A, B size n

$\Downarrow$

C size $2(n-1)+1 = 2n-2+1 = 2n-1$

$P = \{ x_0, x_1, \ldots, x_{2(n-1)\cancel{+1?}} \}$

$\{ C(x_0), C(x_1), \ldots, C(X)_{2(n-1)\cancel{+1?}} \} \Rightarrow$ Determines uniquely C.

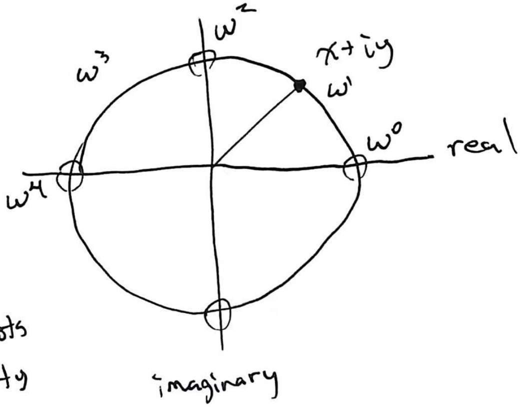
$C(x_i) = A(x_i)\, B(x_i)$

FFT-Based Multiplication

$$\begin{tikzcd} A, B \text{ in coef. form} \arrow[r, dashed] \arrow[d] & C = A \cdot B \text{ in coef. form} \\ A, B \text{ in pt.-val form} \arrow[r, "O(n)"] & C \text{ in pt.-val form} \arrow[u, "DFT^{-1}"'] \end{tikzcd}$$

	$A(x)$	$B(x)$	C
y_0	$A(x_0)$	$B(x_0)$	$C(x_0) = A(x_0)B(x_0)$
$\vdots$		$\vdots$	
x_{2n-2}	$A(x_{2n-2})$		$C(x_{2n-2}) = A(x_{2n-2})B(x_{2n-2})$

DFT "Discrete Fourier Transform"



8th Roots of unity

$$w^2 = w^1 \cdot w^1$$
$$w^3 = w^1 \cdot w^2$$

In general, "n^{th} roots of unity"
are the solutions to

$$w^n = 1$$

$w = 1$ is a solution

$$\omega_n^k = e^{i\frac{2\pi}{n}\cdot k}$$

<u>Properties</u>

1) $\omega_{dn}^{dk} = \omega_n^k$

(n is a power of 2) 2) $\omega_n^{n/2} = \omega_2^1 = -1$

3) <u>Halving Lemma</u> if $n > 0$ (even)

then the squares of the n^{th} roots of unity are the $(n/2)^{th}$ roots of unity.

$0 \le k < n/2$ $\quad (\omega_n^k)^2 = \omega_n^k \cdot \omega_n^k = \omega_n^{2k} = (\omega_n^2)^k$

$$= \omega_{2(n/2)}^{2k}$$

$$= \omega_{n/2}^k$$

$$(\omega_n^{k+n/2})^2 = \omega_n^{(2k+n)} = \omega_n^{2k} \cdot \underbrace{\omega_n^n}_{=1} = \omega_n^{2k} = \omega_{n/2}^k.$$

4) Summation Lemma

Assume $n \geq 1$ $k =$ non-zero integer not divisible by n.

$$\sum_{j=0}^{n-1} (\omega_n^k)^j = 0$$

pf

$$\sum_{j=0}^{n-1} (\omega_n^k)^j = \frac{(\omega_n^k)^n - 1}{\underset{\neq 0}{\omega_n^k - 1}} = 0$$

DFT: We want to evaluate $A(x)$ $x = \omega_n^0, \omega_n^1, \omega_n^2, \ldots, \omega_n^{n-1}$

Let $y = A(\omega_n^k)$.

$$\vec{y} = DFT_n(\vec{a})$$

$$\vec{a} = (a_0, a_1, \ldots, a_{n-1})$$

$$\vec{y} = (y_0, a_1, \ldots, y_{n-1})$$

$$A(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1}$$

$$= a_0 + a_2 x^2 + a_4 x^4 + \ldots$$

$$+ a_1 x + a_3 x^3 + \ldots$$

$$\boxed{= A^{[0]}(x^2) + x \cdot A^{[1]}(x^2)}$$

where

$$A^{[0]} = a_0 + a_2 x + a_4 x^2 + \ldots$$

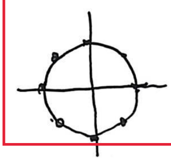
$$A^{[1]} = a_1 + a_3 x + a_5 x^2 + \ldots$$

