Appendix to "Add-On Regimes and Their Relevance for Quantifying the Effects of Opioid-Sparing Treatments"

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A. Identification Proofs in the Absence of Censoring and Competing Events

Throughout this section, we work under the observed data structure, notation, and assumptions presented in Sections 2 and 3 of the main article. In Subsection A.1, we define the counterfactual variables. In Subsection A.2, we argue that the exchangeability condition in Equation (8) of the main article is implied by the corresponding condition used in Theorem 31 of Richardson and Robins (2013, p. 67). In Subsection A.3, we prove the g-formula stated in Theorem 7 of the main article and demonstrate why this aligns with the identification formula in Theorem 31 of Richardson and Robins (2013, p. 67). Finally, in Subsection A.4, we prove the inverse probability weighted identification formula stated in Corollary 7.1 of the main article and demonstrate why this aligns with the identification formula in Corollary 32 of Richardson and Robins (2013, p. 67).

A.1 Definition of Counterfactual Variables

Consider a set of random variables

$$\left(\boldsymbol{L}_k^{\bar{a}_{\min\{k-1,\kappa\}}}, Y_k^{\bar{a}_{\min\{k-1,\kappa\}}}, A_k^{\bar{a}_{\min\{k-1,\kappa\}}}\right)_{k\geq 1, \bar{a}_\kappa \in \{0,1\}^{\kappa+1}}, \tag{1}$$

where $\boldsymbol{L}_k^{\bar{a}_{\min\{k-1,\kappa\}}} \in \mathcal{L}$, $Y_k^{\bar{a}_{\min\{k-1,\kappa\}}} \in \mathcal{Y}$, and $A_k^{\bar{a}_{\min\{k-1,\kappa\}}} \in \{0,1\}$ denote the counterfactual covariate vector, opioid dose, and NSAID indicator, respectively, at time k under a static regime $\bar{a}_\kappa$ for all $k \geq 1$ and all $\bar{a}_\kappa \in \{0,1\}^{\kappa+1}$. For brevity, we often denote $L_k^{\bar{a}_{\min\{k-1,\kappa\}}}$, $Y_k^{\bar{a}_{\min\{k-1,\kappa\}}}$, and $A_k^{\bar{a}_{\min\{k-1,\kappa\}}}$ simply by $\boldsymbol{L}_k^{\bar{a}}$, $Y_k^{\bar{a}}$, and $A_k^{\bar{a}}$, respectively, for all $k \geq 1$. In these abbreviated notations, the index appearing in the superscript is understood to be implicit. An implicit assumption underlying our definition of the counterfactual variables in (1) is the absence of interference. Specifically, we assume that the counterfactual outcomes under a given static regime for any individual do not depend on the treatment assignments received by other individuals. Based on the variables in (1), we define

$$
\begin{aligned}
\boldsymbol{L}_k^{\bar{A}^{g+}_{\min\{k-1,\kappa\}}} &:= \sum_{a_0=0}^{1} \cdots \sum_{a_{\min\{k-1,\kappa\}}=0}^{1} 1\{\bar{A}^{g+}_{\min\{k-1,\kappa\}} = \bar{a}_{\min\{k-1,\kappa\}}\}\boldsymbol{L}_k^{\bar{a}_{\min\{k-1,\kappa\}}} \quad \forall\, k \geq 1,\\
Y_k^{\bar{A}^{g+}_{\min\{k-1,\kappa\}}} &:= \sum_{a_0=0}^{1} \cdots \sum_{a_{\min\{k-1,\kappa\}}=0}^{1} 1\{\bar{A}^{g+}_{\min\{k-1,\kappa\}} = \bar{a}_{\min\{k-1,\kappa\}}\}Y_k^{\bar{a}_{\min\{k-1,\kappa\}}} \quad \forall\, k \geq 1,\\
A_k^{\bar{A}^{g+}_{\min\{k-1,\kappa\}}} &:= \sum_{a_0=0}^{1} \cdots \sum_{a_{\min\{k-1,\kappa\}}=0}^{1} 1\{\bar{A}^{g+}_{\min\{k-1,\kappa\}} = \bar{a}_{\min\{k-1,\kappa\}}\}A_k^{\bar{a}_{\min\{k-1,\kappa\}}} \quad \forall\, k \geq 1.
\end{aligned}
\tag{2}
$$

The variables in (2) represent the counterfactual covariate vector, opioid dose, and NSAID indicator, respectively, at time k under a general regime g specified by $\bar{A}^{g+}_{\kappa}$. For brevity, we often denote $\boldsymbol{L}_k^{\bar{A}^{g+}_{\min\{k-1,\kappa\}}}$, $Y_k^{\bar{A}^{g+}_{\min\{k-1,\kappa\}}}$, and $A_k^{\bar{A}^{g+}_{\min\{k-1,\kappa\}}}$ simply by $\boldsymbol{L}^g_k$, Y^g_k, and A^g_k, respectively, for all $k \geq 1$. Finally, for notational consistency, we sometimes write $\mathbf{L}^g_0$, Y^g_0, and A^g_0 instead of $\mathbf{L}_0$, Y_0, and A_0, respectively.

A.2 Connection Between Our Exchangeability Assumption and That of Richardson and Robins (2013)

In this subsection we argue that the exchangeability condition in Equation (8) of the main article is implied by the corresponding exchangeability condition used in Theorem 31 of Richardson and Robins (2013, p. 67). We first define the following sets

$$
Z^{g_j}_{t,k} = \left(\mathrm{an}_{\mathcal{G}(g_j)}\left(Y^{g_j}_k\right)\right) \setminus \left(\bar{\boldsymbol{L}}^{g_j}_t, \bar{Y}^{g_j}_t, \bar{A}^{g_j}_t, \bar{A}^{g_j+}_{\kappa}\right), \tag{3}
$$

and $Z^{\bar{a}_{k'}}_{j,t,k} = \{V^{\bar{a}} \mid V^{g_j} \in Z^{g_j}_{t,k}\}$ for all $j \in \{0,1\}$, $t \leq k'$, $\bar{a}_{k'} \in \{0,1\}^{k'+1}$, and $k \geq 1$, where $k' = \min\{k-1,\kappa\}$. The following assumption states the exchangeability condition in Theorem 31 of Richardson and Robins (2013, p. 67), adapted to the notation of this article.

Assumption 1.

$$
Z^{\bar{a}_{k'}}_{j,t,k} \perp\!\!\!\perp A^{\bar{a}_{t-1}}_t \mid \bar{\boldsymbol{L}}^{\bar{a}_{t-1}}_t, \bar{Y}^{\bar{a}_{t-1}}_t, \bar{A}^{\bar{a}_{t-2}}_{t-1}, \tag{4}
$$

for all $j \in \{0,1\}$*,* $t \leq k'$*,* $\bar{a}_{k'} \in \{0,1\}^{k'+1}$*, and* $k \geq 1$*, where* $k' = \min\{k-1, \kappa\}$*.*

The following lemma shows that Equation (8) of the main article is implied by Equation 4. Moreover, the proof of the lemma shows that Equation (8) involves fewer variables than Equation 4.

Lemma 2. *Equation 4 implies Equation (8) of the main article.*

Proof. Let $j \in \{0,1\}$, $t \leq k'$, $\bar{a}_{k'} \in \{0,1\}^{k'+1}$, $k \geq 1$, and $k' = \min\{k-1, \kappa\}$. By (3), we have

$$\left(\left(Y_k^{g_j}, \boldsymbol{L}_{t+1}^{g_j}, \ldots, \boldsymbol{L}_{k'}^{g_k}, Y_{t+1}^{g_j}, \ldots, Y_{k'}^{g_j}, A_{t+1}^{g_j}, \ldots, A_{k'}^{g_j}\right) \cap \mathrm{an}_{\mathcal{G}(g_j)}\left(Y_k^{g_j}\right)\right) \subseteq Z_{t,k}^{g_j}, \tag{5}$$

which implies that $W_{j,t,k}^{\bar{a}} \subseteq Z_{j,t,k}^{\bar{a}}$, thereby proving the lemma. $\square$

A.3 G-formula

First, we prove the g-formula in Theorem 7 of the main article.

Theorem 3. *Let* g_j *be an add-on-*j *regime and assume that consistency (Equation (6) of the main article), exchangeability (Equation (8) of the main article), and positivity (Equation (9) of the main article) hold. Then*

$$\begin{aligned} E\left[Y_k^{g_j}\right] = \int \ldots \int & E\left[Y_k \mid \bar{\boldsymbol{L}}_{k'} = \bar{\boldsymbol{l}}_{k'}, \bar{Y}_{k'} = \bar{y}_{k'}, A_0 = g_j(y_0, a_0), \ldots, A_{k'} = g_j(y_{k'}, a_{k'})\right] \\ & \prod_{t=1}^{k'} f_{\boldsymbol{L}_t, Y_t, A_t \mid \bar{\boldsymbol{L}}_{t-1}, \bar{Y}_{t-1}, \bar{A}_{t-1}}(\boldsymbol{l}_t, y_t, a_t \mid \bar{\boldsymbol{l}}_{t-1}, \bar{y}_{t-1}, g_j(y_0, a_0), \ldots, g_j(y_{t-1}, a_{t-1})) \\ & f_{\boldsymbol{L}_0, Y_0, A_0}(\boldsymbol{l}_0, y_0, a_0) \, \mathrm{d}\boldsymbol{l}_{k'} \, \mathrm{d}y_{k'} \, \mathrm{d}a_{k'} \ldots \mathrm{d}\boldsymbol{l}_0 \, \mathrm{d}y_0 \, \mathrm{d}a_0, \end{aligned} \tag{6}$$

for all $j \in \{0,1\}$ *and* $k \geq 1$ *where* $k' = \min\{k-1, \kappa\}$*.*

Proof. Let g_j be an add-on-j regime. Let $k \in \{1, \ldots, K\}$ and denote $k' = \min\{k-1, \kappa\}$.

By notation (see Subsection A.1), we have that

$$E\left[Y_k^{g_j}\right] = E\left[Y_k^{g_j(Y_0,A_0),g_j(Y_1^{g_j},A_1^{g_j}),\dots,g_j(Y_{k'}^{g_j},A_{k'}^{g_j})}\right].$$

First, let $t=0$. It holds that

$$\begin{aligned}
&E\left[Y_k^{g_j(Y_t,A_t),g_j(Y_{t+1}^{g_j},A_{t+1}^{g_j}),\dots,g_j(Y_{k'}^{g_j},A_{k'}^{g_j})}\right]\\
&=\int\dots\int E\left[Y_k^{g_j(Y_t,A_t),g_j(Y_{t+1}^{g_j},A_{t+1}^{g_j}),\dots,g_j(Y_{k'}^{g_j},A_{k'}^{g_j})} \mid \boldsymbol{L}_t=\boldsymbol{l}_t, Y_t=y_t, A_t=a_t\right]\\
&\quad f_{\boldsymbol{L}_t,Y_t,A_t}(\boldsymbol{l}_t,y_t,a_t)\,\mathrm{d}\boldsymbol{l}_t\,\mathrm{d}y_t\,\mathrm{d}a_t\\
&=\int\dots\int E\left[Y_k^{g_j(y_t,a_t),g_j(Y_{t+1}^{g_j},A_{t+1}^{g_j}),\dots,g_j(Y_{k'}^{g_j},A_{k'}^{g_j})} \mid \boldsymbol{L}_t=\boldsymbol{l}_t, Y_t=y_t, A_t=a_t\right]\\
&\quad f_{\boldsymbol{L}_t,Y_t,A_t}(\boldsymbol{l}_t,y_t,a_t)\,\mathrm{d}\boldsymbol{l}_t\,\mathrm{d}y_t\,\mathrm{d}a_t\\
&=\int\dots\int E\left[Y_k^{g_j(y_t,a_t),g_j(Y_{t+1}^{g_j},A_{t+1}^{g_j}),\dots,g_j(Y_{k'}^{g_j},A_{k'}^{g_j})} \mid \boldsymbol{L}_t=\boldsymbol{l}_t, Y_t=y_t, A_t=g_j(y_t,a_t)\right]\\
&\quad f_{\boldsymbol{L}_t,Y_t,A_t}(\boldsymbol{l}_t,y_t,a_t)\,\mathrm{d}\boldsymbol{l}_t\,\mathrm{d}y_t\,\mathrm{d}a_t,
\end{aligned}$$

where the first equality is by the Law of Total Expectation, the second by the definition of counterfactual variables (2), and the third by Exchangeability (Equation (8) of the main article). By analog argumentation, it holds that the above is equal to

