-t_0-lk)} \| x(t_0 + lk) \| \\ &\leq \kappa_\varepsilon \,\mathrm{e}^{\gamma(t-t_0-lk)} \kappa_\varepsilon \,\mathrm{e}^{\gamma(k)} \| x(t_0 + (l-1)k) \| = \kappa_\varepsilon^2 \,\mathrm{e}^{\gamma(t-t_0-(l-1)k)} \| x(t_0 + (l-1)k) \| \\ &\leq \cdots \leq \kappa_\varepsilon^{l+1} \,\mathrm{e}^{\gamma(t-t_0)} \| x(t_0) \| \end{aligned}$$

where $\gamma := -\alpha + \varepsilon + \kappa_\varepsilon \delta$. Thus

$$\| x(t) \| \leq \kappa_\varepsilon \,\mathrm{e}^{l \log \kappa_\varepsilon + \gamma(t-t_0)} \| x(t_0) \|.$$

Because $t - t_0 \geq lk$ we obtain

$$\| x(t) \| \leq \kappa_\varepsilon \exp\left\{\left(\frac{\log \kappa_\varepsilon}{k} + \gamma\right)(t - t_0)\right\} \| x(t_0) \|.$$

Consider

$$\frac{\log \kappa_\varepsilon}{k} + \gamma = \frac{\log(2M/\varepsilon)^{n-1}}{k} + \varepsilon + \kappa_\varepsilon \delta - \alpha.$$

It suffices to find $\varepsilon < 2M$, $k > 0$ such that $(\log \kappa_\varepsilon)/k + \gamma < 0$ or equivalently,

$$(\varepsilon - \alpha) + \frac{\log \kappa_\varepsilon}{k} < -\kappa_\varepsilon \delta,$$

respectively

$$0 < \delta < \frac{1}{\kappa_\varepsilon}\left(\alpha - \varepsilon - \frac{\log \kappa_\varepsilon}{k}\right).$$

However to every $\varepsilon \in (0, \alpha)$ exists $k^* > 0$ such that $\alpha - \varepsilon - (\log \kappa_\varepsilon)/k^* > 0$ and we obtain exponential stability of (3.6) for every $A(\cdot)$ for which

$$\sup_{0 \leq \tau \leq k^*} \| A(t+\tau) - A(t) \| \leq \delta < \frac{1}{\kappa_\varepsilon}\left(\alpha - \varepsilon - \frac{\log \kappa_\varepsilon}{k^*}\right) > 0$$

Now (iii) follows with $\eta = \varepsilon/2M$.

(iv) Assume

$$\left\| \frac{A(t+h)-A(t)}{h} \right\| \le \delta$$

for every $h > 0$. Then by (3.5) we have

$$\begin{aligned}\| x(t) \| &\le \kappa_\varepsilon \exp\left\{(-\alpha+\varepsilon)(t-t_0)+\kappa_3\int_0^{t-t_0} h\delta \,\mathrm{d}h\right\} \| x(t_0) \| \\ &\le \kappa_\varepsilon \exp\left\{\left(\varepsilon-\alpha+\kappa_\varepsilon\frac{t-t_0}{2}\delta\right)(t-t_0)\right\} \| x(t_0) \|.\end{aligned}$$

For $t \in [t_0, t_0+1]$ we obtain

$$\| x(t) \| \le \kappa_\varepsilon \,\mathrm{e}^{\gamma(t-t_0)} \| x(t_0) \|$$

where $\gamma = \varepsilon - \alpha + \frac{1}{2}\kappa_\varepsilon\delta$. For $t \in [t_0+l, t_0+l+1]$, $l \in \mathbb{N}$ arbitrary, we conclude as in the proof of (iii),

$$\| x(t) \| \le \kappa_\varepsilon \,\mathrm{e}^{(\log \kappa_\varepsilon+\gamma)(t-t_0)} \| x(t_0) \|$$

and $\log \kappa_\varepsilon + \gamma < 0$ if

$$\delta < 2\left(\frac{\varepsilon}{2M}\right)^{n-1}\left(\alpha-\varepsilon-\log\left(\frac{2M}{\varepsilon}\right)^{n-1}\right).$$

Then (iv) follows with $\eta = \varepsilon/2M$. □

Note that the proof of (iii) presents a short proof of Lemma 3 in Kreisselmeier [4].

If additional information on the exponential decay of $\mathrm{e}^{A(t)\tau}$ is known, the bounds derived in Theorem 3.2 can be simplified as follows:

3.3. Theorem. *Suppose $A(\cdot) \in \mathcal{S}$ and let*

$$\|\mathrm{e}^{A(t)\sigma}\| \le \kappa \,\mathrm{e}^{-\omega\sigma} \quad \textit{for all } t,\ \sigma \ge 0,$$

for some κ, $\omega > 0$. Let $\beta := \omega/\kappa$. Then the system

$$\dot{x}(t) = A(t)x(t), \quad t \ge 0,$$

is exponentially stable if one of the following conditions holds for all $t \ge 0$:

(i) $\| A(t) \| \le M < \frac{1}{2}\beta$.

(ii) *$A(\cdot)$ is piecewise differentiable and*

$$\| \dot{A}(t) \| \le \delta < 2\beta^2.$$

(iii) *There exists $h > 0$ such that*

$$\sup_{0\le\tau\le h} \| A(t+\tau)-A(t) \| \le \delta < \beta - \frac{1}{h}\frac{\log \kappa}{\kappa}.$$

(iv) $\sup_{h>0}\left\| \frac{A(t+h)-A(t)}{h} \right\| \le \delta < 2\beta - 2\frac{\log \kappa}{\kappa}$.

The proof is similar to that of Theorem 3.2.

References

[1] R.W. Brockett, *Finite Dimensional Linear Systems* (John Wiley, New York, 1970).
[2] W.A. Coppel, *Dichotomies in Stability Theory*, Lecture Notes in Mathematics No. 629 (Springer, Berlin–New York, 1978).
[3] J.M. Krause and K.S.P. Kumar, An alternative stability analysis framework for adaptive control, *Systems Control Lett.* **7** (1986) 19–24.
[4] G. Kreisselmeier, An approach to stable indirect adaptive control, *Automatica* **21** (9) (1985) 425–431.
[5] H.H. Rosenbrock, The stability of linear time-dependent control systems, *Internat. J. Electr. Control* **15** (1) (1963) 73–80.
[6] M.Y. Wu, A note on stability of linear time-varying systems, *IEEE Trans. Automat. Control* **19** (1974) 162.

THE SOLUTIONS OF THE MATHIEU EQUATION WITH A COMPLEX VARIABLE AND AT LEAST ONE PARAMETER LARGE*

BY

RUDOLPH E. LANGER

Introduction. The Mathieu differential equation

$$\frac{d^2u}{dz^2} + \{\Delta - \Omega \cos 2z\} u = 0, \tag{1}$$

also commonly known as the equation of the elliptic cylinder functions, is too well known to require any introduction. Its solutions govern problems of the greatest diversity in astronomy and theoretical physics, and have accordingly been the subjects of a vast number of investigations.†

The differential equation as such depends upon two independent parameters, designated in the form written above by Δ and Ω. In the present discussion these are to be taken real but are to be numerically unrestricted except that at least one is to be large. The variable will be permitted to range over the complex plane.

Since the coefficient of the differential equation is an even simply periodic analytic function of z, it is known from Floquet's theory of such equations that the solutions are in general of the structure

$$u(z) = c_1 e^{\mu z}\phi(z) + c_2 e^{-\mu z}\phi(-z),$$

in which the function $\phi(z)$ is periodic. The *characteristic exponent*, μ, is a constant as to z but depends in an intricate way upon the parameters Δ and Ω. If it is real, the equation obviously possesses a solution which for large real values of the variable becomes exponentially infinite, i.e., a so called *unstable* solution. In the alternative case the exponent is pure imaginary and the solutions remain bounded along the axis of reals, i.e., are of the so called *stable* type. The intermediate case in which $\mu = 0$ is of especial im-

* Presented to the Society, April 6, 1934; received by the editors February 12, 1934.

† Cf. for the literature and for partial enumerations of applications of the equation: Strutt, M. J. O., *Lamésche-Mathieusche und verwandte Funktionen in Physik und Technik*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 1, No. 3, Berlin, 1932; Whittaker and Watson, *A Course in Modern Analysis*, 3d edition, 1920, Cambridge University Press; Humbert, P., *Fonctions de Lamé et Fonctions de Mathieu*, Mémorial des Sciences Máthematiques, X, Paris, 1926; Van der Pol, B., and Strutt, M. J. O., *On the stability of the solutions of Mathieu's equation*, The Philosophical Magazine, vol. 5 (1928), p. 18.

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portance, for the equation then admits one solution known as a *Mathieu function* which is periodic. The second solution, a *Mathieu function of the second kind*, is then not periodic and is of a functional structure distinct from that indicated above.

With either of the parameters Δ and Ω fixed, the relation $\mu=0$ restricts the remaining one to a denumerably infinite set of values called the *characteristic values*. Broadly speaking the determination of these values and of the corresponding Mathieu functions is the matter of prime importance in the applications of the equation which belong more immediately to the domain of physics, while the determination of the characteristic exponent in terms of a fixed set of parameters is generally the peculiar requirement of the applications to astronomy.

When the values of the parameters are small the solution of the differential equation is generally and appropriately essayed through the means of convergent series expansions. When at least one of the parameters is large, on the other hand, the methods of asymptotic representation are adapted and have been generally applied. Though the literature covering investigations of this latter type is large it can hardly be said that the results recorded are by any means complete. Restrictions upon the range of the parameters are generally made and frequently only the forms of the Mathieu functions, i.e., of the solutions with the period 2π, are considered. Again, when forms asymptotic with respect to one parameter are obtained their dependence upon the remaining secondary parameter may not be considered, the results being established, therefore, only for a fixed configuration of the parameters relative to each other. Finally the investigations have almost exclusively been restricted to the case of a real variable. The most recent report on the status of the theory* says on this point: "While we believe that the theory of the Hill and Mathieu differential equations with *real* variables and parameters has to a certain extent been rounded out, it is to be emphasized that no such assertion can be made concerning these equations with *complex* variables and parameters. . . . Only when the problems bearing upon this point have been adequately treated may it be hoped to round out the theory of the Lamé equation as has been done in the case of the equation of Mathieu. Such an investigation would not only throw new light upon many differential equations of mathematical physics, but would make possible the application of certain of the functions obtained to problems of practical importance."

The present investigation is devoted to a general consideration of the asymptotic solutions of the Mathieu equation over the complex plane and for all real configurations of the parameter values in which at least one is

* Strutt, loc. cit. (Vorwort).

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numerically large. The analytic forms which represent the solutions asymptotically are found to differ in essentially different parameter configurations, while in its dependence upon the variable such a representation even for a specific solution and with one and the same configuration of parameters requires the employment of a variety of analytic forms. In general a special form is required for the description in the neighborhood of any point in which the coefficient of the equation vanishes, while outside such neighborhoods several forms again are made necessary by the incidence of the Stokes' phenomenon.

The limitation of the discussion to real parameter values was imposed to keep the extent of the investigation within its present bounds. The method in no way requires such a restriction.* In the matter of the method the present paper is based upon earlier papers of the author† which gave a general derivation of the asymptotic solutions of differential equations of the type

$$\frac{d^2u}{dz^2} + \left\{\rho^2\chi_0^2(z) + \rho\chi_1(z) + \chi_2(z, \rho)\right\}u = 0,$$

in which ρ is a large complex parameter and the coefficient $\chi_0^2(z)$ vanishes at some point of the domain considered. Aside from the considerations peculiar to the Mathieu equation, however, the presence of two independent parameters makes of the present discussion something more than a specialization of the general theory cited. With one parameter assigned to a primary role it must be shown that the hypotheses of the theory cited are met *uniformly* with respect to the secondary parameter which has remained free. This is essential to assure the uniform validity of the conclusions, i.e., that the degree of approximation afforded by the asymptotic representation is maintained during a variation of the parameters within the bounds of a given configuration.

By way of arrangement there have been grouped in chapter 1 such general considerations as are to be subsequently available. Of the following chapters each is given to the deductions peculiar to a specific configuration of parameters. Throughout the paper the forms of two fundamental pairs of solutions are deduced. This is desirable because of the fact that the members of any one pair of solutions may and do become asymptotically indistinguishable in certain regions of the complex plane. Aside from the general asymp-

* An analogous application of the method to a study of the Bessel functions with both the variable and the parameter complex was made by the author in the papers cited below.

† These Transactions, as follows: *On the asymptotic solutions of ordinary differential equations*, etc., vol. 33 (1931), p. 23; *On the asymptotic solutions of differential equations*, etc., vol. 34 (1932), p. 447; *The asymptotic solutions of certain linear ordinary differential equations of the second order*, vol. 36, p. 90. These papers will be referred to in the text by the designations L_1, L_2 and L_3.

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totic forms the special forms which apply to real values of the variable are noted, and the forms of the solutions of the *associated Mathieu equation*,

$$\frac{d^2v}{dz^2} + \{\Omega \cosh 2z - \Delta\}v = 0, \tag{2}$$

are deduced. The asymptotic equations for the characteristic values are given, and the characteristic exponent is asymptotically determined.

Chapter 1

General considerations

1.1. The parameter configurations. The effect of replacing the variable z by $z+\pi/2$ in the equation (1) is merely to alter the sign of the cosine function, i.e., to replace the parameter Ω by its negative. There is, therefore, no loss of generality in assuming, as will henceforth be done, that Ω ranges only over the positive values and zero. The parameter Δ, on the other hand, is to range unrestrictedly over all real values.

For any positive Ω, however small it may be, the term $\Omega \cos 2z$ becomes dominant over Δ when z reaches a domain sufficiently remote from the axis of reals. In any such domain therefore the character of the differential equation is essentially altered if Ω is replaced by zero, and it may accordingly be expected that formulas which are to be valid *uniformly* for $\Omega \geqq 0$ may be obtained only for regions of the z plane in which $|\vartheta(z)|$ is bounded. This fact suggests the grouping into separate configurations of those sets of parameter values in which Ω is relatively small. They are indicated as II and IX in Figure 1 below, the precise specifications to be later determined.

When $\Omega > 0$, the function $\{\Delta - \Omega \cos 2z\}$ vanishes at an infinite set of points in the complex plane. As z moves at a suitable distance about any such point the asymptotic forms which represent a given solution of the differential equation must be altered, i.e., replaced by others, at certain specifiable intervals. This so called Stokes' phenomenon depends quantitatively upon the order of the zero which is encircled, and since this order changes from the first to the second when Ω and $|\Delta|$ become equal, it may be expected that results obtained on the assumption that the parameters are sufficiently different in numerical value may not remain *uniformly* valid when these values are allowed to approach equality. This fact serves as the motivation for considering as distinct configurations those indicated in Figure 1 by the designations IV and VII, in which the parameters numerically approximate each other. They will be precisely defined at appropriate points in the discussion which follows. The division of the half-plane of the coordinates (Δ, Ω) into configurations is, therefore, such as is indicated in the

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figure, the hypothesis that at least one parameter be large having the effect of excluding from consideration a neighborhood of the point O.

1.2. **The hypotheses of the general theory.** The differential equation (1) may be transformed in a variety of ways into an equation of the general form

$$\frac{d^2u}{ds^2} + \{\rho^2\chi_0^2(s, \sigma) + \rho\chi_1(s, \sigma)\}u = 0, \tag{3}$$

in which ρ, the primary parameter, and σ, the secondary parameter, are expressible in terms of Δ and Ω. The particular substitutions and hence the particular equations which result are to depend upon the parameter configuration which obtains, and will therefore be made at appropriate points as the discussion proceeds.

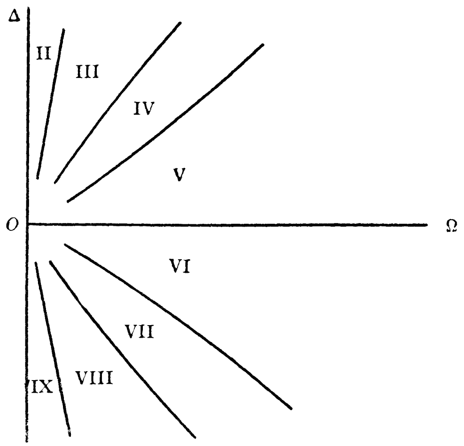

FIG. 1

Equations of the type (3) in which, however, the parameter σ is absent (i.e., fixed) are familiar, the asymptotic forms of their solutions having been deduced* under hypotheses which for the present purposes may be enumerated in the following way:

(i) *The range of the complex variable s is to be a region R_s in which the functions*

$$(s - s_0)^{-\nu}\chi_0^2(s) \text{ and } \chi_1(s)$$

are analytic, s_0 being some point of R_s and ν being some real non-negative constant. Except in some fixed neighborhood of s_0 the several functions

$$\{\chi_0(s)\}^{-1}, \quad \left\{\int_{s_0}^{s} \chi_0 ds\right\}^{-1}, \quad \{\chi_1(s)/\chi_0^2(s)\}, \quad \left\{\int (\chi_1/\chi_0)ds\right\}\left\{\int_{s_0}^{s}\chi_0 ds\right\}^{-1} \tag{4}$$

are to be bounded.

---

* Papers L_2 and L_3 cited above. In the formulas of paper L_3 the variables λ, χ_1, ϕ, and ξ must be replaced by $i\rho$, $i\chi_1$, 2ϕ and $2i\xi$ respectively in order that they may appear as given here.

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It is convenient to have at hand the following definitions:

$$k = \frac{-i\chi_1(s_0)}{4\chi_0'(s_0)}, \qquad \eta(s) = \frac{i\chi_1(s)}{\chi_0(s)} + \frac{2k\chi_0(s)}{\int_{s_0}^{s}\chi_0 ds},$$

$$\phi(s) = \chi_0(s) - \frac{i\eta(s)}{2\rho}, \qquad \Phi(s) = \int_{s_0}^{s}\phi(s)ds, \quad \xi = \rho\Phi. \tag{5}$$

It follows then, as may be shown, from the hypothesis (i) that the functions

$$\omega(\phi) \equiv \frac{3}{4}\left(\frac{\phi'}{\phi}\right)^2 - \frac{1}{2}\left(\frac{\phi''}{\phi}\right) - \frac{\nu(\nu+4)}{4(\nu+2)^2}\left(\frac{\phi}{\Phi}\right)^2,$$

$$\omega_1 \equiv \frac{\eta^2}{4} + \frac{k\chi_0^2\int_{s_0}^{s}\eta ds}{\Phi\int_{s_0}^{s}\chi_0 ds} - \frac{k(4\eta\chi_0 - i\eta^2\rho^{-1})}{2\Phi}, \tag{6}$$

$$\Psi \equiv \frac{\Phi^{\nu/(2\nu+4)}}{\phi^{1/2}}$$

are continuous in the region R_s inclusive of the point $s=s_0$. A second and third hypothesis* made are the following:

(ii) *The differential equation* (3) *is to be in normal form, i.e., such that either* $\chi_1\equiv 0$ *or else* $\nu=2$ *and*

$$\{3\chi_0'\chi_1' - 2\chi_0''\chi_1\}_{s=s_0} = 0.$$

(iii) *Either the region R_s is to be bounded, or else there are to exist constants M and H such that the relations*

$$\int\left|\frac{\omega(\phi)}{\phi}ds\right| < M, \qquad \int\left|\frac{\omega_1(s)}{\phi}ds\right| < M$$

are satisfied for all arcs of integration in R_s on which $|s-s_0|>H$ and on which $\vartheta(\xi)$ varies monotonically with $|\xi|$.

When the secondary parameter σ is not fixed but is permitted to vary, the formulas to be taken from the theory cited will be valid *uniformly* only if the hypotheses stated are satisfied uniformly with respect to σ. Specifically the functions (4) must be uniformly bounded in R_s, the functions (6) must be uniformly bounded in any fixed finite part of R_s, and the hypothesis (iii) must be fulfilled with constants M and H which are independent of σ.

* The hypothesis (iv) of papers L_2 and L_3 is not repeated here. It is obviously satisfied in every case of the present discussion.

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1.3. **The solutions.** When the equation (3) satisfies the several hypotheses and the primary parameter ρ is sufficiently large, the relation defining the variable ξ determines a map of the region R_s upon a corresponding region R_ξ in the complex ξ plane. This map is conformal except possibly at the point corresponding to s_0 where, if $\nu\neq 0$, the region R_ξ has a branch point whose order depends upon ν.

The relations

$$\Xi^{(l)}:\quad (l-1)\pi+\epsilon \leqq \arg \xi \leqq (l+1)\pi-\epsilon, \tag{7}$$

with l an integral index and ϵ an arbitrarily small but fixed positive constant, define in the domain R_ξ the (overlapping) sub-regions $\Xi^{(l)}$. These correspond to respective sub-regions of R_s which will likewise be denoted by $\Xi^{(l)}$.

For any index h the differential equation (3) possesses a fundamental pair of solutions $u_{h,1}(s)$, $u_{h,2}(s)$, which are characterized by the fact that they are of peculiarly simple asymptotic forms as compared with the general solution for values of s which are in the corresponding sub-region $\Xi^{(h)}$ and which are not too near the point s_0. When s passes the bounds of the sub-region $\Xi^{(h)}$ this simplicity is lost and devolves upon a new set of solutions which are in turn associated in the manner indicated with the new sub-region in which s is then to be found. If $\nu\neq 0$ the forms referred to give valid representations of the respective solutions only so long as $|\xi|\geqq N$, where N is a constant whose magnitude is determined by the degree of approximation which the asymptotic representation is required to afford. The excepted region $|\xi|\leqq N$ corresponds in R_s to a neighborhood of the point s_0, and in this region a distinct representation must in general be employed.

The solutions $u_{h,j}(s), j=1, 2$, with a particular index h are thus because of their simplicity especially adapted for use in any deduction in which the associated region $\Xi^{(h)}$ plays a peculiar role. In terms of them, however, any other solutions may be simply expressed. In particular, it will be noted that if the point z_a corresponds to s_a under the correspondence of the variables which relates the equations (1) and (3), then the principal solutions $u(z)$, $U(z)$, of the equation (1) relative to z_a, i.e., those determined by the values

$$u(z_a)=0,\quad \frac{du(z_a)}{dz}=1,\qquad U(z_a)=1,\quad \frac{dU(z_a)}{dz}=0, \tag{8a}$$

are given by the formulas

$$\begin{aligned} u &= \left(\frac{dz}{ds}\right)_{s=s_a}\left\{\frac{u_{h,2}(s_a)u_{h,1}(s)-u_{h,1}(s_a)u_{h,2}(s)}{W}\right\},\\ U &= -\left\{\frac{u'_{h,2}(s_a)u_{h,1}(s)-u'_{h,1}(s_a)u_{h,2}(s)}{W}\right\}, \end{aligned} \tag{8b}$$

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in which h may be any index, the primes denote differentiation with respect to s, and W designates the Wronskian

$$W = u'_{h,1}(s)u_{h,2}(s) - u'_{h,2}(s)u_{h,1}(s),$$

which is a constant.

The principal solutions relative to the origin ($z_a=0$) will be designated throughout the discussion by $u_o(z)$ and $u_e(z)$. Inasmuch as the coefficient of the differential equation is an even function, they will be respectively odd and even functions of z as is to be indicated by the subscripts chosen. The principal solutions relative to the point $z_a=\pi/2$ will be denoted by $u_\alpha(z)$ and $u_\beta(z)$.

1.4. **The asymptotic solutions when $\nu=1$.** The special case of most frequent occurrence in the discussion which follows is that in which $\nu=1$, i.e., in which the zero of the coefficient $\chi_0^2(s)$ is a simple one. It is convenient, therefore, to note at this point for general reference the specific formulas which then apply in the relations of the preceding section, in so far as they are later to be used. Thus, for $h=-1, 0, 1, 2$ the solutions $u_{h,j}(s)$ are described by the following formulas:

When $|\xi| \geqq N$ and s is in $\Xi^{(l)}$,

$$\text{(9a)} \qquad u_{h,j}(s) = \rho^{-1/6}\phi^{-1/2}\{A_{j,1}^{h,l}e^{i\xi} + A_{j,2}^{h,l}e^{-i\xi}\}, \ j = 1, 2,$$

with coefficients to be obtained from the following table:

(9b)

(h, l)	$(-1,-1)$	$(-1,0)$	$(-1,1)$	$(0,-1)$	$(0,0)$	$(0,1)$	$(1,-1)$	$(1,0)$	$(1,1)$	$(2,-1)$	$(2,0)$	$(2,1)$	$(2,2)$
$A_{1,1}^{h,l}$	[1]	[1]	0	[1]	[1]	[1]	[1]	[1]	[1]	0	0	[1]	[1]
$A_{1,2}^{h,l}$	0	$[i]$	$[i]$	$[-i]$	0	0	$[-i]$	0	0	$[-i]$	$[-i]$	$[-i]$	0
$A_{2,1}^{h,l}$	0	0	$[i]$	0	0	$[i]$	$[-i]$	$[-i]$	0	$[-i]$	$[-i]$	0	0
$A_{2,2}^{h,l}$	[1]	[1]	[1]	[1]	[1]	[1]	0	[1]	[1]	0	[1]	[1]	[1]

and, when $|\xi| \leqq N$,

$$\text{(10a)} \quad u_{h,j}(s) = (2\pi/3)^{1/2}\Psi e^{(3/2-j)\pi i/2}[\gamma_{1,j}^{(h)}\xi^{1/3}J_{-1/3}(\xi) + \gamma_{2,j}^{(h)}\xi^{1/3}J_{1/3}(\xi)],$$

with the coefficients

(10b)

h	-1	0	1	2
$\gamma_{1,1}^{(h)}$	1	$e^{-\pi i/3}$	$e^{-\pi i/3}$	$e^{-2\pi i/3}$
$\gamma_{2,1}^{(h)}$	1	$e^{\pi i/3}$	$e^{\pi i/3}$	$e^{2\pi i/3}$
$\gamma_{1,2}^{(h)}$	$e^{\pi i/3}$	$e^{\pi i/3}$	1	1
$\gamma_{2,2}^{(h)}$	$e^{-\pi i/3}$	$e^{-\pi i/3}$	1	1

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The symbols J in these formulas designate Bessel functions in the familiar manner, and the symbol [ ] will be used throughout the discussion in the sense that $[Q]$ designates a quantity which differs from Q by terms of the order of ρ^{-1} and of the order of N^{-1} uniformly in σ.

From formulas thus given the evaluations

$$u_{h,1}(s_a) = \frac{[1]e^{i\xi_a}}{\rho^{1/6}\phi_a^{1/2}}, \qquad u_{h,2}(s_a) = \frac{[1]e^{-i\xi_a}}{\rho^{1/6}\phi_a^{1/2}},$$

when $|\xi_a| \geqq N$ and ξ_a is in $\Xi^{(h)}$, and $W = [2i]\rho^{2/3}$, will be immediately noted. Direct substitution in the relations (8b) leads, therefore, to the following formulas:

When $|\xi| \geqq N$, z is in $\Xi^{(l)}$ and z_a is in $\Xi^{(h)}$,

$$\begin{aligned} u &= \frac{[1]}{2i}\left(\frac{dz}{ds}\right)_{s=s_a}\left(\frac{1}{\rho\phi_a\phi}\right)^{1/2}\Big\{e^{-i\xi_a}(\overset{h,l}{A_{1,1}}e^{i\xi} + \overset{h,l}{A_{1,2}}e^{-i\xi}) \\ &\qquad - e^{i\xi_a}(\overset{h,l}{A_{2,1}}e^{i\xi} + \overset{h,l}{A_{2,2}}e^{-i\xi})\Big\}, \\ U &= \frac{[1]}{2}\left(\frac{\phi_a}{\phi}\right)^{1/2}\left\{e^{-i\xi_a}(\overset{h,l}{A_{1,1}}e^{i\xi} + \overset{h,l}{A_{1,1}}e^{-i\xi}) + e^{i\xi_a}(\overset{h,l}{A_{2,1}}e^{i\xi} + \overset{h,l}{A_{2,2}}e^{-i\xi})\right\}, \end{aligned} \tag{11a}$$

and when $|\xi| \leqq N$ and z_a is in $\Xi^{(h)}$,

$$\begin{aligned} u &= \left(\frac{dz}{ds}\right)_{s=s_a}\left(\frac{\pi}{6\rho\phi_a\phi}\right)^{1/2}\xi^{1/6}\Big\{e^{-i\xi_a - i\pi/4}\big[\overset{(h)}{\gamma_{1,1}}\,\xi^{1/3}J_{-1/3}(\xi) + \overset{(h)}{\gamma_{2,1}}\,\xi^{1/3}J_{1/3}(\xi)\big] \\ &\qquad + e^{i\xi_a + \pi i/4}\big[\overset{(h)}{\gamma_{1,2}}\,\xi^{1/3}J_{-1/3}(\xi) + \overset{(h)}{\gamma_{2,2}}\,\xi^{1/3}J_{1/3}(\xi)\big]\Big\}, \\ U &= \left(\frac{\pi\phi_a}{6\phi}\right)^{1/2}\xi^{1/6}\Big\{e^{i\xi_a + \pi i/4}\big[\overset{(h)}{\gamma_{1,1}}\,\xi^{1/3}J_{-1/3}(\xi) + \overset{(h)}{\gamma_{2,1}}\,\xi^{1/3}J_{1/3}(\xi)\big] \\ &\qquad + e^{i\xi_a - \pi i/4}\big[\overset{(h)}{\gamma_{1,2}}\,\xi^{1/3}J_{-1/3}(\xi) + \overset{(h)}{\gamma_{2,2}}\,\xi^{1/3}J_{1/3}(\xi)\big]\Big\}. \end{aligned} \tag{11b}$$

From these forms certain terms, depending upon the indices, may under certain conditions be omitted as asymptotically negligible in comparison with others. The precise evaluations will be deferred to the points where applications of the formulas are to be made.

1.5. **The "associated" Mathieu equation.** The *associated* Mathieu equation (2) is obtainable from the equation (1) by substituting in the latter iz in place of the variable z. Its solutions may, therefore, be derived from those discussed above by this simple change of variable. In particular it may be observed that the principal solutions relative to the origin, to be denoted by $v_o(z)$ and $v_e(z)$, are respectively odd and even functions of z, and that they are given by the formulas

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$$(12)\qquad \begin{aligned} v_o(z) &\equiv - iu_o(iz), \\ v_e(z) &\equiv u_e(iz). \end{aligned}$$

1.6. **The solutions for general values of z.** The hypotheses stated in §1.2 under which the forms of the solutions of the equation (1) are obtainable through the medium of the equation (3) restrict the variable to a region R_z in which the coefficient $(\Delta - \Omega \cos 2z)$ has at most one zero. It will be found in the subsequent discussion that this region over which the forms are directly deducible is in each case either the strip

$$(13)\qquad 0 \leqq x \leqq \pi/2, \quad \text{where} \quad z = x + iy,$$

or some closely related domain. It remains, therefore, to consider the extension of the asymptotic representations over the remaining parts of the z plane. A method by which this may be done is to be outlined as follows.

Since the coefficient of the differential equation is an even periodic function with the period π, the function $u(n\pi - z)$ is a solution whenever $u(z)$ is such and n is an integer. Hence each member of the several relations

$$(14)\qquad \begin{aligned} &\text{(a)} \qquad u_o(z) \equiv - u_o(\pi - z) + 2u_o(\pi/2)u_\beta(\pi - z), \\ &\text{(b)} \qquad u_e(z) \equiv - u_e(\pi - z) + 2u_e(\pi/2)u_\beta(\pi - z), \\ &\text{(c)} \qquad u_o(z) \equiv u_o(\pi - z) - 2u_o'(\pi/2)u_\alpha(\pi - z), \\ &\text{(d)} \qquad u_e(z) \equiv u_e(\pi - z) - 2u_e'(\pi/2)u_\alpha(\pi - z) \end{aligned}$$

is a solution of the differential equation. The identities are established, therefore, by the fact that in each relation both members and likewise their derivatives take the same values at the point $z = \pi/2$. A similar comparison of values at the point $z = 2^p\pi$, whatever the integer p, establishes the further relations

$$(15)\qquad \begin{aligned} &\text{(a)} \qquad u_o(z) \equiv - u_o(2^{p+1}\pi - z) + 2u_o(2^p\pi)u_e(z - 2^p\pi), \\ &\text{(b)} \qquad u_e(z) \equiv u_e(2^{p+1}\pi - z) + 2u_e'(2^p\pi)u_o(z - 2^p\pi). \end{aligned}$$

Let it be supposed now that the forms of the solutions have been deduced and so are known for all values of the variable which lie in the strip (13). It is to be shown then by the method of induction that they are deducible over the strip $\mathfrak{S}_p$ where p is any integer and $\mathfrak{S}_p$ is defined by the relation

$$(16)\qquad \mathfrak{S}_p\colon \quad 0 \leqq x \leqq 2^p\pi.$$

To begin with, let z lie in the region $\mathfrak{S}_0$. Then either z or $\pi - z$ lies in the strip (13). In the former case the representations of $u_o(z)$ and $u_e(z)$ are known by hypothesis, whereas in the latter they are given by the identities (14) in which the forms of the right-hand members are known. Proceeding, let the

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representations be considered known in the region S_p with any specific p, and let z lie in the strip S_{p+1}. Then either z lies in S_p and the forms are already known, or else both the values $(2^{p+1}\pi - z)$ and $(z - 2^p\pi)$ lie in S_p and the forms of the right-hand members of the relations (15) are known. In the latter event the identities furnish the representations sought in the part of S_{p+1} not included in S_p.

