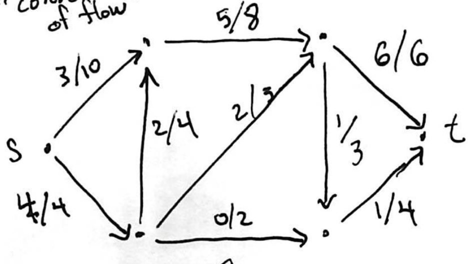
Network Flows

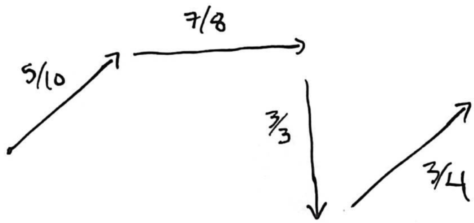
"conservation of flow"



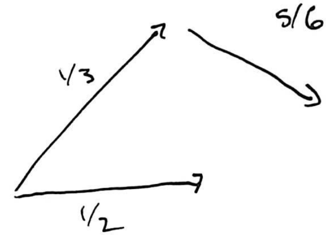
edge's have capacity

Max Flow

Determine a valid flow that maximizes total flow from s to t.



$\Rightarrow$



$\Rightarrow$

Flow Network

$G = (V, E)$ directed graph

$s, t \in V$ $\quad$ s: "source"

t: "sink"

Capacities $c(u,v) = $ capacity of edge $(u,v) \in E$

≥ 0

if $(u,v) \notin E$, then $c(u,v) = 0$.

$\forall v \in V$, $\exists$ a path $p: s \rightsquigarrow v \rightsquigarrow t$

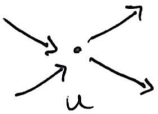
Flow: $f: V \times V \to \mathbb{R}$

Must satisfy: (1) $f(u,v) \leq c(u,v)$ for all $u, v \in V$ (capacity constraint)

(2) $f(u,v) = -f(v,u)$ "skew-symmetry"

(3) Conservation of flow

$\forall u \in V \setminus \{s,t\}$, $\sum_{v \in V} f(u,v) = 0$



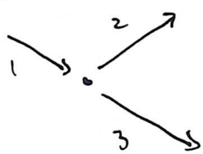
Max Flow Problem

Maximize flow value

flow Value:

$|f| = \sum_{v \in V} f(s,v)$

e.g.



$|f| = 4$

Implicit Summation Notation

Let $X, Y \subseteq V$.

$$f(X, Y) = \sum_{x \in X} \sum_{y \in Y} f(x, y)$$

Suppose $u \in V \setminus \{s, t\}$. Then $f(\{u\}, V) = 0$.

$$|f| = f(\{s\}, V) = f(V, \{t\}).$$

<u>Lemma</u>: Let $G = (V, E, f)$ be a flow network.

Then: 1) $\forall X \subseteq V$, $f(X, X) = 0$

$$f(X, X) = \sum_{x \in X} \sum_{y \in Y} f(x, y)$$

Cancel out

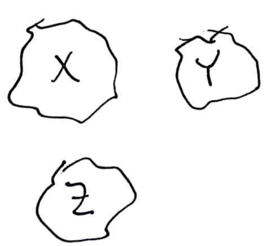
$$a \rightleftarrows b$$

2) $\forall X, Y \subseteq V$, $f(X, Y) = -f(Y, X)$

3) $X, Y, Z \subseteq V$, $X \cap Y = \emptyset$

$$f(X \cup Y, Z) = f(X, Z) + f(Y, Z)$$

$$f(Z, X \cup Y) = f(Z, X) + f(Z, Y)$$



$$|f| = f(\{s\}, V)$$

$$= f(V, V) - f(V \setminus \{s\}, V) \quad \text{by } 3.$$

$$= -f(V \setminus \{s\}, V) \quad \text{by } 1.$$

$$= f(V, V \setminus \{s\}) \quad \text{by } 2.$$

$$= f(V, \{t\}) + \underbrace{f(V, V \setminus \{s,t\})}_{=0} \quad \text{by } 3.$$

<u>Defn</u> Let $(u,v) \in V \times V$, we will define the residual capacity of (u,v) as $c_f(u,v) = c(u,v) - f(u,v)$

$c_f(u,v) = 4$

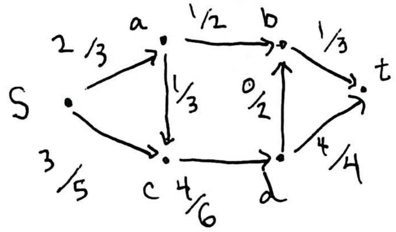
$u \xrightarrow[6/10]{} v$

Suppose $(u,v) \in E$, $(v,u) \notin E$

$c(v,u) = 0$

$c_f(v,u) = c(v,u) - f(v,u) = 0 - (-6) = +6$

"Residual Network"