$$\begin{aligned}
&\int\dots\int E\Big[Y_k^{g_j(y_t,a_t),g_j(Y_{t+1},A_{t+1}),g_j(Y^{g_j},A_2^{g_j}),\dots,g_j(Y_{k'}^{g_j},A_{k'}^{g_j})}\\
&\qquad\mid \bar{\boldsymbol{L}}_{t+1}=\bar{\boldsymbol{l}}_{t+1}, \bar{Y}_{t+1}=\bar{y}_{t+1}, A_t=g_j(y_t,a_t), A_{t+1}=a_{t+1}\Big]\\
&\qquad f_{\boldsymbol{L}_{t+1},Y_{t+1},A_{t+1}\mid\boldsymbol{L}_t,Y_t,A_t}(\boldsymbol{l}_{t+1},y_{t+1},a_{t+1}\mid\boldsymbol{l}_t,y_t,g_j(y_t,a_t))\\
&\qquad f_{\boldsymbol{L}_t,Y_t,A_t}(\boldsymbol{l}_t,y_t,a_t)\,\mathrm{d}\boldsymbol{l}_{t+1}\,\mathrm{d}y_{t+1}\,\mathrm{d}a_{t+1}\,\mathrm{d}\boldsymbol{l}_t\,\mathrm{d}y_t\,\mathrm{d}a_t\\
&=\int\dots\int E\Big[Y_k^{g_j(y_t,a_t),g_j(y_{t+1},a_{t+1}),g_j(Y_2^{g_j},A_2^{g_j}),\dots,g_j(Y_{k'}^{g_j},A_{k'}^{g_j})}\\
&\qquad\mid \bar{\boldsymbol{L}}_{t+1}=\bar{\boldsymbol{l}}_{t+1}, \bar{Y}_{t+1}=\bar{y}_{t+1}, A_t=g_j(y_t,a_t), A_{t+1}=a_{t+1}\Big]\\
&\qquad f_{\boldsymbol{L}_{t+1},Y_{t+1},A_{t+1}\mid\boldsymbol{L}_t,Y_t,A_t}(\boldsymbol{l}_{t+1},y_{t+1},a_{t+1}\mid\boldsymbol{l}_t,y_t,g_j(y_t,a_t))\\
&\qquad f_{\boldsymbol{L}_t,Y_t,A_t}(\boldsymbol{l}_t,y_t,a_t)\,\mathrm{d}\boldsymbol{l}_{t+1}\,\mathrm{d}y_{t+1}\,\mathrm{d}a_{t+1}\,\mathrm{d}\boldsymbol{l}_t\,\mathrm{d}y_t\,\mathrm{d}a_t
\end{aligned}$$

$$= \int \dots \int E\Big[Y_k^{g_j(y_t,a_t),g_j(y_{t+1},a_{t+1}),g_j(Y_2^{g_j},A_2^{g_j}),\dots,g_j(Y_{k'}^{g_j},A_{k'}^{g_j})}$$

$$\mid \bar{\boldsymbol{L}}_{t+1} = \bar{\boldsymbol{l}}_{t+1}, \bar{Y}_{t+1} = \bar{y}_{t+1}, A_t = g_j(y_t,a_t), A_{t+1} = g_j(y_{t+1},a_{t+1})\Big]$$

$$f_{\boldsymbol{L}_{t+1},Y_{t+1},A_{t+1}\mid \boldsymbol{L}_t,Y_t,A_t}(\boldsymbol{l}_{t+1},y_{t+1},a_{t+1} \mid \boldsymbol{l}_t,y_t,g_j(y_t,a_t))$$

$$f_{\boldsymbol{L}_t,Y_t,A_t}(\boldsymbol{l}_t,y_t,a_t)\,\mathrm{d}\boldsymbol{l}_{t+1}\,\mathrm{d}y_{t+1}\,\mathrm{d}a_{t+1}\,\mathrm{d}\boldsymbol{l}_t\,\mathrm{d}y_t\,\mathrm{d}a_t,$$

where the first step is by the Law of Total Expectation, the first equality by the definition of counterfactual variables (2), and the second equality by the definition of counterfactual variables (2), exchangeability (Equation (8) of the main article) and consistency (Equation (6) of the main article). By repeating this argument for $t = 2$ to $t = k'$, we obtain that the above is equal to

$$\int \dots \int E\left[Y_k^{g_j(y_0,a_0),\dots,g_j(y_{k'},a_{k'})} \mid \bar{\boldsymbol{L}}_{k'} = \bar{\boldsymbol{l}}_{k'}, \bar{Y}_{k'} = \bar{y}_{k'}, A_0 = g_j(y_0,a_0),\dots,A_{k'} = g_j(y_{k'},a_{k'})\right]$$

$$\prod_{j=1}^{k'} f_{\boldsymbol{L}_j,Y_j,A_j\mid \bar{\boldsymbol{L}}_{j-1},\bar{Y}_{j-1},\bar{A}_{j-1}}\left(\boldsymbol{l}_j,y_j,a_j \mid \bar{\boldsymbol{l}}_{j-1},\bar{y}_{j-1},g_j(y_0,a_0),\dots,g_j(y_{j-1},a_{j-1})\right)$$

$$f_{\boldsymbol{L}_0,Y_0,A_0}(\boldsymbol{l}_0,y_0,a_0)\,\mathrm{d}\boldsymbol{l}_{k'}\,\mathrm{d}y_{k'}\,\mathrm{d}a_{k'}\dots\mathrm{d}\boldsymbol{l}_0\,\mathrm{d}y_0\,\mathrm{d}a_0.$$

which is equal to (6) under the definition of counterfactual variables (2) and consistency (Equation (6) of the main article). $\square$

A.3.1 Alignment with Richardson and Robins (2013)

Here, we demonstrate that the g-formula in Equation 6 aligns with the g-formula in Equation (67) of Richardson and Robins (2013, p. 67). To this end, assume that Y_k is a discrete variable and that $\boldsymbol{L}_k$ is a vector of discrete variables for every $k \leq K$.

Let g be an add-on-j regime. In Richardson and Robins (2013, p. 65), $q_k^g(a_k^+ \mid y_k, a_k)$ denote the conditional density of $A_k^{g+} = g(Y_k^g, A_k^g)$ given the input variables Y_k^g and A_k^g for $a_k^+, a_k \in \{0,1\}$ and $y_k \in \mathcal{Y}$ for every $k \leq \kappa$. In our paper, this density is denoted by $f_{g(Y_k^g,A_k^g)\mid Y_k^g,A_k^g}(a_k^+ \mid y_k, a_k)$ for $a_k^+, a_k \in \{0,1\}$ and $y_k \in \mathcal{Y}$ for every $k \leq \kappa$. The add-on regime is a deterministic regime, so this conditional density simplifies to

$$f_{g(Y_k^g, A_k^g) \mid Y_k^g, A_k^g}(a_k^+ \mid y_k, a_k) = 1\{g_k(y_k, a_k) = a_k^+\} \quad \text{for } a_k^+, a_k \in \{0,1\},\ y_k \in \mathcal{Y},\ k \leq \kappa. \tag{7}$$

Based on (7), the identification result for the density of the counterfactual outcome presented in Theorem 31 of Richardson and Robins (2013, p. 67), can be formulated as

$$\begin{aligned} f_{Y_k^g}(y) = \sum_{\bar{\boldsymbol{l}}_{k'}, \bar{y}_{k'}, \bar{a}_{k'}, \bar{a}_{k'}^+} & f_{Y_k \mid \bar{\boldsymbol{L}}_{k'}, \bar{Y}_{k'}, \bar{A}_{k'}}(y \mid \bar{\boldsymbol{l}}_{k'}, \bar{y}_{k'}, g(y_0, a_0), \ldots, g(y_{k'}, a_{k'})) \\ & \prod_{j=0}^{k'} f_{\boldsymbol{L}_j, Y_j, A_j \mid \bar{\boldsymbol{L}}_{j-1}, \bar{Y}_{j-1}, \bar{A}_{j-1}}(\boldsymbol{l}_j, y_j, a_j \mid \bar{\boldsymbol{l}}_{j-1}, \bar{y}_{j-1}, g(y_0, a_0), \ldots, g(y_{j-1}, a_{j-1})) \\ & \prod_{t=0}^{k'} 1\{g_t(y_t, a_t) = a_t^+\}. \end{aligned} \tag{8}$$

for $y \in \mathcal{Y}$ and $k \geq 1$ where $k' = \min\{\kappa, k-1\}$, which clearly aligns with the g-formula in Equation 6.

A.4 Inverse probability weighted identification formula

Next, we prove the inverse probability weighted identification formula stated in Corollary 7.1 of the main article.

Corollary 3.1. *Let g_j be an add-on-j regime and assume that consistency (Equation (6) of the main article), exchangeability (Equation (8) of the main article), and positivity (Equation (9) of the main article) hold. Then*

$$E\left[Y_k^{g_j}\right] = E\left[Y_k W_{k'}\right], \tag{9}$$

for all $j \in \{0,1\}$, $k \geq 1$ where $k' = \min\{k-1, \kappa\}$ and

$$W_s = \sum_{a_0=0}^{1} \cdots \sum_{a_s=0}^{1} \prod_{t=0}^{s} \frac{1\{A_t = g_j(Y_t, a_t)\} f_{A_t \mid \bar{\boldsymbol{L}}_t, \bar{Y}_t, \bar{A}_{t-1}}(a_t \mid \bar{\boldsymbol{L}}_t, \bar{Y}_t, \bar{A}_{t-1})}{f_{A_t \mid \bar{\boldsymbol{L}}_t, \bar{Y}_t, \bar{A}_{t-1}}(A_t \mid \bar{\boldsymbol{L}}_t, \bar{Y}_t, \bar{A}_{t-1})} \quad \forall s \geq 0. \tag{10}$$

Proof. Let g_j be an add-on-j regime. Let $k \in \{1, \ldots, K\}$ and denote $k' = \min\{k-1, \kappa\}$. For notational convenience, denote $\boldsymbol{X}_k = (\boldsymbol{L}_k, Y_k)$ and $\boldsymbol{x}_k = (\boldsymbol{l}_k, y_k)$ for every $k \le K$. By Theorem 3, under consistency (Equation (6) of the main article), exchangeability (Equation (8) of the main article), and positivity (Equation (9) of the main article), we have that

$$
\begin{aligned}
E\left[Y_k^{g_j}\right] = \int \ldots \int & E\left[Y_k \mid \bar{\boldsymbol{X}}_{k'} = \bar{\boldsymbol{x}}_{k'}, A_0 = g_j(y_0, a_0), \ldots, A_{k'} = g_j(y_{k'}, a_{k'})\right] \\
& \prod_{s=1}^{k'} f_{\boldsymbol{X}_s, A_s \mid \bar{\boldsymbol{X}}_{s-1}, \bar{A}_{s-1}}(\boldsymbol{x}_s, a_s \mid \bar{\boldsymbol{x}}_{s-1}, g_j(y_0, a_0), \ldots, g_j(y_{s-1}, a_{s-1})) \\
& f_{\boldsymbol{X}_0, A_0}(\boldsymbol{x}_0, a_0) \,\mathrm{d}\boldsymbol{x}_{k'} \,\mathrm{d}a_{k'} \ldots \mathrm{d}\boldsymbol{x}_0 \,\mathrm{d}a_0.
\end{aligned}
$$

Let $t = k'$. By multiplying and dividing by

$$
f_{\boldsymbol{X}_t, A_t \mid \bar{\boldsymbol{X}}_{t-1}, \bar{A}_{t-1}}(\boldsymbol{x}_t, g_j(y_t, a_t) \mid \bar{\boldsymbol{x}}_{t-1}, g_j(y_0, a_0), \ldots, g_j(y_{t-1}, a_{t-1})),
$$

we obtain that the above is equal to

$$
\begin{aligned}
\int \ldots \int E\Bigg[& Y_k \frac{f_{\boldsymbol{X}_t, A_t \mid \bar{\boldsymbol{X}}_{t-1}, \bar{A}_{t-1}}(\boldsymbol{x}_t, a_t \mid \bar{\boldsymbol{x}}_{t-1}, g_j(y_0, a_0), \ldots, g_j(y_{t-1}, a_{t-1}))}{f_{\boldsymbol{X}_t, A_t \mid \bar{\boldsymbol{X}}_{t-1}, \bar{A}_{t-1}}(\boldsymbol{x}_t, g_j(y_t, a_t) \mid \bar{\boldsymbol{x}}_{t-1}, g_j(y_0, a_0), \ldots, g_j(y_{t-1}, a_{t-1}))} \\
& \mid \bar{\boldsymbol{X}}_t = \bar{\boldsymbol{x}}_t, A_0 = g_j(y_0, a_0), \ldots, A_t = g_j(y_t, a_t)\Bigg] \\
& f_{\boldsymbol{X}_t, A_t \mid \bar{\boldsymbol{X}}_{t-1}, \bar{A}_{t-1}}(\boldsymbol{x}_t, g_j(y_t, a_t) \mid \bar{\boldsymbol{x}}_{t-1}, g_j(y_0, a_0), \ldots, g_j(y_{t-1}, a_{t-1})) \\
& \prod_{s=1}^{t-1} f_{\boldsymbol{X}_s, A_s \mid \bar{\boldsymbol{X}}_{s-1}, \bar{A}_{s-1}}(\boldsymbol{x}_s, a_s \mid \bar{\boldsymbol{x}}_{s-1}, g_j(y_0, a_0), \ldots, g_j(y_{s-1}, a_{s-1})) \\
& f_{\boldsymbol{X}_0, A_0}(\boldsymbol{x}_0, a_0) \,\mathrm{d}\boldsymbol{x}_t \,\mathrm{d}a_t \ldots \mathrm{d}\boldsymbol{x}_0 \,\mathrm{d}a_0.
\end{aligned}
$$