Finally the odd and even functional characters of the solutions $u_o(z)$, $u_e(z)$ may be drawn upon to extend their representations into the left-hand half-plane, and with the forms of these solutions at hand the representations of $u_\alpha(z)$ and $u_\beta(z)$ may be drawn from the identities (14).

1.7. **The characteristic values.** With any given value of Ω there are known to be associated specific *characteristic values* of Δ for which the differential equation (1) admits a periodic solution with the period 2π. These periodic solutions are enumerable, and are each either an odd or an even function of z.* With a scheme of enumeration which will become clear as the subsequent quantitative discussion proceeds, the characteristic values for which the odd solution $u_o(z)$ has the period 2π will be denoted by $S_n(\Omega)$, while those for which the period occurs in the even solution $u_e(z)$ will be designated by $C_n(\Omega)$. The equations of which these values are the roots are called *characteristic equations.*

Consider the characteristic equations for the values $S_n(\Omega)$. From the identity (15a) it is seen at once that a necessary and sufficient condition that 2π be a period of $u_o(z)$ is that $u_o(\pi) = 0$, an equation which in virtue of the relation (14c), with $z = \pi$, may be written

$$u_o'(\pi/2)u_\alpha(0) = 0.$$

If the root in question is one for which the factor $u_o'(\pi/2)$ vanishes, it follows from the identity (14c) that $u_o(z)$ admits no smaller period than 2π. On the other hand, if the root is one for which $u_\alpha(0)$ is zero, then the solutions $u_o(z)$ and $u_\alpha(z)$ are linearly dependent. It follows that $u_o(z)$ vanishes at $z = \pi/2$, and hence from the relation (14a) that $u_o(z)$ admits the period π. With the enumeration to be chosen the characteristic equations for odd periodic solutions are accordingly the following:

$$(17)\quad \begin{aligned} &\text{(a)} \qquad u_o(\pi/2) = 0, \quad \text{roots} \quad S_{2n}(\Omega), \\ &\qquad\qquad u_o(z) \text{ periodic with the primitive period } \pi; \\ &\text{(b)} \qquad u_o'(\pi/2) = 0, \quad \text{roots} \quad S_{2n+1}(\Omega), \\ &\qquad\qquad u_o(z) \text{ periodic with the primitive period } 2\pi. \end{aligned}$$

* Cf. Whittaker and Watson, loc. cit., §19.2.

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The characteristic equations for even solutions may be similarly deduced. Thus from the identity (15b), with $p=0$, the condition that 2π be a period of $u_e(z)$ is seen to be $u_e'(\pi)=0$. From the derived relation (14b), taken at $z=\pi$, the condition is found to be

$$u_e(\pi/2)u_\beta'(0) = 0.$$

If for the root in question $u_e(\pi/2)$ is zero, the identity (14b) shows that a smaller period than 2π is precluded. In the alternative the factor $u_\beta'(0)$ is zero, $u_e(z)$ and $u_\beta(z)$ are dependent and hence $u_e'(z)$ vanishes at $z=\pi/2$. It follows from the relation (14d) then that $u_e(z)$ admits the period π. In this instance, therefore, the characteristic equations are

$$(18)\quad \begin{array}{lll} \text{(a)} & u_e'(\pi/2) = 0, \quad \text{roots} \quad C_{2n}(\Omega), & \\ & & u_e(z) \text{ periodic with the primitive period } \pi; \\ \text{(b)} & u_e(\pi/2) = 0, \quad \text{roots} \quad C_{2n+1}(\Omega), & \\ & & u_e(z) \text{ periodic with the primitive period } 2\pi. \end{array}$$

1.8. **The Mathieu functions.** When Δ is a characteristic value $S_n(\Omega)$ or $C_n(\Omega)$, the corresponding periodic solution $u_o(z)$ or $u_e(z)$ is after suitable normalization known as a Mathieu function, and is respectively designated by $\mathrm{se}_n(z, \Omega)$ or $\mathrm{ce}_n(z, \Omega)$. Two modes of normalization have been commonly employed. The first* uses the stipulation that the coefficients of $\sin nz$ and $\cos nz$ in the respective Fourier expansions of $\mathrm{se}_n(z, \Omega)$ and $\mathrm{ce}_n(z, \Omega)$ be unity, i.e.,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \mathrm{se}_n\,(x, \Omega) \sin nx\, dx = 1,$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \mathrm{ce}_n\,(x, \Omega) \cos nx\, dx = 1 + \delta_{0,n}, \qquad \delta_{0,n} = \begin{cases} 1 \text{ for } n = 0, \\ 0 \text{ for } n \neq 0. \end{cases}$$

Since the integrands in these relations are even functions, the intervals of integration may, of course, be reduced to $(0, \pi)$. It may, however, be further observed that in virtue of the equations (17), (18), and (14),

$$(19)\qquad \begin{aligned} \mathrm{se}_n\,(z, \Omega) &\equiv (-1)^{n+1}\,\mathrm{se}_n\,(\pi - z, \Omega), \\ \mathrm{ce}_n\,(z, \Omega) &\equiv (-1)^{n}\,\mathrm{ce}_n\,(\pi - z, \Omega), \end{aligned}$$

i.e., the Mathieu functions are each either even or odd in the variable $z-\pi/2$. The ranges of integration above may, therefore, be reduced further to $(0, \pi/2)$, the formulas which result being

* Cf. Whittaker and Watson, loc. cit.

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$$
(20)\qquad
\begin{aligned}
\operatorname{se}_n(z,\Omega) &= \left\{\frac{\pi u_0(z)}{4\int_0^{\pi/2} u_0(x)\sin nx\,dx}\right\}_{\Delta=S_n(\Omega)},\\
\operatorname{ce}_n(z,\Omega) &= \left\{\frac{\pi u_e(z)}{(4-2\delta_{0,n})\int_0^{\pi/2} u_e(x)\cos nx\,dx}\right\}_{\Delta=C_n(\Omega)}.
\end{aligned}
$$

A second mode of normalization* is based on the requirements

$$
\frac{1}{\pi}\int_{-\pi}^{\pi}\operatorname{se}_n^2(x,\Omega)dx = 1,\quad \frac{1}{\pi}\int_{-\pi}^{\pi}\operatorname{ce}_n^2(x,\Omega)dx = 1+\delta_{0,n}.
$$

In this case the formulas obtained are

$$
(21)\qquad
\begin{aligned}
\operatorname{se}_n(z,\Omega) &= \left\{\frac{\pi^{1/2}u_0(z)}{2\left(\int_0^{\pi/2} u_0^2(x)dx\right)^{1/2}}\right\}_{\Delta=S_n(\Omega)},\\
\operatorname{ce}_n(z,\Omega) &= \left\{\frac{\pi^{1/2}u_e(z)}{2^{1-\delta_{0,n}/2}\left(\int_0^{\pi/2} u_e^2(x)dx\right)^{1/2}}\right\}_{\Delta=C_n(\Omega)}.
\end{aligned}
$$

1.9. **Other periodic solutions.** The characteristic equations for values of Δ which yield periodic solutions with periods other than π or 2π may be deduced by considerations similar to those of §1.7. The identities

$$
(22)\qquad
\begin{aligned}
&\text{(a)}\quad u(z) \equiv -u(2p\pi - z) + 2u(p\pi)u_e(p\pi - z),\\
&\text{(b)}\quad u(z) \equiv u(2p\pi - z) - 2u'(p\pi)u_o(p\pi - z),\\
&\text{(c)}\quad u(z) \equiv -u((2p-1)\pi - z) + 2u((p-\tfrac{1}{2})\pi)u_\beta(p\pi - z),\\
&\text{(d)}\quad u(z) \equiv u((2p-1)\pi - z) - 2u'((p-\tfrac{1}{2})\pi)u_\alpha(p\pi - z).
\end{aligned}
$$

are easily verified when p is any integer and $u(z)$ is an arbitrary solution of the differential equation. With the use of them it can be shown, as is outlined below, that periodic solutions with the periods indicated occur for values of Δ which are roots of the respective equations

$$
(23)\qquad
\begin{aligned}
u_o(n\pi/2) &= 0, &&\text{odd solutions with period } n\pi,\\
u_o'(n\pi/2) &= 0, &&\text{odd solutions with period } 2n\pi,
\end{aligned}
$$

$$
(24)\qquad
\begin{aligned}
u_e'(n\pi/2) &= 0, &&\text{even solutions with period } n\pi,\\
u_e(n\pi/2) &= 0, &&\text{even solutions with period } 2n\pi.
\end{aligned}
$$

---

* Cf. Strutt, loc. cit.

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Moreover, if n is the smallest integer for which an equation is satisfied, the period indicated is primitive.

Consider the equations (23). Their sufficiency for the indicated periodicities may be verified by observing that they imply through the pertinent identities (22) respectively that $u_o(z+n\pi) = \pm u_o(z)$. Conversely, if $2n\pi$ is a period of $u_o(z)$, then $u_o(n\pi) = 0$ by the identity (22a), and this leads when n is even through the relation (22b) to the one or the other of the equations (23). If n is odd the result follows from the identities (22c) and (22d), together with the fact that at least one of the solutions $u_\alpha(z)$ and $u_\beta(z)$ must differ from zero at the point $z = (n+1)\pi/2$.

The necessity and sufficiency of the equations (24) for solutions of their associated types is proved similarly, though in some instances the identities (22) must be differentiated prior to their application.

1.10. **The characteristic exponent.** When the parameters Δ and Ω are both fixed the differential equation in general admits no periodic solution. In this case it is known from Floquet's theory of differential equations with simply periodic coefficients that there are two solutions of the forms

$$e^{\mu z}\phi(z), \text{ and } e^{-\mu z}\phi(-z),$$

in which $\phi(z)$ is a periodic function with the period π, while μ, the so called *characteristic exponent*, is a constant which depends upon Δ and Ω. The equation for μ is*

$$e^{2\pi\mu} - 2\Theta e^{\pi\mu} + 1 = 0,$$

whence

$$\mu = \frac{1}{\pi}\cosh^{-1}\Theta = \frac{i}{\pi}\cos^{-1}\Theta, \tag{25a}$$

with $\Theta = u_e(\pi)$. The alternative evaluation

$$\Theta = 2u_e(\pi/2)u_\beta(0) - 1 \tag{25b}$$

may be obtained from the relation (14b).

It is evident that μ is either real or pure imaginary according as $\Theta > 1$ or $\Theta < 1$. In the former case the solutions noted above become infinite near the one or the other extremity of the axis of reals and are called unstable; in the latter case they remain bounded for real values of z and are called stable.

1.11. **Certain elliptic integrals.** It will be found now and again in the discussion which follows, that the comparison and identification of certain

* Cf. Horn, J., *Gewöhnliche Differentialgleichungen*, Leipzig, 1905, p. 242.

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superficially dissimilar formulas will depend upon the approximate or asymptotic evaluation of certain elliptic integrals of the type

$$G(\tau, h^2) = \int_0^{\pi/2} \frac{1 - \tau \sin^2 \zeta}{\{1 - h^2 \sin^2 \zeta\}^{1/2}} d\zeta. \tag{26}$$

The value of h will in every case be either near zero or near 1, and τ will be either 1 or h^2.

In terms of the standard complete elliptic integrals

$$K = \int_0^{\pi/2} \frac{d\zeta}{\{1 - h^2 \sin^2 \zeta\}^{1/2}}, \quad E = \int_0^{\pi/2} \{1 - h^2 \sin^2 \zeta\} d\zeta,$$

it is evident that

$$G(\tau, h^2) = K + \frac{\tau}{h^2}(E - K).$$

Hence on substituting for these integrals their expansions in powers of h, it is found that when h^2 is nearly zero

$$\begin{aligned} G(1, h^2) &= \frac{\pi}{4}\left\{1 + \frac{h^2}{8} + h^4 O(1)\right\}, \\ G(h^2, h^2) &= \frac{\pi}{2}\left\{1 - \frac{h^2}{4} + h^4 O(1)\right\}. \end{aligned} \tag{26a}$$

On the other hand, when h^2 is nearly 1 the Landen Transformation*

$$h \sin \zeta = \sin (2t - \zeta)$$

yields the form

$$G(\tau, h^2) = \frac{-\tau}{h} + \frac{2}{1+h} \int_0^{t_1} \frac{\left(1 - \frac{\tau}{h}\right) + \frac{2\tau}{h} \cos^2 t}{\cos t \{1 + \epsilon^2 \tan^2 t\}^{1/2}} dt,$$

in which

$$t_1 = \sin^{-1} \left\{\frac{1+h}{2}\right\}^{1/2}, \quad \epsilon = \frac{1-h}{1+h}. \tag{27}$$

The quantity $\epsilon^2 \tan^2 t$ is uniformly small of the order of ϵ. Hence the radical may be replaced by its binomial expansion, whereupon the integration leads to the formula

* Cf. Hancock, H., *Elliptic Integrals*, New York, 1917, p. 84.

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$$G(\tau, h^2) = \frac{-\tau}{h} + \left(\frac{2}{1+h}\right)\left(\frac{2\tau \sin t_1}{h} - \frac{\epsilon^2(h-\tau)\sin t_1}{4h\cos^2 t_1}\right.$$
$$\left. + \left\{\left(\frac{h-\tau}{2h}\right) + \frac{\epsilon^2}{8}\left(1 - \frac{5\tau}{h}\right)\right\} \log \frac{1+\sin t_1}{1-\sin t_1}\right) + o(\epsilon^2).$$

For the special values of τ this reduces to

$$\text{(26b)}\quad \begin{aligned} G(1, h^2) &= \frac{-1}{h} + \frac{2}{h}\left(\frac{2}{1+h}\right)^{1/2} + \frac{1-h}{h(1+h)} \log \frac{1-h}{8} + O(\epsilon^2 \log \epsilon), \\ G(h^2, h^2) &= -h + 2h\left(\frac{2}{1+h}\right)^{1/2} - \frac{1-h}{1+h} \log \frac{1-h}{8} + O(\epsilon^2 \log \epsilon). \end{aligned}$$

Chapter 2

The configuration II

2.1. The differential equation. When the relative values of the parameters Δ and Ω are such that the point (Ω, Δ) in Figure 1 lies in the region II at a sufficient distance from O, i.e., more specifically when Δ is large and positive, and with a constant M_1 (to be specified below) the relation

$$\text{(2.1)}\qquad 0 \leqq \Omega \leqq \frac{1}{M_1}\Delta$$

is fulfilled, the substitutions

$$\text{(2.2)}\qquad \rho = \Delta^{1/2}, \qquad \sigma^2 = \Omega/\Delta, \qquad s = z^*$$

give to the equation (1) the form (3) with

$$\text{(2.3)}\qquad \begin{aligned} &\chi_0 \equiv \phi, \quad \chi_1 \equiv 0, \\ &\phi^2 \equiv 1 - \sigma^2 \cos 2s. \end{aligned}$$

Let the variable z be restricted to any finite region of the complex plane. Then a number M_1 may be determined such that for all admitted values of z

$$\text{(2.4a)}\qquad |y| \leqq \frac{1}{2}\cosh^{-1}\frac{M_1}{2}, \qquad z = x + iy.$$

The constant M_1 of the relation (2.1), which determines the parameter values to be included in the present configuration, is to be one with which the condition (2.4a) is fulfilled. The primary parameter ρ is to be thought of as

* The distinction between s and z, which in the present instance is non-existent, is drawn for the purpose of making the formulas subsequently useful in a case when these variables are not the same.

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bounded below but not above, and the secondary parameter σ is evidently restricted to the range

$$(2.5)\qquad 0 \leqq \sigma^2 \leqq \frac{1}{M_1}\cdot$$

The relation (2.4a), together with

$$(2.4b)\qquad 0 \leqq x \leqq \frac{\pi}{2},$$

defines a strip of the z plane which is to be designated as R_z. The corresponding domain of the variable s is

$$(2.6)\qquad R_s:\quad 0 \leqq s' \leqq \pi/2, \qquad |s''| \leqq \tfrac{1}{2}\cosh^{-1}(M_1/2), \qquad s = s' + is''.$$

This region includes the origin and it is readily verified that with $s_0=0$ the hypothesis (i) of §1.2 is fulfilled uniformly in σ with $\nu=0$. The hypotheses (ii) and (iii) are likewise fulfilled, since $\chi_1\equiv 0$ and R_s is bounded. From the formulas (5) it is seen that in the present instance $\eta(s)\equiv\omega_1(s)\equiv k=0$, in consequence of which

$$\omega(\phi) \equiv 1 + \frac{1}{4\phi^2} - \frac{5(1-\sigma^4)}{4\phi^4}, \qquad \Psi \equiv \phi^{-1/2}.$$

These functions are bounded uniformly in σ and hence the requirements enumerated in §1.2 are completely fulfilled.

2.2. **The solutions.** Since the case in hand is one in which $\nu=0$, there exist solutions of the differential equation which maintain a single asymptotic form over the entire region R_z. Such solutions with their respective forms are

$$(2.7)\qquad \begin{aligned} u_{0,1}(s) &= \phi^{-1/2}e^{i\xi}[1], \\ u_{0,2}(s) &= \phi^{-1/2}e^{-i\xi}[1]. \end{aligned}$$

Their Wronskian has the value $W=[2i]\rho$. The principal solutions relative to the point $z=0$ are accordingly computed directly from the formula (8b), with $h=0$, $s_a=0$, to be

$$(2.8)\qquad \begin{aligned} u_o(z) &= \frac{1}{2i}\left\{\frac{1}{\rho^2\phi_1\phi}\right\}^{1/2}\left\{e^{i\xi}[1] - e^{-i\xi}[1]\right\}, \\ u_e(z) &= \frac{1}{2}\left\{\frac{\phi_1}{\phi}\right\}^{1/2}\left\{e^{i\xi}[1] + e^{-i\xi}[1]\right\}, \end{aligned}$$

with

$$(2.8a)\qquad \begin{aligned} \rho\phi &= \{\Delta - \Omega\cos 2z\}^{1/2}, \quad \rho\phi_1 = \{\Delta - \Omega\}^{1/2}, \\ \xi &= \int_0^z \{\Delta - \Omega\cos 2z\}^{1/2}dz. \end{aligned}$$

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Inasmuch as

$$e^{i\xi}[1] - e^{-i\xi}[1] = [2i] \sin [\xi],$$

with analogous formulas involving the other trigonometric functions, it is seen in particular that for real values of the variable

$$\text{(2.8b)} \qquad \begin{aligned} u_o(x) &= \frac{[1]}{\{(\Delta - \Omega)(\Delta - \Omega \cos 2x)\}^{1/4}} \sin \left[\int_0^x \{\Delta - \Omega \cos 2x\}^{1/2} dx \right], \\ u_e(x) &= \left\{ \frac{\Delta - \Omega}{\Delta - \Omega \cos 2x} \right\}^{1/4} [1] \cos \left[\int_0^x \{\Delta - \Omega \cos 2x\}^{1/2} dx \right]. \end{aligned}$$

The principal solutions relative to $z = \pi/2$ are similarly found to be given by the formulas

$$\text{(2.9)} \qquad \begin{aligned} u_\alpha(z) &= \frac{1}{2i} \left\{ \frac{1}{\rho^2 \phi_2 \phi} \right\}^{1/2} \{ e^{i(\xi - \xi_2)}[1] - e^{-i(\xi - \xi_2)}[1] \}, \\ u_\beta(z) &= \frac{1}{2} \left\{ \frac{\phi_2}{\phi} \right\}^{1/2} \{ e^{i(\xi - \xi_2)}[1] + e^{-i(\xi - \xi_2)}[1] \}, \end{aligned}$$

with

$$\text{(2.9a)} \qquad \rho\phi_2 = \{\Delta + \Omega\}^{1/2}, \quad \xi - \xi_2 = \int_{\pi/2}^z \{\Delta - \Omega \cos 2z\}^{1/2} dz.$$

When z is real they are

$$\text{(2.9b)} \qquad \begin{aligned} u_\alpha(x) &= \frac{-[1]}{\{(\Delta + \Omega)(\Delta - \Omega \cos 2x)\}^{1/4}} \sin \left[\int_x^{\pi/2} \{\Delta - \Omega \cos 2x\}^{1/2} dx \right], \\ u_\beta(x) &= \left\{ \frac{\Delta + \Omega}{\Delta - \Omega \cos 2x} \right\}^{1/4} [1] \cos \left[\int_x^{\pi/2} \{\Delta - \Omega \cos 2x\}^{1/2} dx \right]. \end{aligned}$$

In the special case that $\sigma = 0$ (i.e., $\Omega = 0$) the differential equation (1) is directly integrable, and it is verified immediately that the formulas above are correct when the symbols [ ] are omitted. It may be concluded, therefore, in the discussion of this chapter that the quantities [1] reduce to 1 when $\sigma^2 = 0$.

2.3. The solutions of the associated Mathieu equation. The principal solutions of the associated Mathieu equation (2) relative to the origin may be derived from the functions (2.8) by the substitutions (12) as was noted in §1.5. Their forms so obtained are

$$\text{(2.10)} \qquad \begin{aligned} v_o(z) &= \frac{[1]}{\{(\Delta - \Omega)(\Delta - \Omega \cosh 2z)\}^{1/4}} \sinh \left[\int_0^z \{\Delta - \Omega \cosh 2z\}^{1/2} dz \right], \\ v_e(z) &= \left\{ \frac{\Delta - \Omega}{\Delta - \Omega \cosh 2z} \right\}^{1/4} [1] \cosh \left[\int_0^z \{\Delta - \Omega \cosh 2z\}^{1/2} dz \right], \end{aligned}$$

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the region for z being

$$|x| \leqq \frac{1}{2}\cosh^{-1}\frac{M_1}{2},$$

$$-\pi/2 \leqq y \leqq 0.$$

The solutions (2.10) are evidently asymptotically multiples of each other when z is real and large. A pair, $v_\gamma(z)$, $v_\delta(z)$, not subject to this disadvantage is that obtainable by the substitution of iz for s from the functions (2.7). Their forms are explicitly

$$(2.11)\quad \begin{aligned} v_\gamma(z) &= \frac{[1]}{\{\Delta - \Omega\cosh 2z\}^{1/4}}\exp\left[-\int_0^z\{\Delta - \Omega\cosh 2z\}^{1/2}dz\right],\\ v_\delta(z) &= \frac{[1]}{\{\Delta - \Omega\cosh 2z\}^{1/4}}\exp\left[\int_0^z\{\Delta - \Omega\cosh 2z\}^{1/2}dz\right]. \end{aligned}$$

2.4. **The characteristic values.** If $S_p(\Omega)$ and $C_q(\Omega)$ are a pair of characteristic values, the substitution of the forms (2.8b) into the characteristic equations (17) and (18) shows that each of these values is a root of an equation

$$(2.12)\qquad \left[\int_0^{\pi/2}\{\Delta - \Omega\cos 2x\}^{1/2}dx\right] = \frac{n\pi}{2},$$

with the integer n suitably adjusted to p or q as the case may be. To determine this adjustment, it need merely be observed that when $\Omega=0$ the equation reduces to $\Delta=n^2$, and the corresponding Mathieu functions to $\sin nz$ and $\cos nz$. Since these are by definition the forms of $\mathrm{se}_n(z, 0)$ and $\mathrm{ce}_n(z, 0)$, it must be concluded that $p=n$ and $q=n$, i.e., the form (2.12) is that of the characteristic equation both for $S_n(\Omega)$ and for $C_n(\Omega)$.

The symbol [ ] in the equation (2.12) represents a quantity of the order of $\Delta^{-1/2}$ uniformly in σ, which vanishes when $\sigma=0$. Since it like the equation (1) depends analytically upon σ^2, the equation (2.12) may be written

$$\int_0^{\pi/2}\{\Delta - \Omega\cos 2x\}^{1/2}dx + \sigma^2 O(\Delta^{-1/2}) = \frac{n\pi}{2}.$$

The substitution $x=\pi/2-\zeta$ reduces this to

$$\Delta^{1/2}\{(1+\sigma^2)^{1/2}G(h^2, h^2) + \sigma^2 O(\Delta^{-1/2})\} = \frac{n\pi}{2},$$

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where G is the elliptic integral of (26) with $h^2=2\sigma^2/(1+\sigma^2)$. Since this value of h^2 is small, the evaluation (26a) gives to the equation the form

$$\Delta^{1/2}\{1+\sigma^4O(1)+\sigma^2O(\Delta^{-1})\}=n,$$

from which it follows that

$$(2.13)\qquad \begin{aligned} S_n(\Omega) &= n^2+\frac{\Omega}{n^2}O(1),\\ C_n(\Omega) &= n^2+\frac{\Omega}{n^2}O(1), \end{aligned}$$

the quantities indicated by the symbols $O(1)$ being uniformly bounded as to n and Ω while the configuration with which the present chapter deals is maintained.

2.5. **The characteristic exponent.** The substitution into the formula (25b) of the values given by (2.8b) and (2.9b) yields the evaluation

$$\begin{aligned} \Theta &= [2]\cos[\xi_2]\cos[\xi_2]-1\\ &= \cos 2\xi_2+\sigma^2O(\Delta^{-1/2}). \end{aligned}$$

Accordingly, from (25a) an asymptotic formula for the characteristic exponent is

$$(2.14)\qquad u=\frac{i}{\pi}\cos^{-1}\left\{\cos\left(\int_0^{\pi/2}2\{\Delta-\Omega\cos 2x\}^{1/2}dx\right)+\frac{\Omega^2}{\Delta^{3/2}}O(1)\right\}.$$

When $\Omega=0$ this reduces to $\mu=i\Delta^{1/2}$, a result which may be verified by actual integration of the differential equation.

Inasmuch as the quantity within the brace in the formula (2.14) does not exceed unity, except possibly for very small ranges of the parameters near those values for which the integral is a multiple of π, it follows that the configuration under consideration in this chapter is predominantly one of stable solutions.*

Chapter 3

The configuration III

3.1. **Definitions.** The parameter configuration contiguous with that of the preceding chapter and designated by III in Figure 1 is to be defined by the relation

$$(3.1)\qquad \frac{1}{M_1}\Delta\leqq\Omega\leqq\Delta-M_2\Delta^{1/2},$$

* Cf. the Figure 3 in Strutt, loc. cit.

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in which M_1 is the constant in (2.1), and M_2 is to be momentarily discussed. The substitutions

$$\rho = \frac{\Delta - \Omega}{\Delta^{1/2}}, \qquad \sigma^2 = 1 - \frac{\Omega}{\Delta}, \qquad s = \frac{-iz}{\sigma} \tag{3.2}$$

reduce the differential equation (1) in this case to the form (3) with

$$\begin{aligned} &\chi_0 \equiv \phi, \qquad \chi_1 \equiv 0, \\ &\phi^2 \equiv 2(1 - \sigma^2)\frac{\sinh^2 \sigma s}{\sigma^2} - 1. \end{aligned} \tag{3.3}$$

The parameter ρ is evidently restricted by the relation $\rho \geqq M_2$, and since the degree of approximation which the asymptotic formulas yield depends upon the magnitude of ρ, the constant M_2 is in any specific case to be chosen such that representations which are uniformly suitable to the purposes intended are obtained. The secondary parameter is clearly confined to the fixed closed interval

$$0 \leqq \sigma^2 \leqq 1 - \frac{1}{M_1}, \tag{3.4}$$

in which the lower boundary could in fact more strictly be replaced by $M_2\Delta^{-1/2}$.

Let z be restricted for the discussion of this configuration to the infinite half-strip R_z given by the formulas

$$R_z: \qquad -\pi/2 \leqq x \leqq \pi/2, \quad 0 \leqq y. \tag{3.5}$$

The extension of the solutions from this domain to the entire strip (13) may be accomplished by the use of the identities

$$\begin{aligned} u_\alpha(z) &\equiv u_\alpha(-z) - 2u_\alpha'(0)u_o(-z), \\ u_\beta(z) &\equiv -u_\beta(-z) + 2u_\beta(0)u_e(-z), \end{aligned}$$

and the odd and even characters of $u_o(z)$ and $u_e(z)$. Their extension to general values of z thereupon follows on the lines of §1.6.

3.2. **The variables s, Φ and ξ.** The region R_s corresponding to R_z is the infinite half-strip

$$R_s: \qquad 0 \leqq s', \quad -\frac{\pi}{2\sigma} \leqq s'' \leqq \frac{\pi}{2\sigma}. \tag{3.6}$$

Within this region $\chi_0^2(s)$ has a single zero located on the axis of reals at the point

$$s_0' \equiv \frac{1}{\sigma}\sinh^{-1}\frac{\sigma}{\{2(1-\sigma^2)\}^{1/2}}. \tag{3.7}$$

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Though s_0' depends upon σ it is both bounded and bounded from zero for all admitted values of the parameters.

The relation between s and the quantity Φ maps R_s upon a corresponding region R_Φ conformally except at the point s_0'. The shape of R_Φ may be easily determined by observing the values of Φ when s is either real or on the boundaries of R_s. With R_s thought of as cut along the axis of reals from the origin to s_0' these values for the upper half of R_s are

for $s'' = 0+$ and $0 \leqq s' \leqq s_0'$,

$$\Phi = e^{\pi i} \int_{s'}^{s_0'} i\left\{1 - 2(1-\sigma^2)\frac{\sinh^2 \sigma s'}{\sigma^2}\right\}^{1/2} ds';$$

for $s' = 0$ and $0 \leqq s'' \leqq \pi/(2\sigma)$,

$$\Phi = \Phi(0) + e^{\pi i/2} \int_0^{s''} i\left\{2(1-\sigma^2)\frac{\sin^2 \sigma s''}{\sigma^2} + 1\right\}^{1/2} ds'';$$

for $s'' = 0$ and $s_0' \leqq s'$,

$$\Phi = \int_{s_0'}^{s'} \left\{2(1-\sigma^2)\frac{\sinh^2 \sigma s'}{\sigma^2} - 1\right\}^{1/2} ds';$$

for $s'' = \pi/(2\sigma)$ and $0 \leqq s'$,

$$\Phi = \Phi\left(\frac{\pi i}{2\sigma}\right) + \int_0^{s'} i\left\{2(1-\sigma^2)\frac{\cosh^2 \sigma s'}{\sigma^2} + 1\right\}^{1/2} ds'.$$

The map of the lower half of R_s is obtainable by reflection from that of the upper half, since conjugate complex values of s lead to conjugate values of Φ.

Finally since

$$\left|\frac{\sinh \sigma s}{\sigma}\right| > \frac{2}{\pi}|s|,$$

it follows that when $|s|$ is sufficiently large

$$\begin{aligned} &\phi \sim \{2(1-\sigma^2)\}^{1/2}\frac{\sinh \sigma s}{\sigma}, \\ (3.8) \qquad &\Phi \sim \frac{2}{\sigma^2}\{2(1-\sigma^2)\}^{1/2}\sinh^2\frac{\sigma s}{2}, \end{aligned}$$

the symbolism designating that the ratio of the members of either relation becomes 1 as $|s| \to \infty$. From the second relation it follows that when c is any sufficiently large constant the line $s' = c$ maps upon a simple curve in R_Φ. The uniqueness of the correspondence between points of R_s and R_Φ is thereby assured.* Figure 2 indicates the map.

