Problem 1: 微分的唯一性

考虑映射 $f:\Omega\subset\mathbb{R}^n\to\mathbb{R}^m$，假设存在两个线性映射 $A_i:\mathbb{R}^n\to\mathbb{R}^m$（$i=1,2$），使得对任意 $\boldsymbol{v}\to 0$ 及 $i=1,2$ 均有

$$f(\boldsymbol{x_0}+\boldsymbol{v})=f(\boldsymbol{x_0})+A_i(\boldsymbol{v})+o(\boldsymbol{v}).$$

证明 $A_1=A_2$.

Proof. 假设 $A_1\neq A_2$，取 $\boldsymbol{v}$ 使得 $\|\boldsymbol{v}\|=1$ 且 $(A_1-A_2)\boldsymbol{v}=\boldsymbol{t}\neq 0$. 那么对任意 $\lambda>0$ 总有

$$\begin{aligned} f(\boldsymbol{x_0}+\lambda\boldsymbol{v})&=f(\boldsymbol{x_0})+A_i(\lambda\boldsymbol{v})+o(\lambda).\\ \Rightarrow A_1(\lambda\boldsymbol{v})-A_2(\lambda\boldsymbol{v})&=o(\lambda).\\ \Rightarrow (A_1-A_2)(\boldsymbol{v})&=o(1)=0.\end{aligned}$$

矛盾. 所以 $A_1=A_2$.

Problem 2

考察函数 $f(x,y)=\begin{cases}(x^2+y^2)\sin\dfrac{1}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0),\\ 0, & (x,y)=(0,0).\end{cases}$.
计算 f 的偏导数并证明 f 的两个偏导数在 $(0,0)$ 处均不连续但 f 在 $(0,0)$ 处可微.

Proof. 首先在 $(0,0)$ 处，

$$\lim_{(x,y)\to(0,0)}\left|\frac{(x^2+y^2)\sin\dfrac{1}{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}\right|=\lim_{r\to 0}\left|r\sin r^{-1}\right|\le\lim_{r\to 0}|r|=0.$$

然而在 $y=0$ 上

$$\frac{\partial f}{\partial \boldsymbol{x}}=\begin{cases}2x\sin x^{-1}-\cos x^{-1}, & x>0;\\ 0, & x=0.\end{cases}$$

这个函数在 $x=0$ 处不连续，所以偏导数在 $(0,0)$ 附近不连续.

Problem 3

假设函数 f 在 Ω 上的所有偏导数 $\dfrac{\partial f}{\partial \boldsymbol{x_i}}$ 都存在并且一致有界，证明 f 在 Ω 上连续. f 是否一定在 Ω 上可微？如果是，请给出证明，否则举出反例.

Proof. 设对任意 $i=1,\ldots,n$ 和 $\boldsymbol{x}\in\Omega$ 均有

$$\left|\frac{\partial f}{\partial \boldsymbol{x_i}}(\boldsymbol{x})\right|<M.$$

固定 $i,\boldsymbol{x}$，考虑一元函数 $g(t)=f(\boldsymbol{x}+t\boldsymbol{e_i})$. 根据上述条件 $g'(t)$ 存在且 $|g'(t)|<M$. 因此 $(Mt-g)'>0$ 及 $(g+Mt)'>0$ 恒成立，故 $|g(t)-g(0)|\le Mt$. 这意味着

$$|f(\boldsymbol{x}+t\boldsymbol{e_i})-f(\boldsymbol{x})|\le Mt. \tag{1}$$

现在对任意 $\boldsymbol{x},\boldsymbol{y}\in\Omega$ 满足 $|\boldsymbol{x}-\boldsymbol{y}|<\varepsilon$，设 $\boldsymbol{x}-\boldsymbol{y}=\sum_{i=1}^n a_i\boldsymbol{e_i}$，则由 (1) 可知

$$\left|f(\boldsymbol{y}+\sum_{i=1}^m a_i\boldsymbol{e}_i)-f(\boldsymbol{y}+\sum_{i=1}^{m+1} a_i\boldsymbol{e}_i)\right|\le Ma_{m+1},\quad m=0,\dots,n-1.$$

所以

$$|f(\boldsymbol{x})-f(\boldsymbol{y})|\le M\cdot\sum_{m=1}^n a_m\le Mn\,|\boldsymbol{x}-\boldsymbol{y}|\,.$$

所以 f 在 Ω 上连续. 另一方面，f 不一定可微：构造

$$f(x,y)=\frac{x^3+y^3}{x^2+y^2}.$$

在 $\Omega=(-1,1)\times(-1,1)$ 上计算偏导：

$$\frac{\partial f}{\partial\boldsymbol{x}}=\frac{3x^2(x^2+y^2)-2x(x^3+y^3)}{(x^2+y^2)^2},\quad \frac{\partial f}{\partial\boldsymbol{y}}=\frac{3y^2(x^2+y^2)-2y(x^3+y^3)}{(x^2+y^2)^2}.$$

那么偏导数有界，因为

$$5(x^2+y^2)^2\ge 3x^2(x^2+y^2)+2x^4,(x^2+y^2)^2\ge\left|xy^3\right|.$$

所以

$$\left|\frac{\partial f}{\partial\boldsymbol{x}}\right|\le 6,\quad \frac{\partial f}{\partial\boldsymbol{x}}|_{(0,0)}=\lim_{x\to 0}x=0.$$

同理 f 关于 y 的偏导数也有界. 但是

$$f(x,kx)=\frac{k^3+1}{k^2+1}x\Rightarrow \mathrm{d}f|_{(0,0)}(1,k)=\frac{k^3+1}{k^2+1}.$$

显然这不是一个线性函数，所以 f 在 $(0,0)$ 处不可微.