By standard properties of joint and conditional densities, we have that

$$
\begin{aligned}
& f_{\boldsymbol{X}_t, A_t \mid \bar{\boldsymbol{X}}_{t-1}, \bar{A}_{t-1}}(\boldsymbol{x}_t, g_j(y_t, a_t) \mid \bar{\boldsymbol{x}}_{t-1}, g_j(y_0, a_0), \ldots, g_j(y_{t-1}, a_{t-1})) \\
= & f_{A_t \mid \bar{\boldsymbol{X}}_t, \bar{A}_{t-1}}(g_j(y_t, a_t) \mid \bar{\boldsymbol{x}}_t, g_j(y_0, a_0), \ldots, g_j(y_{t-1}, a_{t-1})) \\
& f_{\boldsymbol{X}_t \mid \bar{\boldsymbol{X}}_{t-1}, \bar{A}_{t-1}}(\boldsymbol{x}_t \mid \bar{\boldsymbol{x}}_{t-1}, g_j(y_0, a_0), \ldots, g_j(y_{t-1}, a_{t-1})),
\end{aligned}
$$

which ensures that the above is equal to

$$\int \dots \int E\Big[Y_k \frac{f_{\boldsymbol{X}_t, A_t \mid \bar{\boldsymbol{X}}_{t-1}, \bar{A}_{t-1}}(\boldsymbol{x}_t, a_t \mid \bar{\boldsymbol{x}}_{t-1}, g_j(y_0, a_0), \dots, g_j(y_{t-1}, a_{t-1}))}{f_{\boldsymbol{X}_t, A_t \mid \bar{\boldsymbol{X}}_{t-1}, \bar{A}_{t-1}}(\boldsymbol{x}_t, g_j(y_t, a_t) \mid \bar{\boldsymbol{x}}_{t-1}, g_j(y_0, a_0), \dots, g_j(y_{t-1}, a_{t-1}))}$$

$$\mid \bar{\boldsymbol{X}}_t = \bar{\boldsymbol{x}}_t, A_0 = g_j(y_0, a_0), \dots, A_t = g_j(y_t, a_t)\Big]$$

$$f_{A_t \mid \bar{\boldsymbol{X}}_t, \bar{A}_{t-1}}(g_j(y_t, a_t) \mid \bar{\boldsymbol{x}}_t, g_j(y_0, a_0), \dots, g_j(y_{t-1}, a_{t-1}))$$

$$f_{\boldsymbol{X}_t \mid \bar{\boldsymbol{X}}_{t-1}, \bar{A}_{t-1}}(\boldsymbol{x}_t \mid \bar{\boldsymbol{x}}_{t-1}, g_j(y_0, a_0), \dots, g_j(y_{t-1}, a_{t-1}))$$

$$\prod_{s=1}^{t-1} f_{\boldsymbol{X}_s, A_s \mid \bar{\boldsymbol{X}}_{s-1}, \bar{A}_{s-1}}(\boldsymbol{x}_s, a_s \mid \bar{\boldsymbol{x}}_{s-1}, g_j(y_0, a_0), \dots, g_j(y_{s-1}, a_{s-1}))$$

$$f_{\boldsymbol{X}_0, A_0}(\boldsymbol{x}_0, a_0) \,\mathrm{d}\boldsymbol{x}_t \,\mathrm{d}a_t \dots \mathrm{d}\boldsymbol{x}_0 \,\mathrm{d}a_0.$$

By Bayes Law, this is equal to

$$\int \dots \int$$

$$E\left[Y_k \frac{f_{\boldsymbol{X}_t, A_t \mid \bar{\boldsymbol{X}}_{t-1}, \bar{A}_{t-1}}(\boldsymbol{x}_t, a_t \mid \bar{\boldsymbol{x}}_{t-1}, g_j(y_0, a_0), \dots, g_j(y_{t-1}, a_{t-1})) 1\{A_t = g_j(y_t, a_t)\}}{f_{\boldsymbol{X}_t, A_t \mid \bar{\boldsymbol{X}}_{t-1}, \bar{A}_{t-1}}(\boldsymbol{x}_t, g_j(y_t, a_t) \mid \bar{\boldsymbol{x}}_{t-1}, g_j(y_0, a_0), \dots, g_j(y_{t-1}, a_{t-1}))}\right.$$

$$\left. \mid \bar{\boldsymbol{X}}_t = \bar{\boldsymbol{x}}_t, A_0 = g_j(y_0, a_0), \dots, A_{t-1} = g_j(y_t, a_{t-1})\right]$$

$$f_{\boldsymbol{X}_t \mid \bar{\boldsymbol{X}}_{t-1}, \bar{A}_{t-1}}(\boldsymbol{x}_t \mid \bar{\boldsymbol{x}}_{t-1}, g_j(y_0, a_0), \dots, g_j(y_{t-1}, a_{t-1}))$$

$$\prod_{s=1}^{t-1} f_{\boldsymbol{X}_s, A_s \mid \bar{\boldsymbol{X}}_{s-1}, \bar{A}_{s-1}}(\boldsymbol{x}_s, a_s \mid \bar{\boldsymbol{x}}_{s-1}, g_j(y_0, a_0), \dots, g_j(y_{s-1}, a_{s-1}))$$

$$f_{\boldsymbol{X}_0, A_0}(\boldsymbol{x}_0, a_0) \,\mathrm{d}\boldsymbol{x}_t \,\mathrm{d}a_t \dots \mathrm{d}\boldsymbol{x}_0 \,\mathrm{d}a_0.$$

By the Law of Total Expectation, the above is equal to

$$\int \dots \int$$

$$E\left[Y_k \frac{f_{\boldsymbol{X}_t, A_t \mid \bar{\boldsymbol{X}}_{t-1}, \bar{A}_{t-1}}(\boldsymbol{X}_t, a_t \mid \bar{\boldsymbol{x}}_{t-1}, g_j(y_0, a_0), \dots, g_j(y_{t-1}, a_{t-1})) 1\{A_t = g_j(Y_t, a_t)\}}{f_{\boldsymbol{X}_t, A_t \mid \bar{\boldsymbol{X}}_{t-1}, \bar{A}_{t-1}}(\boldsymbol{X}_t, g_j(Y_t, a_t) \mid \bar{\boldsymbol{x}}_{t-1}, g_j(y_0, a_0), \dots, g_j(y_{t-1}, a_{t-1}))}\right.$$

$$\left. \mid \bar{\boldsymbol{X}}_{t-1} = \bar{\boldsymbol{x}}_{t-1}, A_0 = g_j(y_0, a_0), \dots, A_{t-1} = g_j(y_t, a_{t-1})\right]$$

$$\prod_{s=1}^{t-1} f_{\boldsymbol{X}_s, A_s \mid \bar{\boldsymbol{X}}_{s-1}, \bar{A}_{s-1}}(\boldsymbol{x}_s, a_s \mid \bar{\boldsymbol{x}}_{s-1}, g_j(y_0, a_0), \dots, g_j(y_{s-1}, a_{s-1}))$$

$$f_{\boldsymbol{X}_0, A_0}(\boldsymbol{x}_0, a_0) \,\mathrm{d}a_t \,\mathrm{d}\boldsymbol{x}_{t-1} \,\mathrm{d}a_{t-1} \dots \mathrm{d}\boldsymbol{x}_0 \,\mathrm{d}a_0,$$

which by standard properties is equal to

$$\int \cdots \int$$
$$E\Bigg[Y_k \sum_{a_t=0}^{1} \frac{f_{A_t|\bar{\boldsymbol{X}}_t,\bar{A}_{t-1}}(a_t \mid \bar{\boldsymbol{x}}_{t-1}, \boldsymbol{X}_t, g_j(y_0,a_0),\ldots,g_j(y_{t-1},a_{t-1}))1\{A_t = g_j(Y_t,a_t)\}}{f_{A_t|\bar{\boldsymbol{X}}_t,\bar{A}_{t-1}}(g_j(Y_t,a_t) \mid \bar{\boldsymbol{x}}_{t-1}, \boldsymbol{X}_t, g_j(y_0,a_0),\ldots,g_j(y_{t-1},a_{t-1}))}$$
$$\mid \bar{\boldsymbol{X}}_{t-1} = \bar{\boldsymbol{x}}_{t-1}, A_0 = g_j(y_0,a_0),\ldots,A_{t-1} = g_j(y_t,a_{t-1})\Bigg]$$
$$\prod_{s=1}^{t-1} f_{\boldsymbol{X}_s,A_s|\bar{\boldsymbol{X}}_{s-1},\bar{A}_{s-1}}(\boldsymbol{x}_s, a_s \mid \bar{\boldsymbol{x}}_{s-1}, g_j(y_0,a_0),\ldots,g_j(y_{s-1},a_{s-1}))$$
$$f_{\boldsymbol{X}_0,A_0}(\boldsymbol{x}_0,a_0)\,\mathrm{d}\boldsymbol{x}_{t-1}\,\mathrm{d}a_{t-1}\ldots\mathrm{d}\boldsymbol{x}_0\,\mathrm{d}a_0.$$

By repeating this argument for $t = k' - 1$ to $t = 0$, we obtain that the above is equal to

$$E\Bigg[Y_k \sum_{a_0=0}^{1} \frac{f_{A_0|\boldsymbol{X}_0}(a_0 \mid \boldsymbol{X}_0)1\{A_0 = g_j(Y_0,a_0)\}}{f_{A_0|\boldsymbol{X}_0}(g_j(Y_0,a_0) \mid \boldsymbol{X}_0)}$$
$$\sum_{a_1=0}^{1} \frac{f_{A_1|\bar{\boldsymbol{X}}_1,A_0}(a_a \mid \bar{\boldsymbol{X}}_1, a_0)1\{A_1 = g_j(Y_1,a_1)\}}{f_{A_1|\bar{\boldsymbol{X}}_1,A_0}(g_j(Y_1,a_1) \mid \bar{\boldsymbol{X}}_1, a_0)}$$
$$\vdots$$
$$\sum_{a_{k'}=0}^{1} \frac{f_{A_{k'}|\bar{\boldsymbol{X}}_{k'},\bar{A}_{k'-1}}(a_{k'} \mid \bar{\boldsymbol{X}}_{k'}, g_j(Y_0,a_0),\ldots,g_j(Y_{k'-1},a_{k'-1}))1\{A_{k'} = g_j(Y_{k'},a_{k'})\}}{f_{A_{k'}|\bar{\boldsymbol{X}}_{k'},\bar{A}_{k'-1}}(g_j(Y_{k'},a_{k'}) \mid \bar{\boldsymbol{X}}_{k'}, g_j(Y_0,a_0),\ldots,g_j(Y_{k'-1},a_{k'-1}))}\Bigg],$$

which, by the indicator functions, is equal to (9). □

A.4.1 Alignment with Richardson and Robins (2013)

Here, we demonstrate that the IPW formula in Equation 9 aligns with the IPW formula presented in Corollary 32 of Richardson and Robins (2013, p. 67). To this end, we show that the weights in Equation 10 are consistent with those in Corollary 32 of Richardson and Robins (2013, p. 67). We implicitly assume the presence of a typographical error in Corollary 32 of Richardson and Robins (2013, p. 67), interpreting the term on the right-hand side of formula (68) as a functional of the distribution of the factual Y, rather than the counterfactual $Y(g)$. This correction ensures that the formula represents a valid identification result.

Let g be an add-on regime. In Richardson and Robins (2013, p. 65), $q_k^g(a_k^+ \mid y_k, a_k)$ denote the conditional density of $A_k^{g+} = g(Y_k^g, A_k^g)$ given the input variables Y_k^g and A_k^g for $a_k^+, a_k \in \{0,1\}$ and $y_k \in \mathcal{Y}$ for every $k \leq \kappa$. In our paper, this density is denoted by $f_{g(Y_k^g,A_k^g)|Y_k^g,A_k^g}(a_k^+ \mid y_k, a_k)$ for $a_k^+, a_k \in \{0,1\}$ and $y_k \in \mathcal{Y}$ for every $k \leq \kappa$. The add-on regime is a deterministic regime, as opposed to a random regime, so this conditional density simplifies to

$$f_{g(Y_k^g,A_k^g)|Y_k^g,A_k^g}(a_k^+ \mid y_k, a_k) = 1\{g_k(y_k, a_k) = a_k^+\} \quad \text{for } a_k^+, a_k \in \{0,1\},\ y_k \in \mathcal{Y},\ k \leq \kappa. \tag{11}$$

Based on the notation presented at the bottom of page 65 of Richardson and Robins (2013, p. 65), define recursively

$$\begin{aligned}
&\tilde{q}_k^{g,p}(a_k^+ \mid \bar{\boldsymbol{l}}_k, \bar{y}_k, \bar{a}_{k-1}^+) \\
=&\frac{1}{\prod_{j=0}^{k-1} \tilde{q}_j^{g,p}(a_k^+ \mid \bar{\boldsymbol{l}}_k, \bar{y}_k, \bar{a}_{k-1}^+)} \\
&\sum_{a_0=0}^{1} \cdots \sum_{a_k=0}^{1} \prod_{j=0}^{k} 1\{g_j(y_j, a_j) = a_j^+\} f_{A_j|\bar{\boldsymbol{L}}_j, \bar{Y}_j, \bar{A}_{j-1}}(a_j \mid \bar{\boldsymbol{l}}_j, \bar{y}_j, \bar{a}_{j-1}^+),
\end{aligned} \tag{12}$$

for every $\bar{a}_k^+ \in \{0,1\}^{k+1}$, $\bar{\boldsymbol{l}}_k \in \mathcal{L}^{k+1}$, $\bar{y}_k \in \mathcal{Y}^{k+1}$, and $k \leq \kappa$. Using (12), we note that

$$\begin{aligned}
&\prod_{j=0}^{k} \tilde{q}_j^{g,p}(a_j^+ \mid \bar{\boldsymbol{l}}_j, \bar{y}_j, \bar{a}_{j-1}^+) \\
=&\left(\prod_{j=0}^{k-1} \tilde{q}_j^{g,p}(a_j^+ \mid \bar{\boldsymbol{l}}_j, \bar{y}_j, \bar{a}_{j-1}^+)\right) \tilde{q}_k^{g,p}(a_k^+ \mid \bar{\boldsymbol{l}}_k, \bar{y}_k, \bar{a}_{k-1}^+) \\
=&\sum_{a_0=0}^{1} \cdots \sum_{a_k=0}^{1} \prod_{j=0}^{k} 1\{g_j(y_j, a_j) = a_j^+\} f_{A_j|\bar{\boldsymbol{L}}_j, \bar{Y}_j, \bar{A}_{j-1}}(a_j \mid \bar{\boldsymbol{l}}_j, \bar{y}_j, \bar{a}_{j-1}^+).
\end{aligned} \tag{13}$$