* Cf. Osgood, W. F., *Lehrbuch der Funktionentheorie*, vol. 1, Leipzig, 1912, p. 377.

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The variables Φ and ξ differ only by the real factor ρ, whence the domains R_ξ and R_Φ differ only in scale. Figure 3 indicates the relation between R_z

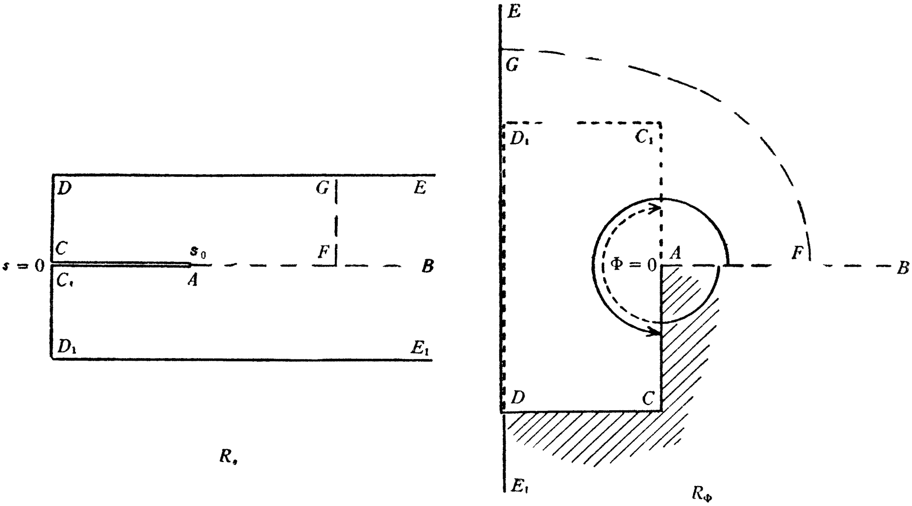

FIG. 2

and R_ξ, each domain being divided into the sub-regions $\Xi^{(l)}$ defined in (7). The lines by which this sub-division is effected need not be determined with

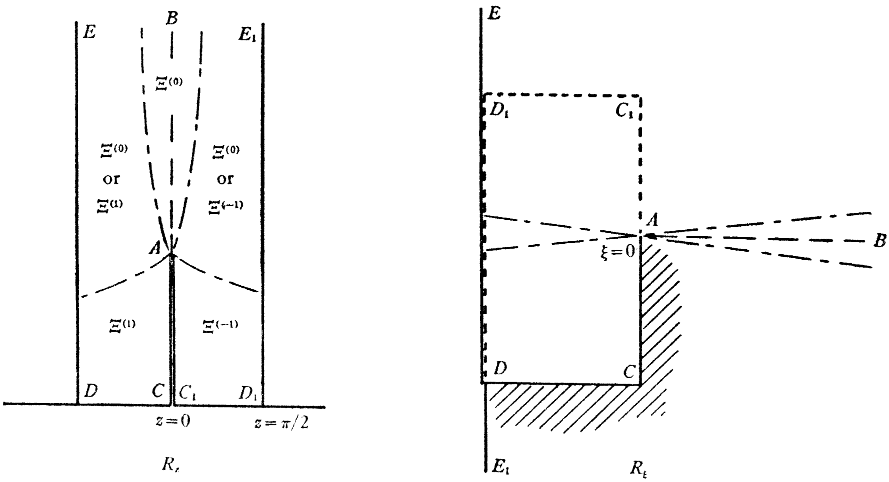

FIG. 3

precision, for due to the overlapping of the regions any displacement of the curves which does not affect the character of the figure is immaterial.

3.3. **Fulfillment of the hypotheses.** The zero of $\chi_0^2(s)$ at s_0' is of the first order. Hence in the hypotheses of §1.2 the values $\nu=1$, $\eta\equiv\omega_1\equiv k=0$ are to

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be used. With the value of ϕ given by the formula (3.3) it is found that the functions (6) are in the present instance

$$(3.9)\qquad \omega(\phi) = \frac{1}{4}\left\{-\frac{5}{9}\left(\frac{\phi}{\Phi}\right)^2 + \sigma^2 + \frac{6}{\phi^2} + \frac{5(2-\sigma^2)}{\phi^4}\right\}, \quad \Psi = \Phi^{1/6}/\phi^{1/2}.$$

Let the region R_s be divided into three parts by the relations

$$\text{(a)} \qquad |s - s_0'| \leqq \delta,$$

$$\text{(b)} \qquad \delta \leqq |s - s_0'| \leqq H,$$

$$\text{(c)} \qquad H \leqq |s - s_0'|,$$

with the constants δ and H as specified in the following. It is to be shown that in each of these parts the hypotheses of §1.2 are uniformly fulfilled.

To begin with let H be chosen so large that in the part (c) the formulas (3.8) may be applied. Then it is a matter of simple computation to show that in this part of R_s the hypotheses (i) and (iii) are uniformly fulfilled.

Next let δ be chosen so small that within the part (a), $|\phi^2| \leqq \frac{1}{2}$ for all admitted values of σ. Then

$$\left\{1 + \frac{\sigma^2\phi^2}{2-\sigma^2}\right\}^{-1/2} \equiv 1 - \frac{\sigma^2\phi^2}{2(2-\sigma^2)} + \phi^4 O(1),$$

$$(1+\phi^2)^{-1/2} \equiv 1 - \frac{\phi^2}{2} + \phi^4 O(1),$$

with $O(1)$ designating functions which are uniformly bounded. Since

$$\Phi = \int_0^{\phi} \frac{\phi}{\phi'}\, d\phi,$$

whereas from the formula (3.3)

$$\frac{\phi}{\phi'} = \frac{\phi^2}{(2-\sigma^2)^{1/2}}\left\{(1+\phi^2)\left(1 + \frac{\sigma^2\phi^2}{2-\sigma^2}\right)\right\}^{-1/2},$$

it is found that

$$\Phi = \frac{\phi^3}{3(2-\sigma^2)^{1/2}}\left\{1 - \frac{3\phi^2}{5(2-\sigma^2)} + \phi^4 O(1)\right\}.$$

With this evaluation it is seen directly that in the part (a) the functions (3.9) are uniformly bounded.

Lastly in the part (b) the formula (3.3) may be written

$$\chi_0^2(s) = 2\left\{(2-\sigma^2)^{1/2}\frac{\sinh 2\sigma(s-s_0')}{2\sigma} + \frac{\sinh^2\sigma(s-s_0')}{\sigma^2}\right\}.$$

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It is evident from this that both

$$\chi_0(s) \quad \text{and} \quad \int_{s_0'}^{s} \chi_0(s)ds$$

are non-vanishing and continuous as functions of the two variables $(s-s_0', \sigma)$ in the closed region determined by (b) and (3.4). Accordingly, they are bounded uniformly in σ and the hypothesis (i) is uniformly fulfilled. Clearly also the functions (3.9) are uniformly bounded and so the requirements of §1.2 upon the differential equation are uniformly met.

3.4. **The forms of the solutions.** Since $\phi^2(s)$ has a simple zero in R_s the asymptotic representation of any solution of the differential equation is subject to the Stokes' phenomenon, and ν being 1 the formulas of §1.4 are applicable. From Figure 3 it is seen that the origin $z=0$ may be regarded as lying in the sub-region $\Xi^{(-1)}$. Hence with $h=-1$ and the subscript a replaced by 1 the formulas (11a) and (11b) yield the representations of the solutions $u_o(z)$ and $u_e(z)$. It may be observed from Figure 3, however, that the value ξ_1 which corresponds to $z=0$ (at C_1 in the figure) is such that $i\xi_1$ is real and negative, so that any quantity multiplied by $e^{i\xi_1}$ is asymptotically negligible in comparison with the same multiplied by $e^{-i\xi_1}$. With the omission of such negligible terms the formulas obtained are the following:

When z is in $\Xi^{(l)}$, and $|\xi| \geqq N$,

$$(3.10) \qquad \begin{aligned} u_o(z) &= \frac{1}{2}\left(\frac{\sigma^2}{\rho^2\phi_1\phi}\right)^{1/2}\left\{K_{0,1}^{-1,l}e^{i\xi} + K_{0,2}^{-1,l}e^{-i\xi}\right\}, \\ u_e(z) &= \frac{1}{2}\left(\frac{\phi_1}{\phi}\right)^{1/2}\left\{K_{e,1}^{-1,l}e^{i\xi} + K_{e,2}^{-1,l}e^{-i\xi}\right\}, \end{aligned}$$

with coefficients

(3.10a)

l	-1	0	1
$K_{0,1}^{-1,l}$	$e^{-i\xi_1}[1]$	$e^{-i\xi_1}[1]$	$-ie^{i\xi_1}[1]$
$K_{0,2}^{-1,l}$	$-e^{i\xi_1}[1]$	$ie^{-i\xi_1}[1]$	$ie^{-i\xi_1}[1]$
$K_{e,1}^{-1,l}$	$e^{-i\xi_1}[1]$	$e^{-i\xi_1}[1]$	$ie^{i\xi_1}[1]$
$K_{e,2}^{-1,l}$	$e^{i\xi_1}[1]$	$ie^{-i\xi_1}[1]$	$ie^{-i\xi_1}[1]$

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When $|\xi| \leqq N$,

$$\text{(3.10b)}\qquad \begin{aligned} u_o(z) &= \left(\frac{\pi i \sigma^2}{6\rho^2\phi_1\phi}\right)^{1/2} \xi^{1/6} e^{-i\xi_1}\left[\xi^{1/3}J_{-1/3}(\xi) + \xi^{1/3}J_{1/3}(\xi)\right], \\ u_e(z) &= \left(\frac{\pi i \phi_1}{6\phi}\right)^{1/2} \xi^{1/6} e^{-i\xi_1}\left[\xi^{1/3}J_{-1/3}(\xi) + \xi^{1/3}J_{1/3}(\xi)\right]. \end{aligned}$$

In the original variables

$$\frac{\rho\phi}{\sigma} = \left\{\Omega \cos 2z - \Delta\right\}^{1/2}, \qquad \frac{\rho\phi_1}{\sigma} = e^{-\pi i/2}\left\{\Delta - \Omega\right\}^{1/2},$$

$$\xi = -i\int_{y_0}^{z}\left\{\Omega\cos 2z - \Delta\right\}^{1/2} dz, \qquad \xi_1 = i\int_0^{y_0}\left\{\Delta - \Omega\cosh 2y\right\}^{1/2} dy,$$

with $y_0 = \frac{1}{2}\cosh^{-1}\Delta/\Omega$. Further, it may be noted that since the values of ϕ on the lines AC and AC_1 in Figure 3 differ only in sign, therefore

$$\xi_1 = \frac{-i\rho}{\sigma}\int_A^{C_1}\phi dz = \frac{i\rho}{\sigma}\int_A^{C}\phi dz,$$

whence the formulas

$$\int_0^z\left\{\Omega\cos 2z - \Delta\right\}^{1/2} dz = \begin{cases} i(\xi - \xi_1), & \text{in } \Xi^{(-1)}, \\ i(\xi + \xi_1), & \text{in } \Xi^{(1)} \end{cases}$$

are also valid provided the entire path of integration is taken in each case in the sub-region indicated.

The formulas (11a), (11b) may likewise be drawn upon to give the representations of the solutions $u_\alpha(z)$, $u_\beta(z)$. If the point corresponding to $z = \pi/2$ is s_2, the subscript a is to be replaced by 2, and since ξ_2 (at D_1 in Figure 3) lies in the region $\Xi^{(-1)}$, h is again to be taken as -1. With the omission of asymptotically negligible terms the formulas obtained are the following:

When z is in $\Xi^{(l)}$, and $|\xi| \geqq N$,

$$\text{(3.11)}\qquad \begin{aligned} u_\alpha(z) &= \frac{1}{2}\left(\frac{\sigma^2}{\rho^2\phi_2\phi}\right)^{1/2}\left\{K_{\alpha,1}^{-1,l}e^{i\xi} + K_{\alpha,2}^{-1,l}e^{-i\xi}\right\}, \\ u_\beta(z) &= \frac{1}{2}\left(\frac{\phi_2}{\phi}\right)^{1/2}\left\{K_{\beta,1}^{-1,l}e^{i\xi} + K_{\beta,2}^{-1,l}e^{-i\xi}\right\}, \end{aligned}$$

with coefficients

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(3.11a)

	-1	0	1
$K^{-1,l}_{\alpha,1}$	$e^{-i\xi_2}[1]$	$e^{-i\xi_2}[1]$	$-ie^{i\xi_2}[1]$
$K^{-1,l}_{\alpha,2}$	$-e^{i\xi_2}[1]$	$ie^{-i\xi_2}[1]$	$ie^{-i\xi_2}[1]$
$K^{-1,l}_{\beta,1}$	$e^{-i\xi_2}[1]$	$e^{-i\xi_2}[1]$	$ie^{i\xi_2}[1]$
$K^{-1,l}_{\beta,2}$	$e^{i\xi_2}[1]$	$ie^{-i\xi_2}[1]$	$ie^{-i\xi_2}[1]$

When $|\xi| \leqq N$,

$$(3.11\text{b})\qquad \begin{aligned} u_\alpha(z) &= \left(\frac{\pi i \sigma^2}{6\rho^2\phi_2\phi}\right)^{1/2} \xi^{1/6} e^{-i\xi_2}\left[\xi^{1/3}J_{-1/3}(\xi) + \xi^{1/3}J_{1/3}(\xi)\right], \\ u_\beta(z) &= \left(\frac{\pi i \phi_2}{6\phi}\right)^{1/2} \xi^{1/6} e^{-i\xi_2}\left[\xi^{1/3}J_{-1/3}(\xi) + \xi^{1/3}J_{1/3}(\xi)\right]. \end{aligned}$$

Again

$$\frac{\rho\phi_2}{\sigma} = e^{-\pi i/2}\{\Delta + \Omega\}^{1/2},$$

$$\xi_2 = \xi_1 - \int_0^{\pi/2} \{\Delta - \Omega \cos 2x\}^{1/2} dx.$$

Figure 3 shows that the segments $-\pi/2 \leqq x \leqq 0$ and $0 \leqq x \leqq \pi/2$ of the axis of reals lie respectively in the sub-regions $\Xi^{(1)}$ and $\Xi^{(-1)}$. The formulas above appropriate to these regions accordingly yield the descriptions of the solutions when z is real. It is found that these formulas are precisely those given in (2.8b) and (2.9b), though it should be noted that with the difference in the definition of the parameter ρ the significance of symbol [ ] is slightly different in this chapter from that in the preceding one.

The pairs of solutions (3.10) and (3.11) have each the defect that in the region about the upper part of the axis of imaginaries the component solutions are asymptotically multiples of each other. The pair of solutions $u_{-1,1}$, $u_{-1,2}$ given in (9) would be one not subject to this particular shortcoming.

3.5. **The solutions of the associated Mathieu equation.** If z lies in any of the domains indicated in Figure 4, the point iz lies in the corresponding

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sub-region of R_z as shown in Figure 3. In accordance with (12) the representations of $iv_o(z)$ and $v_e(z)$ are therefore obtainable in any one of the regions

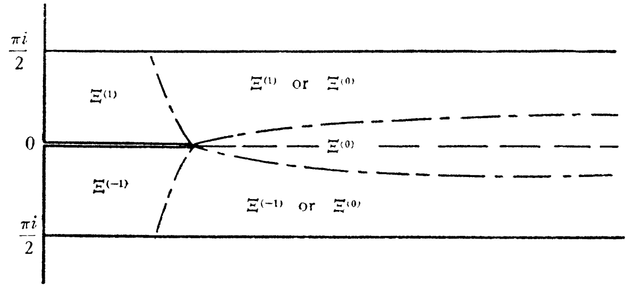

FIG. 4

indicated by the mere substitution in the associated formulas (3.10) of $\bar{\phi}$ and $\bar{\xi}$ in place of ϕ and ξ, the former being the same functions of iz as the latter are of z. Explicitly

$$\frac{\rho\bar{\phi}}{\sigma} = \left\{\Omega \cosh 2z - \Delta\right\}^{1/2},$$

$$\bar{\xi} = \int_{x_0}^{z} \left\{\Omega \cosh 2z - \Delta\right\}^{1/2} dz, \qquad x_0 = \tfrac{1}{2} \cosh^{-1} \Delta/\Omega.$$

In particular, for real values of the variable the formulas so obtained are the following:

For $0 \leqq x < x_0$, $|\bar{\xi}| \geqq N$,

$$\text{(3.12a)} \quad \begin{aligned} v_o(x) &= \frac{[1]}{\left\{(\Delta-\Omega)(\Delta-\Omega\cosh 2x)\right\}^{1/4}} \sinh\left[\int_0^x \left\{\Delta-\Omega\cosh 2x\right\}^{1/2} dx\right], \\ v_e(x) &= \left\{\frac{\Delta-\Omega}{\Delta-\Omega\cosh 2x}\right\}^{1/4} [1] \cosh\left[\int_0^x \left\{\Delta-\Omega\cosh 2x\right\}^{1/2} dx\right]. \end{aligned}$$

For $x \leqq x_0$, $|\bar{\xi}| \leqq N$,

$$\text{(3.12b)} \quad v_o(x) = \frac{(2\pi)^{-1/2} \left|\bar{\xi}\right|^{1/6} e^{-i\xi_1}}{\left\{(\Delta-\Omega)(\Delta-\Omega\cosh 2x)\right\}^{1/4}} \left[\left|\bar{\xi}\right|^{1/3} K_{1/3}(\left|\bar{\xi}\right|)\right].$$

For $x_0 \leqq x$, $|\bar{\xi}| \leqq N$,

$$\text{(3.12c)} \quad v_o(x) = \frac{(\pi/6)^{1/2} \bar{\xi}^{1/6} e^{-i\xi_1}}{\left\{(\Delta-\Omega)(\Omega\cosh 2x - \Delta)\right\}^{1/4}} \left[\bar{\xi}^{1/3} J_{-1/3}(\bar{\xi}) + \bar{\xi}^{1/3} J_{1/3}(\bar{\xi})\right].$$

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For $x<x_0$, $|\bar{\xi}| \geqq N$,

$$(3.12\text{d}) \quad v_o(x)=\frac{[1]e^{-i\xi_1}}{\{(\Delta-\Omega)(\Omega \cosh 2x-\Delta)\}^{1/4}} \cos\left[\int_{x_0}^{x}\{\Omega \cosh 2x-\Delta\}^{1/2}dx-\frac{\pi}{4}\right].$$

For the x ranges concerned in the cases (b), (c) and (d) the representation of $v_e(x)$ has been omitted since it is found to differ in appearance from that of $v_o(x)$ only in that the factor $(\Delta-\Omega)^{-1/4}$ is replaced by $(\Delta-\Omega)^{1/4}$. For the range in case (b) the value of $\bar{\xi}$ is imaginary, i.e., $\bar{\xi}=e^{-3\pi i/2}\,|\bar{\xi}|$, and the relation

$$J_{-1/3}(\bar{\xi}) + J_{1/3}(\bar{\xi}) = \frac{3^{1/2}i}{\pi} K_{1/3}(|\,\bar{\xi}\,|)$$

was used.

As already noted in §3.4, a pair of solutions which unlike those above are not asymptotically multiples of each other for large real values of z would be that obtainable in the manner used above from the functions $u_{-1,j}(z)$ described in (9).

3.6. **The characteristic values and exponent.** The forms of both the exponent μ and the characteristic equations were found in chapter 2 to be determined by the formulas (2.9b). Since these formulas, except for the interpretation of the symbol [ ], remain valid for the configuration at present under discussion, the deductions of §2.5 and §2.4 require but slight modification to apply to the case in hand. The characteristic exponent is thus given by the formula

$$(3.13) \qquad \mu = \frac{i}{\pi}\cos^{-1}\left\{\cos\int_0^{\pi/2} 2\{\Delta-\Omega\cos 2x\}^{1/2}dx + O\left(\frac{\Delta^{1/2}}{\Delta-\Omega}\right)\right\}.$$

The order of the final term within the bracket evidently increases with Ω, from which it is evident that the domain of parameter values for which μ is real, i.e., for which there are unstable solutions, increases in extent as the upper end of the range of values Ω admitted in the configuration of the present chapter is approached.

The characteristic values $S_n(\Omega)$ and $C_n(\Omega)$ are each the root of an equation of the form (2.12) which in the present instance is more explicitly

$$(3.14) \qquad \int_0^{\pi/2}\{\Delta-\Omega\cos 2x\}^{1/2}dx + O\left(\frac{\Delta^{1/2}}{\Delta-\Omega}\right) = \frac{n\pi}{2}.$$

The lower end of the Ω range joins with that of the configuration II, and for such parameter values the formulas (2.13) are again valid as was to be ex-

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pected. To obtain formulas valid near the upper end of the range the following process may be used.

Let k_1 be defined by the relation

$$\Delta - \Omega = 2^{5/2} k_1 \Omega^{1/2}, \tag{3.15}$$

and in the integral of (3.14) replace x by $\pi/2 - \zeta$. Then the equation becomes

$$(\Delta + \Omega)^{1/2} G(h^2, h^2) + O\left(\frac{\Delta^{1/2}}{\Delta - \Omega}\right) = \frac{n\pi}{2},$$

with G the elliptic integral of (26) and

$$h^2 = \left(1 + \frac{4k_1}{\Omega}\right)^{-1}.$$

For the larger of the admitted values of Ω the ratio k_1/Ω is of the order of $\Delta^{-1/2}$ and h^2 is therefore nearly 1. With the use of the formula (26b) the equation may accordingly be written

$$\begin{aligned} (2\Omega)^{1/2} - k_1 \log \frac{k_1}{(32\Omega)^{1/2}} + k_1 \\ + k_1 O\left(\frac{k_1}{\Omega^{1/2}} \log \frac{k_1}{\Omega}\right) + O\left(\frac{\Delta^{1/2}}{\Delta - \Omega}\right) = \frac{n\pi}{2}. \end{aligned} \tag{3.14b}$$

Recalling (3.15), therefore, it follows that

$$\begin{aligned} S_n(\Omega) &= \Omega + 2^{5/2} k_1(n) \Omega^{1/2}, \\ C_n(\Omega) &= \Omega + 2^{5/2} k_1(n) \Omega^{1/2}, \end{aligned} \tag{3.16}$$

with each $k_1(n)$ a root of an equation of the form (3.14b).

Chapter 4

The configuration IV

4.1. **The differential equation.** Let the configuration designated as IV in Figure 1 be defined as that comprising the parameter values (Ω, Δ) in which both are large and

$$- M_2 \Omega^{1/2} \leqq \Delta - \Omega \leqq M_2 \Delta^{1/2}, \tag{4.1}$$

M_2 being the constant in the relation (3.1). Then the substitutions

$$\rho = (32\Omega)^{1/2}, \quad \sigma = \frac{\Delta - \Omega}{(32\Omega)^{1/2}}, \qquad s = z \tag{4.2}$$

determine ρ as a large parameter, while the range of values given to σ is

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bounded. The differential equation (1) takes the form (3) with the coefficients

$$\chi_0 \equiv \tfrac{1}{2}\sin s, \qquad (4.3)$$
$$\chi_1 \equiv \sigma,$$

in virtue of which the functions (5) are in this case explicitly

$$k = -i\sigma,$$
$$\eta(s) = 2i\sigma\tan\frac{s}{2},$$
$$\phi = \sin\frac{s}{2}\left\{\frac{1}{2}\cos\frac{s}{2} + \frac{\sigma}{\rho}\sec\frac{s}{2}\right\}, \qquad (4.4)$$
$$\Phi = \frac{1}{2}\sin^2\frac{s}{2} - \frac{\sigma}{\rho}\log\cos^2\frac{s}{2}.$$

Let R_z be chosen as the strip (13). Then in the region R_s the coefficient $\chi_0{}^2$ has a single zero located at the origin and of the second order. It must be shown that with the appropriate values $s_0=0$, $\nu=2$ the requirements of §1.2 are uniformly fulfilled. The hypotheses (i) and (ii) offer no difficulty in this respect, while the consideration of the functions (6) and the hypothesis (iii) may be made as follows.

The relation

$$e^q = \cos^2\frac{s}{2}$$

defines q, in terms of which

$$\Phi = qe^q\left\{\frac{e^{-q}-1}{2q} - \frac{\sigma}{\rho}e^{-q}\right\},$$

$$\omega_1 = \sigma^2\left\{1 - e^{-q} + \frac{2e^q - 2 - qe^q + \dfrac{2\sigma}{\rho}(1-e^{-q})}{\Phi}\right\},$$

while the various members of the formula

$$\omega(\phi) = \frac{3}{16}\left(\frac{2\phi'}{\phi} - \frac{\phi}{\Phi}\right)\left(\frac{2\phi'}{\phi} + \frac{\phi}{\Phi}\right) - \frac{\phi''}{2\phi}$$

are found to be

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$$\frac{\phi''}{\phi} = - \frac{1 + \dfrac{\sigma}{\rho} e^{-2q}}{1 + \dfrac{\sigma}{\rho} e^{-q}},$$

$$\frac{2\phi'}{\phi} = \cot\frac{s}{2} - \tan\frac{s}{2}\left\{\frac{1 - \dfrac{2\sigma}{\rho} e^{-q}}{1 + \dfrac{2\sigma}{\rho} e^{-q}}\right\},$$

$$\frac{\phi}{\Phi} = \cot\frac{s}{2} - \tan\frac{s}{2}\left\{\frac{2\sigma(q - 1 + e^{-q})}{\rho\Phi(e^{-q} - 1)}\right\}.$$

It is to be observed now that q vanishes with s, that $|e^q| \geqq \frac{1}{2}$ in R_s, and that the ratio σ/ρ will be uniformly as small as desired if Ω is restricted to remain sufficiently large. It is consequently seen that the brace in the formula for Φ is uniformly bounded from zero and hence that both $\omega(\phi)$ and ω_1 are uniformly bounded in any finite part of R_s. Finally, when $|s|$ is great the asymptotic formulas

$$\phi \sim \frac{\pm i}{2} e^q, \qquad \omega(\phi) \sim \frac{-1}{16},$$

$$\omega_1 \sim 2\sigma^2 q, \qquad ds \sim \pm i dq$$

are readily checked and in virtue of them the uniform fulfillment of the hypothesis (iii) becomes evident.

4.2. **The solutions $u_o(z)$ and $u_e(z)$.** The variables Φ and ξ differ only by the real factor ρ, while s and z are identical. Since the values of Φ on the boundaries of R_s are as follows:

for $s' = 0$,

$$\Phi = -\frac{1}{2}\sinh^2\frac{s''}{2} - \frac{\sigma}{\rho}\log\cosh^2\frac{s''}{2},$$

for $s' = \pi/2$,

$$\Phi = \left\{\frac{1}{4} - \frac{\sigma}{\rho}\log\frac{\cosh s''}{2}\right\} + i\left\{\frac{\sinh s''}{4} + \frac{\sigma}{\rho}\tan^{-1}(\sinh s'')\right\},$$

the map of R_z upon R_ξ is as indicated in Figure 5. The figure shows also the partition of these regions into the sub-regions $\Xi^{(l)}$ defined in (7).

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The representation of a pair of solutions $u_1(s)$, $u_2(s)$ which are determined by the initial values

$$u_1(0) = 0, \qquad u_1'(0) = \left(\frac{i\rho}{4}\right)^{1/2}\left(1+\frac{2\sigma}{\rho}\right)^{1/4},$$
$$u_2(0) = \left(1+\frac{2\sigma}{\rho}\right)^{-1/4}, \qquad u_2'(0) = 0$$

is known,* and is expressible in terms of the confluent hypergeometric func-

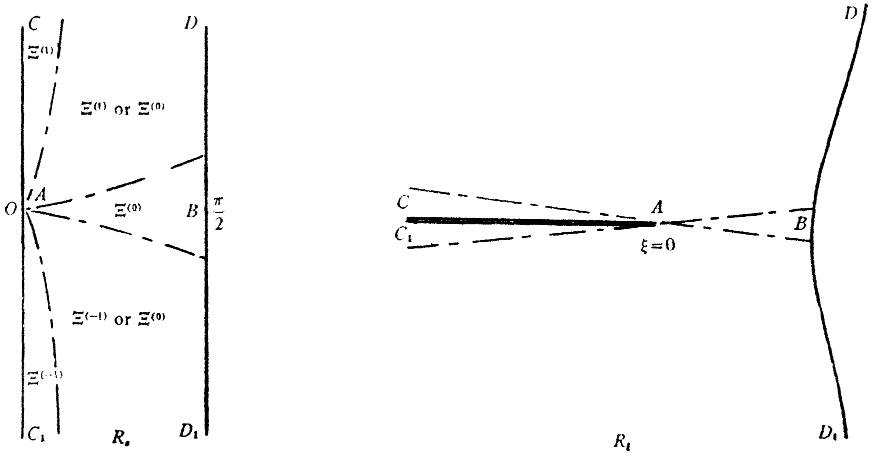

FIG. 5

tions customarily designated by $M_{j,l}$.† With the functions $\mathfrak{M}_j$ defined by the formulas

$$\mathfrak{M}_1(\xi, \sigma) = e^{-3\pi i/8}\xi^{-1/4}\mathfrak{M}_{-i\sigma,1/4}(2i\xi), \tag{4.5}$$
$$\mathfrak{M}_2(\xi, \sigma) = e^{-\pi i/8}\xi^{-1/4}\mathfrak{M}_{-i\sigma,-1/4}(2i\xi),$$

it is found thus that the principal solutions, which are evidently mere multiples of u_1 and u_2, are the following:

For $|\xi| \leqq N$,

$$u_o(z) = \left(\frac{2}{\rho}\right)^{1/2}\Psi[M_1(\xi, \sigma)], \tag{4.6a}$$
$$u_e(z) = \left(\frac{1}{2}\right)^{1/2}\Psi[M_2(\xi, \sigma)].$$

---

* Paper L_3. See, however, the footnote on p. 646 regarding the differences of notation.

† Cf. Whittaker and Watson, loc. cit., chapter XVI.

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On the other hand, when z is not in the neighborhood of the origin the formulas are the following*:

For $|\xi| \geqq N$, and z in $\Xi^{(l)}$,

$$(4.6b)\quad \begin{aligned} u_o(z) &= \left(\frac{\pi}{2\phi}\right)^{1/2}(i\rho)^{-3/4}\left\{\left[\frac{k_{0,1}^{(l)}}{\Gamma(\frac{3}{4}+i\sigma)}\right]_1(2i\xi)^{i\sigma}e^{i\xi} + \left[\frac{k_{0,2}^{(l)}}{\Gamma(\frac{3}{4}-i\sigma)}\right]_1(2i\xi)^{-i\sigma}e^{-i\xi}\right\}, \\ u_e(z) &= \left(\frac{\pi}{2\phi}\right)^{1/2}(i\rho)^{-1/4}\left\{\left[\frac{k_{e,1}^{(l)}}{\Gamma(\frac{1}{4}+i\sigma)}\right]_1(2i\xi)^{i\sigma}e^{i\xi} + \left[\frac{k_{e,2}^{(l)}}{\Gamma(\frac{1}{4}-i\sigma)}\right]_1(2i\xi)^{-i\sigma}e^{-i\xi}\right\}, \end{aligned}$$

with coefficients

(4.6c)

l	-1	0	1
$k_{0,1}^{(l)}$	1	1	$-ie^{2\sigma\pi}$
$k_{0,2}^{(l)}$	$e^{\sigma\pi-3\pi i/4}$	$e^{-\sigma\pi+3\pi i/4}$	$e^{-\sigma\pi+3\pi i/4}$
$k_{e,1}^{(l)}$	1	1	$ie^{2\sigma\pi}$
$k_{e,2}^{(l)}$	$e^{\sigma\pi-\pi i/4}$	$e^{-\sigma\pi+\pi i/4}$	$e^{-\sigma\pi+\pi i/4}$

;

for use in these formulas it is permissible to write in terms of the original variables

$$(4.7)\quad \begin{aligned} &\phi = [\tfrac{1}{4}]\sin z, \quad \xi = (2\Omega)^{1/2}[1](1-\cos z), \\ &e^{i\xi} = \left(\frac{2}{1+\cos z}\right)^{i\sigma} e^{(i\rho/4)(1-\cos z)}, \quad \Psi = \left(\frac{1}{1+\cos z}\right)^{1/4}[1]. \end{aligned}$$

When z is real the same is true of ϕ, Ψ and ξ, and the last of these is positive. For such values the functions $\mathfrak{M}_j$ of (4.5) are real, and the formulas (4.6a) are therefore directly real. From Figure 5 it is seen that such values of z lie in $\Xi^{(0)}$, whence the appropriate formulas (4.6b) reduce to

---

* The symbol $[\,]_1$ is used in the sense that $[Q]_1$ denotes a quantity which differs from Q by terms of the order of $(\log \rho)/\rho$ and terms of the order of N^{-1}.