Let $v \in V \setminus \{s,t\}$: $\sum_{u \in V} f(v,u) = 0$

Residual Network

for $(u,v) \in V \times V$, $c_f(u,v) = c(u,v) - f(u,v)$

"residual capacity"

G_f : residual network $= (V, E_f)$

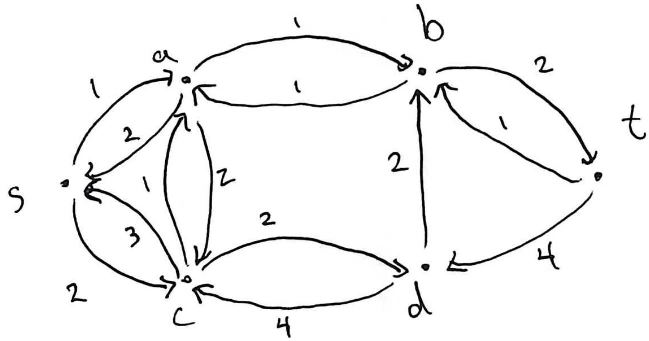
$E_f = \{ (u,v) \in V \times V : c_f(u,v) > 0 \}$

If $\cdot u \quad \cdot v$, $c(u,v) = 0 \quad c(v,u) = 0$

$f(u,v) \leq 0$; $-f(u,v) \leq 0 \Rightarrow f(u,v) = 0$

Thus, $|E_f| \leq 2 \cdot |E|$

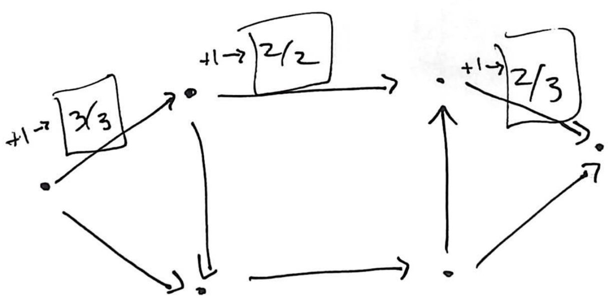
Residual Network

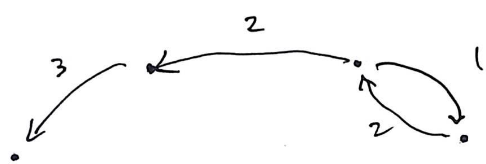


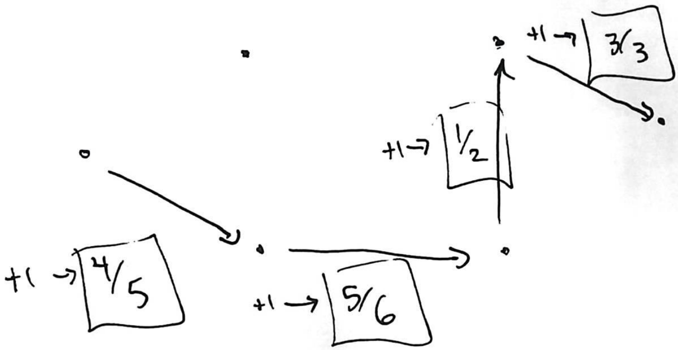
"Augmenting Path" : Any s,t, -Path in G_f

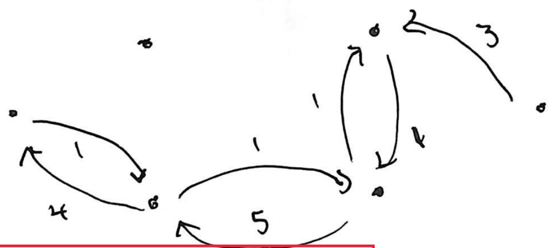
We say $e \in p$ is <u>critical</u>

if $c_f(e) = \min_{e' \in p} c_f(e')$





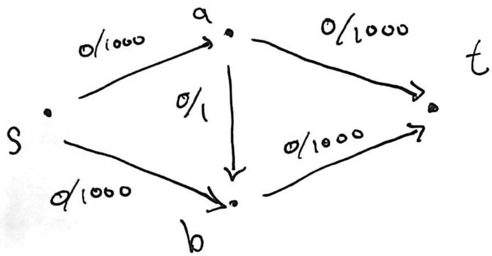


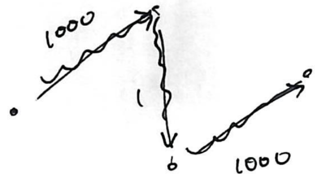


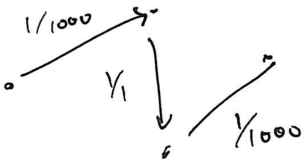
Ford-Fulkerson Algorithm

Start with $f = 0$. While $\exists$ an augmenting path p in G_f, add $c_f(p)$ units of flow along p. Return f.