Problem 4

设 $\Omega\subset\mathbb{R}^2$ 为开集，我们用 (x,y) 来表示 $\mathbb{R}^2$ 上的坐标. 函数 $f:\Omega\to\mathbb{R}$ 的偏导数 $\dfrac{\partial f}{\partial\boldsymbol{x}}$ 和 $\dfrac{\partial f}{\partial\boldsymbol{y}}$ 处处存在. 如果偏导数 $\dfrac{\partial f}{\partial\boldsymbol{y}}$ 在 Ω 上连续，证明 f 在 Ω 上可微.

Proof. 对任意 $(x_0,y_0)\in\Omega$ 及 $(u,v)\in\mathbb{R}^2$，我们证明

$$\lim_{h\to 0}\frac{f(x_0+hu,y_0+hv)-f(x_0,y_0)}{h}=u\frac{\partial f}{\partial\boldsymbol{x}}|_{(x_0,y_0)}+v\frac{\partial f}{\partial\boldsymbol{y}}|_{(x_0,y_0)}.$$

根据定义：

$$\lim_{h\to 0}\frac{f(x_0+hu,y_0)-f(x_0,y_0)}{h}=u\frac{\partial f}{\partial\boldsymbol{x}}|_{(x_0,y_0)}.$$

因此只需证明

$$\lim_{h\to 0}\frac{f(x_0+hu,y_0+hv)-f(x_0+hu,y_0)}{h}=v\frac{\partial f}{\partial \boldsymbol{y}}|_{(x_0,y_0)}.$$

这是因为由于 $g(t)=f(x_0+hu,y_0+tv)$ 是 C^1 的，所以存在 $\xi\in[0,h]$ 使得

$$g(h)-g(0)=g'(\xi)\cdot h=\frac{\partial f}{\partial(\boldsymbol{vy})}|_{(x_0+hu,y_0+\xi v)}\cdot h=v\frac{\partial f}{\partial \boldsymbol{y}}|_{(x_0+hu,y_0+\xi v)}\cdot h.$$

所以

$$\lim_{h\to 0}\frac{f(x_0+hu,y_0+hv)-f(x_0+hu,y_0)}{h}=\lim_{h\to 0}v\frac{\partial f}{\partial \boldsymbol{y}}|_{(x_0+hu,y_0+\xi v)}=v\frac{\partial f}{\partial \boldsymbol{y}}|_{(x_0,y_0)}.$$

最后一步根据 $\dfrac{\partial f}{\partial \boldsymbol{y}}$ 连续及 $\lim_{h\to 0}(x_0+hu,y_0+\xi v)=(x_0,y_0)$ 得到.

Problem 5

考虑在 $\mathbb{R}^{n\times n}$ 上定义的行列式函数

$$\det:\mathbb{R}^{n\times n}\to\mathbb{R},\quad A\mapsto\det A.$$

任意给定 $A\in\mathbb{R}^{n\times n}$，试计算 $\mathrm{d}\det|_{\boldsymbol{x}=A}$.

Proof. 设 $X=(x_{ij})$，如果把每个输入 x_{ij} 看作一个变元，那么

$$\det X=\sum_{\sigma\in S_n}\operatorname{sgn}\sigma\cdot x_{1\sigma(1)}\dots x_{n\sigma(n)}$$

是关于 x_{ij} 的线性多项式. 因此对 x_{ij} 求导后可得：

$$\frac{\partial\det}{\partial \boldsymbol{x_{ij}}}=\sum_{\sigma\in S_n,\sigma(i)=j}\operatorname{sgn}\sigma\cdot\prod_{k\neq i}x_{k\sigma(k)}=C_{ij}.$$

其中 C_{ij} 为 X 中去掉 i 行 j 列后所得的矩阵的行列式. 因此

$$\frac{\partial\det}{\partial \boldsymbol{x_{ij}}}|_{\boldsymbol{x}=A}=A_{ij}.$$

其中 A_{ij} 为 A 关于 i 行 j 列的代数余子式.