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$$
(4.6\mathrm{d})\qquad
\begin{aligned}
u_o(x) &= \rho^{-3/4}\left(\frac{2\pi}{\phi}\right)^{1/2} e^{-\sigma\pi/2}\left[\frac{1}{\Gamma_1}\right]_1 \sin\left[\xi + \sigma\log 2\xi - \gamma_1 + \frac{\pi}{8}\right]_1,\\
u_e(x) &= \rho^{-1/4}\left(\frac{2\pi}{\phi}\right)^{1/2} e^{-\sigma\pi/2}\left[\frac{1}{\Gamma_2}\right]_1 \cos\left[\xi + \sigma\log 2\xi - \gamma_2 - \frac{\pi}{8}\right]_1.
\end{aligned}
$$

The symbols Γ_j and γ_j designate the real values determined by the formulas

$$
(4.8)\qquad \Gamma(\tfrac{3}{4} \pm i\sigma) = \Gamma_1 e^{\pm i\gamma_1}, \qquad \Gamma(\tfrac{1}{4} \pm i\sigma) = \Gamma_2 e^{\pm i\gamma_2},
$$

in which the left-hand members are gamma functions.

4.3. **The solutions $u_\alpha(z)$ and $u_\beta(z)$.** The solutions of the equation (3) especially associated with the sub-region $\Xi^{(0)}$ which by Figure 5 contains the point $z=\pi/2$, are those described by the following formulas:

For $|\xi| \geqq N$, and s in $\Xi^{(l)}$,

$$
(4.9\mathrm{a})\qquad u_{0,j}(s) = (i\rho)^{-1/4}(2\phi)^{-1/2}\left\{ B^{(l)}_{j,1}(2i\xi)^{i\sigma}e^{i\xi} + B^{(l)}_{j,2}(2i\xi)^{-i\sigma}e^{-i\xi}\right\},
$$

with coefficients

(4.9b)

l	-1	0	1
$B^{(l)}_{1,1}$	$[1]_1$	$[1]_1$	$[1]_1$
$B^{(l)}_{1,2}$	$\left[\dfrac{-2\pi i}{\Gamma(\frac{3}{4}-i\sigma)\Gamma(\frac{1}{4}-i\sigma)}\right]_1$	0	0
$B^{(l)}_{2,1}$	0	0	$\left[\dfrac{2\pi i e^{2\sigma\pi}}{\Gamma(\frac{3}{4}+i\sigma)\Gamma(\frac{1}{4}+i\sigma)}\right]_1$
$B^{(l)}_{2,2}$	$[1]_1$	$[1]_1$	$[1]_1$

For $|\xi| \leqq N$,

$$
(4.9\mathrm{c})\qquad
\begin{aligned}
u_{0,1} &= i\left(\frac{\pi}{2}\right)^{1/2} e^{-\sigma\pi}\Psi\left\{\frac{2}{\Gamma(\frac{1}{4}-i\sigma)}[\mathfrak{M}_1(\xi,\sigma)]_1 - \frac{e^{\pi i/4}}{\Gamma(\frac{3}{4}-i\sigma)}[\mathfrak{M}_2(\xi,\sigma)]_1\right\},\\
u_{0,2} &= -\left(\frac{\pi}{2}\right)^{1/2}\Psi\left\{\frac{2e^{\pi i/4}}{\Gamma(\frac{1}{4}+i\sigma)}[\mathfrak{M}_1(\xi,\sigma)]_1 - \frac{1}{\Gamma(\frac{3}{4}+i\sigma)}[\mathfrak{M}_2(\xi,\sigma)]_1\right\}.
\end{aligned}
$$

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The substitution into the formulas (8b) is simple, the Wronskian having the value $W=(i\rho)^{1/2}[1]_1$, and if the subscript 2 is used to designate evaluations at $z=\pi/2$, it is thus found that we have the following:

For $|\xi|\geqq N$, and z in $\Xi^{(l)}$,

$$
\begin{aligned}
u_\alpha(z) &= \frac{1}{2i\rho\phi_2{}^{1/2}\phi^{1/2}}\left\{\left(\frac{\xi}{\xi_2}\right)^{i\sigma} e^{i(\xi-\xi_2)}[1-\theta_1]_1\right. \\
&\qquad \left. - \left(\frac{\xi}{\xi_2}\right)^{-i\sigma} e^{-i(\xi-\xi_2)}[1-\theta_2]_1\right\}, \\
u_\beta(z) &= \frac{\phi_2{}^{1/2}}{2\phi^{1/2}}\left\{\left(\frac{\xi}{\xi_2}\right)^{i\sigma} e^{i(\xi-\xi_2)}[1+\theta_1]_1\right. \\
&\qquad \left. + \left(\frac{\xi}{\xi_2}\right)^{-i\sigma} e^{-i(\xi-\xi_2)}[1+\theta_2]_1\right\},
\end{aligned} \tag{4.10a}
$$

where

$$
\begin{aligned}
\theta_1 &= B^{(l)}_{2,1}(-4\xi_2{}^2)^{i\sigma}e^{2i\xi_2}, \\
\theta_2 &= B^{(l)}_{1,2}(-4\xi_2{}^2)^{-i\sigma}e^{-2i\xi_2}.
\end{aligned}
$$

For $|\xi|\leqq N$,

$$
\begin{aligned}
u_\alpha(z) &= 2\pi^{1/2}\rho^{-3/4}e^{-\sigma\pi/2}\Psi\left\{\frac{2\cos\mathcal{E}_2}{\Gamma_2}[\mathfrak{M}_1(\xi,\sigma)]_1 - \frac{\sin\mathcal{E}_1}{\Gamma_1}[\mathfrak{M}_2(\xi,\sigma)]_1\right\}, \\
u_\beta(z) &= \pi^{1/2}\rho^{1/4}e^{-\sigma\pi/2}\Psi\left\{\frac{\sin\mathcal{E}_2}{\Gamma_2}[\mathfrak{M}_1(\xi,\sigma)]_1 + \frac{\cos\mathcal{E}_1}{\Gamma_1}[\mathfrak{M}_2(\xi,\sigma)]_1\right\},
\end{aligned} \tag{4.10b}
$$

with

$$
\begin{aligned}
\mathcal{E}_1 &= \left[\frac{\rho}{4}+\sigma\log\rho-\gamma_1+\frac{\pi}{8}\right]_1, \\
\mathcal{E}_2 &= \left[\frac{\rho}{4}+\sigma\log\rho-\gamma_2-\frac{\pi}{8}\right]_1.
\end{aligned} \tag{4.11}
$$

For real values of z the formulas (4.10b) are directly real, while the appropriate formulas from (4.10a) reduce to

$$
\begin{aligned}
u_\alpha(x) &= \frac{-[2]_1}{\rho\phi^{1/2}}\sin\left[\frac{\rho}{4}\cos x - 2\sigma\log\tan\frac{x}{2}\right]_1, \\
u_\beta(x) &= \frac{[1]_1}{2\phi^{1/2}}\cos\left[\frac{\rho}{4}\cos x - 2\sigma\log\tan\frac{x}{2}\right]_1.
\end{aligned} \tag{4.10c}
$$

4.4. **The solutions of the associated Mathieu equation.** The representation of the solutions $iv_o(z)$ and $v_e(z)$, of the "associated" differential equation

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(2), are obtainable, as is now familiar, from the formulas of §4.2 by the substitution in place of ϕ, ξ and Ψ of the respective functions of iz, which may be designated by $\bar{\phi}$, $\bar{\xi}$ and $\bar{\Psi}$. Explicitly the evaluations

$$\bar{\phi} = \left[\frac{i}{4}\right] \sinh z,$$

$$\bar{\xi} = (2\Omega)^{1/2}[1](1 - \cosh z),$$

$$e^{i\bar{\xi}} = \left(\frac{2}{1+\cosh z}\right)^{i\sigma} e^{(i\rho/4)(1-\cosh z)},$$

$$\bar{\Psi} = \left(\frac{4}{1+\cosh z}\right)^{1/4} [1]$$

may be used. The sub-regions of the z plane in which the respective formulas so derived are valid are as is shown in Figure 6.

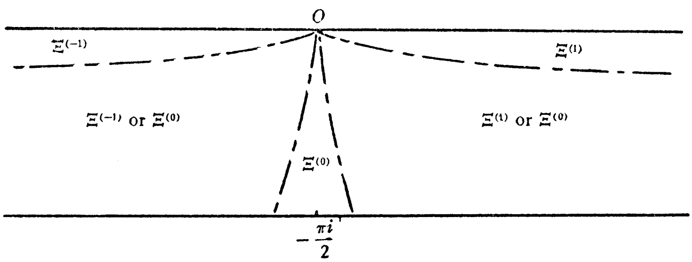

FIG. 6

In particular, when z is real and positive the forms deduced from (4.6b) for $\Xi^{(1)}$ reduce to the following:

for $|\bar{\xi}| \geqq N$,

$$\text{(4.12a)}\quad \begin{aligned} v_o(x) &= \rho^{-3/4}\left(\frac{2\pi}{|\bar{\phi}|}\right)^{1/2} e^{\sigma\pi/2}\left[\frac{1}{\Gamma_1}\right]_1 \sin\left[\,|\bar{\xi}| - \sigma \log 2|\bar{\xi}| + \gamma_1 + \frac{\pi}{8}\right]_1, \\ v_e(x) &= \rho^{-1/4}\left(\frac{2\pi}{|\bar{\phi}|}\right)^{1/2} e^{\sigma\pi/2}\left[\frac{1}{\Gamma_2}\right]_1 \cos\left[\,|\bar{\xi}| - \sigma \log 2|\bar{\xi}| + \gamma_2 - \frac{\pi}{8}\right]_1. \end{aligned}$$

On the other hand, when x is small, i.e.,

for $|\bar{\xi}| \leqq N$,

$$\text{(4.12b)}\quad \begin{aligned} v_o(x) &= \left(\frac{2}{\rho}\right)^{1/2} \Psi[i\mathcal{M}_1(-|\bar{\xi}|, \sigma)], \\ v_e(x) &= \left(\frac{1}{2}\right)^{1/2} \Psi[\mathcal{M}_2(-|\bar{\xi}|, \sigma)]. \end{aligned}$$

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The functions within the brackets may be shown to be explicitly real as they should be.

4.5. **The characteristic values.** The values (4.6d) substituted into the characteristic equations (17) and (18) give to the latter the forms

$$(2\Omega)^{1/2} + \frac{\sigma}{2}\log(32\Omega) - \gamma_1 + \frac{\pi}{8} + O(\Omega^{-1/2}\log\Omega) = \frac{n\pi}{2},$$

for an odd Mathieu function,

(4.13)

$$(2\Omega)^{1/2} + \frac{\sigma}{2}\log(32\Omega) - \gamma_2 - \frac{\pi}{8} + O(\Omega^{-1/2}\log\Omega) = \frac{n\pi}{2},$$

for an even Mathieu function.

These equations may be given a somewhat more detailed form when σ is near either the one or the other extreme or the middle of its admitted range of values. The indices of the characteristic values which satisfy the equations with a specific integer n on the right may also be determined as will be shown.

The theory of the gamma function supplies, in particular when $c_1 = 3/4$ and $c_2 = 1/4$, the formulas*

$$\gamma_j = \sigma\frac{\Gamma'(c_j)}{\Gamma(c_j)} + \sum_{r=1}^{\infty}\left(\frac{\sigma}{c_j + r} - \tan^{-1}\frac{\sigma}{c_j + r}\right),$$

(4.14)

$$\log\Gamma(c_j + i\sigma) = \tfrac{1}{2}\log 2\pi + (c_j - \tfrac{1}{2} + i\sigma)\log(c_j + i\sigma) - (c_j + i\sigma) + O\left(\frac{1}{|\sigma|}\right),$$

$$\Gamma(\zeta)\Gamma(1 - \zeta) = \pi\csc\pi\zeta,$$

and from the first of these it is readily seen that with Ω fixed the left members of the equations (4.13) vary monotonically with σ so that the roots for any integer n are unique.

When σ is near the upper end of its admitted range of values, it is large and positive, and the second of the formulas (4.14) gives the evaluations

$$\gamma_j = \sigma\log\sigma - \sigma + (2c_j - 1)(\pi/4) + O(1/\sigma).$$

Both the equations (4.13) thus become

$$\text{(4.13a)}\quad (2\Omega)^{1/2} - \frac{\sigma}{2}\log\frac{\sigma^2}{32\Omega} + \sigma + O(\Omega^{1/2}\log\Omega) + O\left(\frac{\Omega^{1/2}}{\Delta - \Omega}\right) = \frac{n\pi}{2},$$

* Cf. Nielsen, N., *Handbuch der Theorie der Gammafunktion*, Leipzig, 1906, p. 23 and pp. 94 and 209.

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which is, therefore, the form of the characteristic equations when Ω is near the lower end of the range of values admitted for it in the present configuration. Since for these values the configurations of the present and the preceding chapter abut, the indices of the characteristic values concerned may be determined by a comparison of the equations (4.13a) and (3.16), k_1 in the latter having been defined precisely as σ is in the former. With a given value of n the roots of the equations (4.13) are thus seen to be precisely $S_n(\Omega)$ and $C_n(\Omega)$ respectively.

Near the middle of its range σ is small, and the left members of the equations (4.13) are essentially represented by the early terms of their expansions in powers of σ. Thus the equations become

$$\left\{\left(n+\frac{1}{2}-c_j\right)\frac{\pi}{2}-(2\Omega)^{1/2}+O(\Omega^{-1/2}\log\Omega)\right\}$$
$$+\sigma\left\{\frac{\Gamma'(c_j)}{\Gamma(c_j)}-\frac{1}{2}\log\,(32\Omega)\right\}+O(\sigma^2)=0,$$

the values of n concerned being such as make the initial term small. The formulas which are valid in this case, i.e., when Δ and Ω are nearly equal, are thus

$$\text{(4.13b)}\qquad \begin{aligned} S_n(\Omega) &= \Omega+(32\Omega)^{1/2}\left\{\frac{(n-\frac{1}{4})\pi-(8\Omega)^{1/2}}{\log\,(32\Omega)-2\Gamma'(\frac{3}{4})/\Gamma(\frac{3}{4})}\right\}+O(1),\\ C_n(\Omega) &= \Omega+(32\Omega)^{1/2}\left\{\frac{(n+\frac{1}{4})\pi-(8\Omega)^{1/2}}{\log\,(32\Omega)-2\Gamma'(\frac{1}{4})/\Gamma(\frac{1}{4})}\right\}+O(1).\end{aligned}$$

In particular, the values of Ω for which $\sigma=0$ is a root, i.e., for which there is a characteristic value equal to Ω, are found to be as follows:

$$\text{(4.15)}\qquad \begin{aligned} &\text{If } S_n(\Omega)=\Omega, \text{ then } (2\Omega)^{1/2}=\left(n-\frac{1}{4}\right)\frac{\pi}{2}+O\left(\frac{\log n}{n}\right),\\ &\text{If } C_n(\Omega)=\Omega, \text{ then } (2\Omega)^{1/2}=\left(n+\frac{1}{4}\right)\frac{\pi}{2}+O\left(\frac{\log n}{n}\right).^{*}\end{aligned}$$

---

* These values were considered by Goldstein, S., in *A note on certain approximate solutions of linear differential equations*, etc., Proceedings of the London Mathematical Society, (2), vol. 28 (1928), p. 87, where the results are stated in the following form:

$$\text{If } S_n(\Omega)=\Omega, \text{ then } \begin{cases} 2^{1/2}\cos\,(8\Omega)^{1/2}\sim(-1)^n,\\ 2^{1/2}\sin\,(8\Omega)^{1/2}\sim(-1)^{n+1}.\end{cases}$$
$$\text{If } C_n(\Omega)=\Omega, \text{ then } \begin{cases} 2^{1/2}\cos\,(8\Omega)^{1/2}\sim(-1)^n,\\ 2^{1/2}\sin\,(8\Omega)^{1/2}\sim(-1)^{n}.\end{cases}$$

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Finally near the lower end of its permitted range σ is large but negative, and the second of formulas (4.14) gives

$$\gamma_j = -\sigma + \sigma \log |\sigma| - (2c_j - 1)\frac{\pi}{4} + O\left(\frac{1}{|\sigma|}\right).$$

The characteristic equations (4.13) accordingly become respectively

$$(2\Omega)^{1/2} + \frac{\sigma}{2}\log\frac{32\Omega}{\sigma^2} + \sigma + \frac{\pi}{4} + O(\Omega^{-1/2}\log\Omega) + O\left(\frac{\Omega^{1/2}}{\Omega - \Delta}\right) = \frac{n\pi}{2},$$

for the characteristic value $S_n(\Omega)$;

(4.13c)

$$(2\Omega)^{1/2} + \frac{\sigma}{2}\log\frac{32\Omega}{\sigma^2} + \sigma - \frac{\pi}{4} + O(\Omega^{-1/2}\log\Omega) + O\left(\frac{\Omega^{1/2}}{\Omega - \Delta}\right) = \frac{n\pi}{2},$$

for the characteristic value $C_n(\Omega)$.

These are, therefore, the forms which are valid when Ω is near the upper end of its permitted range of values, or, in other words, when Δ is near the lower end of its possible range.

4.6. **The characteristic exponent.** The formulas (4.6d) and (4.10b) yield for the evaluation of Θ in (25b)

$$\Theta = 4e^{-\sigma\pi}\left[\frac{\pi}{\Gamma_1\Gamma_2}\right]_1 \cos \mathcal{E}_1 \cos \mathcal{E}_2 - 1,$$

where $\mathcal{E}_1$ and $\mathcal{E}_2$ are as defined in (4.11). The third of the formulas (4.14) may be made to give further

$$\frac{\pi}{\Gamma_1\Gamma_2} = \left\{\frac{\cosh 2\sigma\pi}{2}\right\}^{1/2},$$

whence

$$\Theta = 2\{1 + e^{-4\sigma\pi}\}^{1/2}[1]_1 \cos \mathcal{E}_2 \cos \mathcal{E}_1 - 1, \tag{4.16}$$

and the characteristic exponent is obtainable from the appropriate formula (25a).

The (Ω, Δ) sub-regions of the domain IV of Figure 1 which comprise parameter values for which the differential equation has stable solutions are those for which the value of Θ is less than unity. It is evident from the formula (4.16) that these sub-regions become more and more attenuated as σ decreases, i.e., as the right-hand boundary of the configuration IV is approached.

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CHAPTER 5

THE CONFIGURATION V

5.1. **Preliminaries.** Abutting the configuration of the preceding chapter is that denoted by V in Figure 1, in which Ω is taken to be large and

$$0 \leqq \Delta \leqq \Omega - M_2\Omega^{1/2}. \tag{5.1}$$

In this case the substitutions

$$\rho = \frac{\Omega - \Delta}{\Omega^{1/2}}, \qquad \sigma^2 = 1 - \frac{\Delta}{\Omega}, \qquad s = \frac{z}{\sigma} \tag{5.2}$$

reduce the differential equation (1) to the form (3) with

$$\begin{aligned} \chi_0{}^2(s, \sigma) &\equiv \frac{2 \sin^2 \sigma s}{\sigma^2} - 1, \\ \chi_1 &\equiv 0. \end{aligned} \tag{5.3}$$

The parameter ρ is bounded below by the constant M_2 while σ^2 is confined to the fixed closed range $0 \leqq \sigma^2 \leqq 1$, its smallest possible value being in fact $M_2\Omega^{-1/2}$.

With the strip (13) chosen as R_z, the region R_s is

$$R_s: \qquad 0 \leqq s' \leqq \frac{\pi}{2\sigma}, \tag{5.4}$$

and within this $\chi_0{}^2$ admits just one zero which is simple and is located on the axis of reals at the point

$$s_0' = \frac{1}{\sigma} \sin^{-1} \frac{\sigma}{2^{1/2}}.$$

The position of s_0' varies with σ but is restricted to the fixed interval $(2^{-1/2}, \pi/4)$.

If R_s is thought of as cut along the axis of reals from $s=0$ to $s=s_0'$, the values of Φ on its boundaries are the following:

For $s''=0+$, $0 \leqq s' \leqq s_0'$,

$$\Phi = e^{\pi i} \int_{s'}^{s_0'} i \left\{ 1 - \frac{2 \sin^2 \sigma s'}{\sigma^2} \right\}^{1/2} ds'.$$

For $s'=0$, $s'' \geqq 0$,

$$\Phi = \Phi(0) + e^{\pi i/2} \int_0^{s'} i \left\{ \frac{2 \sinh^2 \sigma s''}{\sigma^2} + 1 \right\}^{1/2} ds'.$$

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For $s' = \pi/(2\sigma)$, $s'' \geqq 0$,

$$\Phi = \Phi\left(\frac{\pi}{2\sigma}\right) + e^{\pi i/2}\int_0^{s''}\left\{\frac{2\cosh^2\sigma s''}{\sigma^2} - 1\right\}^{1/2} ds''.$$

The maps of R_s upon R_Φ, and hence of R_z upon R_ξ, are thus revealed, the latter being as indicated in Figure 7:

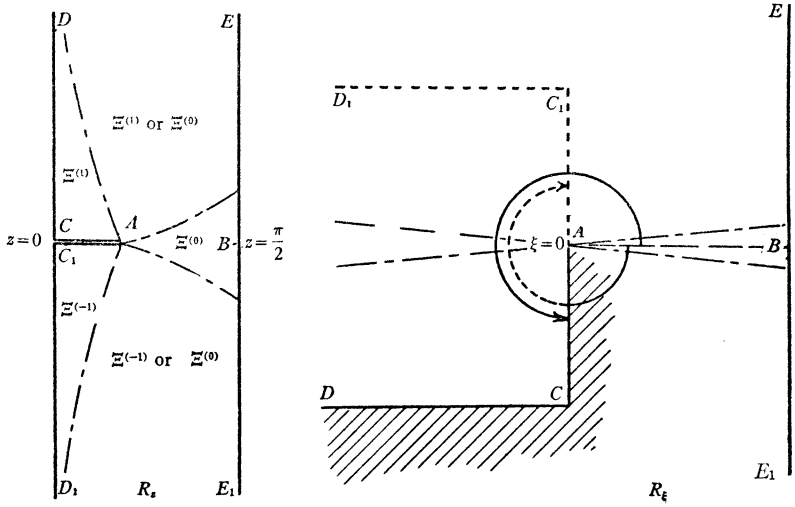

FIG. 7

5.2. **The hypotheses.** The discussion by which the uniform fulfillment of the requirements of §1.2 by the present differential equation may be established, will be omitted as to detail inasmuch as it proceeds almost entirely like that of §3.3. In virtue of the values (5.3) the functions (6) are in this case explicitly

$$\omega(\phi) = \frac{1}{4}\left\{-\frac{5}{9}\left(\frac{\phi}{\Phi}\right)^2 - \sigma^2 + \frac{6(1-\sigma^2)}{\phi^2} + \frac{5(2-\sigma^2)}{\phi^4}\right\},$$

$$\Psi = \Phi^{1/6}/\phi^{1/2}, \qquad \omega_1 \equiv 0.$$

When $|s - s_0'|$ is great the formulas

$$\phi \sim 2^{1/2}\frac{\sin\sigma s}{\sigma}, \qquad \Phi \sim \frac{2^{3/2}}{\sigma^2}\sin^2\frac{\sigma s}{2}$$

may be used, while for intermediate values the first of the formulas (5.3) may be written

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$$\phi^2 = 2(1-\sigma^2)\frac{\sin^2\sigma(s-s_0')}{\sigma^2} + (2-\sigma^2)\frac{\sin 2\sigma(s-s_0')}{\sigma}.$$

For small values of $|s-s_0'|$ it may be shown that

$$\Phi = \frac{\phi^3}{3(2-\sigma^2)^{1/2}}\left\{1 - \frac{3(1-\sigma^2)}{5(2-\sigma^2)}\phi^2 + \phi^4 O(1)\right\},$$

and with these formulas at hand the arguments of §3.3 may be paralleled.

5.3. **The solutions relative to $z=0$.** The point $z=0$ may, as is seen from Figure 7, be regarded as lying in the sub-region $\Xi^{(1)}$. Moreover, the zero of χ_0^2 being simple the formulas of §1.4 are applicable, with $h=1$ as the appropriate value. The formulas (11b) and (11a) thus become, in the manner now familiar, the following:

When $|\xi| \leqq N$,

$$\text{(5.5a)}\quad \begin{aligned} u_o(z) &= \left\{\frac{\pi\sigma^2 i}{6\rho^2\phi_1\phi}\right\}^{1/2} \xi^{1/6}e^{i\xi_1}[\xi^{1/3}J_{-1/3}(\xi) + \xi^{1/3}J_{1/3}(\xi)], \\ u_e(z) &= \left\{\frac{\pi\phi_1}{6i\phi}\right\}^{1/2} \xi^{1/6}e^{i\xi_1}[\xi^{1/3}J_{-1/3}(\xi) + \xi^{1/3}J_{1/3}(\xi)]; \end{aligned}$$

and when z lies in $\Xi^{(l)}$, and $|\xi| \geqq N$,

$$\text{(5.5b)}\quad \begin{aligned} u_o(z) &= \frac{1}{2}\left\{\frac{\sigma^2}{\rho^2\phi_1\phi}\right\}^{1/2}\{K_{0,1}^{(l)}e^{i\xi} + K_{0,2}^{(l)}e^{-i\xi}\}, \\ u_e(z) &= \frac{1}{2}\left\{\frac{\phi_1}{\phi}\right\}^{1/2}\{K_{e,1}^{(l)}e^{i\xi} + K_{e,2}^{(l)}e^{-i\xi}\}, \end{aligned}$$

with coefficients

(5.5c)

l	-1	0	1
$K_{0,1}^{(l)}$	$e^{i\xi_1}[1]$	$e^{i\xi_1}[1]$	$-ie^{-i\xi_1}[1]$
$K_{0,2}^{(l)}$	$-e^{-i\xi_1}[1]$	$ie^{i\xi_1}[1]$	$ie^{i\xi_1}[1]$
$K_{e,1}^{(l)}$	$-ie^{i\xi_1}[1]$	$-ie^{i\xi_1}[1]$	$e^{-i\xi_1}[1]$
$K_{e,2}^{(l)}$	$-ie^{-i\xi_1}[1]$	$e^{i\xi_1}[1]$	$e^{i\xi_1}[1]$

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The symbols involved would have the evaluations

$$\frac{\rho\phi}{\sigma} = \{\Delta - \Omega\cos 2z\}^{1/2}, \qquad \frac{\rho\phi_1}{\sigma} = e^{\pi i/2}\{\Omega - \Delta\}^{1/2},$$

$$\xi = \int_{x_0}^{z}\{\Delta - \Omega\cos 2z\}^{1/2}dz, \qquad x_0 = \tfrac{1}{2}\cos^{-1}\Delta/\Omega,$$

$$\xi_1 = e^{3\pi i/2}\int_0^{x_0}\{\Omega\cos 2x - \Delta\}^{1/2}dx.$$

When z is real and less than x_0 the relation $\xi = |\xi|\,e^{3\pi i/2}$ is valid and hence

$$J_{-1/3}(\xi) + J_{1/3}(\xi) = \frac{3^{1/2}}{\pi i}K_{1/3}(|\xi|).$$

The formulas given in (5.5) thus reduce when the variable is real to the following:

When $0 \leqq x < x_0$, and $|\xi| \geqq N$,

$$\text{(5.6a)}\quad \begin{aligned} u_o(x) &= \frac{[1]}{\{(\Omega - \Delta)(\Omega\cos 2x - \Delta)\}^{1/4}}\sinh\left[\int_0^x \{\Omega\cos 2x - \Delta\}^{1/2}dx\right], \\ u_e(x) &= \left\{\frac{\Omega - \Delta}{\Omega\cos 2x - \Delta}\right\}^{1/4}[1]\cosh\left[\int_0^x \{\Omega\cos 2x - \Delta\}^{1/2}dx\right]. \end{aligned}$$

When $x \leqq x_0$, and $|\xi| \leqq N$,

$$\text{(5.6b)}\quad \begin{aligned} u_o(x) &= \frac{|\xi|^{1/6}e^{|\xi_1|}}{\{4\pi^2(\Omega - \Delta)(\Omega\cos 2x - \Delta)\}^{1/4}}[|\xi|^{1/3}K_{1/3}(|\xi|)], \\ &\qquad \textbf{with}\quad |\xi| = \int_x^{x_0}\{\Omega\cos 2x - \Delta\}^{1/2}dx. \end{aligned}$$

When $x_0 \leqq x$, and $|\xi| \leqq N$,

$$\text{(5.6c)}\quad u_o(x) = \frac{(\pi/6)^{1/2}\xi^{1/6}e^{|\xi_1|}}{\{(\Omega - \Delta)(\Delta - \Omega\cos 2x)\}^{1/4}}[\xi^{1/3}J_{-1/3}(\xi) + \xi^{1/3}J_{1/3}(\xi)].$$

When $x_0 < x \leqq \pi/2$, and $|\xi| \geqq N$,

$$\text{(5.6d)}\quad \begin{aligned} u_o(x) &= \frac{[1]e^{|\xi|}}{\{(\Omega - \Delta)(\Delta - \Omega\cos 2x)\}^{1/4}} \\ &\qquad \cdot\sin\left[\frac{\pi}{4} + \int_{x_0}^{x}\{\Delta - \Omega\cos 2x\}^{1/2}dx\right]. \end{aligned}$$

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In the cases (b), (c) and (d) the representation of $u_e(x)$ may be formally obtained from that of $u_o(x)$ by replacing the factor $(\Omega-\Delta)^{-1/4}$ by $(\Omega-\Delta)^{1/4}$.

