ORI 397 Optimization Under Uncertainty Spring 2020

Assignment 2

Due: Tuesday, February 11

Problem 1. Derive the dual quadratic program to the mean-variance portfolio optimization problem

$$\begin{aligned} \sup \quad & (1-\lambda)\boldsymbol{\mu}^\top \boldsymbol{x} - \lambda \boldsymbol{x}^\top \boldsymbol{\Sigma} \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{x} \in \mathbb{R}^N_+ \\ & \mathbf{e}^\top \boldsymbol{x} = 1. \end{aligned}$$

Here, $\boldsymbol{\mu} \in \mathbb{R}^N_+$ is the mean vector and $\boldsymbol{\Sigma} \in \mathbb{S}^N_+$ is the covariance matrix of the asset returns. The parameter $\lambda \in [0,1]$ describes the risk aversion level of the investor. Implement the quadratic program in a MATLAB function `portfolio_dual.m` defined as follows:

```
[obj]=portfolio_dual(mu,Sigma,lambda)
```

The function takes the mean vector $\boldsymbol{\mu}$, covariance matrix $\boldsymbol{\Sigma}$, and risk aversion level λ as inputs. It outputs the optimal objective value of the dual problem. Submit your implementation of `portfolio_dual.m` to **Canvas**.

Problem 2 (Uncertainty Quantification)**.** The magnitude of an earthquake in Fukushima, Japan, can be represented by a univariate random variable $\tilde{\xi}$ that takes values in the set $\{1,2,3,4,5,6,7,8,9\}$. This random variable has an unknown probability distribution $\mathbb{P}$. However, it is known that the random variable has mean 3 and variance 4. Given an input $t \in \mathbb{N}$, formulate two linear programs that find the upper and lower bounds on the probability

$$\mathbb{P}(\tilde{\xi} \geq t). \tag{1}$$

Implement both linear programs in a MATLAB function `bounds.m` defined as follows:

```
[lower,upper]=bounds(t)
```

Here, the function takes the number t as input and outputs the lower and upper bounds on the probability (1). Submit your implementation of `bounds.m` to **Canvas**.

Problem 3 (Conditional Value-at-Risk)**.** Let $\tilde{\boldsymbol{\xi}}$ be a random vector supported on Ξ. The loss function $\ell(\boldsymbol{x}, \boldsymbol{\xi})$ is known to be convex in $\boldsymbol{x}$ for any fixed $\boldsymbol{\xi} \in \Xi$. Show that the function

$$g(\boldsymbol{x}) = \inf_{\beta \in \mathbb{R}} \beta + \frac{1}{\epsilon} \mathbb{E}_{\mathbb{P}} \left[\max\{\ell(\boldsymbol{x}, \tilde{\boldsymbol{\xi}}) - \beta, 0\} \right]$$

is convex in $\boldsymbol{x}$. *Hint: Use the convexity preserving operations that we discussed in class.*