Problem 6

计算下列函数的偏导数：

$$(1)\ f(x,y,z)=x^{y^z},\quad (2)\ f(x,y,z)=\tan\frac{xy}{z^2}.$$

Proof. (1) 化简表达式：

$$f(x,y,z)=x^{y^z}=e^{y^z\log x}=e^{e^{z\log y}\log x}.$$

计算偏导数

$$\frac{\partial f}{\partial \boldsymbol{x}}=y^zx^{y^z-1};\ \frac{\partial f}{\partial \boldsymbol{y}}=e^{y^z\log x}zy^{z-1}=x^{y^z}zy^{z-1}.$$

$$\frac{\partial f}{\partial \boldsymbol{z}}=e^{e^{z\log y}\log x}\cdot e^{z\log y}\log x\cdot\log y=x^{y^z}y^z\log x\cdot\log y.$$

(2)

$$\frac{\partial f}{\partial \boldsymbol{x}} = \sec^2 \frac{xy}{z^2} \cdot \frac{y}{z^2}, \quad \frac{\partial f}{\partial \boldsymbol{y}} = \sec^2 \frac{xy}{z^2} \cdot \frac{x}{z^2};$$
$$\frac{\partial f}{\partial \boldsymbol{z}} = \sec^2 \frac{xy}{z^2} \cdot -2xyz^{-3}.$$

Problem 7

假设 f 为可微函数，用 f 的偏导数表示下列函数的偏导数：

$$(1)\ u(x,y,z) = f(x, xy, xyz); \quad (2)\ u(x,y) = f(\log x + \frac{1}{y}).$$

Proof. 设 $f = f(u,v,w)$，它对应的偏导数为 $\frac{\partial f}{\partial \boldsymbol{u}}, \frac{\partial f}{\partial \boldsymbol{v}}, \frac{\partial f}{\partial \boldsymbol{w}}$.

(1) 设 $p(x,y,z) = (x, xy, xyz)$，则

$$\begin{aligned}
\frac{\partial u}{\partial \boldsymbol{x}}|_{(x,y,z)} &= \mathrm{d}(f\circ p)|_{(x,y,z)}(\boldsymbol{x}) = \mathrm{d}f|_{(x,xy,xyz)}(1, y, yz) \\
&= \frac{\partial f}{\partial \boldsymbol{u}}|_{(x,xy,xyz)} + y\frac{\partial f}{\partial \boldsymbol{v}}|_{(x,xy,xyz)} + yz\frac{\partial f}{\partial \boldsymbol{w}}|_{(x,xy,xyz)}. \\
\frac{\partial u}{\partial \boldsymbol{y}}|_{(x,y,z)} &= \mathrm{d}(f\circ p)|_{(x,y,z)}(\boldsymbol{y}) = \mathrm{d}f|_{(x,xy,xyz)}(0, x, xz) \\
&= x\frac{\partial f}{\partial \boldsymbol{v}}|_{(x,xy,xyz)} + xz\frac{\partial f}{\partial \boldsymbol{w}}|_{(x,xy,xyz)}. \\
\frac{\partial u}{\partial \boldsymbol{z}}|_{(x,y,z)} &= \mathrm{d}(f\circ p)|_{(x,y,z)}(\boldsymbol{z}) = \mathrm{d}f|_{(x,xy,xyz)}(0, 0, xy) \\
&= xy\frac{\partial f}{\partial \boldsymbol{w}}|_{(x,xy,xyz)}.
\end{aligned}$$

(2) 设 $p(x,y) = \log x + \frac{1}{y}$，则

$$\frac{\partial u}{\partial \boldsymbol{x}}|_{(x,y)} = \mathrm{d}(f\circ p)|_{(x,y)}(\boldsymbol{x}) = \mathrm{d}f|_{\log x + y^{-1}}(x^{-1}) = x^{-1}f'(\log x + y^{-1}).$$
$$\frac{\partial u}{\partial \boldsymbol{y}}|_{(x,y)} = \mathrm{d}(f\circ p)|_{(x,y)}(\boldsymbol{y}) = \mathrm{d}f|_{\log x + y^{-1}}(-y^{-2}) = -y^{-2}f'(\log x + y^{-1}).$$

Problem 8

求如下坐标变换的 Jacobi 矩阵 $J(f)$ 并计算 $\det J(f)$：

$$(1)\ f: \mathbb{R}^2 \to \mathbb{R}^2, \quad (u,v) \mapsto (e^u \cos v, e^u \sin v);$$
$$(2)\ f: \mathbb{R}^2\backslash(0,0) \to \mathbb{R}^2\backslash(0,0), \quad (u,v) \mapsto (\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}).$$

Proof. (1)

$$\frac{\partial f}{\partial \boldsymbol{u}} = (e^u \cos v, e^u \sin v), \quad \frac{\partial f}{\partial \boldsymbol{v}} = (-e^u \sin v, e^u \cos v).$$

所以它确定的 Jacobi 矩阵就是

$$J(f)=\begin{pmatrix} e^u\cos v & -e^u\sin v\\ e^u\sin v & e^u\cos v\end{pmatrix},\quad \det J(f)=e^{2u}(\cos^2 v+\sin^2 v)=e^{2u}.$$