Claim: f is a max flow.







Inefficient!

Choosing a shortest augmenting (in terms of # edges) is good. $\Longrightarrow$ Edmonds - Karp Max flow Algorithm

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<u>Lemma</u>: Let f be a flow in G, and let $(S, V \setminus S)$ be an s,t-cut in G ($s \in S$, $t \in V \setminus S$).

Then $|f| = f(S, V \setminus S)$.

<u>Proof</u> $f(S \setminus \{s\}, V) = 0$ (last time)

$$
\begin{aligned}
f(S, V \setminus S) &= f(S,V) - f(S,S) \quad \text{(last time)} \\
&= f(S,V) \\
&= f(\{s\}, V) + \underbrace{f(S - \{s\}, V)}_{=0} \\
&= |f|
\end{aligned}
$$

Corollary: $|f| \le c(S, V \setminus S)$

Pf

$$|f| = \sum_{\substack{x,y \\ x \in S, y \in V \setminus S}} f(x,y) \le \sum_{x,y} c(x,y) = c(S, V \setminus S)$$

Theorem "max-flow-min-cut"

Let f be a flow in G.

TFAE: "The following are equivalent."

(1) f is a max flow.

(2) G_f contains no augmenting path.

(3) $\exists$ s,t-cut $(S, V \setminus S)$ s.t. $|f| = c(S, V \setminus S)$.

Pf (1) $\Rightarrow$ (2): Clearly if there was some augmenting path, we could increase f.

(2) $\Rightarrow$ (3): Suppose G_f does not have augmenting path.

Let $S = \{ v \in V \mid \exists$ a path from s to v in $G_f \}$

$(S, V \setminus S)$ is a s,t-cut.

Suppose (u,v) crosses this cut.

Then, $f(u,v) = c(u,v)$.

$|f| = f(S, V \setminus S) = c(S, V \setminus S)$

$(3) \Rightarrow (1)$: Suppose $(S, V \setminus S)$ is an s.t. cut such that $f(S, V \setminus S) = c(S, V \setminus S)$

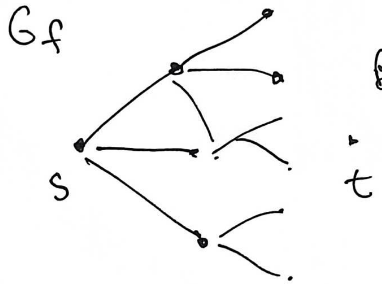
by previous corollary,

$|\text{any flow}| \leq c(S, V \setminus S)$

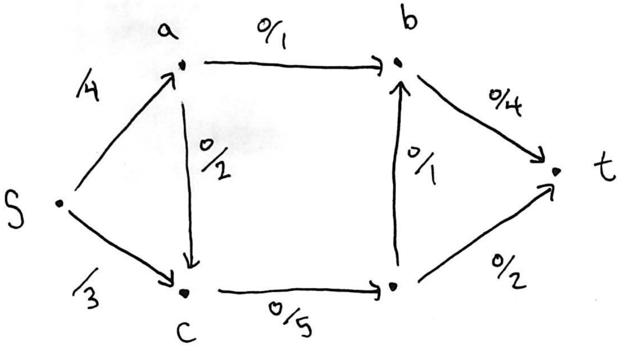
Therefore, f must be a max flow since $|f| = c(S, V \setminus S)$.

<u>Corollary</u>: ford-fulkerson finds a max flow.

Edmonds-Karp: F-F but choose shortest aug. path in G_f.



BFS $O(|E|)$

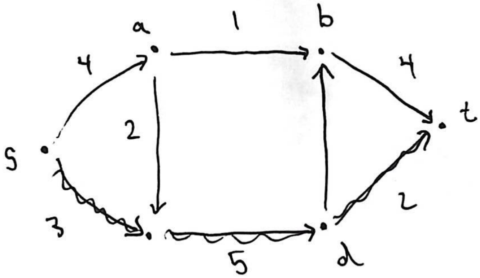


Edmonds-Karp : augment via shortest paths.