The weights in Corollary 32 of Richardson and Robins (2013, p. 67) are formulated as

$$W_k = \frac{\prod_{j=0}^{k} \tilde{q}_j^{g,p}(A_j \mid \bar{\boldsymbol{L}}_j, \bar{Y}_j, \bar{A}_{j-1})}{\prod_{j=0}^{k} f_{A_j|\bar{\boldsymbol{L}}_j, \bar{Y}_j, \bar{A}_{j-1}}(A_j \mid \bar{\boldsymbol{L}}_j, \bar{Y}_j, \bar{A}_{j-1})} \quad \forall k \leq \kappa. \tag{14}$$

Using (12) and (13), these weights can be reformulated as

$$W_k = \sum_{a_0=0}^{1} \cdots \sum_{a_k=0}^{1} \prod_{s=0}^{k} \frac{1\{g(Y_s, a_s) = A_s\} f_{A_s|\bar{\boldsymbol{L}}_s, \bar{Y}_s, \bar{A}_{s-1}}(a_s \mid \bar{\boldsymbol{L}}_s, \bar{Y}_s, \bar{A}_{s-1})}{f_{A_s|\bar{A}_{s-1}, \bar{\boldsymbol{L}}_s, \bar{Y}_s}(A_s \mid \bar{A}_{s-1}, \bar{\boldsymbol{L}}_s, \bar{Y}_s)} \quad \forall k \le \kappa. \tag{15}$$

B. Identification Proofs in the Presence of Censoring and Competing Events

Throughout this section, we work under the observed data structure, notation, and assumptions presented in Section 5 of the main article. In Subsection B.1, we define the counterfactual variables. In Subsection B.2, we present sufficient conditions for identifying the add-on effect in the presence of censoring and competing events. In Subsection B.3, we prove a version of the g-formula in the presence of censoring and competing events. Finally, in Subsection B.4, we prove a version of the inverse probability weighted identification formula in the presence of censoring and competing events.

B.1 Definition of the Counterfactual Variables

Consider a set of random variables

$$\begin{aligned}&\Big(C_k^{\bar{a}_{\min\{k-1,\kappa\}}, \bar{c}_{k-1}=\bar{0}}, D_k^{\bar{a}_{\min\{k-1,\kappa\}}, \bar{c}_{k-1}=\bar{0}}, \boldsymbol{L}_k^{\bar{a}_{\min\{k-1,\kappa\}}, \bar{c}_k=\bar{0}}, \\ &\quad Y_k^{\bar{a}_{\min\{k-1,\kappa\}}, \bar{c}_k=\bar{0}}, A_k^{\bar{a}_{\min\{k-1,\kappa\}}, \bar{c}_k=\bar{0}}\Big)_{k\ge 1, \bar{a}_\kappa \in \{0,1\}^{\kappa+1}},\end{aligned} \tag{16}$$

where $C_k^{\bar{a}_{\min\{k-1,\kappa\}}, \bar{c}_{k-1}=\bar{0}}$, $D_k^{\bar{a}_{\min\{k-1,\kappa\}}, \bar{c}_{k-1}=\bar{0}}$, $A_k^{\bar{a}_{\min\{k-1,\kappa\}}, \bar{c}_k=\bar{0}} \in \{0,1\}$, denote the counterfactual censoring indicator, competing event indicator, NSAID indicator at time k, and $L_k^{\bar{a}_{\min\{k-1,\kappa\}}, \bar{c}_k=\bar{0}} \in \mathcal{L}$, and $Y_k^{\bar{a}_{\min\{k-1,\kappa\}}, \bar{c}_k=\bar{0}} \in \mathcal{Y}$ denote the covariate vector, and opioid dose, respectively, at time k under a static regime g specified by $\bar{A}_\kappa^{g+} = \bar{a}_\kappa$ for all $k \ge 1$ and all $\bar{a}_\kappa \in \{0,1\}^{\kappa+1}$ and an additional intervention that eliminates censoring. For brevity, we often denote the variables in (16) simply by

$$\left(C_k^{\bar{a},\bar{c}=\bar{0}}, D_k^{\bar{a},\bar{c}=\bar{0}}, \boldsymbol{L}_k^{\bar{a},\bar{c}=\bar{0}}, Y_k^{\bar{a},\bar{c}=\bar{0}}, A_k^{\bar{a},\bar{c}=\bar{0}}\right)_{k\geq 1, \bar{a}_\kappa \in \{0,1\}^{\kappa+1}},$$

In these abbreviated notations, the indices appearing in the superscripts are understood to be implicit. An implicit assumption underlying our definition of the counterfactual variables in (16) is the absence of interference. Specifically, we assume that the counterfactual outcomes under a given static regime for any individual do not depend on the treatment assignments received by other individuals. Based on the variables in (16), we define

$$\begin{aligned}
C_k^{\bar{A}^{g+}_{\min\{k-1,\kappa\}}, \bar{c}_{k-1}=\bar{0}} &:= \sum_{a_0=0}^{1} \cdots \sum_{a_{\min\{k-1,\kappa\}}=0}^{1} 1\{\bar{A}^{g+}_{\min\{k-1,\kappa\}} = \bar{a}_{\min\{k-1,\kappa\}}\} C_k^{\bar{a}_{\min\{k-1,\kappa\}}, \bar{c}_{k-1}=\bar{0}}, \\
D_k^{\bar{A}^{g+}_{\min\{k-1,\kappa\}}, \bar{c}_{k}=\bar{0}} &:= \sum_{a_0=0}^{1} \cdots \sum_{a_{\min\{k-1,\kappa\}}=0}^{1} 1\{\bar{A}^{g+}_{\min\{k-1,\kappa\}} = \bar{a}_{\min\{k-1,\kappa\}}\} D_k^{\bar{a}_{\min\{k-1,\kappa\}}, \bar{c}_{k}=\bar{0}}, \\
\boldsymbol{L}_k^{\bar{A}^{g+}_{\min\{k-1,\kappa\}}, \bar{c}_{k}=\bar{0}} &:= \sum_{a_0=0}^{1} \cdots \sum_{a_{\min\{k-1,\kappa\}}=0}^{1} 1\{\bar{A}^{g+}_{\min\{k-1,\kappa\}} = \bar{a}_{\min\{k-1,\kappa\}}\} \boldsymbol{L}_k^{\bar{a}_{\min\{k-1,\kappa\}}, \bar{c}_{k}=\bar{0}}, \\
Y_k^{\bar{A}^{g+}_{\min\{k-1,\kappa\}}, \bar{c}_{k}=\bar{0}} &:= \sum_{a_0=0}^{1} \cdots \sum_{a_{\min\{k-1,\kappa\}}=0}^{1} 1\{\bar{A}^{g+}_{\min\{k-1,\kappa\}} = \bar{a}_{\min\{k-1,\kappa\}}\} Y_k^{\bar{a}_{\min\{k-1,\kappa\}}, \bar{c}_{k}=\bar{0}}, \\
A_k^{\bar{A}^{g+}_{\min\{k-1,\kappa\}}, \bar{c}_{k}=\bar{0}} &:= \sum_{a_0=0}^{1} \cdots \sum_{a_{\min\{k-1,\kappa\}}=0}^{1} 1\{\bar{A}^{g+}_{\min\{k-1,\kappa\}} = \bar{a}_{\min\{k-1,\kappa\}}\} A_k^{\bar{a}_{\min\{k-1,\kappa\}}, \bar{c}_{k}=\bar{0}},
\end{aligned} \tag{17}$$

for every $k \geq 1$. The variables in (17) represent the counterfactual censoring indicator, competing event indicator, covariate vector, opioid dose, and NSAID indicator, respectively, at time k under a general regime g specified by $\bar{A}_k^{g+}$ and an additional intervention that eliminates censoring. For brevity, we often denote the variables in (17) simply by $C_k^{g,\bar{c}=\bar{0}}$, $D_k^{g,\bar{c}=\bar{0}}$, $\boldsymbol{L}_k^{g,\bar{c}=\bar{0}}$, $Y_k^{g,\bar{c}=\bar{0}}$, and $A_k^{g,\bar{c}=\bar{0}}$, respectively, for all $k \geq 1$.

B.2 Identifiability conditions

Assumption 4 (Consistency).

$$\begin{aligned}&\textit{If } \bar{A}_{k'} = \bar{a}_{k'} \textit{ and } \bar{C}_k = \bar{0} \textit{ then } D_k = D_k^{\bar{a},\bar{c}=\bar{0}}, \boldsymbol{L}_k = \boldsymbol{L}_k^{\bar{a},\bar{c}=\bar{0}}, Y_k = Y_k^{\bar{a},\bar{c}=\bar{0}}, \textit{ and } A_k = A_k^{\bar{a},\bar{c}=\bar{0}},\\ &\textit{If } \bar{A}_{k'} = \bar{a}_{k'} \textit{ and } \bar{C}_{k-1} = \bar{0} \textit{ then } C_k = C_k^{\bar{a},\bar{c}=\bar{0}},\end{aligned} \tag{18}$$

for every $k \geq 1$ *where* $k' = \min\{k-1, \kappa\}$.

We first define the following sets

$$W_{t,k}^{g_j,\bar{c}=\bar{0}} = (Y_k^{g_j,\bar{c}=\bar{0}}, Y_{t+1}^{g_j,\bar{c}=\bar{0}}, \ldots, Y_{k'}^{g_j,\bar{c}=\bar{0}}, A_{t+1}^{g_j,\bar{c}=\bar{0}}, \ldots, A_{k'}^{g_j,\bar{c}=\bar{0}}) \cap \mathrm{an}_{\mathcal{G}(g_j,\bar{c}_k=\bar{0})}(Y_k^{g_j,\bar{c}=\bar{0}}), \tag{19}$$

and $W_{j,t,k}^{\bar{a}_{k'},\bar{c}=\bar{0}} = \{V^{\bar{a},\bar{c}=\bar{0}} \mid V^{g_j,\bar{c}=\bar{0}} \in W_{t,k}^{g_j,\bar{c}=\bar{0}}\}$, the subset of vertices in $\mathcal{G}(\bar{a}_{k'}, \bar{c}_k = \bar{0})$ that correspond to the vertices in (19) for all $j \in \{0,1\}$, $t \leq k'$, $\bar{a}_{k'} \in \{0,1\}^{k'+1}$, and $k \geq 1$, where $k' = \min\{k-1, \kappa\}$.

Assumption 5 (Exchangeability).

$$W_{j,t,k}^{\bar{a}_{k'},\bar{c}=\bar{0}} \perp\!\!\!\perp A_t^{\bar{a},\bar{c}=\bar{0}} \mid \bar{C}_t^{\bar{a},\bar{c}=\bar{0}}, \bar{D}_t^{\bar{a},\bar{c}=\bar{0}}, \bar{\boldsymbol{L}}_t^{\bar{a},\bar{c}=\bar{0}}, \bar{Y}_t^{\bar{a},\bar{c}=\bar{0}}, \bar{A}_{t-1}^{\bar{a},\bar{c}=\bar{0}}, \tag{20}$$

and

$$W_{j,t,k}^{\bar{a}_{k'},\bar{c}=\bar{0}} \perp\!\!\!\perp C_{t+1}^{\bar{a},\bar{c}=\bar{0}} \mid \bar{C}_t^{\bar{a},\bar{c}=\bar{0}}, \bar{D}_t^{\bar{a},\bar{c}=\bar{0}}, \bar{\boldsymbol{L}}_t^{\bar{a},\bar{c}=\bar{0}}, \bar{Y}_t^{\bar{a},\bar{c}=\bar{0}}, \bar{A}_t^{\bar{a},\bar{c}=\bar{0}}, \tag{21}$$

for all $j \in \{0,1\}$, $t \leq k'$, $\bar{a}_{k'} \in \{0,1\}^{k'+1}$, *and* $k \geq 1$, *where* $k' = \min\{k-1, \kappa\}$.