(2)

$$\frac{\partial f}{\partial \boldsymbol{u}}=\Big(\frac{v^2-u^2}{(u^2+v^2)^2},\frac{-2uv}{(u^2+v^2)^2}\Big),\quad \frac{\partial f}{\partial \boldsymbol{v}}=\Big(\frac{-2uv}{(u^2+v^2)^2},\frac{u^2-v^2}{(u^2+v^2)^2}\Big).$$

所以它确定的 Jacobi 矩阵就是

$$J(f)=\begin{pmatrix} \frac{v^2-u^2}{(u^2+v^2)^2} & \frac{-2uv}{(u^2+v^2)^2}\\ \frac{-2uv}{(u^2+v^2)^2} & \frac{u^2-v^2}{(u^2+v^2)^2}\end{pmatrix},\quad \det J(f)=-\frac{(u^2-v^2)^2}{(u^2+v^2)^4}-\frac{4u^2v^2}{(u^2+v^2)^4}$$
$$=-\frac{1}{(u^2+v^2)^2}.$$

Problem 9

我们考虑 $\mathbb{R}^3$ 上的柱面坐标系：$x=r\sin\theta, y=r\sin\theta, z=z$. 这个坐标变换用映射来写就是

$$\Phi:\mathbb{R}_{>0}\times(0,2\pi)\times\mathbb{R}\to\mathbb{R}^3,\quad (r,\theta,z)\mapsto(r\cos\theta,r\sin\theta,z).$$

设 f 是 $\mathbb{R}^3$ 上的二次可微函数. 当 $(x,y)\neq(0,0)$ 时，试通过计算来证明：

$$\frac{\partial f}{\partial \boldsymbol{r}}=\cos\theta\frac{\partial f}{\partial \boldsymbol{x}}+\sin\theta\frac{\partial f}{\partial \boldsymbol{y}},\quad \frac{\partial f}{\partial \boldsymbol{\theta}}=-r\sin\theta\frac{\partial f}{\partial \boldsymbol{x}}+r\cos\theta\frac{\partial f}{\partial \boldsymbol{y}},$$
$$\left(\frac{\partial f}{\partial \boldsymbol{x}}\right)^2+\left(\frac{\partial f}{\partial \boldsymbol{y}}\right)^2+\left(\frac{\partial f}{\partial \boldsymbol{z}}\right)^2=\left(\frac{\partial f}{\partial \boldsymbol{r}}\right)^2+\frac{1}{r^2}\left(\frac{\partial f}{\partial \boldsymbol{\theta}}\right)^2+\left(\frac{\partial f}{\partial \boldsymbol{z}}\right)^2,$$
$$\frac{\partial^2 f}{\partial \boldsymbol{x}^2}+\frac{\partial^2 f}{\partial \boldsymbol{y}^2}+\frac{\partial^2 f}{\partial \boldsymbol{z}^2}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial \boldsymbol{r}}\right)+\frac{1}{r^2}\frac{\partial^2 f}{\partial \boldsymbol{\theta}^2}+\frac{\partial^2 f}{\partial \boldsymbol{z}^2}.$$

Proof. 计算一阶偏导：

$$\frac{\partial f}{\partial \boldsymbol{r}}|_{(r,\theta,z)}=\mathrm{d}f|_{(x,y,z)}(\cos\theta,\sin\theta,0)=\cos\theta\frac{\partial f}{\partial \boldsymbol{x}}|_{(x,y,z)}+\sin\theta\frac{\partial f}{\partial \boldsymbol{y}}|_{(x,y,z)}.$$
$$\frac{\partial f}{\partial \boldsymbol{\theta}}|_{(r,\theta,z)}=\mathrm{d}f|_{(x,y,z)}(-r\sin\theta,r\cos\theta,0)=-r\sin\theta\frac{\partial f}{\partial \boldsymbol{x}}|_{(x,y,z)}+r\cos\theta\frac{\partial f}{\partial \boldsymbol{y}}|_{(x,y,z)}.$$
$$\frac{\partial f}{\partial \boldsymbol{z}}|_{(r,\theta,z)}=\mathrm{d}f|_{(x,y,z)}(0,0,1)=\frac{\partial f}{\partial \boldsymbol{z}}|_{(x,y,z)}.$$

$$\left(\frac{\partial f}{\partial \boldsymbol{r}}\right)^2+\frac{1}{r^2}\left(\frac{\partial f}{\partial \boldsymbol{\theta}}\right)^2+\left(\frac{\partial f}{\partial \boldsymbol{z}}\right)^2=(\cos^2\theta+\sin^2\theta)\left(\frac{\partial f}{\partial \boldsymbol{x}}\right)^2$$
$$+(\sin^2\theta+\cos^2\theta)\left(\frac{\partial f}{\partial \boldsymbol{y}}\right)^2+\left(\frac{\partial f}{\partial \boldsymbol{z}}\right)^2=\left(\frac{\partial f}{\partial \boldsymbol{x}}\right)^2+\left(\frac{\partial f}{\partial \boldsymbol{y}}\right)^2+\left(\frac{\partial f}{\partial \boldsymbol{z}}\right)^2.$$