5.4. **The solutions relative to $z=\pi/2$.** The formulas (11) with the subscripts a replaced by 2, where the latter denote values corresponding to $z=\pi/2$, may be made to yield also the solutions $u_\alpha(z)$ and $u_\beta(z)$. Since the point $z=\pi/2$ lies in the sub-region $\Xi^{(0)}$ the value $h=0$ is appropriate and the formulas obtained are the following:

When $|\xi|\leqq N$,

$$
\begin{aligned}
u_\alpha(z) &= \left(\frac{2\pi\sigma^2}{3\rho^2\phi_2\phi}\right)^{1/2}\xi^{1/6}\left\{\cos\left(\xi_2-\frac{\pi}{12}\right)[\xi^{1/3}J_{1/3}(\xi)]\right. \\
&\qquad\qquad \left. -\sin\left(\xi_2+\frac{\pi}{12}\right)[\xi^{1/3}J_{-1/3}(\xi)]\right\}, \\
u_\beta(z) &= \left(\frac{2\pi\phi_2}{3\phi}\right)^{1/2}\xi^{1/6}\left\{\sin\left(\xi_2-\frac{\pi}{12}\right)[\xi^{1/3}J_{1/3}(\xi)]\right. \\
&\qquad\qquad \left. +\cos\left(\xi_2+\frac{\pi}{12}\right)[\xi^{1/3}J_{-1/3}(\xi)]\right\}.
\end{aligned} \tag{5.7a}
$$

When z is in $\Xi^{(l)}$, and $|\xi|\geqq N$,

$$
\begin{aligned}
u_\alpha(z) &= \frac{1}{2}\left(\frac{\sigma^2}{\rho^2\phi_2\phi}\right)^{1/2}\left\{K_{\alpha,1}^{(l)}e^{i\xi}+K_{\alpha,2}^{(l)}e^{-i\xi}\right\}, \\
u_\beta(z) &= \frac{1}{2}\left(\frac{\phi_2}{\phi}\right)^{1/2}\left\{K_{\beta,1}^{(l)}e^{i\xi}+K_{\beta,2}^{(l)}e^{-i\xi}\right\},
\end{aligned} \tag{5.7b}
$$

with coefficients

(5.7c)

l	-1	0	1
$K_{\alpha,1}^{(l)}$	$-ie^{-i\xi_2}[1]$	$-ie^{i\xi_2}[1]$	$2e^{-3\pi i/4}\left[\cos\left(\xi_2-\frac{\pi}{4}\right)\right]$
$K_{\alpha,2}^{(l)}$	$2e^{3\pi i/4}\left[\cos\left(\xi_2-\frac{\pi}{4}\right)\right]$	$ie^{i\xi_2}[1]$	$ie^{i\xi_2}[1]$
$K_{\beta,1}^{(l)}$	$2e^{3\pi i/4}\left[\sin\left(\xi_2-\frac{\pi}{4}\right)\right]$	$e^{i\xi_2}[1]$	$e^{i\xi_2}[1]$
$K_{\beta,2}^{(l)}$	$e^{-i\xi_2}[1]$	$e^{-i\xi_2}[1]$	$2e^{-3\pi i/4}\left[\sin\left(\xi_2-\frac{\pi}{4}\right)\right]$

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In terms of the original variables

$$\frac{\rho\phi_2}{\sigma} = \{\Omega + \Delta\}^{1/2},$$

$$\xi_2 = \int_{x_0}^{\pi/2} \{\Delta - \Omega \cos 2x\}^{1/2} dx.$$

The forms obtained for real values of the variable are the following: When $0 \leqq x < x_0$, and $|\xi| \geqq N$,

$$\begin{aligned} u_\alpha(x) &= \frac{-\left[\cos\left(\frac{\pi}{4} - \xi_2\right)\right]}{\{(\Omega+\Delta)(\Omega\cos 2x - \Delta)\}^{1/4}} \exp\left(\int_x^{x_0} \{\Omega \cos 2x - \Delta\}^{1/2} dx\right), \\ u_\beta(x) &= \left\{\frac{\Omega+\Delta}{\Omega \cos 2x - \Delta}\right\}^{1/4} \left[\sin\left(\frac{\pi}{4} - \xi_2\right)\right] \\ &\qquad \cdot \exp\left(\int_x^{x_0} \{\Omega \cos 2x - \Delta\}^{1/2} dx\right). \end{aligned} \tag{5.8a}$$

When $x \leqq x_0$, and $|\xi| \leqq N$,

$$\begin{aligned} u_\alpha(x) &= \frac{-(2\pi/3)^{1/2}|\xi|^{1/6}}{\{(\Omega+\Delta)(\Omega\cos 2x - \Delta)\}^{1/4}} \left\{\cos\left(\xi_2 - \frac{\pi}{12}\right)[|\xi|^{1/3} I_{1/3}(|\xi|)] \right. \\ &\qquad \left. + \sin\left(\xi_2 + \frac{\pi}{12}\right)[|\xi|^{1/3} I_{-1/3}(|\xi|)]\right\}, \\ u_\beta(x) &= \left(\frac{2\pi}{3}\right)^{1/2} |\xi|^{1/6} \left\{\frac{\Omega+\Delta}{\Omega\cos 2x - \Delta}\right\}^{1/4} \left\{\sin\left(\xi_2 - \frac{\pi}{12}\right)[|\xi|^{1/3} I_{1/3}(|\xi|)] \right. \\ &\qquad \left. - \cos\left(\xi_2 + \frac{\pi}{12}\right)[|\xi|^{1/3} I_{-1/3}(|\xi|)]\right\}. \end{aligned} \tag{5.8b}$$

When $x_0 \leqq x$, and $|\xi| \leqq N$,

$$\begin{aligned} u_\alpha(x) &= \frac{(2\pi/3)^{1/2}\xi^{1/6}}{\{(\Omega+\Delta)(\Delta - \Omega\cos 2x)\}^{1/4}} \left\{\cos\left(\xi_2 - \frac{\pi}{12}\right)[\xi^{1/3} J_{1/3}(\xi)] \right. \\ &\qquad \left. - \sin\left(\xi_2 + \frac{\pi}{12}\right)[\xi^{1/3} J_{-1/3}(\xi)]\right\}, \\ u_\beta(x) &= \left(\frac{2\pi}{3}\right)^{1/2} \xi^{1/6} \left\{\frac{\Omega+\Delta}{\Delta - \Omega\cos 2x}\right\}^{1/4} \left\{\sin\left(\xi_2 - \frac{\pi}{12}\right)[\xi^{1/3} J_{1/3}(\xi)] \right. \\ &\qquad \left. + \cos\left(\xi_2 + \frac{\pi}{12}\right)[\xi^{1/3} J_{-1/3}(\xi)]\right\}. \end{aligned} \tag{5.8c}$$

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When $x_0 < x \leqq \pi/2$, and $|\xi| \geqq N$,

$$
(5.8\text{d})\qquad
\begin{aligned}
u_\alpha(x) &= \frac{[1]}{\{(\Omega+\Delta)(\Delta-\Omega\cos 2x)\}^{1/4}} \sin\left[\int_{\pi/2}^{x} \{\Delta - \Omega\cos 2x\}^{1/2} dx\right],\\
u_\beta(x) &= \left\{\frac{\Omega+\Delta}{\Delta-\Omega\cos 2x}\right\}^{1/4} [1] \cos\left[\int_{\pi/2}^{x} \{\Delta-\Omega\cos 2x\}^{1/2} dx\right].
\end{aligned}
$$

5.5. **The solutions of the associated equation.** The positive axis of imaginaries in Figure 7 lies in the sub-region $\Xi^{(1)}$. The formulas (5.5) appropriate to this region are to be used, therefore, in obtaining the solutions of the equation (2) for real values of the variable by the substitutions (12). The formulas thus found are

$$
(5.9)\qquad
\begin{aligned}
v_o(x) &= \frac{[1]}{\{(\Omega-\Delta)(\Omega\cosh 2x-\Delta)\}^{1/4}} \sin\left[\int_0^x \{\Omega\cosh 2x - \Delta\}^{1/2} dx\right],\\
v_e(x) &= \left\{\frac{\Omega-\Delta}{\Omega\cosh 2x-\Delta}\right\}^{1/4} [1] \cos\left[\int_0^x \{\Omega\cosh 2x-\Delta\}^{1/2} dx\right].
\end{aligned}
$$

5.6. **The characteristic values and exponent.** The forms (5.6d) show that the characteristic values for both even and odd Mathieu functions are in this case determined by equations

$$
(5.10\text{a})\qquad \left[\frac{\pi}{4} + \int_{x_0}^{\pi/2} \{\Delta - \Omega\cos 2x\}^{1/2} dx\right] = \frac{n\pi}{2},
$$

the proper correlation of the indices of the roots with the integer n being duly regarded.

If k_1 is defined in terms of Δ and Ω by the same formula as is the σ of chapter 4, i.e., by (3.15), the substitutions

$$
\cos x = h \sin \zeta, \qquad h^2 = 1 + \frac{4k_1}{(2\Omega)^{1/2}}
$$

reduce the equation (5.10a) to the form

$$
(5.10\text{b})\qquad \frac{\pi}{4} + (2\Omega)^{1/2} h^2 G(1, h^2) + O\left(\frac{\Omega^{1/2}}{\Omega-\Delta}\right) = \frac{n\pi}{2},
$$

in which G is the elliptic integral of (26). In the range of transition from the configuration of chapter 4 to that of the present chapter, k_1 is negative and h^2 accordingly little less than unity. The evaluation of (5.10b) to the form

$$
\frac{\pi}{4} + (2\Omega)^{1/2} + \frac{k_1}{2}\log\frac{32\Omega}{k_1{}^2} + k_1 + O(\Omega^{-1/2}\log\Omega) + O\left(\frac{\Omega^{1/2}}{\Omega-\Delta}\right) = \frac{n\pi}{2}
$$

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may, therefore, be obtained by the use of (26b), and a comparison of the result with the equations (4.13c) shows that the characteristic values which occur as the roots of equations representable by (5.10a) are respectively $S_n(\Omega)$ and $C_{n-1}(\Omega)$. In other words the characteristic equations are as follows:

$$(5.11)\qquad \begin{aligned} &\int_{x_0}^{\pi/2}\{\Delta - \Omega\cos 2x\}^{1/2}dx + O\left(\frac{\Omega^{1/2}}{\Omega - \Delta}\right) = \left(n - \frac{1}{2}\right)\frac{\pi}{2}, \\ &\qquad\qquad \text{for the characteristic value } S_n(\Omega); \\ &\int_{x_0}^{\pi/2}\{\Delta - \Omega\cos 2x\}^{1/2}dx + O\left(\frac{\Omega^{1/2}}{\Omega - \Delta}\right) = \left(n + \frac{1}{2}\right)\frac{\pi}{2}, \\ &\qquad\qquad \text{for the characteristic value } C_n(\Omega). \end{aligned}$$

In the consideration of the characteristic exponent the formulas (5.6d) and (5.8a) in conjunction with (25b) are found to lead to the evaluation

$$(5.12)\qquad \Theta = e^{2|\xi_1|}[\cos 2\xi_2] - 1,$$

and with this the value of μ is given by the formula (25a). Since the right-hand member of (5.12) can be exceeded by unity only when the cosine is very small, it is evident that the unstable solutions greatly predominate in the present configuration.

An evaluation of the several elliptic integrals involved may be made to show that the transition from the formula (4.16) to (5.12) is a continuous one.

Chapter 6

The configuration VI

6.1. **Remarks.** The configuration designated by VI in Figure 1 is to be that in which Δ is negative and

$$(6.1)\qquad -\{\Omega - M_2\Omega^{1/2}\} \leqq \Delta \leqq 0.$$

It clearly differs from that of the preceding chapter only in the sign of Δ. The distinction between the two configurations is indeed largely an artificial one, entered into primarily for the purpose of utilizing the discussion of §5.2 without modification when parameter values admitted by (6.1) are concerned. For in this latter case the substitutions

$$(6.2)\qquad \rho = \frac{\Omega + \Delta}{\Omega^{1/2}}e^{3\pi i/2}, \qquad \sigma^2 = 1 + \frac{\Delta}{\Omega}, \qquad s = \frac{1}{\sigma}\left(\frac{\pi}{2} - z\right)$$

transform the differential equation (1) into the form (3) with precisely the coefficients (5.3), with σ restricted precisely as in the earlier case. The de-

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ductions of §5.2, therefore, serve again to show that the requirements of the general theory are uniformly fulfilled.

By their definitions the intermediate variables s, ϕ, and Φ, and the parameters σ and ρ, differ from the corresponding quantities in chapter 5. The ultimate variables z and ξ are, however, found to have the same relation to each other, so that Figure 7 continues to remain valid in the present configuration. It is found as a consequence that the various formulas deduced in §5.3, §5.4, and §5.5 apply also in the present instance, provided they are expressed entirely in terms of the original variables z, Δ, and Ω.

6.2. **The characteristic values and exponent.** With the prevailing forms of the solutions exactly those of chapter 5 the characteristic equations of course remain of the form (5.11). It is of interest, however, to obtain from these equations more explicit formulas which are valid near the lower end of the admitted range of values for Δ. For such values h^2, which may be written

$$h^2 = \frac{\Omega - |\Delta|}{2\Omega},$$

is small of the order of $\Omega^{-1/2}$, and in the equation (5.10b) the evaluation given by (26a) is appropriate. The equation thus becomes

$$(2\Omega)^{1/2}h^2\left\{1 + \frac{h^2}{8} + h^4 O(1)\right\} = 2n - 1.$$

It is evident that the integers n concerned are those of a bounded set, the equation being expressible for such n in the form

$$\Omega + \Delta = (2n - 1)(2\Omega)^{1/2} + O(1).$$

Inasmuch as the characteristic equations represented by (5.10b) were found to be those for $S_n(\Omega)$ and $C_{n-1}(\Omega)$, it follows that for the algebraically smaller of the presently admitted values of Δ the characteristic values are described by formulas

$$\begin{aligned} S_n(\Omega) &= -\Omega + (2n - 1)(2\Omega)^{1/2} + O(1), \\ C_n(\Omega) &= -\Omega + (2n + 1)(2\Omega)^{1/2} + O(1). \end{aligned} \tag{6.3}$$

The characteristic exponent is again given by (25a) and (5.12).

Chapter 7

The configuration VII

7.1. **The transformed differential equation.** When Δ is large and negative and

$$-M_2|\Delta|^{1/2} \leqq \Omega + \Delta \leqq M_2\Omega^{1/2}, \tag{7.1}$$

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the configuration is that designated by VII, Figure 1. In this case the substitutions

$$(7.2) \qquad \rho = (32\Omega)^{1/2}e^{-\pi i/2}, \qquad \sigma = \frac{\Omega + \Delta}{(32\Omega)^{1/2}}, \qquad s = \frac{\pi}{2} - z$$

bring the differential equation (1) into the form (3) with

$$(7.3) \qquad \begin{aligned} \chi_0(s, \sigma) &\equiv \tfrac{1}{4} \sin s, \\ \chi_1 &\equiv i\sigma. \end{aligned}$$

The transformed equation thus differs from that obtained in chapter 4 only to the extent that σ is replaced by $i\sigma$. The formulas for ϕ and Φ given in (4.4) are adaptable to the present case by the substitution of $-\sigma/|\rho|$ in place of σ/ρ, a change which is easily seen to affect in no way the validity of the arguments of §4.1. That the differential equation in the present instance uniformly satisfies the hypotheses of §1.2 may, therefore, be accepted without further consideration.

The regions R_s and R_Φ which correspond to the strip (13) are, both as to outline and relative orientation, precisely like the z and ξ regions shown in Figure 5. Since under the relations (7.2) the region R_z is a reflection of R_s in the point $s=\pi/4$, whereas R_ξ is obtainable from R_Φ by a rotation besides the change of scale, the figure which relates the ultimate regions R_z and R_ξ for the chapter at hand is as indicated in Figure 8. The division of these regions into the sub-regions $\Xi^{(l)}$ is also as shown.

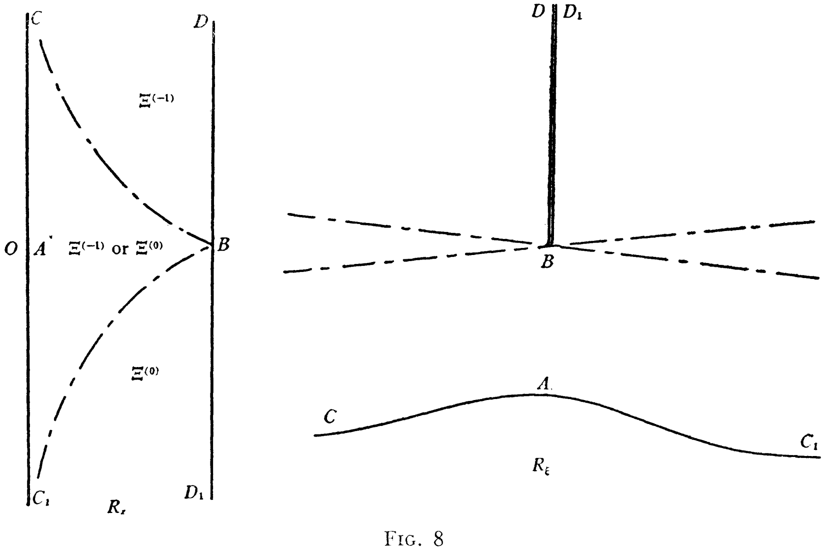

FIG. 8

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7.2. The solutions. The origin $z=0$ corresponds to $s_1=\pi/2$ and lies in the sub-region $\Xi^{(0)}$. The principal solutions relative to this point may accordingly be deduced by the substitution of the values (4.9) (with σ replaced by $i\sigma$) into the formulas (8b), the subscript a being taken as 1. To this extent the process coincides with that by which the forms (4.10) were deduced. In the present instance, however, certain terms may be dropped from the resulting formulas for, as may be seen from Figure 8, the quantity $i\xi_1$ is real and positive and $e^{-i\xi_1}$ therefore asymptotically negligible in comparison with $e^{i\xi_1}$. It is found thus that the following formulas hold:

When z is (anywhere) in R_z, and $|\xi| \geqq N$,

$$
\begin{aligned}
u_o(z) &= \frac{-1}{2i\rho\phi_1^{1/2}\phi^{1/2}}\left\{\left(\frac{\xi}{\xi_1}\right)^{-\sigma} e^{i(\xi-\xi_1)}[1]_1 - \left(\frac{\xi}{\xi_1}\right)^{\sigma} e^{-i(\xi-\xi_1)}[1]_1\right\},\\
u_e(z) &= \frac{\phi_1^{1/2}}{2\phi^{1/2}}\left\{\left(\frac{\xi}{\xi_1}\right)^{-\sigma} e^{i(\xi-\xi_1)}[1]_1 + \left(\frac{\xi}{\xi_1}\right)^{\sigma} e^{-i(\xi-\xi_1)}[1]_1\right\};
\end{aligned}
\tag{7.4a}
$$

when $|\xi| \leqq N$,

$$
\begin{aligned}
u_o(z) &= \left(\frac{\pi}{\phi_1\phi}\right)^{1/2} \frac{(2i\xi_1)^{-\sigma}e^{i\xi_1}(i\xi)^{1/4}}{2i\rho}\left\{\frac{2e^{\pi i/4}}{\Gamma(\frac{1}{4}-\sigma)}[\mathfrak{M}_1(\xi, i\sigma)]_1 \right.\\
&\qquad \left. - \frac{1}{\Gamma(\frac{3}{4}-\sigma)}[\mathfrak{M}_2(\xi, i\sigma)]\right\},\\
u_e(z) &= -\left(\frac{\pi\phi_1}{\phi}\right)^{1/2} \frac{(2i\xi_1)^{-\sigma}e^{i\xi_1}(i\xi)^{1/4}}{2}\left\{\frac{2e^{\pi i/4}}{\Gamma(\frac{1}{4}-\sigma)}[\mathfrak{M}_1(\xi, i\sigma)]_1 \right.\\
&\qquad \left. - \frac{1}{\Gamma(\frac{3}{4}-\sigma)}[\mathfrak{M}_2(\xi, i\sigma)]_1\right\}.
\end{aligned}
\tag{7.4b}
$$

In these as in subsequent formulas any term is to be omitted if σ is such that the gamma function involved is infinite.

The point $z=\pi/2$ corresponds to $s_2=0$ and the principal solutions relative to this point are therefore to be obtained precisely as were the solutions of §4.2. The formulas found are as follows:

When $|\xi| \leqq N$,

$$
\begin{aligned}
u_\alpha(z) &= -\left(\frac{2}{i\rho}\right)^{1/2} \Psi[(i\xi)^{-1/4}M_{\sigma,1/4}(2i\xi)],\\
u_\beta(z) &= \left(\frac{1}{2}\right)^{1/2} \Psi[(i\xi)^{-1/4}M_{\sigma,-1/4}(2i\xi)];
\end{aligned}
\tag{7.5a}
$$

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when $|\xi| \geqq N$, and z is in $\Xi^{(l)}$,

$$
\begin{aligned}
u_\alpha(z) &= -\left(\frac{\pi}{2\phi}\right)^{1/2}(i\rho)^{-1/4}\left\{\left[\frac{1}{\Gamma(\frac{3}{4}-\sigma)}\right]_1 (2i\xi)^{-\sigma}e^{i\xi}\right. \\
&\qquad\qquad \left. + \left[\frac{h_\alpha^{(l)}}{\Gamma(\frac{3}{4}+\sigma)}\right]_1 (2i\xi)^{\sigma}e^{-i\xi}\right\},
\end{aligned}
\tag{7.5b}
$$

$$
\begin{aligned}
u_\beta(z) &= \left(\frac{\pi}{2\phi}\right)^{1/2}(i\rho)^{-1/4}\left\{\left[\frac{1}{\Gamma(\frac{1}{4}-\sigma)}\right]_1 (2i\xi)^{-\sigma}e^{i\xi}\right. \\
&\qquad\qquad \left. + \left[\frac{h_\beta^{(l)}}{\Gamma(\frac{1}{4}+\sigma)}\right]_1 (2i\xi)^{\sigma}e^{-i\xi}\right\},
\end{aligned}
$$

with coefficients given by the table

l	-1	0
$h_\alpha^{(l)}$	$e^{(\sigma-3/4)\pi i}$	$e^{-(\sigma-3/4)\pi i}$
$h_\beta^{(l)}$	$e^{(\sigma-1/4)\pi i}$	$e^{-(\sigma-1/4)\pi i}$

In terms of the original variables

$$
\phi = \frac{\cos z}{4} - \frac{\Omega+\Delta}{32\Omega}\tan\left(\frac{\pi}{4}-\frac{z}{2}\right),
$$

$$
i\xi = (2\Omega)^{1/2}(1-\sin z) + \frac{\Omega+\Delta}{(32\Omega)^{1/2}}\log\frac{1+\sin z}{2},
$$

which permits the abbreviated relations

$$
\phi = \left[\frac{\cos z}{4}\right], \qquad \phi_1 = \left[\frac{1}{4}\right],
$$

$$
i\xi = \frac{|\rho|}{4}[1-\sin z], \qquad i\xi_1 = \frac{|\rho|}{4}[1],
$$

$$
e^{i\xi} = \left(\frac{1+\sin z}{2}\right)^{\sigma} e^{(|\rho|/4)(1-\sin z)} \qquad e^{i\xi_1} = \left(\frac{1}{2}\right)^{\sigma} e^{|\rho|/4}.
$$

The specialization of the various formulas to the case in which the variable is real may be made as usual, it being noted that then $i\xi = |\xi|$. The representations which result are as follows:

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When $|\xi| \geqq N$

$$
(7.6\mathrm{a})\qquad
\begin{aligned}
u_o(x) &= \left(\frac{\sec x}{2\Omega}\right)^{1/2}[1]_1 \sinh\left[(2\Omega)^{1/2}\sin x + \sigma \log\frac{1+\sin x}{1-\sin x}\right]_1,\\
u_e(x) &= (\sec x)^{1/2}[1]_1 \cosh\left[(2\Omega)^{1/2}\sin x + \sigma \log\frac{1+\sin x}{1-\sin x}\right]_1;
\end{aligned}
$$

when $|\xi| \leqq N$,

$$
(7.6\mathrm{b})\qquad
\begin{aligned}
u_o(x) &= \frac{-(2\pi)^{1/2}e^{(2\Omega)^{1/2}}}{(32\Omega)^{\sigma/2+3/8}(1+\sin x)^{1/4}}\left\{\frac{2}{\Gamma(\frac{1}{4}-\sigma)}[\,|\xi|^{-1/4}M_{\alpha,1/4}(2|\xi|)\,]_1\right.\\
&\qquad\left. - \frac{1}{\Gamma(\frac{3}{4}-\sigma)}[\,|\xi|^{-1/4}M_{\alpha,-1/4}(2|\xi|)\,]_1\right\},\\
u_e(x) &= \frac{-(2\pi)^{1/2}e^{(2\Omega)^{1/2}}}{(32\Omega)^{\sigma/2-1/8}(1+\sin x)^{1/4}}\left\{\frac{2}{\Gamma(\frac{1}{4}-\sigma)}[\,|\xi|^{-1/4}M_{\alpha,1/4}(2|\xi|)\,]_1\right.\\
&\qquad\left. - \frac{1}{\Gamma(\frac{3}{4}-\sigma)}[\,|\xi|^{-1/4}M_{\alpha,-1/4}(2|\xi|)\,]_1\right\},
\end{aligned}
$$

the symbols M representing the confluent hypergeometric functions which occur in the formulas (4.5).

When $|\xi| \geqq N$,

$$
(7.7\mathrm{a})\qquad
\begin{aligned}
u_\alpha(x) &= \frac{-(2\pi \sec x)^{1/2}}{(32\Omega)^{\sigma/2+3/8}}\left(\frac{1+\sin x}{1-\sin x}\right)^{\sigma}\left[\frac{1}{\Gamma(\frac{3}{4}-\sigma)}\right]_1 e^{(2\Omega)^{1/2}(1-\sin x)},\\
u_\beta(x) &= \frac{(2\pi \sec x)^{1/2}}{(32\Omega)^{\sigma/2+1/8}}\left(\frac{1+\sin x}{1-\sin x}\right)^{\sigma}\left[\frac{1}{\Gamma(\frac{1}{4}-\sigma)}\right]_1 e^{(2\Omega)^{1/2}(1-\sin x)}.
\end{aligned}
$$

When $|\xi| \leqq N$,

$$
(7.7\mathrm{b})\qquad
\begin{aligned}
u_\alpha(x) &= \frac{-1}{(8\Omega)^{1/2}(1+\sin x)^{1/4}}[\,|\xi|^{-1/4}M_{\alpha,1/4}(2|\xi|)\,],\\
u_\beta(x) &= \frac{1}{(1+\sin x)^{1/4}}[\,|\xi|^{-1/4}M_{\alpha,-1/4}(2|\xi|)\,].
\end{aligned}
$$

The solutions of the associated Mathieu equation, as obtained from the forms (7.4a) by the method of §1.5, are for real values of the variable represented thus:

$$
(7.8)\qquad
\begin{aligned}
v_o(x) &= \frac{[1]_1}{(2\Omega\cosh x)^{1/2}} \sin\,[(2\Omega)^{1/2}\sinh x - 2\sigma \tan^{-1}(\sinh x)]_1,\\
v_e(x) &= \frac{[1]_1}{(\cosh x)^{1/2}} \cos\,[(2\Omega)^{1/2}\sinh x - 2\sigma \tan^{-1}(\sinh x)]_1.
\end{aligned}
$$

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7.3. **The characteristic values and exponent.** The characteristic equations (17) and (18) may obviously if desired be rewritten in the forms $u_\alpha(0)=0$, $u_\beta(0)=0$ and $u_\beta'(0)=0$, $u_\alpha'(0)=0$. It accordingly follows from the formulas (7.7a) that any characteristic value must be a root of the one or the other of the equations

$$(7.9)\qquad \left[\frac{1}{\Gamma(\frac{3}{4}-\sigma)}\right]_1 = 0, \qquad \left[\frac{1}{\Gamma(\frac{1}{4}-\sigma)}\right]_1 = 0.$$

If σ is not positive the relations (7.9) are manifestly impossible. Hence no characteristic values exist when $\Delta \leqq -\Omega$, a fact which may be simply concluded from a direct perusal of the differential equation. When σ is positive and of suitable magnitude, on the other hand, a relation (7.9) may be satisfied in virtue of the gamma function becoming infinite. The appropriate values are clearly those for which

$$\left[\sigma + \frac{1}{4}\right]_1 = \frac{n}{2},$$

whence the characteristic equations are found to be of the form

$$\Delta = -\Omega + (2n-1)(2\Omega)^{1/2} - O(\log \Omega).$$

This result when compared with the formulas (6.3), with which it must be in accord for suitable values of Δ and Ω, shows that the characteristic values in the present configuration are given by formulas

$$(7.10)\qquad \begin{aligned} S_n(\Omega) &= -\Omega + (2n-1)(2\Omega)^{1/2} + O(\log \Omega), \\ C_n(\Omega) &= -\Omega + (2n+1)(2\Omega)^{1/2} + O(\log \Omega). \end{aligned}$$

Finally the computation of the characteristic exponent depends only upon the evaluation of the quantity Θ given in (25b). This evaluation from the forms (7.6b) and (7.7a) is found in the present case to be

$$(7.11)\qquad \Theta = \frac{\pi e^{(8\Omega)^{1/2}}}{(32\Omega)^\sigma}\left[\frac{1}{\Gamma(\frac{1}{4}-\sigma)\Gamma(\frac{3}{4}-\sigma)}\right]_1 - 1.$$

Chapter 8

The configuration VIII

8.1. **The change of variables.** The configuration numbered VIII in Figure 1 is to be defined as that in which Δ is negative and numerically large, while

$$(8.1)\qquad -|\Delta| + M_2|\Delta|^{1/2} \leqq -\Omega \leqq \frac{-1}{M_1}|\Delta|.$$

The substitutions for the transformation of the equation (1) are to be

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$$\rho = \frac{|\Delta| - \Omega}{|\Delta|^{1/2}} e^{\pi i/2}, \qquad \sigma^2 = 1 - \frac{\Omega}{|\Delta|}, \qquad s = \frac{i}{\sigma}\left(\frac{\pi}{2} - z\right), \tag{8.2}$$

in which case the resulting equation of the form (3) has precisely the coefficients (3.3). The value of σ is again confined to the range (3.4), and if the half-strip

$$R_z: \quad \begin{array}{l} 0 \leqq x \leqq \pi, \\ 0 \leqq y, \end{array} \tag{8.3}$$

is chosen as the domain of z, the corresponding region R_s is precisely that of (3.6). In terms of s, therefore, the present equation coincides entirely with that of chapter 3. The hypotheses are in consequence uniformly fulfilled and Figure 2 again applies. The latter evidently leads in the present instance to Figure 9.

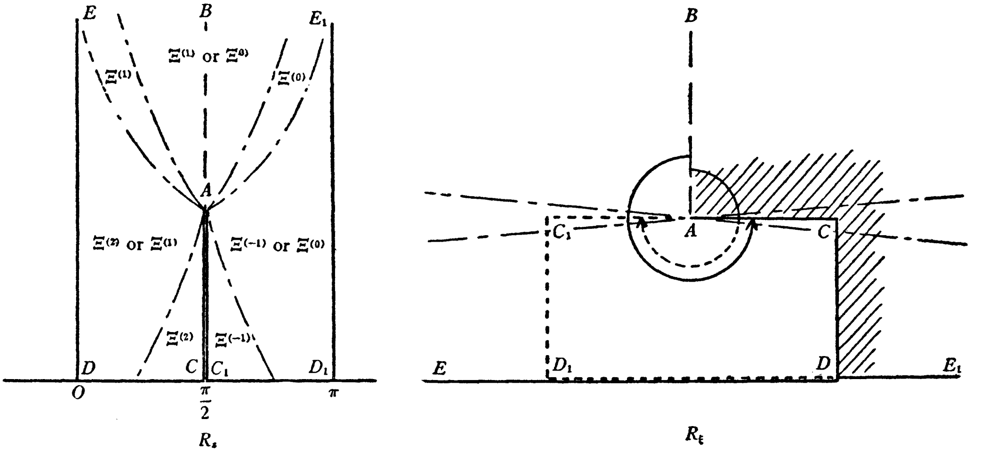

FIG. 9

The extension of the representations which are to be obtained from R_z into the entire strip (13) may be made directly by observing that $u_\alpha(z)$ and $u_\beta(z)$ are respectively odd and even as functions of the variable $(z-\pi/2)$, and by applying the identities (14a) and (14b) to the formulas for $u_o(z)$ and $u_e(z)$. With this accomplished the further considerations of §1.6 are, of course, applicable.

8.2. **The solutions.** The zero of χ_0^2 is of the first order, and, as may be seen from Figure 9, both the points $z=0$ and $z=\pi/2$ lie in the sub-region $\Xi^{(2)}$. The formulas (11a) and (11b) may, therefore, be drawn upon, with $h=2$, and lead to the formulas which follow.