Assumption 6 (Positivity).

$$\begin{aligned}&f_{\bar{C}_k, \bar{D}_k, \bar{\boldsymbol{L}}_k, \bar{Y}_k, \bar{A}_k}(\bar{0}, \bar{d}_k, \bar{\boldsymbol{L}}_k, \bar{y}_k, \bar{a}_k) > 0\\ \Rightarrow\ &f_{A_k \mid \bar{C}_k, \bar{D}_k, \bar{\boldsymbol{L}}_k, \bar{Y}_k, \bar{A}_{k-1}}(g_j(y_k, a_k) \mid \bar{0}, \bar{d}_k, \bar{\boldsymbol{l}}_k, \bar{y}_k, \bar{a}_{k-1}) > 0,\end{aligned} \tag{22}$$

and

$$f_{\bar{C}_k,\bar{D}_k,\bar{\boldsymbol{L}}_k,\bar{Y}_k,\bar{A}_k}(\bar{0},\bar{d}_k,\bar{\boldsymbol{l}}_k,\bar{y}_k,\bar{a}_k) > 0$$
$$\Rightarrow f_{C_{k+1}\mid\bar{C}_k,\bar{D}_k,\bar{\boldsymbol{L}}_k,\bar{Y}_k,\bar{A}_k}(0 \mid \bar{0},\bar{d}_k,\bar{\boldsymbol{l}}_k,\bar{y}_k,\bar{a}_k) > 0, \tag{23}$$

for all $j \in \{0,1\}$, $\bar{\boldsymbol{l}}_k \in \mathcal{L}^{k+1}$, $\bar{y}_k \in \mathcal{Y}^{k+1}$, $\bar{a}_k \in \{0,1\}^{k+1}$, *and* $k < K$.

B.3 G-formula

Under these assumptions, we prove a g-formula identification formula.

Theorem 7. *Let* g_j *be an add-on-*j *regime and assume that consistency (18), exchangeability (20) - (21), and positivity (22) - (23) hold. Then*

$$\begin{aligned}
&E\left[Y_k^{g_j,\bar{c}=\bar{0}}\right] \\
&= \int \cdots \int E\Big[\tilde{Y}_k \mid \bar{C}_k = \bar{0}, \bar{\tilde{D}}_{k-1} = \bar{d}_{k-1}, \bar{\tilde{L}}_{k-1} = \bar{\boldsymbol{l}}_{k-1}, \bar{\tilde{Y}}_{k-1} = \bar{y}_{k-1}, \\
&\tilde{A}_0 = g_j(y_0,a_0),\ldots,\tilde{A}_{k'} = g_j(y_{k'},a_{k'}), \tilde{A}_{k'+1} = a_{k'+1},\ldots,\tilde{A}_{k-1} = a_{k-1}\Big] \\
&\prod_{t=k'+1}^{k-1} f_{\tilde{D}_t,\tilde{L}_t,\tilde{Y}_t,\tilde{A}_t\mid\bar{C}_t,\bar{\tilde{D}}_{t-1},\bar{\tilde{L}}_{t-1},\bar{\tilde{Y}}_{t-1},\bar{\tilde{A}}_{t-1}}(d_t,\boldsymbol{l}_t,y_t,a_t \\
&\mid \bar{0},\bar{d}_{t-1},\bar{\boldsymbol{l}}_{t-1},\bar{y}_{t-1},g_j(y_0,a_0),\ldots,g_j(y_{k'},a_{k'}),a_{k'+1},\ldots,a_{t-1}) \\
&\prod_{t=0}^{k'} f_{\tilde{D}_t,\tilde{L}_t,\tilde{Y}_t,\tilde{A}_t\mid\bar{C}_t,\bar{\tilde{D}}_{t-1},\bar{\tilde{L}}_{t-1},\bar{\tilde{Y}}_{t-1},\bar{\tilde{A}}_{t-1}}(d_t,\boldsymbol{l}_t,y_t,a_t \\
&\mid \bar{0},\bar{d}_{t-1},\bar{\boldsymbol{l}}_{t-1},\bar{y}_{t-1},g_j(y_0,a_0),\ldots,g_j(y_{t-1},a_{t-1})) \\
&\mathrm{d}\,d_{k-1}\,\mathrm{d}\,\boldsymbol{l}_{k-1}\,\mathrm{d}\,y_{k-1}\,\mathrm{d}\,a_{k-1}\ldots\mathrm{d}\,d_0\,\mathrm{d}\,\boldsymbol{l}_0\,\mathrm{d}\,y_0\,\mathrm{d}\,a_0,
\end{aligned} \tag{24}$$

for all $j \in \{0,1\}$ *and* $k \geq 1$ *where* $k' = \min\{k-1,\kappa\}$.

Proof. Let g_j be an add-on-j regime. Let $k \in \{1,\ldots,K\}$ and denote $k' = \min\{k-1,\kappa\}$. For notational convenience, denote $\boldsymbol{X}_k = (D_k,\boldsymbol{L}_k,Y_k)$ and $\boldsymbol{x}_k = (d_k,\boldsymbol{l}_k,y_k)$ for every $k \leq$

K. By notation (16), we have that

$$E\left[Y_k^{g_j,\bar{c}=\bar{0}}\right] = E\left[Y_k^{g_j(Y_0,A_0),g_j(Y_1^{g_j,\bar{c}=\bar{0}},A_1^{g_j,\bar{c}=\bar{0}}),\ldots,g_j(Y_{k'}^{g_j,\bar{c}=\bar{0}},A_{k'}^{g_j,\bar{c}=\bar{0}}),\bar{c}=\bar{0}}\right].$$

First, let $t = 0$. Since $P(C_t = 0) = 1$, we can condition on the event $(C_t = 0)$ without changing the expectation. Hence, the above is equal to

$$E\left[Y_k^{g_j(Y_t,A_t),g_j(Y_{t+1}^{g_j,\bar{c}=\bar{0}},A_{t+1}^{g_j,\bar{c}=\bar{0}}),\ldots,g_j(Y_{k'}^{g_j,\bar{c}=\bar{0}},A_{k'}^{g_j,\bar{c}=\bar{0}}),\bar{c}=\bar{0}} \mid C_0 = 0\right].$$

By the Law of Total Expectation, it holds that this is equal to

$$\int\ldots\int E\Big[Y_k^{g_j(Y_t,A_t),g_j(Y_{t+1}^{g_j,\bar{c}=\bar{0}},A_{t+1}^{g_j,\bar{c}=\bar{0}}),\ldots,g_j(Y_{k'}^{g_j,\bar{c}=\bar{0}},A_{k'}^{g_j,\bar{c}=\bar{0}}),\bar{c}=\bar{0}} \mid C_t = 0, \boldsymbol{X}_t = \boldsymbol{x}_t, A_t = a_t\Big]$$
$$f_{\boldsymbol{X}_t,A_t}(\boldsymbol{x}_t, a_t)\,\mathrm{d}\,\boldsymbol{x}_t\,\mathrm{d}\,a_t.$$

By the definition of the counterfactual variables (17), it holds that this is equal to

$$\int\ldots\int E\Big[Y_k^{g_j(y_t,a_t),g_j(Y_{t+1}^{g_j,\bar{c}=\bar{0}},A_{t+1}^{g_j,\bar{c}=\bar{0}}),\ldots,g_j(Y_{k'}^{g_j,\bar{c}=\bar{0}},A_{k'}^{g_j,\bar{c}=\bar{0}}),\bar{c}=\bar{0}} \mid C_t = 0, \boldsymbol{X}_t = \boldsymbol{x}_t, A_t = a_t\Big]$$
$$f_{\boldsymbol{X}_t,A_t}(\boldsymbol{x}_t, a_t)\,\mathrm{d}\,\boldsymbol{x}_t\,\mathrm{d}\,a_t,$$

It follows from exchangeability (20), the definition of the counterfactual variables (17), and consistency (18), that

$$Y_k^{g_j(y_t,a_t),g_j(Y_{t+1}^{g_j,\bar{c}=\bar{0}},A_{t+1}^{g_j,\bar{c}=\bar{0}}),\ldots,g_j(Y_{k'}^{g_j,\bar{c}=\bar{0}},A_{k'}^{g_j,\bar{c}=\bar{0}}),\bar{c}=\bar{0}} \perp\!\!\!\perp A_t \mid \bar{C}_t, \bar{\boldsymbol{X}}_t.$$

Using this, the above is equal to

$$\int\ldots\int E\Big[Y_k^{g_j(y_t,a_t),g_j(Y_{t+1}^{g_j,\bar{c}=\bar{0}},A_{t+1}^{g_j,\bar{c}=\bar{0}}),\ldots,g_j(Y_{k'}^{g_j,\bar{c}=\bar{0}},A_{k'}^{g_j,\bar{c}=\bar{0}}),\bar{c}=\bar{0}} \mid C_t = 0, \boldsymbol{X}_t = \boldsymbol{x}_t, A_t = g_j(y_t, a_t)\Big]$$
$$f_{\boldsymbol{X}_t,A_t}(\boldsymbol{x}_t, a_t)\,\mathrm{d}\,\boldsymbol{x}_t\,\mathrm{d}\,a_t.$$

We now repeat this argument for $t+1$. I follows from exchangeability (21), the definition of the counterfactual variables (17), and consistency (18) that

$$Y_k^{g_j(y_t,a_t),g_j(Y_{t+1}^{g_j,\bar{c}=\bar{0}},A_{t+1}^{g_j,\bar{c}=\bar{0}}),\ldots,g_j(Y_{k'}^{g_j,\bar{c}=\bar{0}},A_{k'}^{g_j,\bar{c}=\bar{0}}),\bar{c}=\bar{0}} \perp\!\!\!\perp C_{t+1} \mid \bar{C}_t, \bar{\boldsymbol{X}}_t, \bar{A}_t.$$

Using this, we obtain that the above is equal to

$$\int \cdots \int E\Big[Y_k^{g_j(y_t,a_t),g_j(Y_{t+1}^{g_j,\bar{c}=\bar{0}},A_{t+1}^{g_j,\bar{c}=\bar{0}}),\ldots,g_j(Y_{k'}^{g_j,\bar{c}=\bar{0}},A_{k'}^{g_j,\bar{c}=\bar{0}}),\bar{c}=\bar{0}} \mid \bar{C}_{t+1}=\bar{0}, \boldsymbol{X}_t=\boldsymbol{x}_t, A_t=g_j(y_t,a_t)\Big]$$
$$f_{\boldsymbol{X}_t,A_t}(\boldsymbol{x}_t,a_t)\,\mathrm{d}\,\boldsymbol{x}_t\,\mathrm{d}\,a_t.$$

By the Law of Total Expectation, this is equal to

$$\int \cdots \int E\Big[Y_k^{g_j(y_t,a_t),g_j(Y_{t+1}^{g_j,\bar{c}=\bar{0}},A_{t+1}^{g_j,\bar{c}=\bar{0}}),\ldots,g_j(Y_{k'}^{g_j,\bar{c}=\bar{0}},A_{k'}^{g_j,\bar{c}=\bar{0}}),\bar{c}=\bar{0}}$$
$$\mid \bar{C}_{t+1}=\bar{0}, \bar{\boldsymbol{X}}_{t+1}=\bar{\boldsymbol{x}}_{t+1}, A_t=g_j(y_t,a_t), A_{t+1}=a_{t+1}\Big]$$
$$f_{\boldsymbol{X}_{t+1},A_{t+1}\mid\bar{C}_{t+1},\boldsymbol{X}_t,A_t}(\boldsymbol{x}_{t+1},a_{t+1}\mid\bar{0},\boldsymbol{x}_t,g_j(y_t,a_t))f_{\boldsymbol{X}_t,A_t}(\boldsymbol{x}_t,a_t)$$
$$\mathrm{d}\,\boldsymbol{x}_{t+1}\,\mathrm{d}\,a_{t+1}\,\mathrm{d}\,\boldsymbol{x}_t\,\mathrm{d}\,a_t.$$

By consistency (18) and the definition of the counterfactual variables (17), this is equal to

$$\int \cdots \int E\Big[Y_k^{g_j(y_t,a_t),g_j(y_{t+1},a_{t+1}),g_j(Y_{t+2}^{g_j,\bar{c}=\bar{0}},A_{t+2}^{g_j,\bar{c}=\bar{0}}),\ldots,g_j(Y_{k'}^{g_j,\bar{c}=\bar{0}},A_{k'}^{g_j,\bar{c}=\bar{0}}),\bar{c}=\bar{0}}$$
$$\mid \bar{C}_{t+1}=\bar{0}, \bar{\boldsymbol{X}}_{t+1}=\bar{\boldsymbol{x}}_{t+1}, A_t=g_j(y_t,a_t), A_{t+1}=a_{t+1}\Big]$$
$$f_{\boldsymbol{X}_{t+1},A_{t+1}\mid\bar{C}_{t+1},\boldsymbol{X}_t,A_t}(\boldsymbol{x}_{t+1},a_{t+1}\mid\bar{0},\boldsymbol{x}_t,g_j(y_t,a_t))f_{\boldsymbol{X}_t,A_t}(\boldsymbol{x}_t,a_t)$$
$$\mathrm{d}\,\boldsymbol{x}_{t+1}\,\mathrm{d}\,a_{t+1}\,\mathrm{d}\,\boldsymbol{x}_t\,\mathrm{d}\,a_t.$$

By exchangeability (20) and consistency (18), this is equal to

$$\int \cdots \int E\Big[Y_k^{g_j(y_t,a_t),g_j(y_{t+1},a_{t+1}),g_j(Y_{t+2}^{g_j,\bar{c}=\bar{0}},A_{t+2}^{g_j,\bar{c}=\bar{0}}),\ldots,g_j(Y_{k'}^{g_j,\bar{c}=\bar{0}},A_{k'}^{g_j,\bar{c}=\bar{0}}),\bar{c}=\bar{0}}$$
$$\mid \bar{C}_{t+1}=\bar{0}, \bar{\boldsymbol{X}}_{t+1}=\bar{\boldsymbol{x}}_{t+1}, A_t=g_j(y_t,a_t), A_{t+1}=g_j(t_{t+1},a_{t+1})\Big]$$
$$f_{\boldsymbol{X}_{t+1},A_{t+1}\mid\bar{C}_{t+1},\boldsymbol{X}_t,A_t}(\boldsymbol{x}_{t+1},a_{t+1}\mid\bar{0},\boldsymbol{x}_t,g_j(y_t,a_t))f_{\boldsymbol{X}_t,A_t}(\boldsymbol{x}_t,a_t)$$

$$\mathrm{d}\boldsymbol{x}_{t+1}\,\mathrm{d}a_{t+1}\,\mathrm{d}\boldsymbol{x}_t\,\mathrm{d}a_t.$$