计算二阶偏导：

$$\begin{aligned}\frac{1}{r}\frac{\partial}{\partial \boldsymbol{r}}\left(r\frac{\partial f}{\partial \boldsymbol{r}}\right) &= \frac{1}{r}\left(\cos\theta\frac{\partial f}{\partial \boldsymbol{x}} + r\cos\theta(\cos\theta\frac{\partial^2 f}{\partial \boldsymbol{x^2}} + \sin\theta\frac{\partial^2 f}{\partial \boldsymbol{x}\partial \boldsymbol{y}})\right)\\ &+\frac{1}{r}\left(\sin\theta\frac{\partial f}{\partial \boldsymbol{y}} + r\sin\theta(\cos\theta\frac{\partial^2 f}{\partial \boldsymbol{x}\partial \boldsymbol{y}} + \sin\theta\frac{\partial^2 f}{\partial \boldsymbol{y^2}})\right).\\ \frac{1}{r^2}\frac{\partial^2 f}{\partial \boldsymbol{\theta^2}} &= \frac{1}{r^2}\left(-r\cos\theta\frac{\partial f}{\partial \boldsymbol{x}} - r\sin\theta(-r\sin\theta\frac{\partial^2 f}{\partial \boldsymbol{x^2}} + r\cos\theta\frac{\partial^2 f}{\partial \boldsymbol{x}\partial \boldsymbol{y}})\right)\\ &+\frac{1}{r^2}\left(-r\sin\theta\frac{\partial f}{\partial \boldsymbol{y}} + r\cos\theta(-r\sin\theta\frac{\partial^2 f}{\partial \boldsymbol{x}\partial \boldsymbol{y}} + r\cos\theta\frac{\partial^2 f}{\partial \boldsymbol{y^2}})\right)\end{aligned}$$

上述两者相加即得

$$\frac{1}{r}\frac{\partial}{\partial \boldsymbol{r}}\left(r\frac{\partial f}{\partial \boldsymbol{r}}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \boldsymbol{\theta^2}} = \frac{\partial^2 f}{\partial \boldsymbol{x^2}} + \frac{\partial^2 f}{\partial \boldsymbol{y^2}}.$$

再加上 $\boldsymbol{z}$ 的贡献即得结论.

Problem 10

我们考虑 $\mathbb{R}^3$ 上的球坐标系

$$(x, y, z) = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta).$$

证明：

$$\begin{aligned}\left(\frac{\partial f}{\partial \boldsymbol{x}}\right)^2 + \left(\frac{\partial f}{\partial \boldsymbol{y}}\right)^2 + \left(\frac{\partial f}{\partial \boldsymbol{z}}\right)^2 &= \left(\frac{\partial f}{\partial \boldsymbol{r}}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \boldsymbol{\theta}}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial f}{\partial \boldsymbol{\varphi}}\right)^2,\\ \frac{\partial^2 f}{\partial \boldsymbol{x^2}} + \frac{\partial^2 f}{\partial \boldsymbol{y^2}} + \frac{\partial^2 f}{\partial \boldsymbol{z^2}} &= \frac{1}{r^2}\frac{\partial}{\partial \boldsymbol{r}}\left(r^2\frac{\partial f}{\partial \boldsymbol{r}}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \boldsymbol{\theta}}\left(\sin\theta\frac{\partial f}{\partial \boldsymbol{\theta}}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \boldsymbol{\varphi^2}}.\end{aligned}$$