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When z is in $\Xi^{(l)}$, and $|\xi| \geqq N$,

$$(8.4a)\qquad \begin{aligned} u_o(z) &= \frac{1}{2}\left(\frac{\sigma^2}{\rho^2\phi_1\phi}\right)^{1/2}\left\{K^{2,l}_{0,1}e^{i\xi} + K^{2,l}_{0,2}e^{-i\xi}\right\}, \\ u_e(z) &= \frac{1}{2}\left(\frac{\phi_1}{\phi}\right)^{1/2}\left\{K^{2,l}_{e,1}e^{i\xi} + K^{2,l}_{e,2}e^{-i\xi}\right\}, \end{aligned}$$

$$(8.5a)\qquad \begin{aligned} u_\alpha(z) &= \frac{1}{2}\left(\frac{\sigma^2}{\rho^2\phi_2\phi}\right)^{1/2}\left\{K^{2,l}_{\alpha,1}e^{i\xi} + K^{2,l}_{\alpha,2}e^{-i\xi}\right\}, \\ u_\beta(z) &= \frac{1}{2}\left(\frac{\phi_2}{\phi}\right)^{1/2}\left\{K^{2,l}_{\beta,1}e^{i\xi} + K^{2,l}_{\beta,2}e^{-i\xi}\right\}, \end{aligned}$$

with coefficients

l	-1	0	1	2
$K^{2,l}_{0,1}$	$[i]e^{i\xi_1}$	$[i]e^{i\xi_1}$	$[1]e^{-i\xi_1}$	$[1]e^{-i\xi_1}$
$K^{2,l}_{0,2}$	$[-i]e^{-i\xi_1}$	$[-1]e^{i\xi_1}$	$[-1]e^{i\xi_1}$	$[-1]e^{i\xi_1}$
$K^{2,l}_{e,1}$	$[-i]e^{i\xi_1}$	$[-i]e^{i\xi_1}$	$[1]e^{-i\xi_1}$	$[1]e^{-i\xi_1}$
$K^{2,l}_{e,2}$	$[-i]e^{i\xi_1}$	$[1]e^{i\xi_1}$	$[1]e^{i\xi_1}$	$[1]e^{i\xi_1}$

,

l	-1	0	1	2
$K^{2,l}_{\alpha,1}$	$[i]e^{i\xi_2}$	$[i]e^{i\xi_2}$	$[1]e^{-i\xi_2}$	$[1]e^{-i\xi_2}$
$K^{2,l}_{\alpha,2}$	$[-i]e^{-i\xi_2}$	$\left[-2e^{\pi i/4}\cos\left(\xi_2 - \frac{\pi}{4}\right)\right]$	$\left[-2e^{\pi i/4}\cos\left(\xi_2 - \frac{\pi}{4}\right)\right]$	$[-1]e^{i\xi_2}$
$K^{2,l}_{\beta,1}$	$[-i]e^{i\xi_2}$	$[-i]e^{i\xi_2}$	$[1]e^{-i\xi_2}$	$[1]e^{-i\xi_2}$
$K^{2,l}_{\beta,2}$	$[-i]e^{i\xi_2}$	$\left[2e^{-\pi i/4}\cos\left(\xi_2 + \frac{\pi}{4}\right)\right]$	$\left[2e^{-\pi i/4}\cos\left(\xi_2 + \frac{\pi}{4}\right)\right]$	$[1]e^{i\xi_2}$

.

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When $|\xi| \leqq N$,

$$(8.4b)\quad \begin{aligned} u_o(z) &= \left(\frac{\pi\sigma^2}{6\rho^2\phi_1\phi}\right)^{1/2} \xi^{1/6} e^{i\xi_1+3\pi i/4}\left[\xi^{1/3}J_{-1/3}(\xi) + \xi^{1/3}J_{-1/3}(\xi)\right], \\ u_e(z) &= \left(\frac{\pi\phi_1}{6\phi}\right)^{1/2} \xi^{1/6} e^{i\xi_1-\pi i/4}\left[\xi^{1/3}J_{-1/3}(\xi) + \xi^{1/3}J_{1/3}(\xi)\right], \end{aligned}$$

$$(8.5b)\quad \begin{aligned} u_\alpha(z) = -\left(\frac{2\pi\sigma^2}{3\rho^2\phi_2\phi}\right)^{1/2} \xi^{1/6}\Big\{ & e^{\pi i/6}\sin\left(\xi_2 + \frac{\pi}{12}\right)\left[\xi^{1/3}J_{-1/3}(\xi)\right] \\ & + e^{-\pi i/6}\cos\left(\xi_2 - \frac{\pi}{12}\right)\left[\xi^{1/3}J_{1/3}(\xi)\right]\Big\}, \\ u_\beta(z) = \left(\frac{2\pi\phi_2}{3\phi}\right)^{1/2} \xi^{1/6}\Big\{ & e^{-\pi i/3}\cos\left(\xi_2 + \frac{\pi}{12}\right)\left[\xi^{1/3}J_{-1/3}(\xi)\right] \\ & + e^{\pi i/3}\sin\left(\xi_2 - \frac{\pi}{12}\right)\left[\xi^{1/3}J_{1/3}(\xi)\right]\Big\}. \end{aligned}$$

For use in these formulas,

$$\frac{\rho\phi}{\sigma} = -\left\{|\Delta| + \Omega\cos 2z\right\}^{1/2},$$

$$\frac{\rho\phi_1}{\sigma} = -\left\{|\Delta| + \Omega\right\}^{1/2}, \qquad \frac{\rho\phi_2}{\sigma} = -\left\{|\Delta| - \Omega\right\}^{1/2},$$

$$\xi = \int_{z_0}^{z}\left\{\Delta - \Omega\cos 2z\right\}^{1/2}dz, \qquad z_0 = \tfrac{1}{2}\cos^{-1}\Delta/\Omega,$$

$$\xi_2 = \int_0^{y_0}\left\{|\Delta| - \Omega\cosh 2y\right\}^{1/2}dy, \qquad y_0 = \tfrac{1}{2}\cosh^{-1}|\Delta|/\Omega,$$

$$\xi_1 = \xi_2 - i\int_0^{\pi/2}\left\{|\Delta| + \Omega\cos 2x\right\}^{1/2}dx.$$

It is found that for all real values of z on the interval $(0, \pi)$ the respective formulas are

$$(8.4c)\quad \begin{aligned} u_o(x) &= \frac{[1]}{\left\{(|\Delta|+\Omega)(|\Delta|+\Omega\cos 2x)\right\}^{1/4}}\sinh\left[\int_0^x\left\{|\Delta|+\Omega\cos 2x\right\}^{1/2}dx\right], \\ u_e(x) &= \left\{\frac{|\Delta|+\Omega}{|\Delta|+\Omega\cos 2x}\right\}^{1/4}[1]\cosh\left[\int_0^x\left\{|\Delta|+\Omega\cos 2x\right\}^{1/2}dx\right], \end{aligned}$$

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$$
(8.5c)\qquad
\begin{aligned}
u_\alpha(x) &= \frac{[1]}{\{(|\Delta|-\Omega)(|\Delta|+\Omega\cos 2x)\}^{1/4}}\sinh\left[\int_{\pi/2}^{x}\{|\Delta|+\Omega\cos 2x\}^{1/2}dx\right],\\
u_\beta(x) &= \left\{\frac{|\Delta|-\Omega}{|\Delta|+\Omega\cos 2x}\right\}^{1/4}[1]\cosh\left[\int_{\pi/2}^{x}\{|\Delta|+\Omega\cos 2x\}^{1/2}dx\right].
\end{aligned}
$$

The axis of imaginaries in Figure 9 likewise lies in the sub-region $\Xi^{(2)}$, and the formulas for the solutions of the associated Mathieu equation are accordingly found to be

$$
(8.6)\qquad
\begin{aligned}
v_o(x) &= \frac{[1]}{\{(|\Delta|+\Omega)(|\Delta|+\Omega\cosh 2x)\}^{1/4}}\sin\left[\int_{0}^{x}\{|\Delta|-\Omega\cosh 2x\}^{1/2}dx\right],\\
v_e(x) &= \left\{\frac{|\Delta|+\Omega}{|\Delta|+\Omega\cosh 2x}\right\}^{1/4}[1]\cos\left[\int_{0}^{x}\{|\Delta|+\Omega\cosh 2x\}^{1/2}dx\right].
\end{aligned}
$$

8.3. **The characteristic exponent.** It is evident that the present configuration admits no characteristic values. The formulas (8.4c) and (8.5c) yield the evaluation

$$
\Theta = [2]\cosh^2\left[\int_0^{\pi/2}\{|\Delta| + \Omega\cos 2x\}^{1/2}dx\right] - 1,
$$

and the formula (25a) accordingly gives the characteristic exponent in the form

$$
(8.7)\qquad \mu = \left[\frac{2}{\pi}\int_0^{\pi/2}\{|\Delta| + \Omega\cos 2x\}^{1/2}dx\right].
$$

Clearly, the configuration is one of unstable solutions.

Chapter 9

The configuration IX

9.1. **The differential equation.** In this final configuration to be considered, i.e., IX of Figure 1, the parameter Δ is large and negative while

$$
(9.1)\qquad -\frac{1}{M_1}|\Delta| \leqq -\Omega \leqq 0.
$$

The variable is to be restricted to any region in which a relation (2.4a) is fulfilled with some constant M_1, and this constant is that which figures in (9.1). The substitutions

$$
(9.2)\qquad \rho = |\Delta|^{1/2}e^{\pi i/2},\qquad \sigma^2 = \frac{\Omega}{|\Delta|},\qquad s = \frac{\pi}{2} - z
$$

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reduce the differential equation to the form (3) with the coefficients (2.3), the parameter σ being confined as in (2.5). As was remarked in chapter 2, the Stokes' phenomenon is absent and a single formula serves to describe a solution over the entire strip given by (2.4a) and (2.4b).

9.2. The solutions. The solutions (2.7) apply to the present differential equation (3) and may be used in the formulas (8b). It is found thus that

$$(9.3)\qquad \begin{aligned} u_o(z) &= \frac{i}{2}\left(\frac{1}{\rho^2\phi_1\phi}\right)^{1/2}\left\{e^{i(\xi-\xi_1)}[1] - e^{-i(\xi-\xi_1)}[1]\right\}, \\ u_e(z) &= \frac{1}{2}\left(\frac{\phi_1}{\phi}\right)^{1/2}\left\{e^{i(\xi-\xi_1)}[1] + e^{-i(\xi-\xi_1)}[1]\right\}, \end{aligned}$$

$$(9.4)\qquad \begin{aligned} u_\alpha(z) &= \frac{i}{2}\left(\frac{1}{\rho^2\phi_2\phi}\right)^{1/2}\left\{e^{i\xi}[1] - e^{-i\xi}[1]\right\}, \\ u_\beta(z) &= \frac{1}{2}\left(\frac{\phi_2}{\phi}\right)^{1/2}\left\{e^{i\xi}[1] + e^{-i\xi}[1]\right\}, \end{aligned}$$

with the symbols evaluated by the relations

$$\rho\phi = i\left\{\,|\Delta| + \Omega\cos 2z\right\}^{1/2},$$

$$\rho\phi_1 = i\left\{\,|\Delta| + \Omega\right\}^{1/2}, \qquad \rho\phi_2 = i\left\{\,|\Delta| - \Omega\right\}^{1/2},$$

$$i\xi = \int_{\pi/2}^{z}\left\{\,|\Delta| + \Omega\cos 2z\right\}^{1/2}dz,$$

$$i(\xi-\xi_1) = \int_0^{z}\left\{\,|\Delta| + \Omega\cos 2z\right\}^{1/2}dz.$$

For real values of z these formulas are found to reduce precisely to the forms (8.4c) and (8.5c), while the forms which describe the solutions of the equation (2) are again found to be those of (8.6). As in the case of chapter 2 the conclusion is possible that the symbols [ ] may be dropped from the formulas when $\Omega=0$.

Lastly, the formula for the characteristic exponent is that already given in (8.7), and there are, of course, no characteristic values.

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produce the succeeding state $X_i = [f_{i-1}A_{i-1}/(1 + C)] + CX^S$. Thus

$$V_i = f_{i-1}A_{i-1}^2 + A_{i-1}[B_i^0 - CB_{i-1}^0]$$
$$= f_{i-1}A_{i-1}^2 + V_i^0.$$

If a special test signal is employed having the property $A_i = \pm A$, then $V_i = f_{i-1}R_0 + V_i^0$, where $R_0 = A^2$.

If A_i and B_i^0 are statistically independent the spectral density of V_i^0 is [10]

$$S^{v^0v^0}(\omega) = \int_{-\pi}^{\pi} S^{u^0u^0}(\omega - \nu)S^{AA}(\nu)d\nu,$$

where ω is the normalized radian frequency and

$$S^{u^0u^0}(\omega) = |1 - C \exp - j\omega|^2 S^{B^0B^0}(\omega).$$

The Wiener filter is thus required to extract a signal f_i of given spectral density from an unknown noise V_i^0 whose spectral density is directly related to that of the process external perturbations, B_i^0.

IV. Conclusions

A generalization of the concept of correlation has been introduced that facilitates the evaluation of process parameters from observation of their input and output signals. In general, these parameters are estimated by means of a nonlinear filter of relatively complex structure. However, in a specific example considered, the parameter estimator was reduced to an easily derivable Wiener filter.

The assumption of the nonsingularity of E can always be satisfied in the case of observable processes where the output vector can be augmented if necessary by estimated process state variables.

Tests have shown that the method gives good matching of heuristic models in the form of (1) to linear processes of higher order, in terms of their responses to specific inputs. It is conjectured that this may extend to the modeling of distributed parameter processes satisfying a more general linear functional relationship of the form of (2). Extensions to the modeling of specific nonlinearities are being considered.

References

[1] Hughes and Noton, "The measurement of control system characteristics by means of a cross correlator," *Proc. Inst. Elec. Eng.*, Jan. 1962.
[2] Van der Grinten, "Application of random test signal in process optimization," *2nd Int. Fed. Automatic Control Congr.*, 1963.
[3] Izawa and Furuta, "Apparatus to measure dynamic characteristics," *3rd Int. Fed. Automatic Control Congr.*, London, 1966.
[4] Isobe *et al.*, "Impulse response determination with on-line cross-correlation," *3rd Int. Fed. Automatic Control Congr.*, London, 1966.
[5] Hill and McMurtry, "An application of digital computers to linear system identification," *IEEE Trans. Automat. Contr.* (Short Papers), vol. AC-9, Oct. 1964, pp. 536–538.
[6] Briggs and Godfrey, "Pseudo-random signals for dynamic analysis of multivariable systems," *Proc. Inst. Elec. Eng.*, vol. 113, July 1966, pp. 1259–1267.
[7] Davies, "Using the binary maximum length sequence for the identification of system dynamics," *Proc. Inst. Elec. Eng.*, vol. 114, Oct. 1967, pp. 1582–1584.
[8] Otterman, "The properties and methods for computation of exponentially-mapped-past statistical variables," *IRE Trans. Automat. Contr.*, vol. AC-5, Jan. 1960, pp. 11–17.
[9] Wierwille and Knight, "Off-line correlation analysis of nonstationary signals," *IEEE Trans. Comput.*, vol. C-17, June 1968, pp. 525–536.
[10] Rubin, "Time optimal control by a novel adaptive system," D.Sc. dissertation, Univ. Pretoria, Pretoria, South Africa, May 1969.
[11] Lindenlaub and Cooper, "Noise limitations of system identification techniques," *IEEE Trans. Automat. Contr.*, vol. AC-8, Jan. 1963, pp. 43–48.
[12] Aoki, "Optimization of Stochastic Systems," New York: Academic Press, 1967.

Technical Notes and Correspondence

An Improved Stability Criterion for the Damped Mathieu Equation

***Abstract*—An improved stability criterion is derived for the damped Mathieu equation using periodic Lyapunov functions.**

Considering the damped Mathieu equation

$$\ddot{x} + 0.5\delta\dot{x} + (\omega^2 + \epsilon \cos 2t)x = 0 \tag{1}$$

(with $\delta > 0$) Michael [1] has shown that for large ω and small δ, stability is assured if $|\epsilon| < \delta$. In this correspondence, an improved stability criterion is obtained using periodic Lyapunov functions.

Putting $x = x_1$ and $\dot{x} = x_2$, (1) can be written as

$$\dot{x}_1 = x_2 \tag{2}$$

$$\dot{x}_2 = -0.5\delta x_2 - (\omega^2 + \epsilon \cos 2t)\, x_1.$$

Consider the Lyapunov function

$$V_1 = (\omega^2 + 0.5k\delta + k\epsilon \sin 2t)x_1^2 + 2kx_1x_2 + x_2^2 \tag{3}$$

where k is to be chosen. Differentiating (3) with respect to time and using (2),

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$$\dot{V}_1 = -2\omega^2 k x_1^2 - 2\epsilon(\cos 2t - k \sin 2t)x_1x_2 - (\delta - 2k)x_2^2.$$

For $\dot{V}_1$ to be negative definite, we require

$$0 < 2k < \delta \tag{4}$$

and

$$\epsilon^2 < \frac{2\omega^2 k(\delta - 2k)}{(1 + k^2)}. \tag{5}$$

Now choose the value of k as

$$k = \frac{\sqrt{\delta^2 + 4} - 2}{\delta}, \tag{6}$$

which maximizes the right-hand side of (5) and also satisfies (4). It can be easily proved that, for the above value of k, the function V_1 is positive definite. Substitution of the value of k given in (6) into (5) yields the stability criterion as

$$|\epsilon| < [\sqrt{\delta^2 + 4} - 2]^{1/2}\omega. \tag{7}$$

Now consider another Lyapunov function given by

$$V_2 = (\omega^2 + \epsilon \cos 2t)x_1^2 + (0.5\delta x_1 + x_2)^2. \tag{8}$$

Differentiating (8) and using (2),

$$\dot{V}_2 = -[\delta\omega^2 + \delta\epsilon \cos 2t + 2\epsilon \sin 2t]x_1^2.$$

Here we see that $\dot{V}_2$ can only be negative semidefinite since $\dot{V}_2 \equiv 0$

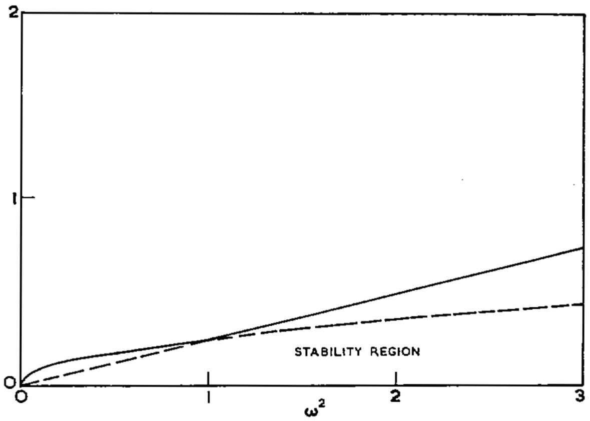

Fig. 1. Stability region for $\delta = 0.5$.

when $x_1 \equiv 0$. It is still possible to prove asymptotic stability in the large (ASIL) of the system (2) by using the theorem of Krasovskii and Barbasin [2] for periodic Lyapunov functions if we can prove that $\dot{V}_2 \equiv 0$ only when $x_1 \equiv x_2 \equiv 0$. For the system (2), it can be easily seen that when $x_1 \equiv 0$, x_2 is also identically zero. Hence it is sufficient if $\dot{V}_2$ is negative semidefinite which requires that

$$|\epsilon| < \frac{\delta\omega^2}{\sqrt{\delta^2 + 4}}. \tag{9}$$

Combining (7) and (9), the stability criterion can be written as

$$|\epsilon| < \max \left[\{\sqrt{\delta^2 + 4} - 2\}^{1/2}\omega, \frac{\delta\omega^2}{\sqrt{\delta^2 + 4}} \right]. \tag{10}$$

The stability region using (10) for $\delta = 0.5$ is shown in Fig. 1. It may be noted that the stability region obtained by Michael [1] is contained in the stability region obtained here.

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References

[1] G. J. Michael, "Explicit stability criteria for the damped Mathieu equation," *IEEE Trans. Automat. Contr.* (Corresp.), vol. AC-12, June 1967, pp. 337–338.
[2] T. A. Burton, "Some Liapunov theorems," *J. SIAM Contr.*, vol. 4, 1966, pp. 460–465.

A Simplified Criterion for the Controllability of Linear Systems with Delay in Control

Abstract—Chyung has recently given a necessary and sufficient condition for the controllability of linear time-invariant systems with a time delay in control [1]. This correspondence is intended to complement his criterion and to present a useful remark.[1]

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[1] The authors are grateful to a referee for pointing out that related results were recently reported by H. T. Banks, M. Q. Jacobs, and M. R. Latina, Center for Dynamical Systems, Brown University, Providence, R. I., Rep. 70-4.

Lemma: Let A, B, and C be constant matrices of dimensions $n \times n$, $n \times m$ and $n \times m$, respectively. Then

$$\text{Rank } [B,\cdots,A^{n-1}B,C,\cdots,A^{n-1}C] = n$$

if and only if

$$\text{Rank } [B,\cdots,A^{n-1}B, \exp(A\tau)C,\cdots,A^{n-1}\exp(A\tau)C] = n$$

for any finite τ.

Proof: $\Longrightarrow$ Assume the contrary, which is to say that

$$\text{Rank } [B,\cdots,A^{n-1}B, \exp(A\tau)C,\cdots,A^{n-1}\exp(A\tau)C] < n.$$

Then there exists a nonzero n-dimensional vector ν such that

$$\nu'B = \nu'AB = \cdots = \nu'A^{n-1}B = 0 \tag{1}$$

and

$$\nu' \exp(A\tau)C = \nu'A \exp(A\tau)C = \cdots = \nu'A^{n-1}\exp(A\tau)C = 0.$$

By the Cayley–Hamilton theorem

$$\begin{aligned}\nu' \exp(A\tau)C &= \cdots = \nu'A^{n-1}\exp(A\tau)C = \cdots \\ &= \nu'A^{p}\exp(A\tau)C = 0, \qquad p > n-1, \end{aligned} \tag{2}$$

but

$$\exp(-A\tau) = \sum_{i=0}^{k-1} \alpha_i(-\tau)A^i$$

where k is the degree of the minimal polynomial of the matrix A [2]. Multiplying the first equation of (2) by α_0, the second by α_1, and so on, and adding these k equations, one gets $\nu'C = 0$. Repeating the same steps for the next k equations of (2) one gets $\nu'AC = 0$. Similarly, one obtains

$$\nu'C = \nu'AC = \cdots = \nu'A^{n-1}C = 0. \tag{3}$$

But (1) and (3) contradict the hypothesis that

$$\text{Rank } [B,\cdots,A^{n-1}B,C,\cdots,A^{n-1}C] = n.$$

Hence the assumption that

$$\text{Rank } [B,\cdots,A^{n-1}B, \exp(A\tau)C,\cdots,A^{n-1}\exp(A\tau)C] < n$$

is false.

$\Longleftarrow$ Applying the result of the preceding part,

$$\text{Rank } [B,\cdots,A^{n-1}B, \exp(A\tau)C,\cdots,A^{n-1}\exp(A\tau)C] = n$$

implies that

$$\begin{aligned}\text{Rank } [B,\cdots,A^{n-1}B, &\exp(-A\tau)\exp(A\tau)C,\cdots, \\ &\cdot A^{n-1}\exp(-A\tau)\exp(A\tau)C] = n,\end{aligned}$$

i.e.,

$$\text{Rank } [B,\cdots,A^{n-1}B,C,\cdots,A^{n-1}C] = n.$$

This completes the proof of the lemma.

Consider the linear time-invariant system with a time delay in the control function

$$\dot{x}(t) = Ax(t) + Bu(t) + Cu(t-h)$$

where $x(t)$ is the n-dimensional vector state variable, $u(t)$ is the m-dimensional vector control function, $h > 0$ is the time delay, and A, B, and C are constant matrices of compatible dimensions.

The system is called controllable if for any given pair of points x_0, x_1 and initial control function $u_0(t)$ on $[-h,0]$ there exists a control function $u(t) \subset R^m$ on $[0,T]$ for some finite time T that is measurable and bounded on every finite time interval and that steers the response of the system from x_0 at $t = 0$ to x_1 at $t = T$.

Mathieu's Equation

Richard H. Rand
Cornell University

The differential equation

$$\frac{d^2x}{dt^2} + (\delta + \epsilon \cos t)\ x = 0 \tag{1}$$

is called Mathieu's equation. It is a linear differential equation with variable (periodic) coefficients. It commonly occurs in nonlinear vibration problems in two different ways: (i) in systems in which there is periodic forcing, and (ii) in stability studies of periodic motions in nonlinear *autonomous* systems.

As an example of (i), take the case of a pendulum whose support is periodically forced in a vertical direction. The governing differential equation is

$$\frac{d^2x}{dt^2} + \left(\frac{g}{L} - \frac{A\omega^2}{L}\cos\omega t\right)\sin x = 0 \tag{2}$$

where the vertical motion of the support is $A\cos\omega t$, and where g is the acceleration of gravity, L is the pendulum's length, and x is its angle of deflection. In order to investigate the stability of one of the equilibrium solutions $x = 0$ or $x = \pi$, we would linearize (2) about the desired equilibrium, giving, after suitable rescaling of time, an equation of the form of (1).

As an example of (ii), we consider a system known as "the particle in the plane". This consists of a particle of unit mass which is constrained to move in the x-y plane, and is restrained by two linear springs, each with spring constant of $\frac{1}{2}$. The anchor points of the two springs are located on the x axis at $x = 1$ and $x = -1$. Each of the two springs has unstretched length L. This autonomous two degree of freedom system exhibits an exact solution corresponding to a mode of vibration in which the particle moves along the x axis:

$$x = A\cos t, \qquad y = 0 \tag{3}$$

In order to determine the stability of this motion, one must first derive the equations of motion, then substitute $x = A\cos t + u,\ \ y = 0 + v$, where u and v are small deviations from the motion (3), and then linearize in u and v. The result is two linear differential equations on u and v. The u equation turns out to be the simple harmonic oscillator, and cannot produce instability. The v equation is:

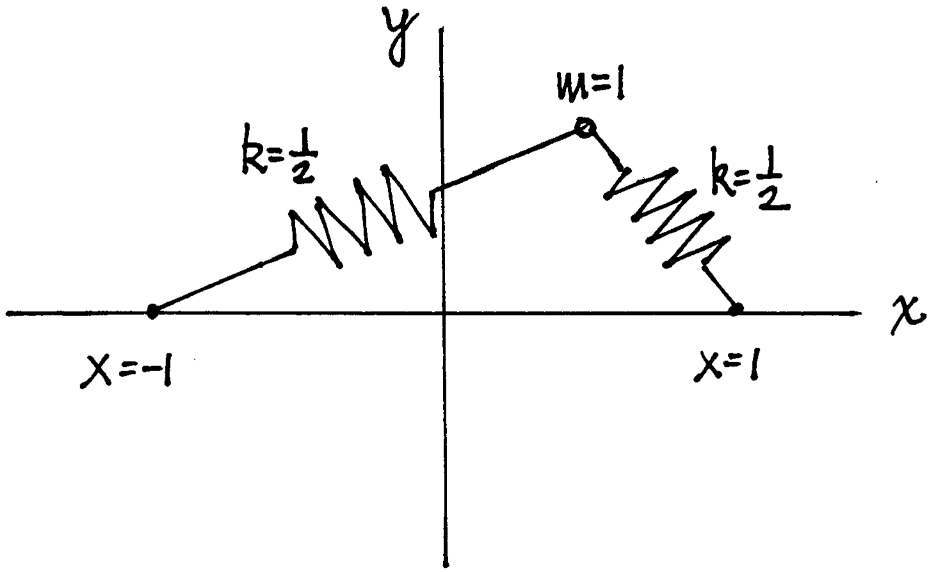
$$\frac{d^2v}{dt^2} + \left(\frac{1 - L - A^2\cos^2 t}{1 - A^2\cos^2 t}\right) v = 0 \tag{4}$$

Expanding (4) for small A and setting $\tau = 2t$, we obtain

$$\frac{d^2v}{d\tau^2} + \left(\frac{2 - 2L - A^2L}{8} - \frac{A^2L}{8}\cos\tau + O(A^4)\right) v = 0 \tag{5}$$

which is, to $O(A^4)$, in the form of Mathieu's eq.(1) with $\delta = \dfrac{2 - 2L - A^2L}{8}$ and $\epsilon = -\dfrac{A^2L}{8}$.

The particle in the plane



$$\frac{d^2x}{dt^2} + f_1(x,y)\ (x+1) + f_2(x,y)\ (x-1) = 0$$

$$\frac{d^2y}{dt^2} + f_1(x,y)\ y + f_2(x,y)\ y = 0$$

$$f_1(x,y) = \frac{1}{2}\left(1 - \frac{L}{\sqrt{(1+x)^2+y^2}}\right)$$

$$f_2(x,y) = \frac{1}{2}\left(1 - \frac{L}{\sqrt{(1-x)^2+y^2}}\right)$$

The chief concern with regard to Mathieu's equation is whether or not all solutions are bounded for given values of the parameters δ and ϵ. If all solutions are bounded then the corresponding point in the δ-ϵ parameter plane is said to be stable. A point is called unstable if an unbounded solution exists.

Perturbations

In this section we will use the two variable expansion method to look for a general solution to Mathieu's eq.(1) for small ϵ. Since (1) is linear, there is no need to stretch time, and we set $\xi = t$ and $\eta = \epsilon t$, giving

$$\frac{\partial^2 x}{\partial \xi^2} + 2\epsilon \frac{\partial^2 x}{\partial \xi \partial \eta} + \epsilon^2 \frac{\partial^2 x}{\partial \eta^2} + (\delta + \epsilon \cos \xi)\ x = 0 \tag{6}$$

Next we expand x in a power series:

$$x(\xi, \eta) = x_0(\xi, \eta) + \epsilon x_1(\xi, \eta) + \cdots \tag{7}$$

Substituting (7) into (1) and neglecting terms of $O(\epsilon^2)$, gives, after collecting terms:

$$\frac{\partial^2 x_0}{\partial \xi^2} + \delta\ x_0 = 0 \tag{8}$$

$$\frac{\partial^2 x_1}{\partial \xi^2} + \delta\ x_1 = -2\frac{\partial^2 x_0}{\partial \xi \partial \eta} - x_0 \cos \xi \tag{9}$$

We take the general solution to eq.(8) in the form:

$$x_0(\xi, \eta) = A(\eta) \cos \sqrt{\delta}\ \xi + B(\eta) \sin \sqrt{\delta}\ \xi \tag{10}$$

Substituting (10) into (9), we obtain

$$\begin{aligned} \frac{\partial^2 x_1}{\partial \xi^2} + \delta\ x_1 \quad = \quad & 2\sqrt{\delta}\frac{dA}{d\eta} \sin \sqrt{\delta}\ \xi - 2\sqrt{\delta}\frac{dB}{d\eta} \cos \sqrt{\delta}\ \xi \\ & -A \cos \sqrt{\delta}\ \xi \cos \xi - B \sin \sqrt{\delta}\ \xi \cos \xi \end{aligned} \tag{11}$$

Using some trig identities, this becomes

$$\begin{aligned} \frac{\partial^2 x_1}{\partial \xi^2} + \delta\ x_1 \quad = \quad & 2\sqrt{\delta}\frac{dA}{d\eta} \sin \sqrt{\delta}\ \xi - 2\sqrt{\delta}\frac{dB}{d\eta} \cos \sqrt{\delta}\ \xi \\ & -\frac{A}{2}\left(\cos(\sqrt{\delta}\ + 1)\xi + \cos(\sqrt{\delta}\ - 1)\xi\right) \\ & -\frac{B}{2}\left(\sin(\sqrt{\delta}\ + 1)\xi + \sin(\sqrt{\delta}\ - 1)\xi\right) \end{aligned} \tag{12}$$

For a general value of δ, removal of resonance terms gives the trivial slow flow:

$$\frac{dA}{d\eta} = 0, \qquad \frac{dB}{d\eta} = 0 \tag{13}$$

This means that for general δ, the $\cos t$ driving term in Mathieu's eq.(1) has no effect. However, if we choose $\delta = \frac{1}{4}$, eq.(12) becomes

$$\begin{aligned} \frac{\partial^2 x_1}{\partial \xi^2} + \frac{1}{4}\, x_1 &= \frac{dA}{d\eta}\sin\frac{\xi}{2} - \frac{dB}{d\eta}\cos\frac{\xi}{2} \\ &\quad -\frac{A}{2}\left(\cos\frac{3\xi}{2} + \cos\frac{\xi}{2}\right) \\ &\quad -\frac{B}{2}\left(\sin\frac{3\xi}{2} - \sin\frac{\xi}{2}\right) \end{aligned} \tag{14}$$

Now removal of resonance terms gives the slow flow:

$$\frac{dA}{d\eta} = -\frac{B}{2}, \qquad \frac{dB}{d\eta} = -\frac{A}{2} \qquad \Rightarrow \qquad \frac{d^2A}{d\eta^2} = \frac{A}{4} \tag{15}$$

Thus $A(\eta)$ and $B(\eta)$ involve exponential growth, and the parameter value $\delta = \frac{1}{4}$ causes instability. This corresponds to a 2:1 subharmonic resonance in which the driving frequency is twice the natural frequency.