By repeating this argument for $t = 2$ to $t = k'$, we obtain that the above is equal to

$$\int \ldots \int E\Big[Y_k^{g_j(y_0,a_0),\ldots,g_j(y_{k'},a_{k'}),\bar{c}=\bar{0}} \mid \bar{C}_{k'} = \bar{0}, \bar{\boldsymbol{X}}_{k'} = \bar{\boldsymbol{x}}_{k'}, A_0 = g_j(y_0,a_0),\ldots, A_{k'} = g_j(y_{k'},a_{k'})\Big]$$
$$\prod_{s=0}^{k'} f_{\boldsymbol{X}_s,A_s|\bar{C}_s,\bar{\boldsymbol{X}}_{s-1},\bar{A}_{s-1}}(\boldsymbol{x}_s, a_s \mid \bar{0}, \bar{\boldsymbol{x}}_{s-1}, g_j(y_0,a_0),\ldots,g_j(y_{s-1},a_{s-1}))\,\mathrm{d}\boldsymbol{x}_{k'}\,\mathrm{d}a_{k'}\ldots\mathrm{d}\boldsymbol{x}_0\,\mathrm{d}a_0.$$

Now, let $t = k' + 1$. By exchangeability (21) and consistency (18), the above is equal to

$$\int \ldots \int E\Big[Y_k^{g_j(y_0,a_0),\ldots,g_j(y_{k'},a_{k'}),\bar{c}=\bar{0}} \mid \bar{C}_t = \bar{0}, \bar{\boldsymbol{X}}_{k'} = \bar{\boldsymbol{x}}_{k'}, A_0 = g_j(y_0,a_0),\ldots, A_{k'} = g_j(y_{k'},a_{k'})\Big]$$
$$\prod_{s=0}^{k'} f_{\boldsymbol{X}_s,A_s\,|\,\bar{C}_s,\bar{\boldsymbol{X}}_{s-1},\bar{A}_{s-1}}(\boldsymbol{x}_s, a_s \mid \bar{0}, \bar{\boldsymbol{x}}_{s-1}, g_j(y_0,a_0),\ldots,g_j(y_{s-1},a_{s-1}))$$
$$\mathrm{d}\boldsymbol{x}_{k'}\,\mathrm{d}a_{k'}\ldots\mathrm{d}\boldsymbol{x}_0\,\mathrm{d}a_0.$$

By the Law of Total Expectation, this is equal to

$$\int \ldots \int E\Big[Y_k^{g_j(y_0,a_0),\ldots,g_j(y_{k'},a_{k'}),\bar{c}=\bar{0}} \mid \bar{C}_t = \bar{0}, \bar{\boldsymbol{X}}_t = \bar{\boldsymbol{x}}_t, A_0 = g_j(y_0,a_0),\ldots, A_{k'} = g_j(y_{k'},a_{k'}), A_t = a_t\Big]$$
$$f_{\boldsymbol{X}_t,A_t|\bar{C}_t,\bar{\boldsymbol{X}}_{t-1},\bar{A}_{t-1}}(\boldsymbol{x}_t, a_t \mid \bar{0}, \bar{\boldsymbol{x}}_{t-1}, g_j(y_0,a_0),\ldots,g_j(y_{t-1},a_{t-1}))$$
$$\prod_{s=0}^{k'} f_{\boldsymbol{X}_s,A_s|\bar{C}_s,\bar{\boldsymbol{X}}_{s-1},\bar{A}_{s-1}}(\boldsymbol{x}_s, a_s \mid \bar{0}, \bar{\boldsymbol{x}}_{s-1}, g_j(y_0,a_0),\ldots,g_j(y_{s-1},a_{s-1}))$$
$$\mathrm{d}\boldsymbol{x}_t\,\mathrm{d}a_t\ldots\mathrm{d}\boldsymbol{x}_0\,\mathrm{d}a_0.$$

We now repeat this argument for $t + 1$. By exchangeability (21) and consistency (18), the above is equal to

$$\int \ldots \int E\Big[Y_k^{g_j(y_0,a_0),\ldots,g_j(y_{k'},a_{k'}),\bar{c}=\bar{0}} \mid \bar{C}_{t+1} = \bar{0}, \bar{\boldsymbol{X}}_t = \bar{\boldsymbol{x}}_t, A_0 = g_j(y_0,a_0),\ldots, A_{k'} = g_j(y_{k'},a_{k'}), A_t = a_t\Big]$$
$$f_{\boldsymbol{X}_t,A_t|\bar{C}_t,\bar{\boldsymbol{X}}_{t-1},\bar{A}_{t-1}}(\boldsymbol{x}_t, a_t \mid \bar{0}, \bar{\boldsymbol{x}}_{t-1}, g_j(y_0,a_0),\ldots,g_j(y_{t-1},a_{t-1}))$$
$$\prod_{s=0}^{k'} f_{\boldsymbol{X}_s,A_s|\bar{C}_s,\bar{\boldsymbol{X}}_{s-1},\bar{A}_{s-1}}(\boldsymbol{x}_s, a_s \mid \bar{0}, \bar{\boldsymbol{x}}_{s-1}, g_j(y_0,a_0),\ldots,g_j(y_{s-1},a_{s-1}))$$
$$\mathrm{d}\boldsymbol{x}_t\,\mathrm{d}a_t\ldots\mathrm{d}\boldsymbol{x}_0\,\mathrm{d}a_0.$$

By the Law of Total Expectation, this is equal to

$$\int \dots \int E\Big[Y_k^{g_j(y_0,a_0),\dots,g_j(y_{k'},a_{k'}),\bar{c}=\bar{0}} \mid \bar{C}_{t+1} = \bar{0}, \bar{\boldsymbol{X}}_{t+1} = \bar{\boldsymbol{l}}_{t+1},$$
$$A_0 = g_j(y_0,a_0),\dots,A_{k'} = g_j(y_{k'},a_{k'}), A_t = a_t, A_{t+1} = a_{t+1}\Big]$$
$$f_{\boldsymbol{X}_{t+1},A_{t+1}|\bar{C}_{t+1},\bar{\boldsymbol{X}}_t,\bar{A}_t}(\boldsymbol{x}_{t+1}, a_{t+1} \mid \bar{0}, \bar{\boldsymbol{x}}_t, g_j(y_0,a_0),\dots,g_j(y_{k'},a_{k'}),a_t)$$
$$f_{\boldsymbol{X}_t,A_t|\bar{C}_t,\bar{\boldsymbol{X}}_{t-1},\bar{A}_{t-1}}(\boldsymbol{x}_t, a_t \mid \bar{0}, \bar{\boldsymbol{x}}_{t-1}, g_j(y_0,a_0),\dots,g_j(y_{t-1},a_{t-1}))$$
$$\prod_{s=0}^{k'} f_{\boldsymbol{X}_s,A_s|\bar{C}_s,\bar{\boldsymbol{X}}_{s-1},\bar{A}_{s-1}}(\boldsymbol{x}_s, a_s \mid \bar{0}, \bar{\boldsymbol{x}}_{s-1}, g_j(y_0,a_0),\dots,g_j(y_{s-1},a_{s-1}))$$
$$\mathrm{d}\boldsymbol{x}_{t+1}\,\mathrm{d}a_{t+1}\dots\mathrm{d}\boldsymbol{x}_0\,\mathrm{d}a_0.$$

By repeating this argument for $t = k'+3$ to $t = k-1$, we obtain that the above is equal to

$$\int \dots \int E\Big[Y_k^{g_j(y_0,a_0),\dots,g_j(y_{k'},a_{k'}),\bar{c}=\bar{0}} \mid \bar{C}_{k-1} = \bar{0}, \bar{\boldsymbol{X}}_{k-1} = \bar{\boldsymbol{x}}_{k-1},$$
$$A_0 = g_j(y_0,a_0),\dots,A_{k'} = g_j(y_{k'},a_{k'}), A_{k'+1} = a_{k'+1},\dots,A_{k-1} = a_{k-1}\Big]$$
$$\prod_{t=k'+1}^{k-1} f_{\boldsymbol{X}_t,A_t|\bar{C}_t,\bar{\boldsymbol{X}}_{t-1},\bar{A}_{t-1}}(\boldsymbol{x}_t, a_t \mid \bar{0}, \bar{\boldsymbol{x}}_{t-1}, g_j(y_0,a_0),\dots,g_j(y_{k'},a_{k'}), a_{k'+1},\dots,a_{t-1})$$
$$\prod_{s=0}^{k'} f_{\boldsymbol{X}_s,A_s|\bar{C}_s,\bar{\boldsymbol{X}}_{s-1},\bar{A}_{s-1}}(\boldsymbol{x}_s, a_s \mid \bar{0}, \bar{\boldsymbol{x}}_{s-1}, g_j(y_0,a_0),\dots,g_j(y_{s-1},a_{s-1}))$$
$$\mathrm{d}\boldsymbol{x}_{k-1}\,\mathrm{d}a_{k-1}\dots\mathrm{d}\boldsymbol{x}_0\,\mathrm{d}a_0.$$

By consistency (18) and exchangeability (21), this is equal to

$$\int \dots \int E\Big[Y_k^{g_j(y_0,a_0),\dots,g_j(y_{k'},a_{k'}),\bar{c}=\bar{0}} \mid \bar{C}_k = \bar{0}, \bar{\boldsymbol{X}}_{k-1} = \bar{\boldsymbol{x}}_{k-1},$$
$$A_0 = g_j(y_0,a_0),\dots,A_{k'} = g_j(y_{k'},a_{k'}), A_{k'+1} = a_{k'+1},\dots,A_{k-1} = a_{k-1}\Big]$$
$$\prod_{s=k'+1}^{k-1} f_{\boldsymbol{X}_s,A_s|\bar{C}_s,\bar{\boldsymbol{X}}_{s-1},\bar{A}_{s-1}}(\boldsymbol{x}_s, a_s \mid \bar{0}, \bar{\boldsymbol{x}}_{s-1}, g_j(y_0,a_0),\dots,g_j(y_{k'},a_{k'}), a_{k'+1},\dots,a_{s-1})$$
$$\prod_{s=0}^{k'} f_{\boldsymbol{X}_s,A_s|\bar{C}_s,\bar{\boldsymbol{X}}_{s-1},\bar{A}_{s-1}}(d_s, \boldsymbol{l}_s, y_s, a_s \mid \bar{0}, \bar{\boldsymbol{x}}_{s-1}, g_j(y_0,a_0),\dots,g_j(y_{s-1},a_{s-1}))$$
$$\mathrm{d}\boldsymbol{x}_{k-1}\,\mathrm{d}a_{k-1}\dots\mathrm{d}\boldsymbol{x}_0\,\mathrm{d}a_0.$$

By consistency (18), this is equal to

$$\int \dots \int E\Big[Y_k \mid \bar{C}_k = \bar{0}, \bar{\boldsymbol{X}}_{k-1} = \bar{\boldsymbol{x}}_{k-1},$$
$$A_0 = g_j(y_0, a_0), \dots, A_{k'} = g_j(y_{k'}, a_{k'}), A_{k'+1} = a_{k'+1}, \dots, A_{k-1} = a_{k-1}\Big]$$
$$\prod_{s=k'+1}^{k-1} f_{\boldsymbol{X}_s, A_s \mid \bar{C}_s, \bar{\boldsymbol{X}}_{s-1}, \bar{A}_{s-1}}(\boldsymbol{x}_s, a_s \mid \bar{0}, \bar{\boldsymbol{x}}_{s-1}, g_j(y_0, a_0), \dots, g_j(y_{k'}, a_{k'}), a_{k'+1}, \dots, a_{s-1})$$
$$\prod_{s=0}^{k'} f_{\boldsymbol{X}_s, A_s \mid \bar{C}_s, \bar{\boldsymbol{X}}_{s-1}, \bar{A}_{s-1}}(\boldsymbol{x}_s, a_s \mid \bar{0}, \bar{\boldsymbol{x}}_{s-1}, g_j(y_0, a_0), \dots, g_j(y_{s-1}, a_{s-1}))$$
$$\mathrm{d}\boldsymbol{x}_{k-1}\, \mathrm{d}a_{k-1} \dots \mathrm{d}\boldsymbol{x}_0\, \mathrm{d}a_0.$$

The conditional expectation and all the density functions are conditional on a history of no censoring. Hence, by definition of censoring (Definition 9 of the main article), the above is equal to

$$\int \dots \int E\Big[\tilde{Y}_k \mid \bar{C}_k = \bar{0}, \bar{\tilde{X}}_{k-1} = \bar{\boldsymbol{x}}_{k-1},$$
$$\tilde{A}_0 = g_j(y_0, a_0), \dots, \tilde{A}_{k'} = g_j(y_{k'}, a_{k'}), \tilde{A}_{k'+1} = a_{k'+1}, \dots, \tilde{A}_{k-1} = a_{k-1}\Big]$$
$$\prod_{s=k'+1}^{k-1} f_{\tilde{X}_s, \tilde{A}_s \mid \bar{C}_s, \bar{\tilde{X}}_{s-1}, \bar{\tilde{A}}_{s-1}}(\boldsymbol{x}_s, a_s \mid \bar{0}, \bar{\boldsymbol{x}}_{s-1}, g_j(y_0, a_0), \dots, g_j(y_{k'}, a_{k'}), a_{k'+1}, \dots, a_{s-1})$$
$$\prod_{s=0}^{k'} f_{\tilde{X}_s, \tilde{A}_s \mid \bar{C}_s, \bar{\tilde{X}}_{s-1}, \bar{\tilde{A}}_{s-1}}(\boldsymbol{x}_s, a_s \mid \bar{0}, \bar{\boldsymbol{x}}_{s-1}, g_j(y_0, a_0), \dots, g_j(y_{s-1}, a_{s-1}))$$
$$\mathrm{d}\boldsymbol{x}_{k-1}\, \mathrm{d}a_{k-1} \dots \mathrm{d}\boldsymbol{x}_0\, \mathrm{d}a_0.$$

□

B.4 Inverse probability weighted identification formula

Next, we prove an inverse probability weighted identification formula in the presence of censoring and competing events.