Proof. 算：

$$\begin{aligned}
\frac{\partial f}{\partial \boldsymbol{r}} &= \sin\theta\cos\varphi\frac{\partial f}{\partial \boldsymbol{x}} + \sin\theta\sin\varphi\frac{\partial f}{\partial \boldsymbol{y}} + \cos\theta\frac{\partial f}{\partial \boldsymbol{z}};\\
\frac{\partial f}{\partial \boldsymbol{\theta}} &= r\cos\theta\cos\varphi\frac{\partial f}{\partial \boldsymbol{x}} + r\cos\theta\sin\varphi\frac{\partial f}{\partial \boldsymbol{y}} - r\sin\theta\frac{\partial f}{\partial \boldsymbol{z}};\\
\frac{\partial f}{\partial \boldsymbol{\varphi}} &= -r\sin\theta\sin\varphi\frac{\partial f}{\partial \boldsymbol{x}} + r\sin\theta\cos\varphi\frac{\partial f}{\partial \boldsymbol{y}}.\\
\frac{\partial}{\partial \boldsymbol{r}}\left(r^2\frac{\partial f}{\partial \boldsymbol{r}}\right) &= 2r\sin\theta\cos\varphi\frac{\partial f}{\partial \boldsymbol{x}} + 2r\sin\theta\sin\varphi\frac{\partial f}{\partial \boldsymbol{y}} + 2r\cos\theta\frac{\partial f}{\partial \boldsymbol{z}}\\
&\quad + r^2\sin\theta\cos\varphi(\sin\theta\cos\varphi\frac{\partial^2 f}{\partial \boldsymbol{x^2}} + \sin\theta\sin\varphi\frac{\partial^2 f}{\partial \boldsymbol{x}\partial \boldsymbol{y}} + \cos\theta\frac{\partial^2 f}{\partial \boldsymbol{x}\partial \boldsymbol{z}})\\
&\quad + r^2\sin\theta\sin\varphi(\sin\theta\cos\varphi\frac{\partial^2 f}{\partial \boldsymbol{x}\partial \boldsymbol{y}} + \sin\theta\sin\varphi\frac{\partial^2 f}{\partial \boldsymbol{y^2}} + \cos\theta\frac{\partial^2 f}{\partial \boldsymbol{y}\partial \boldsymbol{z}})\\
&\quad + r^2\cos\theta(\sin\theta\cos\varphi\frac{\partial^2 f}{\partial \boldsymbol{x}\partial \boldsymbol{z}} + \sin\theta\sin\varphi\frac{\partial^2 f}{\partial \boldsymbol{y}\partial \boldsymbol{z}} + \cos\theta\frac{\partial^2 f}{\partial \boldsymbol{z^2}}).\\
\frac{\partial}{\partial \boldsymbol{\theta}}\left(\sin\theta\frac{\partial f}{\partial \boldsymbol{\theta}}\right) &= r(\cos^2\theta - \sin^2\theta)\cos\varphi\frac{\partial f}{\partial \boldsymbol{x}} + r(\cos^2\theta - \sin^2\theta)\sin\varphi\frac{\partial f}{\partial \boldsymbol{y}} - 2r\sin\theta\cos\theta\frac{\partial f}{\partial \boldsymbol{z}}\\
&\quad + r\cos\theta\sin\theta\cos\varphi(r\cos\theta\cos\varphi\frac{\partial^2 f}{\partial \boldsymbol{x^2}} + r\cos\theta\sin\varphi\frac{\partial^2 f}{\partial \boldsymbol{x}\partial \boldsymbol{y}} - r\sin\theta\frac{\partial^2 f}{\partial \boldsymbol{x}\partial \boldsymbol{z}})\\
&\quad + r\cos\theta\sin\theta\sin\varphi(r\cos\theta\cos\varphi\frac{\partial^2 f}{\partial \boldsymbol{x}\partial \boldsymbol{y}} + r\cos\theta\sin\varphi\frac{\partial^2 f}{\partial \boldsymbol{y^2}} - r\sin\theta\frac{\partial^2 f}{\partial \boldsymbol{y}\partial \boldsymbol{z}})\\
&\quad - r\sin^2\theta(r\cos\theta\cos\varphi\frac{\partial^2 f}{\partial \boldsymbol{x}\partial \boldsymbol{z}} + r\cos\theta\sin\varphi\frac{\partial^2 f}{\partial \boldsymbol{y}\partial \boldsymbol{z}} - r\sin\theta\frac{\partial^2 f}{\partial \boldsymbol{z^2}}).\\
\frac{\partial^2 f}{\partial \boldsymbol{\varphi^2}} &= -r\sin\theta\cos\varphi\frac{\partial f}{\partial \boldsymbol{x}} - r\sin\theta\sin\varphi\frac{\partial f}{\partial \boldsymbol{y}}\\
&\quad - r\sin\theta\sin\varphi(-r\sin\theta\sin\varphi\frac{\partial^2 f}{\partial \boldsymbol{x^2}} + r\sin\theta\cos\varphi\frac{\partial^2 f}{\partial \boldsymbol{x}\partial \boldsymbol{y}})\\
&\quad + r\sin\theta\cos\varphi(-r\sin\theta\sin\varphi\frac{\partial^2 f}{\partial \boldsymbol{x}\partial \boldsymbol{y}} + r\sin\theta\cos\varphi\frac{\partial^2 f}{\partial \boldsymbol{y^2}}).
\end{aligned}$$

所以

$$\begin{aligned}
&\left(\frac{\partial f}{\partial \boldsymbol{r}}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \boldsymbol{\theta}}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial f}{\partial \boldsymbol{\varphi}}\right)^2\\
=&(\sin\theta\cos\varphi\frac{\partial f}{\partial \boldsymbol{x}} + \sin\theta\sin\varphi\frac{\partial f}{\partial \boldsymbol{y}} + \cos\theta\frac{\partial f}{\partial \boldsymbol{z}})^2 + (\cos\theta\cos\varphi\frac{\partial f}{\partial \boldsymbol{x}} + \cos\theta\sin\varphi\frac{\partial f}{\partial \boldsymbol{y}}\\
&- \sin\theta\frac{\partial f}{\partial \boldsymbol{z}})^2 + (\sin\varphi\frac{\partial f}{\partial \boldsymbol{x}} - \cos\varphi\frac{\partial f}{\partial \boldsymbol{y}})^2\\
=&\sin^2\theta(\cos\varphi\frac{\partial f}{\partial \boldsymbol{x}} + \sin\varphi\frac{\partial f}{\partial \boldsymbol{y}})^2 + \cos^2\theta(\cos\varphi\frac{\partial f}{\partial \boldsymbol{x}} + \sin\varphi\frac{\partial f}{\partial \boldsymbol{y}})^2\\
&+ (\sin\varphi\frac{\partial f}{\partial \boldsymbol{x}} - \cos\varphi\frac{\partial f}{\partial \boldsymbol{y}})^2 + \left(\frac{\partial f}{\partial \boldsymbol{z}}\right)^2\\
=&\left(\frac{\partial f}{\partial \boldsymbol{x}}\right)^2 + \left(\frac{\partial f}{\partial \boldsymbol{y}}\right)^2 + \left(\frac{\partial f}{\partial \boldsymbol{z}}\right)^2.
\end{aligned}$$