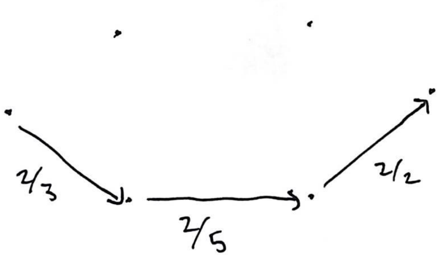
Residual Network G_f

$$c_f(u,v) = c(u,v) - f(u,v)$$

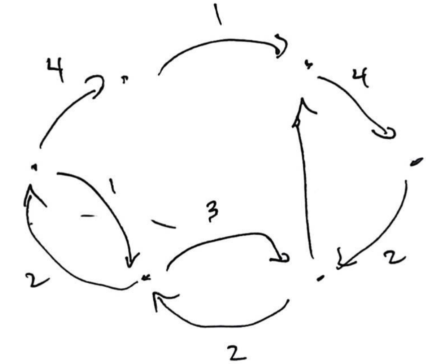
$$c_f(p) = \min_{e \in p} c_f(e)$$



$\Downarrow$



$\Rightarrow$



Edmonds-Karp: Each augmenting path / augmentation can be done in $O(|E|)$ time (BFS)

To prove: bound the # of flow augmentations (at most $|V| \cdot |E|$) $\Rightarrow$ total time: $O(|V| \cdot |E|^2)$

Defn: Let $\delta_f(u,v)$ be the shortest path distance from u to v.

Lemma $\forall v \in V \setminus \{s,t\}$, $\delta_f(s,v)$ is non-decreasing with each flow augmentation. (defn $\delta_f(v) = \delta_f(s,v)$)

Pf (by contradiction)

Suppose that for some $v \in V \setminus \{s,t\}$, there is a flow augmentation that causes $\delta_f(v)$ to decrease. Let f be the flow before, and f' be flow after.

$\delta_{f'}(v) < \delta_f(v)$.

If there are multiple such v's, choose v with min $\delta_{f'}(v)$.

Let p be the/a shortest path in $G_{f'}$ from s to v.

$$p : s \rightsquigarrow u \to v$$

Then, $\delta_{f'}(u) = \delta_{f'}(v) - 1$

But, $\delta_{f'}(u) \geq \delta_f(u)$ (by choice of v)

Claim: $(u, v) \notin E_f$

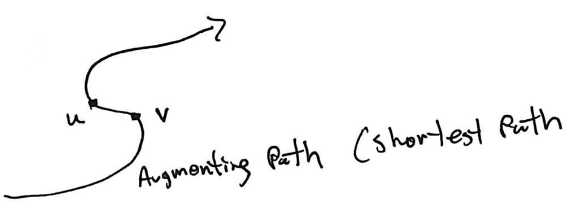
Suppose $(u,v) \in E_f$. Then, $\delta_f(v) \leq \delta_f(u) + 1$.

Thus, $\delta_f(v) \leq \delta_f(u) + 1 \leq \delta_{f'}(u) + 1 = \delta_{f'}(v)$.

This, is a contradiction since we assumed $\delta_{f'}(v) < \delta_f(v)$.

We have $(u,v) \notin E_f$ and $(u,v) \in E_{f'}$

Then,



$f'(v,u) > f(u,v)$

So, $\delta_f(v) = \delta_f(u) - 1 \leq \delta_{f'}(u) - 1 = \delta_{f'}(v) - 2$.

$\delta_{f'}(v) \geq \delta_f(v) + 2$, which contradicts that $\delta_{f'}(v) < \delta_f(v)$.

Back to E-K analysis:

Suppose (u,v) is a critical edge

G_f $\quad s \rightsquigarrow u \to v \to t$

so, $\delta_f(v) = \delta_f(u) + 1 \qquad (1)$

if (u,v) is critical again, it's because there was an augmenting path that used (v,u) in $G_{f'}$.

By lemma, $\delta_{f'}(v) \geq \delta_f(v). \qquad (2)$

Also, $\delta_{f'}(u) = \delta_{f'}(v) + 1 \qquad (3)$

$$
\begin{aligned}
\text{So,}\quad \delta_f(u) &= \delta_f(v) - 1, && \text{by } (1)\\
&\leq \delta_{f'}(v) - 1 && \text{by } (2)\\
&= \delta_{f'}(u) - 2, && \text{by } (3)
\end{aligned}
$$

$\Rightarrow \delta_{f'}(u) \geq \delta_f(u) + 2$

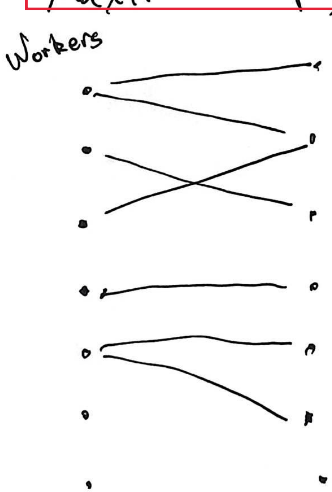
Observe $\delta_f(u) \leq |V| - 1.$

So, (u,v) is critical at most $\frac{|V|}{2}$ times.

On the other hand, every iteration makes some edge critical.

$\Rightarrow$ # of iterations of E-K $\leq |E| \cdot \frac{|V|}{2}$. $\square$

Maximum Bipartite Matching



$\Rightarrow$