A, B $\qquad\qquad\qquad\qquad$ $C = A \cdot B$

$(a_0, a_1, a_2, \ldots, a_{n-1})$ $\qquad$ $(c_0, c_1, \ldots, c_{2(n-1)})$

$(b_0, b_1, b_2, \ldots, b_{n-1})$

$\downarrow$ FFT



pt. value form $\xrightarrow{\text{pointwise multiplication}}$ C pt. val form

(2n points)

$$\omega_n^k = k^{th} \; n^{th} \text{ root of unity} = e^{i \frac{2\pi}{n} k}$$

$$\omega_n^0 = 1$$

<u>Summation Lemma</u>

$n \geq 1$, k = non-zero integer not divisible by n.

$$\sum_{j=0}^{n-1} (\omega_n^k)^j = 0$$

$$\text{DFT} \quad \text{"discrete fourier transform"}$$

$$y_k = A(\omega_n^k)$$

$$\vec{y} = \text{DFT}_n(\vec{a})$$

$$A(x) = A^{[0]}(x^2) + x A^{[1]}(x^2)$$

RECURSIVE - DFT(a)

Let n = length(a)

if $(n_i = 1)$ return a

$$\omega_n = e^{i\frac{2\pi}{n}}$$

$$\omega = 1 \quad (\text{current root})$$

$$a^{[0]} = (a_0, a_2, a_4, \ldots, a_{n-2})$$

$$a^{[1]} = (a_1, a_3, a_5, \ldots, a_{n-1})$$

$$y^{[0]} = \text{RECURSIVE-DFT}(a^{[0]})$$

$$y^{[1]} = \text{RECURSIVE-DFT}(a^{[1]})$$

for $k = 0$ to $\frac{n}{2} - 1$ {

$$y_k = y_k^{[0]} + \omega \cdot y_k^{[1]}$$

$$y_{k+\frac{n}{2}} = y_k^{[0]} \cancel{+} - \omega \cdot y_k^{[1]}$$

}

return y

<u>running time</u>

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n)$$

$$= \Theta(n \lg(n)) = \tilde{\Theta}(n)$$

"Inverse DFT"

We have

$$y_k = \sum_{j=0}^{n-1} a_j \cdot (w_n^k)^j$$

$$\vec{y} = V_n \cdot \vec{a}$$

$$V_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w_n & w_n^2 & \cdots & w_n^{n-1} \\ 1 & w_n^2 & w_n^2 & \cdots & (w_n^2)^{n-1} \\ \vdots & & & & \\ 1 & w_n^{n-1} & (w_n^{n-1})^2 & \cdots & (w_n^{n-1})^{n-1} \end{bmatrix}$$

$$\begin{pmatrix} y_0 \\ \\ \\ \end{pmatrix}^{\checkmark} = \begin{pmatrix} \quad & \quad \\ & \\ & \end{pmatrix} \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

$$V_n^{-1} \vec{y} = V_n^{-1} V_n \cdot \vec{a}$$

$$V_n^{-1} \vec{y} = \vec{a}$$

V_n is invertible

$$\left[V_n^{-1}\right]_{jk} = \frac{1}{n}\,\omega_n^{-kj}$$

check $V_n^{-1} V_n = I_n$

$$\left[V_n^{-1} \cdot V_n\right]_{ij} = \sum_{k=0}^{n-1} \frac{\omega_n^{-ik}}{n}\,\omega_n^{kj}$$

$$= \frac{1}{n} \sum_{k=0}^{n-1} \omega_n^{k(j-i)}$$

$$= \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

✓

$$a_j = \frac{1}{n} \sum_{k=0}^{n-1} y_k \cdot (\omega_n^{-1})^{kj}$$

Modify RECURSIVE-DFT as follows:

- switch $a_k \longleftrightarrow y_k$ variables
- replace ω_n with ω_n^{-1}
- divide a_k's by n.

$\Downarrow$ DFT_n^{-1}

Selection Problem

input: * A array (unsorted, distinct values) length $= n$.

* k $(0 \leq k \leq n-1)$

output: $A_{sorted}[k] = ?$