$$
\begin{aligned}
&\frac{1}{r^2}\frac{\partial}{\partial \boldsymbol{r}}\left(r^2\frac{\partial f}{\partial \boldsymbol{r}}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \boldsymbol{\theta}}\left(\sin\theta\frac{\partial f}{\partial \boldsymbol{\theta}}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \boldsymbol{\varphi}^2}\\
=&(\frac{2\sin\theta\cos\varphi}{r}+\frac{(1-2\sin^2\theta)\cos\varphi}{r\sin\theta}-\frac{\cos\varphi}{r\sin\theta})\frac{\partial f}{\partial \boldsymbol{x}}\\
&+(\frac{2\sin\theta\sin\varphi}{r}+\frac{(1-2\sin^2\theta)\sin\varphi}{r\sin\theta}-\frac{\sin\varphi}{r\sin\theta})\frac{\partial f}{\partial \boldsymbol{y}}\\
&+(\frac{2\cos\theta}{r}-\frac{2\cos\theta}{r})\frac{\partial f}{\partial \boldsymbol{z}}\\
&+(\sin^2\theta\cos^2\varphi+\cos^2\theta\cos^2\varphi+\sin^2\varphi)\frac{\partial^2 f}{\partial \boldsymbol{x}^2}\\
&+(\sin^2\theta\sin\varphi\cos\varphi+\cos^2\theta\sin\varphi\cos\varphi-\sin\varphi\cos\varphi)\frac{\partial^2 f}{\partial \boldsymbol{x}\partial \boldsymbol{y}}\\
&+(\sin\theta\cos\theta\cos\varphi-\cos\theta\sin\theta\cos\varphi)\frac{\partial^2 f}{\partial \boldsymbol{x}\partial \boldsymbol{z}}\\
&+(\sin^2\theta\sin\varphi\cos\varphi+\cos^2\theta\sin\varphi\cos\varphi+\sin\varphi\cos\varphi)\frac{\partial^2 f}{\partial \boldsymbol{x}\partial \boldsymbol{y}}\\
&+(\sin^2\theta\sin^2\varphi+\cos^2\theta\sin^2\varphi+\cos^2\varphi)\frac{\partial^2 f}{\partial \boldsymbol{y}^2}\\
&+(\sin\theta\cos\theta\sin\varphi-\cos\theta\sin\theta\sin\varphi)\frac{\partial^2 f}{\partial \boldsymbol{y}\partial \boldsymbol{z}}\\
&+(\cos\theta\sin\theta\cos\varphi-\sin\theta\cos\theta\cos\varphi)\frac{\partial^2 f}{\partial \boldsymbol{x}\partial \boldsymbol{z}}\\
&+(\cos\theta\sin\theta\sin\varphi-\sin\theta\cos\theta\sin\varphi)\frac{\partial^2 f}{\partial \boldsymbol{y}\partial \boldsymbol{z}}\\
&+(\cos^2\theta+\sin^2\theta)\frac{\partial^2 f}{\partial \boldsymbol{z}^2}\\
=&\frac{\partial^2 f}{\partial \boldsymbol{x}^2}+\frac{\partial^2 f}{\partial \boldsymbol{y}^2}+\frac{\partial^2 f}{\partial \boldsymbol{z}^2}.
\end{aligned}
$$

Problem 11

对于常数 $k\in\mathbb{R}$，我们称 $\mathbb{R}^n$ 上的函数是 k 次齐次的，如果对任意的 $\lambda>0, x\neq 0$，有 $f(\lambda x)=\lambda^k f(x)$. 证明线性函数是齐次函数.

Proof. 线性性蕴含 $f(\lambda x)=\lambda f(x)$.

Problem 12

假设 f 是可微的，证明：f 是 k 次齐次函数当且仅当它满足 Euler 等式

$$\sum_{i=1}^n x_i\frac{\partial f}{\partial \boldsymbol{x_i}}=kf.$$

Proof. 如果 f 是 k 次齐次函数，对 $\boldsymbol{x}=(x_1,\dots,x_n)\in\mathbb{R}^n$，我们有

$$f(\lambda\boldsymbol{x})=\lambda^k f(\boldsymbol{x}).$$

对 λ 求导并在 $\boldsymbol{x}$ 处取值即得

$$\frac{\mathrm{d}f}{\mathrm{d}\lambda}|_{\lambda=1} = kf(\boldsymbol{x}).$$

而当 $\boldsymbol{x} \neq 0$ 时，

$$\frac{\mathrm{d}f}{\mathrm{d}\lambda}|_{\lambda=1} = \lim_{t\to 0}\frac{|f(\boldsymbol{x}+t\boldsymbol{x})-f(\boldsymbol{x})|}{t} = \mathrm{d}f|_{\boldsymbol{x}}(\boldsymbol{x}) = \sum_{i=1}^{n} x_i\frac{\partial f}{\partial \boldsymbol{x_i}}.$$

当 $\boldsymbol{x}=0$ 时，需证式退化为 $kf(\mathbf{0})=\mathbf{0}$，显然成立.