Corollary 7.1 (IPW)**.** *Let g_j be an add-on-j regime and assume that consistency (18),* *exchangeability (20) - (21), and positivity (22) - (23) hold. Then*

$$E\Big[Y_k^{g_j,\bar{c}=\bar{0}}\Big] = E\Big[\tilde{Y}_k W_k^C W_{k'}^A\Big], \tag{25}$$

for all $k \geq 1$ where $k' = \min\{k-1, \kappa\}$ and

$$W_s^C = \prod_{t=0}^{s} \frac{1\{C_t = 0\}}{f_{C_t|\bar{C}_{t-1},\bar{\tilde{D}}_{t-1},\bar{\tilde{L}}_{t-1},\bar{\tilde{Y}}_{t-1},\bar{\tilde{A}}_{t-1}}(0 \mid \bar{0}, \bar{\tilde{D}}_{t-1}, \bar{\tilde{L}}_{t-1}, \bar{\tilde{Y}}_{t-1}, \bar{\tilde{A}}_{t-1})} \quad \forall s \leq K, \tag{26}$$

$$W_s^A = \sum_{a_0=0}^{1} \cdots \sum_{a_s=0}^{1} \prod_{t=0}^{s} \frac{f_{\tilde{A}_t|\bar{C}_t,\bar{\tilde{D}}_t,\bar{\tilde{L}}_t,\bar{\tilde{Y}}_t,\bar{\tilde{A}}_{t-1}}(a_t \mid \bar{0}, \bar{\tilde{D}}_t, \bar{\tilde{L}}_t, \bar{\tilde{Y}}_t, \bar{\tilde{A}}_{t-1}\}}{f_{\tilde{A}_t|\bar{C}_t,\bar{\tilde{D}}_t,\bar{\tilde{L}}_t,\bar{\tilde{Y}}_t,\bar{\tilde{A}}_{t-1}}(\tilde{A}_t \mid \bar{0}, \bar{\tilde{D}}_{t-1}, \bar{\tilde{L}}_{t-1}, \bar{\tilde{Y}}_{t-1}, \bar{\tilde{A}}_{t-1})} \quad \forall s \leq \kappa, \tag{27}$$

Proof. Let g_j be an add-on-j regime. Let $k \in \{1, \ldots, K\}$ and denote $k' = \min\{k-1, \kappa\}$. For notational convenience, denote $\boldsymbol{X}_k = (D_k, \boldsymbol{L}_k, Y_k)$ and $\boldsymbol{x}_k = (d_k, \boldsymbol{l}_k, y_k)$ for every $k \leq K$. By Theorem 7, under consistency (18), exchangeability (20) - (21), and positivity (22) - (23), we have that

$$\begin{aligned}
&\int \cdots \int E\Big[Y_k \mid \bar{C}_k = \bar{0}, \bar{\boldsymbol{X}}_{k-1} = \bar{\boldsymbol{x}}_{k-1},\\
&A_0 = g_j(y_0, a_0), \ldots, A_{k'} = g_j(y_{k'}, a_{k'}), A_{k'+1} = a_{k'+1}, \ldots, A_{k-1} = a_{k-1}\Big]\\
&\prod_{s=k'+1}^{k-1} f_{\boldsymbol{X}_s,A_s|\bar{C}_s,\bar{\boldsymbol{X}}_{s-1},\bar{A}_{s-1}}(\boldsymbol{x}_s, a_s \mid \bar{0}, \bar{\boldsymbol{x}}_{s-1}, g_j(y_0, a_0), \ldots, g_j(y_{k'}, a_{k'}), a_{k'+1}, \ldots, a_{s-1})\\
&\prod_{s=0}^{k'} f_{\boldsymbol{X}_s,A_s|\bar{C}_s,\bar{\boldsymbol{X}}_{s-1},\bar{A}_{s-1}}(\boldsymbol{x}_s, a_s \mid \bar{0}, \bar{\boldsymbol{x}}_{s-1}, g_j(y_0, a_0), \ldots, g_j(y_{s-1}, a_{s-1}))\\
&\mathrm{d}\boldsymbol{x}_{k-1}\,\mathrm{d}a_{k-1} \ldots \mathrm{d}\boldsymbol{x}_0\,\mathrm{d}a_0.
\end{aligned}$$

Let $t = k$. By Bayes' Law, the conditional expectation can be rewritten, so the above equals

$$\begin{aligned}
&\int \cdots \int E\Big[Y_t \frac{1\{C_t = 0\}}{f_{C_t|\bar{C}_{t-1},\bar{\boldsymbol{X}}_{t-1},\bar{A}_{t-1}}(0 \mid \bar{0}, \bar{\boldsymbol{x}}_{t-1}, g_j(y_0, a_0), \ldots, g_j(y_{k'}, a_{k'}), a_{k'+1}, \ldots \bar{a}_{t-1})}\\
&\mid \bar{C}_{t-1} = \bar{0}, \bar{\boldsymbol{X}}_{t-1} = \bar{\boldsymbol{x}}_{t-1}, A_0 = g_j(y_0, a_0), \ldots, A_{k'} = g_j(y_{k'}, a_{k'}), A_{k'+1} = a_{k'+1}, \ldots, A_{t-1} = a_{t-1}\Big]\\
&\prod_{s=0}^{t-1} f_{\boldsymbol{X}_s,A_s|\bar{C}_s,\bar{\boldsymbol{X}}_{s-1},\bar{A}_{s-1}}(\boldsymbol{x}_s, a_s \mid \bar{0}, \bar{\boldsymbol{x}}_{s-1}, g_j(y_0, a_0), \ldots, g_j(y_{k'}, a_{k'}), a_{k'+1}, \ldots, a_{s-1})
\end{aligned}$$

$\mathrm{d}\boldsymbol{x}_{t-1}\,\mathrm{d}a_{t-1}\ldots\mathrm{d}\boldsymbol{x}_0\,\mathrm{d}a_0.$

By the Law of Total expectation, this is equal to

$$\int\cdots\int E\Big[Y_t\frac{1\{C_t=0\}}{f_{C_t|\bar{C}_{t-1},\bar{\boldsymbol{X}}_{t-1},A_{t-1}}(0\mid\bar{0},\bar{\boldsymbol{x}}_{t-2},\boldsymbol{X}_{t-1},g_j(y_0,a_0),\ldots,g_j(y_{k'},a_{k'}),a_{k'+1},\ldots a_{t-2},A_{t-1})}$$

$$\mid\bar{C}_{t-1}=\bar{0},\bar{\boldsymbol{X}}_{t-2}=\bar{\boldsymbol{x}}_{t-1},A_0=g_j(y_0,a_0),\ldots,A_{k'}=g_j(y_{k'},a_{k'}),A_{k'+1}=a_{k'+1},\ldots,A_{t-2}=a_{t-2}\Big]$$

$$\prod_{s=0}^{t-2}f_{\boldsymbol{X}_s,A_s|\bar{C}_s,\bar{\boldsymbol{X}}_{s-1},\bar{A}_{s-1}}(\boldsymbol{x}_s,a_s\mid\bar{0},\bar{\boldsymbol{x}}_{s-1},g_j(y_0,a_0),\ldots,g_j(y_{k'},a_{k'}),a_{k'+1},\ldots,a_{s-1})$$

$\mathrm{d}\boldsymbol{x}_{t-2}\,\mathrm{d}a_{t-2}\ldots\mathrm{d}\boldsymbol{x}_0\,\mathrm{d}a_0.$

By repeating this argument for $t=k-1$ to $t=k'+2$, we obtain

$$\int\cdots\int E\Big[Y_t\prod_{s=k'+2}^{k}\frac{1\{C_s=0\}}{f_{C_s|\bar{C}_{s-1},\bar{\boldsymbol{X}}_{s-1},A_{s-1}}(0\mid\bar{0},\bar{\boldsymbol{x}}_{s-1},g_j(y_0,a_0),\ldots,g_j(y_{k'},a_{k'}),A_{k'+1},\ldots A_{s-1})}$$

$$\mid\bar{C}_{k'+1}=\bar{0},\bar{\boldsymbol{X}}_{k'+1}=\bar{\boldsymbol{x}}_{k'},A_0=g_j(y_0,a_0),\ldots,A_{k'}=g_j(y_{k'},a_{k'})\Big]$$

$$\prod_{s=0}^{k'}f_{\boldsymbol{X}_s,A_s|\bar{C}_s,\bar{\boldsymbol{X}}_{s-1},\bar{A}_{s-1}}(\boldsymbol{x}_s,a_s\mid\bar{0},\bar{\boldsymbol{x}}_{s-1},g_j(y_0,a_0),\ldots,g_j(y_{s-1},a_{s-1}))$$

$\mathrm{d}\boldsymbol{x}_{k'}\,\mathrm{d}a_{k'}\ldots\mathrm{d}\boldsymbol{x}_0\,\mathrm{d}a_0.$

Now, let $t=k'+1$. Again, by Bayes Law, the above is equal to

$$\int\cdots\int E\Big[Y_t\prod_{s=t}^{k}\frac{1\{C_s=0\}}{f_{C_s|\bar{C}_{s-1},\bar{\boldsymbol{X}}_{s-1},A_{s-1}}(0\mid\bar{0},\bar{\boldsymbol{x}}_{s-1},g_j(y_0,a_0),\ldots,g_j(y_{t-1},a_{t-1}),A_t,\ldots A_{s-1})}$$

$$\mid\bar{C}_{t-1}=\bar{0},\bar{\boldsymbol{X}}_{t-1}=\bar{\boldsymbol{x}}_{t-1},A_0=g_j(y_0,a_0),\ldots,A_{t-1}=g_j(y_{t-1},a_{t-1})\Big]$$

$$\prod_{s=0}^{t-1}f_{\boldsymbol{X}_s,A_s|\bar{C}_s,\bar{\boldsymbol{X}}_{s-1},\bar{A}_{s-1}}(\boldsymbol{x}_s,a_s\mid\bar{0},\bar{\boldsymbol{x}}_{s-1},g_j(y_0,a_0),\ldots,g_j(y_{s-1},a_{s-1}))$$

$\mathrm{d}\boldsymbol{x}_{t-1}\,\mathrm{d}a_{t-1}\ldots\mathrm{d}\boldsymbol{x}_0\,\mathrm{d}a_0.$

By multiplying and dividing by

$$f_{\boldsymbol{X}_{t-1},A_{t-1}|\bar{C}_{t-1},\bar{\boldsymbol{X}}_{t-2},\bar{A}_{t-2}}(\boldsymbol{x}_{t-1},g_j(y_{t-1},a_{t-1})\mid\bar{0},\bar{\boldsymbol{x}}_{t-2},g_j(y_0,a_0),\ldots,g_j(y_{t-2},a_{t-2})),$$

we obtain that the above is equal to

$$\int \dots \int E\Big[Y_t \prod_{s=t}^{k} \frac{1\{C_s = 0\}}{f_{C_s|\bar{C}_{s-1},\bar{\boldsymbol{X}}_{s-1},A_{s-1}}(0 \mid \bar{0}, \bar{\boldsymbol{x}}_{s-1}, g_j(y_0,a_0),\dots,g_j(y_{t-1},a_{t-1}), A_t, \dots A_{s-1})}$$
$$\frac{f_{\boldsymbol{X}_{t-1},A_{t-1}|\bar{C}_{t-1},\bar{\boldsymbol{X}}_{t-2},\bar{A}_{t-2}}(\boldsymbol{x}_{t-1}, a_{t-1} \mid \bar{0}, \bar{\boldsymbol{x}}_{t-2}, g_j(y_0,a_0),\dots,g_j(y_{t-2},a_{t-2}))}{f_{\boldsymbol{X}_{t-1},A_{t-1}|\bar{C}_{t-1},\bar{\boldsymbol{X}}_{t-2},\bar{A}_{t-2}}(\boldsymbol{x}_{t-1}, g_j(y_{t-1},a_{t-1}) \mid \bar{0}, \bar{\boldsymbol{x}}_{t-2}, g_j(y_0,a_0),\dots,g_j(y_{t-2},a_{t-2}))}$$
$$\mid \bar{C}_{t-1} = \bar{0}, \bar{\boldsymbol{X}}_{t-1} = \bar{\boldsymbol{x}}_{t-1}, A_0 = g_j(y_0,a_0),\dots,A_{t-1} = g_j(y_{t-1},a_{t-1})\Big]$$
$$f_{\boldsymbol{X}_{t-1},A_{t-1}|\bar{C}_{t-1},\bar{\boldsymbol{X}}_{t-2},\bar{A}_{t-2}}(\boldsymbol{x}_{t-1}, g_j(y_{t-1},a_{t-1}) \mid \bar{0}, \bar{\boldsymbol{x}}_{t-2}, g_j(y_0,a_0),\dots,g_j(y_{t-2},a_{t-2}))$$
$$\prod_{s=0}^{t-2} f_{\boldsymbol{X}_s,A_s|\bar{C}_s,\bar{\boldsymbol{X}}_{s-1},\bar{A}_{s-1}}(\boldsymbol{x}_s, a_s \mid \bar{0}, \bar{\boldsymbol{x}}_{s-1}, g_j(y_0,a_0),\dots,g_j(y_{s-1},a_{s-1}))$$
$$\mathrm{d}\boldsymbol{x}_{t-1}\,\mathrm{d}a_{t-1}\dots\mathrm{d}\boldsymbol{x}_0\,\mathrm{d}a_0.$$