This discussion may be generalized by "detuning" the resonance, that is, by expanding δ in a power series in ϵ:

$$\delta = \frac{1}{4} + \delta_1\epsilon + \delta_2\epsilon^2 + \cdots \tag{16}$$

Now eq.(9) gets an additional term:

$$\frac{\partial^2 x_1}{\partial \xi^2} + \frac{1}{4}\, x_1 = -2\frac{\partial^2 x_0}{\partial \xi \partial \eta} - x_0\cos\xi - \delta_1 x_0 \tag{17}$$

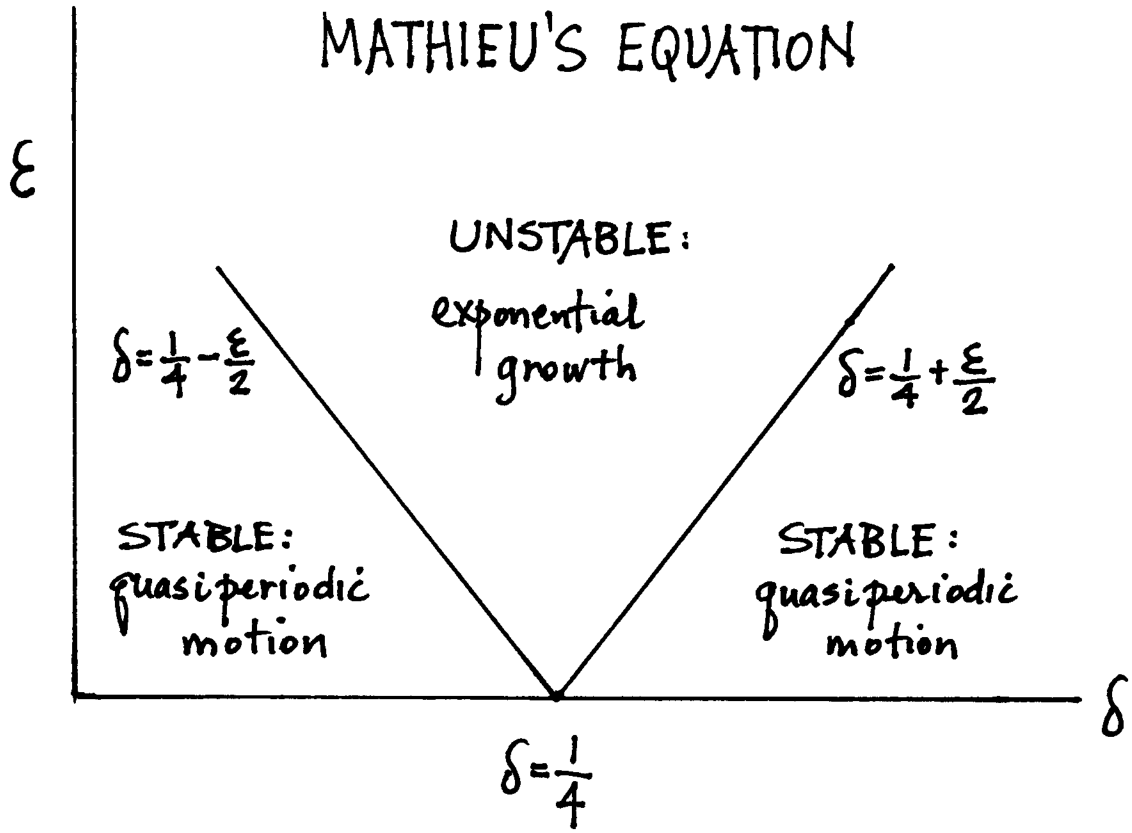
which results in the following additional terms in the slow flow eqs.(15):

$$\frac{dA}{d\eta} = \left(\delta_1 - \frac{1}{2}\right)B, \qquad \frac{dB}{d\eta} = -\left(\delta_1 + \frac{1}{2}\right)A \qquad \Rightarrow \qquad \frac{d^2A}{d\eta^2} + \left(\delta_1^2 - \frac{1}{4}\right)\, A = 0 \tag{18}$$

Here we see that $A(\eta)$ and $B(\eta)$ will be sine and cosine functions of slow time η if $\delta_1^2 - \dfrac{1}{4} > 0$, that is, if either $\delta_1 > \dfrac{1}{2}$ or $\delta_1 < -\dfrac{1}{2}$. Thus the following two curves in the δ-ϵ plane represent stability changes, and are called *transition curves:*

$$\delta = \frac{1}{4} \pm \frac{\epsilon}{2} + O(\epsilon^2) \tag{19}$$

These two curves emanate from the point $\delta = \frac{1}{4}$ on the δ axis and define a region of instability called a *tongue.* Inside the tongue, for small ϵ, x grows exponentially in time. Outside the tongue, from (10) and (18), x is the sum of terms each of which is the product of two periodic (sinusoidal) functions with generally incommensurate frequencies, that is, x is a quasiperiodic function of t.


MATHIEU'S EQUATION
ε
UNSTABLE:
exponential growth
$\delta=\frac{1}{4}-\frac{\varepsilon}{2}$
$\delta=\frac{1}{4}+\frac{\varepsilon}{2}$
STABLE:
quasiperiodic motion
STABLE:
quasiperiodic motion
δ
$\delta=\frac{1}{4}$

Floquet Theory

In this section we present Floquet theory, that is, the general theory of linear differential equations with periodic coefficients. Our goal is to apply this theory to Mathieu's equation (1).

Let x be an $n \times 1$ column vector, and let A be an $n \times n$ matrix with time-varying coefficients which have period T. Floquet theory is concerned with the following system of first order differential equations:

$$\frac{dx}{dt} = A(t)\, x, \qquad A(t+T) = A(t) \tag{20}$$

Notice that if the independent variable t is replaced by $t+T$, the system (20) remains invariant. This means that if $x(t)$ is a solution (vector) of (20), and if in the vector function $x(t)$, t is replaced everywhere by $t+T$, then new vector, $x(t+T)$, which in general will be completely different from $x(t)$, is also a solution of (20). This observation may be stated conveniently in terms of fundamental solution matrices.

Let $X(t)$ be a fundamental solution matrix of (20). $X(t)$ is then an $n \times n$ matrix, with each of its columns consisting of a linearly independent solution vector of (20). In particular, we choose the i^{th} column vector to satisfy an initial condition for which each of the scalar components of $x(0)$ is zero, except for the i^{th} scalar component of $x(0)$, which is unity. This gives $X(0) = I$, where I is the $n \times n$ identity matrix. Since the columns of $X(t)$ are linearly independent, they form a basis for the n-dimensional solution space of (20), and thus any other fundamental solution matrix $Z(t)$ may be written in the form $Z(t) = X(t)\, C$, where C is a nonsingular $n \times n$ matrix. This means that each of the columns of $Z(t)$ may be written as a linear combination of the columns of $X(t)$.

From our previous observations, replacing t by $t+T$ in $X(t)$ produces a new fundamental solution matrix $X(t+T)$. Each of the columns of $X(t+T)$ may be written as a linear combination of the columns of $X(t)$, so that

$$X(t+T) = X(t)\, C \tag{21}$$

Note that at $t = 0$, (21) becomes $X(T) = X(0)C = IC = C$, that is,

$$C = X(T) \tag{22}$$

Eq.(22) says that the matrix C (about which we know nothing up to now) is in fact equal to the value of the fundamental solution matrix $X(t)$ evaluated at time T, that is, after one forcing period. Thus C could be obtained by numerically integrating (20) from $t = 0$ to $t = T$, n times, once for each of the n initial conditions satisfied by the i^{th} column of $X(0)$.

Eq.(21) is a key equation here. It has replaced the original system of o.d.e.'s with an iterative equation. For example, if we were to consider eq.(21) for the set of t values $t = 0, T, 2T, 3T, \cdots$, we would be generating the successive iterates of a Poincare map corresponding to the surface of section $\Sigma : t = 0 \pmod{2\pi}$. This immediately gives the result that $X(nT) = C^n$, which shows that the question of the boundedness of solutions is intimately connected to the matrix C.

In order to solve eq.(21), we transform to normal coordinates. Let $Y(t)$ be another fundamental solution matrix, as yet unknown. Each of the columns of $Y(t)$ may be written as a linear combination of the columns of $X(t)$:

$$Y(t) = X(t)\ R \tag{23}$$

where R is an as yet unknown $n \times n$ nonsingular matrix. Combining eqs.(21) and (23), we obtain

$$Y(t+T) = Y(t)\ R^{-1}CR \tag{24}$$

Now let us suppose that the matrix C has n linearly independent eigenvectors. If we choose the columns of R as these n eigenvectors, then the matrix product $R^{-1}CR$ will be a diagonal matrix with the eigenvalues λ_i of C on its main diagonal. With $R^{-1}CR$ diagonal, the matrix $Y(t)$ satisfying (24) will also be diagonal. This can be shown by construction: Let $y_i(t)$ represent the i^{th} scalar component on the main diagonal of $Y(t)$. Then assuming $Y(t)$ is diagonal, (24) can be written:

$$y_i(t+T) = \lambda_i\ y_i(t) \tag{25}$$

Eq.(25) is a linear functional equation. Let us look for a solution to it in the form

$$y_i(t) = \lambda_i^{kt}\ p_i(t) \tag{26}$$

where k is an unknown constant and $p_i(t)$ is an unknown function. Substituting (26) into (25) gives:

$$y_i(t+T) = \lambda_i^{k(t+T)}\ p_i(t+T) = \lambda_i(\lambda_i^{kt}\ p_i(t)) = \lambda_i\ y_i(t) \tag{27}$$

Eq.(27) is satisfied if we take $k = 1/T$ and $p_i(t)$ a periodic function of period T:

$$y_i(t) = \lambda_i^{t/T}\ p_i(t), \qquad p_i(t+T) = p_i(t) \tag{28}$$

Here eq.(28) is the general solution to eq.(25). The arbitrary periodic function $p_i(t)$ plays the same role here that an arbitrary constant plays in the case of a linear first order o.d.e.

Since we are interested in the question of boundedness of solutions, we can see from eq.(28) that if $|\lambda_i| > 1$, then $y_i \to \infty$ as $t \to \infty$, whereas if $|\lambda_i| < 1$, then $y_i \to 0$ as $t \to \infty$. Thus we see that the original system (20) will be stable (all solutions bounded) if every eigenvalue λ_i of $C = X(T)$ has modulus less than unity. If any one eigenvalue λ_i has modulus greater than unity, then (20) will be unstable (an unbounded solution exists).

Note that our assumption that C has n linearly independent eigenvectors could be relaxed, in which case we would have to deal with Jordan canonical form. The reader is referred to "Asymptotic Behavior and Stability Problems in Ordinary Differential Equations" by L.Cesari, Springer Verlag, 1963, section 4.1 for a complete discussion of this case.

Hill's Equation

In this section we apply Floquet theory to a generalization of Mathieu's equation (1), called Hill's equation:

$$\frac{d^2x}{dt^2} + f(t)\ x = 0, \qquad f(t+T) = f(t) \tag{29}$$

Here x and f are scalars, and $f(t)$ represents a general periodic function with period T. Eq.(29) includes examples such as eq.(4).

We begin by defining $x_1 = x$ and $x_2 = \frac{dx}{dt}$ so that (29) can be written as a system of two first order o.d.e.'s:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -f(t) & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{30}$$

Next we construct a fundamental solution matrix out of two solution vectors, $\begin{bmatrix} x_{11}(t) \\ x_{12}(t) \end{bmatrix}$ and $\begin{bmatrix} x_{21}(t) \\ x_{22}(t) \end{bmatrix}$, which satisfy the initial conditions:

$$\begin{bmatrix} x_{11}(0) \\ x_{12}(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} x_{21}(0) \\ x_{22}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{31}$$

As we saw in the previous section, the matrix C is the evaluation of the fundamental solution matrix at time T:

$$C = \begin{bmatrix} x_{11}(T) & x_{21}(T) \\ x_{12}(T) & x_{22}(T) \end{bmatrix} \tag{32}$$

From Floquet theory we know that stability is determined by the eigenvalues of C:

$$\lambda^2 - (\mathrm{tr}C)\lambda + \det C = 0 \tag{33}$$

where tr$C$ and det$C$ are the trace and determinant of C. Now Hill's eq.(29) has the special property that det$C$=1. This may be shown by defining W (the Wronskian) as:

$$W(t) = \det C = x_{11}(t)\ x_{22}(t) - x_{12}(t)\ x_{21}(t) \tag{34}$$

Taking the time derivative of W and using eq.(30) gives that $\frac{dW}{dt} = 0$, which implies that $W(t) =$ constant $= W(0) = 1$. Thus eq.(33) can be written:

$$\lambda^2 - (\mathrm{tr}C)\lambda + 1 = 0 \tag{35}$$

which has the solution:

$$\lambda = \frac{\mathrm{tr}C \pm \sqrt{\mathrm{tr}C^2 - 4}}{2} \tag{36}$$

Floquet theory showed that instability results if either eigenvalue has modulus larger than unity.

Thus if $|\mathrm{tr}C| > 2$, then (36) gives real roots. But the product of the roots is unity, so if one root has modulus less than unity, the other has modulus greater than unity, with the result that this case is UNSTABLE and corresponds to exponential growth in time.

On the other hand, if $|\mathrm{tr}C| < 2$, then (36) gives a pair of complex conjugate roots. But since their product must be unity, they must both lie on the unit circle, with the result that this case is STABLE. Note that the stability here is neutral stability not asymptotic stability, since Hill's eq.(29) has no damping. This case corresponds to quasiperiodic behavior in time.

Thus the transition from stable to unstable corresponds to those parameter values which give $|\mathrm{tr}C| = 2$. From (36), if $\mathrm{tr}C = 2$ then $\lambda = 1, 1$, and from eq.(28) this corresponds to a periodic solution with period T. On the other hand, if $\mathrm{tr}C = -2$ then $\lambda = -1, -1$, and from eq.(28) this corresponds to a periodic solution with period $2T$. This gives the important result that *on the transition curves in parameter space between stable and unstable, there exist periodic motions of period T or $2T$.*

The theory presented in this section can be used as a practical numerical procedure for determining stability of a Hill's equation. Begin by numerically integrating the o.d.e. for the two initial conditions (31). Carry each numerical integration out to time $t = T$ and so obtain $\mathrm{tr}C = x_{11}(T) + x_{22}(T)$. Then $|\mathrm{tr}C| > 2$ is unstable, while $|\mathrm{tr}C| < 2$ is stable. Note that this approach allows you to draw conclusions about large time behavior after numerically integrating for only one forcing period. Without Floquet theory you would have to numerically integrate out to large time in order to determine if a solution was growing unbounded, especially for systems which are close to a transition curve, in which case the asymptotic growth is very slow.

The reader is referred to "Nonlinear Vibrations in Mechanical and Electrical Systems" by J.Stoker, Wiley, 1950, Chapter 6, for a brief treatment of Floquet theory and Hill's equation. See "Hill's Equation" by W.Magnus and S.Winkler, Dover, 1979 for a complete treatment.

Harmonic Balance

In this section we apply Floquet theory to Mathieu's equation (1). Since the period of the forcing function in (1) is $T = 2\pi$, we may apply the result obtained in the previous section to conclude that on the transition curves in the δ-ϵ parameter plane there exist solutions of period 2π or 4π. This motivates us to look for such a solution in the form of a Fourier series:

$$x(t) = \sum_{n=0}^{\infty} a_n \cos \frac{nt}{2} + b_n \sin \frac{nt}{2} \tag{37}$$

This series represents a general periodic function with period 4π, and includes functions with period 2π as a special case (when a_{odd} and b_{odd} are zero). Substituting (37) into Mathieu's equation (1), simplifying the trig and collecting terms (a procedure called *harmonic balance)* gives four sets of algebraic equations on the coefficients a_n and b_n. Each set deals exclusively with a_{even}, b_{even}, a_{odd} and b_{odd}, respectively. Each set is homogeneous and of infinite order, so for a nontrivial solution the determinants must vanish. This gives four infinite determinants (called Hill's determinants):

$$a_{even}: \qquad \begin{vmatrix} \delta & \epsilon/2 & 0 & 0 & \\ \epsilon & \delta-1 & \epsilon/2 & 0 & \cdots \\ 0 & \epsilon/2 & \delta-4 & \epsilon/2 & \\ & & \cdots & & \end{vmatrix} = 0 \tag{38}$$

$$b_{even}: \qquad \begin{vmatrix} \delta-1 & \epsilon/2 & 0 & 0 & \\ \epsilon/2 & \delta-4 & \epsilon/2 & 0 & \cdots \\ 0 & \epsilon/2 & \delta-9 & \epsilon/2 & \\ & & \cdots & & \end{vmatrix} = 0 \tag{39}$$

$$a_{odd}: \qquad \begin{vmatrix} \delta-1/4+\epsilon/2 & \epsilon/2 & 0 & 0 & \\ \epsilon/2 & \delta-9/4 & \epsilon/2 & 0 & \cdots \\ 0 & \epsilon/2 & \delta-25/4 & \epsilon/2 & \\ & & \cdots & & \end{vmatrix} = 0 \tag{40}$$

$$b_{odd}: \qquad \begin{vmatrix} \delta-1/4-\epsilon/2 & \epsilon/2 & 0 & 0 & \\ \epsilon/2 & \delta-9/4 & \epsilon/2 & 0 & \cdots \\ 0 & \epsilon/2 & \delta-25/4 & \epsilon/2 & \\ & & \cdots & & \end{vmatrix} = 0 \tag{41}$$

In all four determinants the typical row is of the form:

$$\cdots \qquad 0 \qquad \epsilon/2 \qquad \delta - n^2/4 \qquad \epsilon/2 \qquad 0 \qquad \cdots$$

(except for the first one or two rows).

Each of these four determinants represents a functional relationship between δ and ϵ, which plots as a set of transition curves in the δ-ϵ plane. By setting $\epsilon = 0$ in these determinants it is easy to see where the associated curves intersect the δ axis. The transition curves obtained from the a_{even} and b_{even} determinants intersect the δ axis at $\delta = n^2$, $n = 0, 1, 2, \cdots$, while those obtained from the a_{odd} and b_{odd} determinants intersect the δ axis at $\delta = \dfrac{(2n+1)^2}{4}$, $n = 0, 1, 2, \cdots$. For $\epsilon > 0$, each of these points on the δ axis gives rise to two transition curves, one coming from the associated a determinant, and the other from the b determinant. Thus there is a tongue of instability emanating from each of the following points on the δ axis:

$$\delta = \frac{n^2}{4}, \qquad n = 0, 1, 2, 3, \cdots \tag{42}$$

The $n = 0$ case is an exception as only one transition curve emanates from it, as a comparison of eq.(38) with eq.(39) will show.

Note that the transition curves (19) found earlier in this Chapter by using the two variable expansion method correspond to $n = 1$ in eq.(42). Why did the perturbation method miss the other tongues of instability? It was because we truncated the perturbation method, neglecting terms of $O(\epsilon^2)$. The other tongues of instability turn out to emerge at higher order truncations in the various perturbation methods (two variable expansion, averaging, Lie transforms, normal forms, even regular perturbations). In all cases these methods deliver an expression for a particular transition curve in the form of a power series expansion:

$$\delta = \frac{n^2}{4} + \delta_1 \epsilon + \delta_2 \epsilon^2 + \cdots \tag{43}$$

As an alternative method of obtaining such an expansion, we can simply substitute (43) into any of the determinants (38)-(41) and collect terms, in order to obtain values for the coefficients δ_i. As an example, let us substitute (43) for $n = 1$ into the a_{odd} determinant (40). Expanding a 3×3 truncation of (40), we get (using computer algebra):

$$-\frac{\epsilon^3}{8} - \frac{\delta\,\epsilon^2}{2} + \frac{13\,\epsilon^2}{8} + \frac{\delta^2\,\epsilon}{2} - \frac{17\,\delta\,\epsilon}{4} + \frac{225\,\epsilon}{32} + \delta^3 - \frac{35\,\delta^2}{4} + \frac{259\,\delta}{16} - \frac{225}{64} \tag{44}$$

Substituting (43) with $n = 1$ into (44) and collecting terms gives:

$$(12\,\delta_1 + 6)\,\epsilon + \frac{\left(24\,\delta_2 - 16\,{\delta_1}^2 - 8\,\delta_1 + 3\right)\,\epsilon^2}{2} + \cdots \tag{45}$$

Requiring the coefficients of ϵ and ϵ^2 in (45) to vanish gives:

$$\delta_1 = -\frac{1}{2}, \qquad \delta_2 = -\frac{1}{8} \tag{46}$$

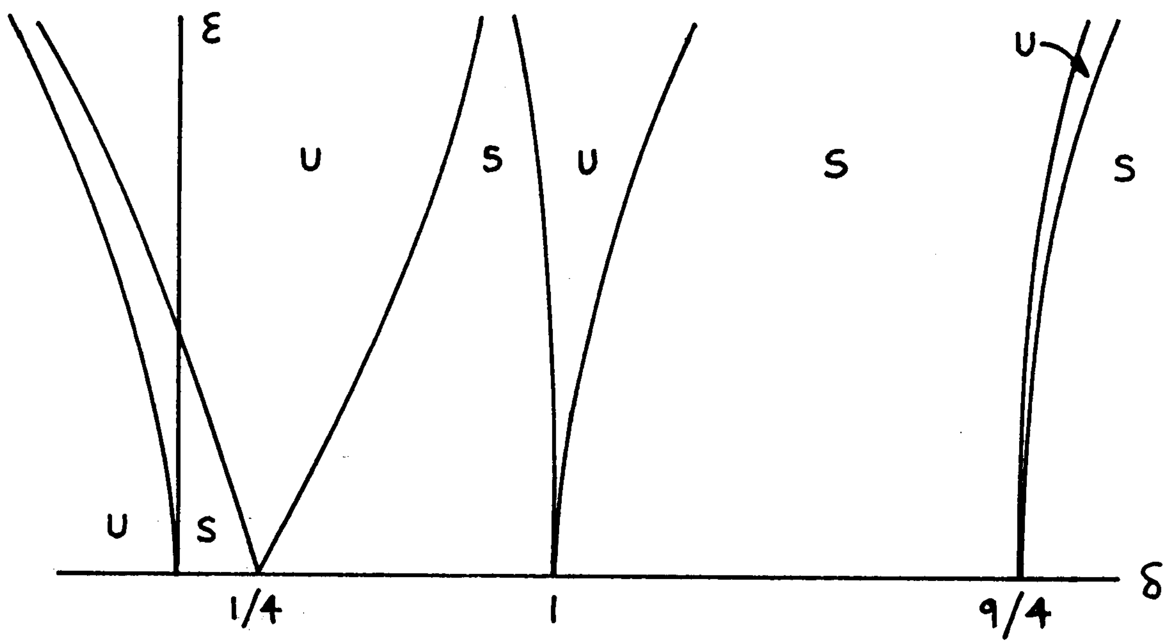
This process can be continued to any order of truncation. Here are the expansions of the first few transition curves:

$$\delta = -\frac{\epsilon^2}{2} + \frac{7\,\epsilon^4}{32} - \frac{29\,\epsilon^6}{144} + \frac{68687\,\epsilon^8}{294912} - \frac{123707\,\epsilon^{10}}{409600} + \frac{8022167579\,\epsilon^{12}}{19110297600} + \cdots \tag{47}$$

$$\begin{aligned}\delta \;=\;& \frac{1}{4} - \frac{\epsilon}{2} - \frac{\epsilon^2}{8} + \frac{\epsilon^3}{32} - \frac{\epsilon^4}{384} - \frac{11\,\epsilon^5}{4608} + \frac{49\,\epsilon^6}{36864} - \frac{55\,\epsilon^7}{294912} - \frac{83\,\epsilon^8}{552960} \\ &+ \frac{12121\,\epsilon^9}{117964800} - \frac{114299\,\epsilon^{10}}{6370099200} - \frac{192151\,\epsilon^{11}}{15288238080} + \frac{83513957\,\epsilon^{12}}{8561413324800} + \cdots\end{aligned} \tag{48}$$

$$\begin{aligned}\delta \;=\;& \frac{1}{4} + \frac{\epsilon}{2} - \frac{\epsilon^2}{8} - \frac{\epsilon^3}{32} - \frac{\epsilon^4}{384} + \frac{11\,\epsilon^5}{4608} + \frac{49\,\epsilon^6}{36864} + \frac{55\,\epsilon^7}{294912} - \frac{83\,\epsilon^8}{552960} \\ &- \frac{12121\,\epsilon^9}{117964800} - \frac{114299\,\epsilon^{10}}{6370099200} + \frac{192151\,\epsilon^{11}}{15288238080} + \frac{83513957\,\epsilon^{12}}{8561413324800} + \cdots\end{aligned} \tag{49}$$

$$\begin{aligned}\delta \;=\;& 1 - \frac{\epsilon^2}{12} + \frac{5\,\epsilon^4}{3456} - \frac{289\,\epsilon^6}{4976640} + \frac{21391\,\epsilon^8}{7166361600} \\ &- \frac{2499767\,\epsilon^{10}}{14447384985600} + \frac{1046070973\,\epsilon^{12}}{97086427103232000} + \cdots\end{aligned} \tag{50}$$



Transition curves in Mathieu's equation. S=stable, U=unstable.

$$
\begin{aligned}
\delta &= 1+\frac{5\,\epsilon^2}{12}-\frac{763\,\epsilon^4}{3456}+\frac{1002401\,\epsilon^6}{4976640}-\frac{1669068401\,\epsilon^8}{7166361600} \\
&\quad +\frac{4363384401463\,\epsilon^{10}}{14447384985600}-\frac{40755179450909507\,\epsilon^{12}}{97086427103232000}+\cdots
\end{aligned}
\tag{51}
$$

Effect of Damping

In this section we investigate the effect that damping has on the transition curves of Mathieu's equation by applying the two variable expansion method to the following equation, known as the damped Mathieu equation:

$$\frac{d^2x}{dt^2}+c\frac{dx}{dt}+(\delta+\epsilon\cos t)\;x=0 \tag{52}$$

In order to facilitate the perturbation method, we scale the damping coefficient c to be $O(\epsilon)$:

$$c=\epsilon\mu \tag{53}$$

We can use the same setup that we did earlier in this Chapter, whereupon eq.(6) becomes:

$$\frac{\partial^2 x}{\partial\xi^2}+2\epsilon\frac{\partial^2 x}{\partial\xi\partial\eta}+\epsilon^2\frac{\partial^2 x}{\partial\eta^2}+\epsilon\mu\left(\frac{\partial x}{\partial\xi}+\epsilon\frac{\partial x}{\partial\eta}\right)+(\delta+\epsilon\cos\xi)\;x=0 \tag{54}$$

Now we expand x as in eq.(7) and δ as in eq.(16), and we find that eq.(17) gets an additional term:

$$\frac{\partial^2 x_1}{\partial\xi^2}+\frac{1}{4}\;x_1=-2\frac{\partial^2 x_0}{\partial\xi\partial\eta}-x_0\cos\xi-\delta_1 x_0-\mu\frac{\partial x_0}{\partial\xi} \tag{55}$$

which results in two additional terms appearing in the slow flow eqs.(18):

$$\frac{dA}{d\eta}=-\frac{\mu}{2}A+\left(\delta_1-\frac{1}{2}\right)B,\qquad \frac{dB}{d\eta}=-\left(\delta_1+\frac{1}{2}\right)A-\frac{\mu}{2}B \tag{56}$$

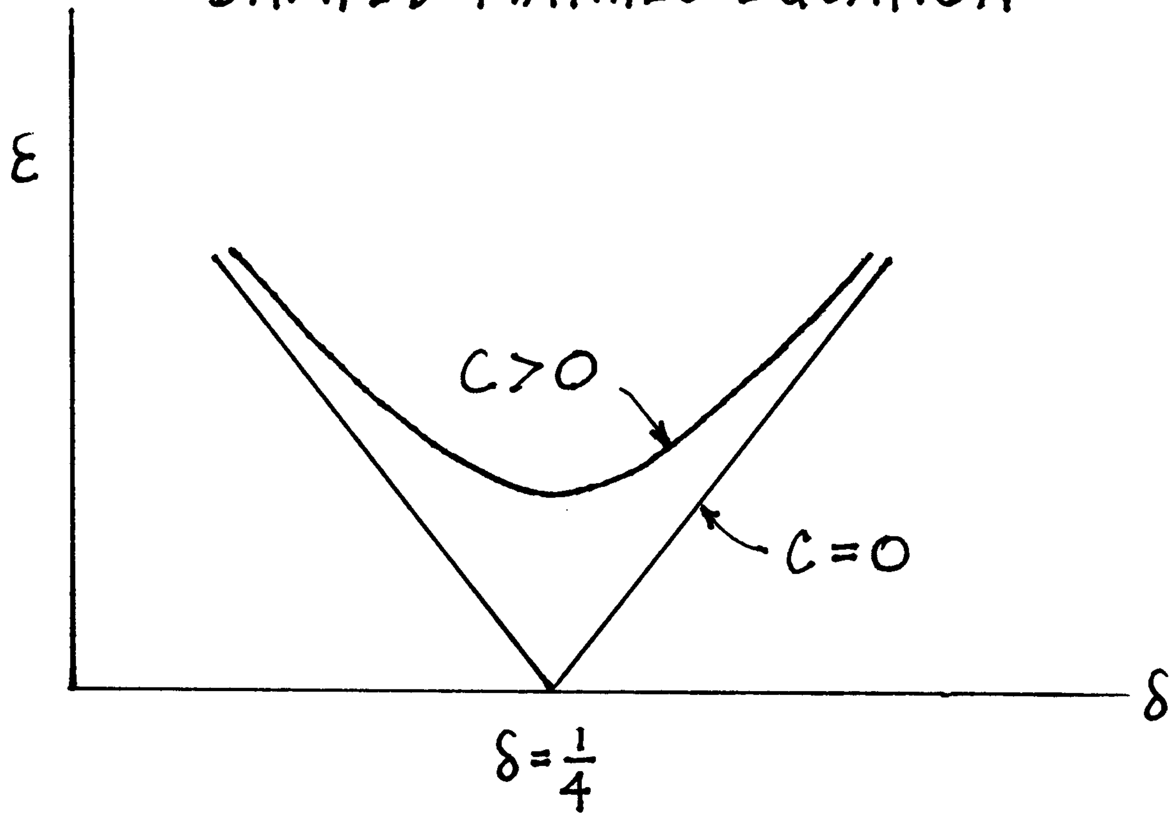
Eqs.(56) are a linear constant coefficient system which may be solved by assuming a solution in the form $A(\eta)=A_0\exp(\lambda\eta)$, $B(\eta)=B_0\exp(\lambda\eta)$. For nontrivial constants A_0 and B_0, the following determinant must vanish:

$$\begin{vmatrix} -\frac{\mu}{2}-\lambda & -\frac{1}{2}+\delta_1 \\ -\frac{1}{2}-\delta_1 & -\frac{\mu}{2}-\lambda \end{vmatrix}=0 \qquad\Rightarrow\qquad \lambda=-\frac{\mu}{2}\pm\sqrt{-\delta_1^2+\frac{1}{4}} \tag{57}$$

For the transition between stable and unstable, we set $\lambda=0$, giving the following value for δ_1:

$$\delta_1=\pm\frac{\sqrt{1-\mu^2}}{2} \tag{58}$$

DAMPED MATHIEU EQUATION



This gives the following expressions for the $n = 1$ transition curves:

$$\delta = \frac{1}{4} \pm \epsilon \frac{\sqrt{1-\mu^2}}{2} + O(\epsilon^2) = \frac{1}{4} \pm \frac{\sqrt{\epsilon^2 - c^2}}{2} + O(\epsilon^2) \tag{59}$$

Eq.(59) predicts that for a given value of c there is a minimum value of ϵ which is required for instability to occur. The $n = 1$ tongue, which for $c = 0$ emanates from the δ axis, becomes detached from the δ axis for $c > 0$. This prediction is verified by numerically integrating eq.(52) for fixed c, while δ and ϵ are permitted to vary.

Effect of Nonlinearity

In the previous sections of this Chapter we have seen how unbounded solutions to Mathieu's equation (1) can result from resonances between the forcing frequency and the oscillator's unforced natural frequency. However, real physical systems do not exhibit unbounded behavior. The difference lies in the fact that the Mathieu equation is linear. The effects of nonlinearity can be explained as follows: as the resonance causes the amplitude of the motion to increase, the relation between period and amplitude (which is a characteristic effect of nonlinearity) causes the resonance to detune, decreasing its tendency to produce large motions.