另一方面，若已知 Euler 等式，则对任意 $\boldsymbol{x}\neq 0$ 均有

$$\mathrm{d}f|_{\boldsymbol{x}}(\boldsymbol{x}) = kf(\boldsymbol{x}).$$

则固定任何一个方向，考虑该方向上的单位向量记为 $\boldsymbol{e}$，那么将 $\lambda\boldsymbol{e}$ 代入上式可得

$$kf(\lambda\boldsymbol{e}) = \mathrm{d}f|_{\lambda\boldsymbol{e}}(\lambda\boldsymbol{e}) = \lambda\,\mathrm{d}f|_{\lambda\boldsymbol{e}}(\boldsymbol{e}) = \lambda\frac{\partial f}{\partial \boldsymbol{e}}(\lambda\boldsymbol{e}).$$

考虑一元函数 $g(\lambda)=f(\lambda\boldsymbol{e})$，则 $kg(\lambda)=\lambda g'(\lambda)$，即 $(\lambda^{-k}g)'=0$，所以 $g=C\cdot\lambda^k=g(1)\cdot\lambda^k$. 这就证明了 $f(\lambda\boldsymbol{e})=\lambda^k f(\boldsymbol{e})$，从而对任意向量结论均成立.

Problem 13

假设可微函数 f 是 k 阶线性函数. 证明对任意 $\boldsymbol{v}\in\mathbb{R}^n$，$\nabla_{\boldsymbol{v}}f$ 是 $k-1$ 次齐次函数.

Proof. 对表达式关于方向 $\boldsymbol{v}$ 求偏导：

$$\begin{aligned}\nabla_{\boldsymbol{v}}f(\lambda\boldsymbol{x}) = \lim_{t\to 0}\frac{f(\lambda\boldsymbol{x}+t\boldsymbol{v})-f(\lambda\boldsymbol{x})}{t} &= \lim_{t\to 0}\frac{\lambda^k(f(\boldsymbol{x}+t\lambda^{-1}\boldsymbol{v})-f(\boldsymbol{x}))}{t}\\ &= \lambda^{k-1}\nabla_{\boldsymbol{v}}f(\boldsymbol{x}).\end{aligned}$$

即得结论.

Problem 14

令 $f(x_1,\dots,x_n)=\det\begin{pmatrix}1 & \dots & 1\\ x_1 & \dots & x_n\\ \vdots & \ddots & \vdots\\ x_1^{n-1} & \dots & x_n^{n-1}\end{pmatrix}$. 证明：

$$\sum_{i=1}^{n} x_i\frac{\partial f}{\partial \boldsymbol{x_i}} = \frac{n(n-1)}{2}f,\quad \sum_{i=1}^{n}\frac{\partial f}{\partial \boldsymbol{x_i}} = 0.$$

Proof. 根据行列式的性质：

$$f(\lambda x_1,\lambda x_2,\dots,\lambda x_n) = \det\begin{pmatrix}1 & \dots & 1\\ \lambda x_1 & \dots & \lambda x_n\\ \vdots & \ddots & \vdots\\ \lambda^{n-1}x_1^{n-1} & \dots & \lambda^{n-1}x_n^{n-1}\end{pmatrix} = \lambda^{\frac{n(n-1)}{2}}f(x_1,\dots,x_n).$$

根据 Euler 公式就可得到前一个等式. 后一个等式来源于 Vandemonde 行列式

$$\det\begin{pmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_n \\ \vdots & \ddots & \vdots \\ x_1^{n-1} & \dots & x_n^{n-1} \end{pmatrix} = \prod_{i<j}(x_j - x_i).$$

所以实际上有

$$\sum_{i=1}^{n}\frac{\partial f}{\partial \boldsymbol{x_i}} = \sum_{i=1}^{n}\left(\sum_{p<q}\chi_{i,p,q}\cdot\prod_{\substack{i<j\\(i,j)\neq(p,q)}}(x_j - x_i)\right).$$

其中

$$\chi_{i,p,q} = \begin{cases} 1, & i = q; \\ -1, & i = p; \\ 0, & \text{else}. \end{cases}$$

因此交换求和后可得到

$$\sum_{i=1}^{n}\frac{\partial f}{\partial \boldsymbol{x_i}} = \sum_{p<q}\prod_{\substack{i<j\\(i,j)\neq(p,q)}}(x_j - x_i)\cdot\left(\sum_{i=1}^{n}\chi_{i,p,q}\right) = 0.$$