A more realistic model can be obtained by including nonlinear terms in the Mathieu equation. For example, in the case of the vertically driven pendulum, eq.(2), if we expand $\sin x$ in a Taylor series, we get:

$$\frac{d^2x}{dt^2} + \left(\frac{g}{L} - \frac{A\omega^2}{L}\cos\omega t\right)\left(x - \frac{x^3}{6} + \cdots\right) = 0 \tag{60}$$

Now if we rescale time by $\tau = \omega t$ and set $\delta = \dfrac{g}{\omega^2 L}$ and $\epsilon = \dfrac{A}{L}$, we get:

$$\frac{d^2x}{d\tau^2} + (\delta - \epsilon\cos\tau)\left(x - \frac{x^3}{6} + \cdots\right) = 0 \tag{61}$$

Next, if we scale x by $x = \sqrt{\epsilon}\, y$ and neglect terms of $O(\epsilon^2)$, we get:

$$\frac{d^2y}{d\tau^2} + (\delta - \epsilon\cos\tau)\, y - \epsilon\frac{\delta}{6}y^3 + O(\epsilon^2) = 0 \tag{62}$$

Motivated by this example, in this section we study the following nonlinear Mathieu equation:

$$\frac{d^2x}{dt^2} + (\delta + \epsilon\cos t)\, x + \epsilon\alpha x^3 = 0 \tag{63}$$

We once again use the two variable expansion method to treat this equation. Using the same setup that we did earlier in this Chapter, eq.(6) becomes:

$$\frac{\partial^2 x}{\partial\xi^2} + 2\epsilon\frac{\partial^2 x}{\partial\xi\partial\eta} + \epsilon^2\frac{\partial^2 x}{\partial\eta^2} + (\delta + \epsilon\cos\xi)\; x + \epsilon\alpha x^3 = 0 \tag{64}$$

We expand x as in eq.(7) and δ as in eq.(16), and we find that eq.(17) gets an additional term:

$$\frac{\partial^2 x_1}{\partial \xi^2} + \frac{1}{4}\, x_1 = -2\frac{\partial^2 x_0}{\partial \xi \partial \eta} - x_0 \cos\xi - \delta_1 x_0 - \alpha x_0^3 \tag{65}$$

where x_0 is of the form:

$$x_0(\xi,\eta) = A(\eta)\cos\frac{\xi}{2} + B(\eta)\sin\frac{\xi}{2} \tag{66}$$

Removal of resonant terms in (65) results in the appearance of some additional cubic terms in the slow flow eqs.(18):

$$\frac{dA}{d\eta} = \left(\delta_1 - \frac{1}{2}\right) B + \frac{3\alpha}{4} B(A^2+B^2), \qquad \frac{dB}{d\eta} = -\left(\delta_1 + \frac{1}{2}\right) A - \frac{3\alpha}{4} A(A^2+B^2) \tag{67}$$

In order to more easily work with the slow flow (67), we transform to polar coordinates in the A-B phase plane:

$$A = R\ \cos\theta, \qquad B = R\ \sin\theta \tag{68}$$

Note that eqs.(68) and (66) give the following alternate expression for x_0:

$$x_0(\xi,\eta) = R(\eta)\cos\left(\frac{\xi}{2} - \theta(\eta)\right) \tag{69}$$

Substitution of (68) into the slow flow (67) gives:

$$\frac{dR}{d\eta} = -\frac{R}{2}\ \sin 2\theta, \qquad \frac{d\theta}{d\eta} = -\delta_1 - \frac{\cos 2\theta}{2} - \frac{3\alpha}{4} R^2 \tag{70}$$

We seek equilibria of the slow flow (70). From (69), a solution in which R and θ are constant in slow time η represents a periodic motion of the nonlinear Mathieu equation (63) which has one-half the frequency of the forcing function, that is, such a motion is a 2:1 subharmonic. Such slow flow equilibria satisfy the equations:

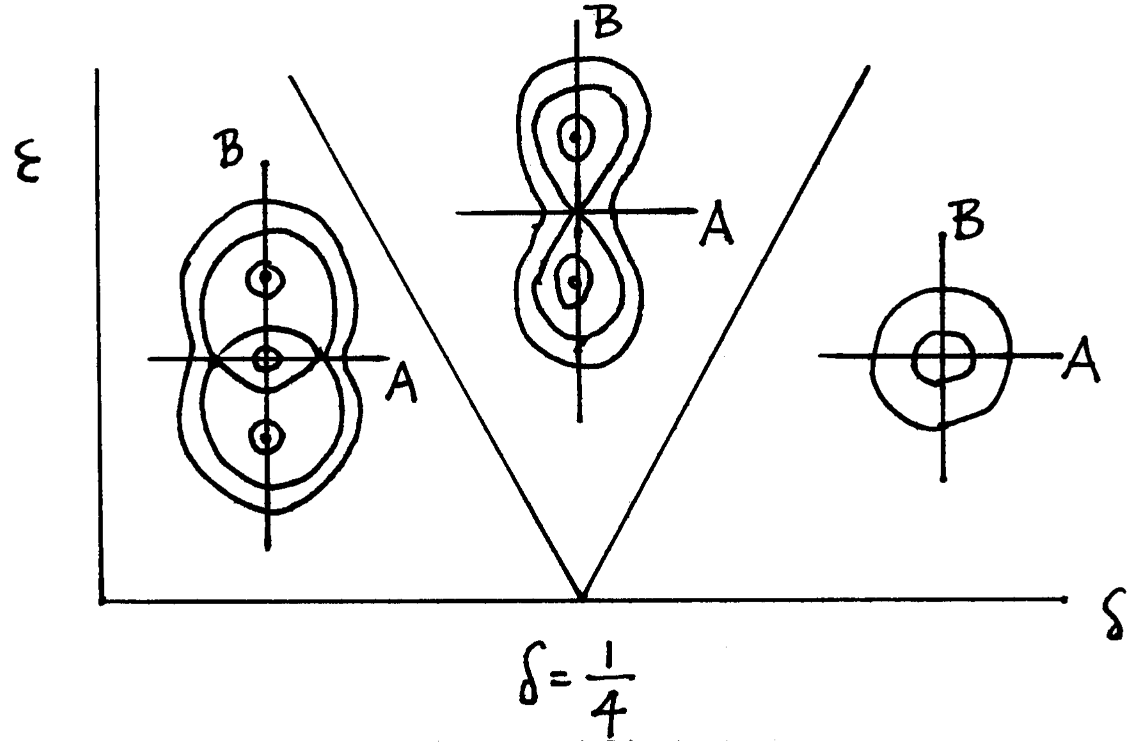
$$-\frac{R}{2}\ \sin 2\theta = 0, \qquad -\delta_1 - \frac{\cos 2\theta}{2} - \frac{3\alpha}{4} R^2 = 0 \tag{71}$$

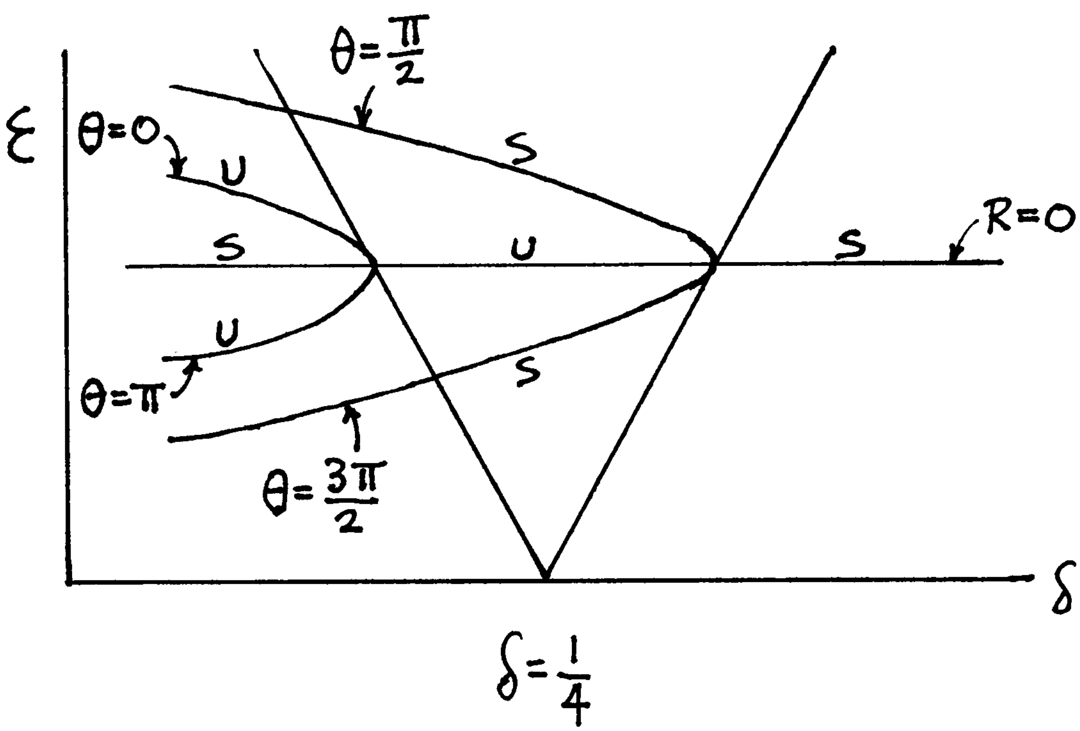
Ignoring the trivial solution $R = 0$, the first eq. of (71) requires $\sin 2\theta = 0$ or $\theta = 0, \frac{\pi}{2}, \pi$ or $\frac{3\pi}{2}$. Solving the second eq. of (71) for R^2, we get:

$$R^2 = -\frac{4}{3\alpha}\left(\frac{\cos 2\theta}{2} + \delta_1\right) \tag{72}$$

For a nontrivial real solution, $R^2 > 0$. Let us assume that the nonlinearity parameter $\alpha > 0$. Then in the case of $\theta = 0$ or π, $\cos 2\theta = 1$ and nontrivial equilibria exist only for $\delta_1 < -\frac{1}{2}$. On the other hand, for $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, $\cos 2\theta = -1$ and nontrivial equilibria require $\delta_1 < \frac{1}{2}$.

Nonlinear Mathieu Equation ($\alpha > 0$)





Since $\delta_1 = \pm\dfrac{1}{2}$ corresponds to transition curves for the stability of the trivial solution, the analysis predicts that bifurcations occur as we cross the transition curves in the δ-ϵ plane. That is, imagine quasistatically decreasing the parameter δ while ϵ is kept fixed, and moving through the $n = 1$ tongue emanating from the point $\delta = \dfrac{1}{4}$ on the δ axis. As δ decreases across the right transition curve, the trivial solution $x = 0$ becomes unstable and simultaneously a stable 2:1 subharmonic motion is born. This motion grows in amplitude as δ continues to decrease. When the left transition curve is crossed, the trivial solution becomes stable again, and an unstable 2:1 subharmonic is born. This scenario can be pictured as involving two pitchfork bifurcations.

If the nonlinearity parameter $\alpha < 0$, a similar sequence of bifurcations occurs, except in this case the subharmonic motions are born as δ *increases* quasistatically through the $n = 1$ tongue.

Problems

Problem 6.1

Alternatives to Floquet theory. As we saw in this Chapter, Floquet theory offers an approach to determining the stablity (that is the boundedness of all solutions) of the n-dimensional linear system with periodic coefficients:

$$\frac{dx}{dt} = A(t)\, x, \qquad A(t+T) = A(t) \tag{73}$$

where x is an n-vector and $A(t)$ is an $n \times n$ matrix.

This problem involves three alternative approaches. For each one, decide whether or not it is valid. If you think a method is valid, offer a line of reasoning showing why it works. If you think it is wrong, explain why it doesn't work or find a counterexample.

1. Set $x = Ty$ where y is an n-vector and T is an $n \times n$ matrix. Then $\frac{dy}{dt} = T^{-1}ATy$. Choose T such that $T^{-1}AT = D$ is diagonal (or more generally in Jordan canonical form). Then study the uncoupled system $\frac{dy}{dt} = Dy$.

2. Consider $\frac{dx}{dt} = A(t^*)\, x$ for t^* a fixed value of t. Examine the eigenvalues of $A(t^*)$. If the real parts of these eigenvalues remain negative for all positive t^*, then the solutions are asymptotically stable.

3. Replace the given equations by the averaged equations, $\frac{dx}{dt} = B\, x$, where $B = \frac{1}{T}\int_0^T A(t)dt$. Note that B is a constant coefficient matrix. Use the usual stability criteria on $\frac{dx}{dt} = B\, x$.

Problem 6.2
Nonlinear parametric resonance. This problem concerns the following differential equation:

$$\frac{d^2x}{dt^2} + \left(\frac{1}{4} + \epsilon k_1\right) x + \epsilon x^3 \cos t = 0, \qquad \epsilon << 1 \tag{74}$$

a) Use the two variable expansion method to derive a slow flow, neglecting terms of $O(\epsilon^2)$.

b) Analyze the slow flow. In particular, determine all slow flow equilibria and their stability. Make a sketch of the slow flow phase portrait for $k_1 = 0$ and for $k_1 = 0.1$.

Problem 6.3
The particle in the plane. Earlier in this Chapter we showed that the stability of the x-mode of the particle in the plane is governed by eq.(4) which may be written in the form:

$$\frac{d^2v}{dt^2} + \left(\frac{\delta - \epsilon \cos^2 t}{1 - \epsilon \cos^2 t}\right) v = 0 \tag{75}$$

where $\delta = 1 - L$ and $\epsilon = A^2$. Using the method of harmonic balance, obtain an approximate expression for the transition curve in the δ-ϵ plane which passes through the origin ($\delta = 0,\ \epsilon = 0$). Neglect terms of $O(\epsilon^4)$.

Problem 6.4
Damped Mathieu equation and Floquet theory. This question concerns eq.(52) for δ=1/4, exact 2:1 resonance (no detuning):

$$\frac{d^2x}{dt^2} + c\frac{dx}{dt} + (\frac{1}{4} + \epsilon \cos t)\ x = 0 \tag{76}$$

a. Find an approximate expression for the transition curve separating stable regions from unstable regions in the c-ϵ parameter plane, valid for small ϵ.
b. Compare your answer with results obtained by numerically integrating eq.(76) in conjunction with Floquet theory.
Hint: For a given pair of parameters (c, ϵ), numerically integrate (76) twice, respectively for initial conditions x=1, dx/dt=0 and x=0, $dx/dt = 1$. Evaluate the two resulting solution vectors at time $t = 2\pi$, and use them as the columns in the fundamental solution matrix $X(T)$ referred to in eq.(22). Compute the eigenvalues λ_1, λ_2 of this matrix. As discussed in the text, stability requires that both eigenvalues satisfy $|\lambda_i| < 1$.

the control law (3) does not only improves the transient performance, but also provides smoother outputs. It is important to stress that a better performance can still be achieved with (28) and (30). However, as discussed in Remark IV.1, increasing gains would not only cause more peaks in the outputs, but it might also yield saturation, especially for τ_2.

V. Conclusion

The tracking control problem for rigid robots with model parameter uncertainty has been studied in this note. In order to improve the parameter error convergence to zero, robust control techniques have been used. It was shown that with only parameter-dependent persistent excitation, the transient response of the parameter and tracking errors can be improved notably. Unlike other existing algorithms, the improvement of the transient performance is not achieved by increasing control gains, but by achieving a fast parameter adaptation, what results in smoother control outputs. Under the assumption of known bounds for the real parameters, it is also guaranteed that even in the absence of excitation the estimated parameter will remain bounded. By using the model of a two-link robot available in the literature, the proposed algorithm was tested in simulation. It was shown that the transient performance of this new adaptive algorithm is better in comparison with other well-known algorithms in the literature.

References

[1] M. Corless, "Tracking controllers for uncertain systems: Application to a manutec r3 robot," *ASME J. Dyna. Syst., Meas., Control*, vol. 111, pp. 609–618, 1989.

[2] C. H. An, C. G. Atkeson, and J. M. Hollerbach, "Estimation of inertial parameters of rigid body links of manipulators," in *Proc. 24th IEEE Conf. Decision Control*, Ft. Lauderdale, FL, Dec. 1985, pp. 990–995.

[3] P. K. Khosla and T. Kanade, "Parameter identification of robot dynamics," in *Proc. 24th IEEE Conf. Decision and Control, IEEE*, Ft. Lauderdale, FL, Dec. 1985, pp. 1754–1760.

[4] R. H. Middleton and G. C. Goodwin, "Adaptive computed torque control for rigid link manipulators," *Syst. Control Lett.*, vol. 10, pp. 9–16, 1988.

[5] J. J. E. Slotine and W. Li, "Adaptive manipulator control: A case study," *IEEE Trans. Automat. Contr.*, vol. 33, pp. 995–1003, Nov. 1988.

[6] Z. Qu, D. M. Dawson, and J. F. Dorsey, "Exponentially stable trajectory following of robotic manipulators under a class of adaptive controls," *Automatica*, vol. 28, no. 3, pp. 579–586, 1992.

[7] S. Nader and R. Horowitz, "Stability and robustness analysis of a class of adaptive controllers for robotic manipulators," *Int. J. Robot. Res.*, vol. 9, no. 3, pp. 74–92, 1990.

[8] ——, "An exponentially stable adaptive control law for robotic manipulators," *IEEE Trans. Robot. Automat.*, vol. 6, pp. 491–496, Apr. 1990.

[9] M. S. de Queiroz, D. M. Dawson, and M. Agarwal, "Adaptive control of robot manipulators with controller/update law modularity," *Automatica*, vol. 35, pp. 1379–1390, 1999.

[10] R. Ortega and M. W. Spong, "Adaptive motion control of rigid robots: A tutorial," *Automatica*, vol. 25, no. 6, pp. 877–888, 1989.

[11] L. Sciavicco and B. Siciliano, *Modeling and Control of Robot Manipulators*, 2nd ed. London, U.K.: Springer-Verlag, 2000.

[12] J. J. E. Slotine and W. Li, "On the adaptive control of robot manipulators," *Int. J. Robot. Res.*, vol. 6, no. 3, pp. 49–59, 1987.

[13] ——, "Composite adaptive control of robot manipulators," *Automatica*, vol. 25, no. 4, pp. 509–519, 1989.

[14] R. Lozano and C. C. de Wit, "Passivity based adaptive control for mechanical manipulators using ls-type estimation," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 1363–1365, Dec. 1990.

[15] B. Brogliato, I. D. Landau, and R. L. Real, "Adaptive motion control of robot manipulators: A unified approach based on passivity," *Int. J. Robust Nonlinear Control*, vol. 1, pp. 187–202, 1991.

[16] Y. Tang and M. A. Arteaga Pérez, "Adaptive control of robot manipulators based on passivity," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 1871–1875, Sept. 1994.

[17] A. Yao and M. Tomizuka, "Robust desired compensation adaptive control of robot manipulators with guaranteed transient performance," in *Proc. IEEE Int. Conf. Robotics Automation*, vol. 3, 1994, pp. 1830–1836.

[18] ——, "Smooth robust adaptive sliding mode control of manipulators with guaranteed transient performance," in *Proc. Amer. Control Conf.*, Baltimore, MD, June 1994, pp. 1176–1180.

[19] P. Tomei, "Robust adaptive control of robots with arbitrary transient performance and disturbance attenuation," *IEEE Trans. Automat. Contr.*, vol. 44, pp. 654–658, MAr. 1999.

[20] M. A. Arteaga Pérez, "On the properties of a dynamic model of flexible robot manipulators," *ASME J. Dyna. Syst., Meas., Control*, vol. 120, pp. 8–14, 1998.

[21] J. J. E. Slotine and W. Li, *Applied Nonlinear Control*. Upper Saddle River, NJ: Prentice-Hall, 1991.

[22] F. Reyes and R. Kelly, "Experimental evaluation of model-based controllers on a direct-drive robot arm," *Mechatronics*, vol. 11, pp. 267–282, 2001.

A Continuous-Time Observer Which Converges in Finite Time

Robert Engel and Gerhard Kreisselmeier

***Abstract*—It is shown that a continuous-time observer, which comprises two standard nth order observers and a delay D, can observe the state of an nth order linear system in finite time D exactly. In particular, (almost) any convergence time D can be assigned, independent of the observer eigenvalues.**

***Index Terms*—Convergence, delay time, linear systems, observer.**

I. Introduction

Consider an observable linear system in continuous time

$$\dot{x} = Ax + Bu \quad x(t_0) = x_0, \quad t \geq t_0 \tag{1a}$$

$$y = Cx \tag{1b}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$ and output $y \in \mathbb{R}^p$.

The theory of observers for such systems, which reconstruct the state x from measurements of the input and output, is well established, see, e.g., [2], [6], and [7].

In a continuous-time setting, the convergence of the state observation to zero is always *asymptotic* with time. The convergence rate is exponential and can be assigned by suitably choosing the observer eigenvalues [6].

In contrast, the observation problem in a discrete-time setting allows the choice of zero eigenvalues and thereby a dead-beat response, i.e., a transient evolution which converges in *finite* time. The guaranteed convergence time is then n times the sampling time and can be assigned by choosing the latter [1], [3], [5].

Convergence in finite time is an attractive feature and, as this note shows, not restricted to the use of sampled-data or discrete-time techniques. This note presents a purely continuous-time observer which

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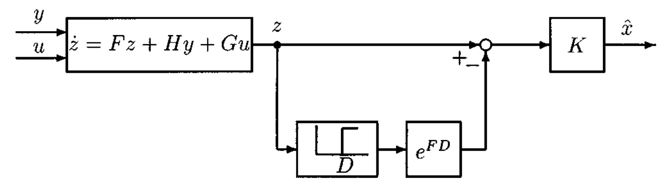

Fig. 1. Finite time observer structure.

converges in finite time. To the authors' knowledge, such a result has not been reported in the literature thus far.

II. Main Result

It is assumed that the pair (A, C) is observable. The equations

$$\dot{z}_i = (A - H_iC)\, z_i + H_i y + Bu, \quad (i = 1, 2)$$

represent two standard identity obervers for the system (1a)–1(b).[1] By assuming $F_i := A - H_iC$, $i = 1, 2$ and

$$F := \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}, \quad H := \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}$$
$$G := \begin{bmatrix} B \\ B \end{bmatrix} \quad T := \begin{bmatrix} I_{n,n} \\ I_{n,n} \end{bmatrix} \quad z := \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

we combine these two observers in one equation and use a delay D to generate a new state estimate $\hat{x}$ by

$$\dot{z} = Fz + Hy + Gu, \quad t \geq t_0 \tag{2a}$$
$$\hat{x}(t) = K\left[z(t) - e^{FD} z(t - D)\right]. \tag{2b}$$

Because of the delay, this observer has initial conditions $z(t), t \in [t_0 - D, t_0]$.

Theorem: Let H and D be chosen such that

i) F is stable;
ii) $\det[T, e^{FD}T] \neq 0$.

Then, (2a) and (2b) with $K := [I_{n,n}, 0_{n,n}][T, e^{FD}T]^{-1}$ are an observer for (1), whose state estimate $\hat{x}$ converges to x in finite time D.

Proof: For $t \geq t_0$ we have

$$\begin{aligned} \frac{d}{dt}(z - Tx) &= Fz + Hy + Gu - T(Ax + Bu) \\ &= F(z - Tx) + [FT - TA + HC]x + [G - TB]u \\ &= F(z - Tx) \end{aligned}$$

and, therefore

$$z(t) - Tx(t) = e^{FD}[z(t - D) - Tx(t - D)], \quad t \geq t_0 + D.$$

Using the fact that $KT = I$ and $Ke^{FD}T = 0$ by the definition of K, gives

$$\begin{aligned} \hat{x}(t) &= K\left[z(t) - e^{FD}z(t - D)\right] \\ &= x(t) + K[z(t) - Tx(t)] \\ &\quad - Ke^{FD}[z(t - D) - Tx(t - D)]. \end{aligned}$$

[1]To see this, note that $d/dt(z_i - x) = (A - H_iC)(z_i - x)$. This implies $z_i(t) - x(t) \to 0$ as $t \to \infty$, provided $A - H_iC$ is stable.

This implies $\hat{x}(t) = x(t)$ for $t \geq t_0 + D$.

The observer is also consistent, i.e., results in $\hat{x}(t) = x(t)$ for all $t \geq t_0$, if the initial conditions are $z_0 = Tx_0$ and $z(t) = T\xi(t)$ with any choice of $\xi(t), t \in [t_0 - D, t_0)$, because $Ke^{FD}T = 0$. □

The new observer has a simple open-loop structure, which is illustrated in Fig. 1.

It remains to be shown that a suitable choice of H and D to satisfy the assumptions of the Theorem is possible. The following Lemma shows that whith a choice of H, which is slightly stronger than assumption i), almost any choice of D satisfies assumption ii).

Lemma: If H is chosen such that

$$Re\, \lambda_j(F_2) < \sigma < Re\, \lambda_j(F_1), \quad j = 1, 2, \ldots, n$$

for some $\sigma < 0$, then $\det[T, e^{FD}T] \neq 0$ for almost all $D \in \mathbb{R}^+$.

Proof: H can be chosen as required, because (C, A) is observable. We have

$$\left[T, e^{FD}T\right] = \begin{bmatrix} I & e^{F_1D} \\ I & e^{F_2D} \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} I & e^{F_1D} \\ 0 & e^{F_1D} - e^{F_2D} \end{bmatrix}$$

and thus obtain

$$\begin{aligned} \det\left[T, e^{FD}T\right] &= (-1)^n \det\left(e^{F_1D} - e^{F_2D}\right) \\ &= (-1)^n \det\left(e^{F_1D}\right) \det\left(I - e^{-F_1D}e^{F_2D}\right). \end{aligned}$$

The first determinant on the right hand side is nonzero for each $D \in \mathbb{R}^+$. The second one tends to $+1$ as $D \to \infty$ because, due to the assumption of the Lemma, $e^{-F_1D}e^{F_2D} = e^{(-\sigma I - F_1)D}e^{(\sigma I + F_2)D} \to 0$ as $D \to \infty$. Consequently the overall determinant does not vanish identically. Since it is an analytic function of D, it can have only isolated zeros (Principle of Isolated Zeros [4]). □

Since $\det[T, e^{FD}T]$ is zero at $D = 0$ and the zeros are isolated points, it is nonzero for all sufficiently small values of D. Therefore, the observer can be made to converge exactly within an arbitrarily short time interval, in theory.

The condition of the Lemma is sufficient to prove that an appropriate choice of H and D in the theorem can be made. In numerical tests a choice of H such that the eigenvalues of F_1 are different from those of F_2, but not necessarily ordered as in the condition of the Lemma was also found appropriate with almost any choice of D.

To give an idea of how the transient of the state observation error of the new observer evolves, consider any initial condition $x(0) = x_0$ for the system, any initial state estimate $\hat{x}_0$ and initial condition $z(t) = T\hat{x}_0, t \in [-D, 0]$ for the observer. Then (2a) gives $z(t) = Tx(t) + e^{Ft}(T\hat{x}_0 - Tx_0)$ and from (2b) it follows for $t \in [0, D]$ that:

$$\begin{aligned} \hat{x}(t) &= Kz(t) - Ke^{FD}z(t - D) \\ &= K\left[Tx(t) + e^{Ft}T(\hat{x}_0 - x_0)\right] - Ke^{FD}T\hat{x}_0, \quad t \in [0, D]. \end{aligned}$$

The fact that $KT = I$ and $Ke^{FD}T = 0$ by the definition of K, finally results in

$$\hat{x}(t) - x(t) = Ke^{Ft}T(\hat{x}_0 - x_0), \quad t \in [0, D].$$

It is seen that, in the transient phase, the state observation error is a linear combination of those $2n$ exponentials pertaining to the two individual observers, which becomes zero at time $t = D$ because $Ke^{FD}T = 0$.

III. Concluding Remarks

A continuous-time observer, which converges in finite time, is accomplished by using the redundancy of two standard observers and a delay. What enables this result is that the individual two observers with state estimate $z_i(t)$ and state observation error $\varepsilon_i(t)$, ($i = 1, 2$), respectively, give rise to the relations

$$\begin{aligned} z_1(t) &= x(t) + \varepsilon_1(t) \\ z_2(t) &= x(t) + \varepsilon_2(t) \\ z_1(t-D) &= x(t-D) + e^{-F_1 D} \cdot \varepsilon_1(t) \\ z_2(t-D) &= x(t-D) + e^{-F_2 D} \cdot \varepsilon_2(t) \end{aligned}$$

i.e., a set of four equations with four unknowns $x(t)$, $x(t-D)$, $\varepsilon_1(t)$, $\varepsilon_1(t)$. The state estimate $\hat{x}(t)$ is just taken to be the result, which arises from solving these equations for $x(t)$, given $z_i(t)$ and $z_i(t-D)$, ($i = 1, 2$).

The convergence time D and the observer eigenvalues (resp. the observer gains H_i) are independent quantities to be chosen or designed. They have clearly a joint (filtering) effect on the state estimate. In particular, after the transient is over, one has from (2) that

$$\hat{x}(t) = [I_{n,n}, 0_{n,n}] \left[T, e^{FD}T\right]^{-1} \cdot \int_{t-D}^{t} e^{F(t-\tau)} \{Hy(\tau) + Gu(\tau)\} \, d\tau$$

i.e., the state estimate is generated using measurements from the finite interval $[t-D, t]$ only.

References

[1] J. Ackermann, *Abtastregelung, Band 1*. Berlin, Germany: Springer-Verlag, 1983.
[2] S. Barnett, *Introduction to Mathematical Control Theory*. Oxford, U.K.: Clarendon, 1975.
[3] F. F. Franklin and J. D. Powell, *Digital Control Systems*. Reading, MA: Addison-Wesley, 1980.
[4] J. Dieudonné, *Foundations of Modern Analysis*. New York: Academic, 1969, p. 200.
[5] R. Isermann, *Digital Control Theory*. Berlin, Germany: Springer-Verlag, 1981.
[6] T. Kailath, *Linear Systems*. Upper Saddle River, NJ: Prentice-Hall, 1985.
[7] J. O'Reilly, *Observers for Linear Systems*. New York: Academic, 1983.

Stochastic Stability of Jump Linear Systems

Yuguang Fang and Kenneth A. Loparo

***Abstract*—In this note, some testable conditions for mean square (i.e., second moment) stability for discrete-time jump linear systems with time-homogenous and time-inhomogenous finite state Markov chain form processes are presented.**

***Index Terms*—Jump linear systems, Kronecker product, Lyapunov equation, mean square stability, stability, stochastic systems.**

I. Introduction

In this note, second moment (mean square) stability for the jump linear system (1.1) whose form process $\{\sigma_k\}$ is a finite state time-homogenous or time-inhomogenous Markov chain is studied.

$$x_{k+1} = H(\sigma_k)x_k \tag{1.1}$$

A stochastic version of Lyapunov's second method is used to obtain a necessary and sufficient condition for second moment exponential stability if the probability transition matrix is periodic in time. This is a general result in which the results of Morozan [1] and Ji *et al.* [2] for the time-homogenous case and Krtolica *et al.* [3] can be recovered as special cases. In order to apply these results, a coupled system of Lyapunov equations needs to be solved for which Kronecker product techniques will be used and a very general sufficient condition is presented. For one-dimensional systems, this sufficient condition is also necessary.

A second moment stabilization problem for systems of type (1.1) is investigated by Ji *et al.* [2] and Feng *et al.* [4], where the equivalence between some second moment stability concepts were also proved. Mariton also studied stochastic controllability, observability, stabilizability and linear quadratic optimal control problems for continuous-time jump linear control systems, the details can be found in [6]. Krtolica *et al.* [3] applied the Kalman–Bertram decomposition to study closed-loop control systems with communication delays. The system is modeled as a jump linear system with an inhomogenous Markov chain and they obtained a necessary and sufficient condition for exponential stability. Wonham [9] systematically studied linear quadratic optimal control problems for these types of systems. Other work related to the stability of jump linear systems is summarized in [7].

Before we present the main results, some preliminaries are necessary. Suppose that $\{\sigma_k\}$ is a finite state Markov chain with state space $\underline{N}$, transition probability matrix $P = (p_{ij})_{N \times N}$ and initial distribution $p = (p_1, \ldots, p_N)$. For simplicity, assume that the initial state $x_0 \in \mathcal{R}^n$ is a (nonrandom) constant vector. Let $(\Omega, \mathcal{F}, P)$ denote the underlying probability space and let Ξ be the collection of all probabiltiy distribution on $\underline{N}$. Let $e_i \in \Xi$ be the initial distribution concentrated at the ith state, i.e., given by $P\{\sigma_0 = i\} = 1$. If properties depend on the choice of the initial distribution of the Markov form process $\{\sigma_k\}$, for each $\xi \in \Xi$, let P_ξ denote the probability measure for the Markov chain

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