By standard properties of joint and conditional densities, we have that

$$\int \dots \int E\Big[Y_t \prod_{s=t}^{k} \frac{1\{C_s = 0\}}{f_{C_s|\bar{C}_{s-1},\bar{\boldsymbol{X}}_{s-1},A_{s-1}}(0 \mid \bar{0}, \bar{\boldsymbol{x}}_{s-1}, g_j(y_0,a_0),\dots,g_j(y_{t-1},a_{t-1}), A_t, \dots A_{s-1})}$$
$$\frac{f_{\boldsymbol{X}_{t-1},A_{t-1}|\bar{C}_{t-1},\bar{\boldsymbol{X}}_{t-2},\bar{A}_{t-2}}(\boldsymbol{x}_{t-1}, a_{t-1} \mid \bar{0}, \bar{\boldsymbol{x}}_{t-2}, g_j(y_0,a_0),\dots,g_j(y_{t-2},a_{t-2}))}{f_{\boldsymbol{X}_{t-1},A_{t-1}|\bar{C}_{t-1},\bar{\boldsymbol{X}}_{t-2},\bar{A}_{t-2}}(\boldsymbol{x}_{t-1}, g_j(y_{t-1},a_{t-1}) \mid \bar{0}, \bar{\boldsymbol{x}}_{t-2}, g_j(y_0,a_0),\dots,g_j(y_{t-2},a_{t-2}))}$$
$$\mid \bar{C}_{t-1} = \bar{0}, \bar{\boldsymbol{X}}_{t-1} = \bar{\boldsymbol{x}}_{t-1}, A_0 = g_j(y_0,a_0),\dots,A_{t-1} = g_j(y_{t-1},a_{t-1})\Big]$$
$$f_{A_{t-1}|\bar{C}_{t-1},\bar{\boldsymbol{X}}_{t-1},\bar{A}_{t-2}}(g_j(y_{t-1},a_{t-1}) \mid \bar{0}, \bar{\boldsymbol{x}}_{t-1}, g_j(y_0,a_0),\dots,g_j(y_{t-2},a_{t-2}))$$
$$f_{\boldsymbol{X}_{t-1},|\bar{C}_{t-1},\bar{\boldsymbol{X}}_{t-2},\bar{A}_{t-2}}(\boldsymbol{x}_{t-1} \mid \bar{0}, \bar{\boldsymbol{x}}_{t-2}, g_j(y_0,a_0),\dots,g_j(y_{t-2},a_{t-2}))$$
$$\prod_{s=0}^{t-2} f_{\boldsymbol{X}_s,A_s|\bar{C}_s,\bar{\boldsymbol{X}}_{s-1},\bar{A}_{s-1}}(\boldsymbol{x}_s, a_s \mid \bar{0}, \bar{\boldsymbol{x}}_{s-1}, g_j(y_0,a_0),\dots,g_j(y_{s-1},a_{s-1}))$$
$$\mathrm{d}\boldsymbol{x}_{t-1}\,\mathrm{d}a_{t-1}\dots\mathrm{d}\boldsymbol{x}_0\,\mathrm{d}a_0.$$

By Bayes Law, this is equal to

$$\int \dots \int E\Big[Y_t \prod_{s=t}^{k} \frac{1\{C_s = 0\}}{f_{C_s|\bar{C}_{s-1},\bar{\boldsymbol{X}}_{s-1},A_{s-1}}(0 \mid \bar{0}, \bar{\boldsymbol{x}}_{s-1}, g_j(y_0,a_0),\dots,g_j(y_{t-1},a_{t-1}), A_t, \dots A_{s-1})}$$
$$1\{A_{t-1} = g_j(y_{t-1},a_{t-1})\}$$
$$\frac{f_{\boldsymbol{X}_{t-1},A_{t-1}|\bar{C}_{t-1},\bar{\boldsymbol{X}}_{t-2},\bar{A}_{t-2}}(\boldsymbol{x}_{t-1}, a_{t-1} \mid \bar{0}, \bar{\boldsymbol{x}}_{t-2}, g_j(y_0,a_0),\dots,g_j(y_{t-2},a_{t-2}))}{f_{\boldsymbol{X}_{t-1},A_{t-1}|\bar{C}_{t-1},\bar{\boldsymbol{X}}_{t-2},\bar{A}_{t-2}}(\boldsymbol{x}_{t-1}, g_j(y_{t-1},a_{t-1}) \mid \bar{0}, \bar{\boldsymbol{x}}_{t-2}, g_j(y_0,a_0),\dots,g_j(y_{t-2},a_{t-2}))}$$

$$\mid \bar{C}_{t-1} = \bar{0}, \bar{\boldsymbol{X}}_{t-1} = \bar{\boldsymbol{x}}_{t-1}, A_0 = g_j(y_0, a_0), \ldots, A_{t-2} = g_j(y_{t-2}, a_{t-2})\Big]$$

$$f_{\boldsymbol{X}_{t-1}|\bar{C}_{t-1}, \bar{\boldsymbol{X}}_{t-2}, \bar{A}_{t-2}}(\boldsymbol{x}_{t-1} \mid \bar{0}, \bar{\boldsymbol{x}}_{t-2}, g_j(y_0, a_0), \ldots, g_j(y_{t-2}, a_{t-2}))$$

$$\prod_{s=0}^{t-2} f_{\boldsymbol{X}_s, A_s|\bar{C}_s, \bar{\boldsymbol{X}}_{s-1}, \bar{A}_{s-1}}(\boldsymbol{x}_s, a_s \mid \bar{0}, \bar{\boldsymbol{x}}_{s-1}, g_j(y_0, a_0), \ldots, g_j(y_{s-1}, a_{s-1}))$$

$$\mathrm{d}\,\boldsymbol{x}_{t-1}\,\mathrm{d}\,a_{t-1} \ldots \mathrm{d}\,\boldsymbol{x}_0\,\mathrm{d}\,a_0.$$

By the Law of Total Expectation, the above is equal to

$$\int \ldots \int E\Big[Y_t$$

$$\prod_{s=t}^{k} \frac{1\{C_s = 0\}}{f_{C_s|\bar{C}_{s-1}, \bar{\boldsymbol{X}}_{s-1}, \bar{A}_{s-1}}(0 \mid \bar{0}, \bar{\boldsymbol{x}}_{t-2}, \boldsymbol{X}_{t-1}, \ldots, \boldsymbol{X}_{s-1}, g_j(y_0, a_0), \ldots, g_j(Y_{t-1}, a_{t-1}), A_t, \ldots A_{s-1})}$$

$$1\{A_{t-1} = g_j(Y_{t-1}, a_{t-1})\}$$

$$\frac{f_{\boldsymbol{X}_{t-1}, A_{t-1}|\bar{C}_{t-1}, \bar{\boldsymbol{X}}_{t-2}, \bar{A}_{t-2}}(\boldsymbol{X}_{t-1}, a_{t-1} \mid \bar{0}, \bar{\boldsymbol{x}}_{t-2}, g_j(y_0, a_0), \ldots, g_j(y_{t-2}, a_{t-2}))}{f_{\boldsymbol{X}_{t-1}, A_{t-1}|\bar{C}_{t-1}, \bar{\boldsymbol{X}}_{t-2}, \bar{A}_{t-2}}(\boldsymbol{X}_{t-1}, g_j(Y_{t-1}, a_{t-1}) \mid \bar{0}, \bar{\boldsymbol{x}}_{t-2}, g_j(y_0, a_0), \ldots, g_j(y_{t-2}, a_{t-2}))}$$

$$\mid \bar{C}_{t-1} = \bar{0}, \bar{\boldsymbol{X}}_{t-1} = \bar{\boldsymbol{x}}_{t-1}, A_0 = g_j(y_0, a_0), \ldots, A_{t-2} = g_j(y_{t-2}, a_{t-2})\Big]$$

$$\prod_{s=0}^{t-2} f_{\boldsymbol{X}_s, A_s|\bar{C}_s, \bar{\boldsymbol{X}}_{s-1}, \bar{A}_{s-1}}(\boldsymbol{x}_s, a_s \mid \bar{0}, \bar{\boldsymbol{x}}_{s-1}, g_j(y_0, a_0), \ldots, g_j(y_{s-1}, a_{s-1}))$$

$$\mathrm{d}\,a_{t-1}\,\mathrm{d}\,\boldsymbol{x}_{t-2}\,\mathrm{d}\,a_{t-2} \ldots \mathrm{d}\,\boldsymbol{x}_0\,\mathrm{d}\,a_0.$$

By standard properties and linearity of conditional expectations, this is equal to

$$\int \ldots \int E\Big[Y_t \sum_{a_{t-1}=0}^{1} \prod_{s=t}^{k}$$

$$\frac{1\{C_s = 0\}}{f_{C_s|\bar{C}_{s-1}, \bar{\boldsymbol{X}}_{s-1}, \bar{A}_{s-1}}(0 \mid \bar{0}, \bar{\boldsymbol{x}}_{t-2}, \boldsymbol{X}_{t-1}, \ldots, \boldsymbol{X}_{s-1}, g_j(y_0, a_0), \ldots, g_j(Y_{t-1}, a_{t-1}), A_t, \ldots A_{s-1})}$$

$$\frac{f_{A_{t-1}|\bar{C}_{t-1}, \bar{\boldsymbol{X}}_{t-1}, \bar{A}_{t-2}}(a_{t-1} \mid \bar{0}, \bar{\boldsymbol{x}}_{t-2}, g_j(y_0, a_0), \ldots, g_j(y_{t-2}, a_{t-2}))1\{A_{t-1} = g_j(Y_{t-1}, a_{t-1})\}}{f_{A_{t-1}|\bar{C}_{t-1}, \bar{\boldsymbol{X}}_{t-1}, \bar{A}_{t-2}}(g_j(Y_{t-1}, a_{t-1}) \mid \bar{0}, \bar{\boldsymbol{x}}_{t-2}, g_j(y_0, a_0), \ldots, g_j(y_{t-2}, a_{t-2}))}$$

$$\mid \bar{C}_{t-1} = \bar{0}, \bar{\boldsymbol{X}}_{t-2} = \bar{\boldsymbol{x}}_{t-2}, A_0 = g_j(y_0, a_0), \ldots, A_{t-2} = g_j(y_{t-2}, a_{t-2})\Big]$$

$$\prod_{s=0}^{t-2} f_{\boldsymbol{X}_s, A_s|\bar{C}_s, \bar{\boldsymbol{X}}_{s-1}, \bar{A}_{s-1}}(\boldsymbol{x}_s, a_s \mid \bar{0}, \bar{\boldsymbol{x}}_{s-1}, g_j(y_0, a_0), \ldots, g_j(y_{s-1}, a_{s-1}))$$

$$\mathrm{d}\,\boldsymbol{x}_{t-2}\,\mathrm{d}\,a_{t-2} \ldots \mathrm{d}\,\boldsymbol{x}_0\,\mathrm{d}\,a_0.$$

By repeating this argument for $t = k'$ to $t = 0$, we obtain

$$
E\Bigg[Y_k \sum_{a_0=0}^{1} \cdots \sum_{a_{k'}=0}^{1}
\prod_{s=0}^{k} \frac{1\{C_s = 0\}}{f_{C_s|\bar{C}_{s-1},\bar{\boldsymbol{X}}_{s-1},\bar{A}_{s-1}}(0 \mid \bar{0}, \bar{\boldsymbol{X}}_{s-1}, g_j(Y_0, a_0), \ldots, g_j(Y_{k'}, a_{k'}), A_{k'+1}, \ldots A_{s-1})}
\prod_{s=0}^{k'} \frac{f_{A_s|\bar{C}_s,\bar{\boldsymbol{X}}_s,\bar{A}_{s-1}}(a_s \mid \bar{0}, \bar{\boldsymbol{X}}_s, g_j(Y_0, a_0), \ldots, g_j(Y_{s-1}, a_{s-1}))1\{A_s = g_j(Y_s, a_s)\}}{f_{A_s|\bar{C}_s,\bar{\boldsymbol{X}}_s,\bar{A}_{s-1}}(g_j(Y_s, a_s) \mid \bar{0}, \bar{\boldsymbol{X}}_{s-1}, g_j(Y_0, a_0), \ldots, g_j(Y_{s-1}, a_{s-1}))}\Bigg].
$$

By the indicator functions, this is equal to

$$
E\Bigg[Y_k \prod_{s=0}^{k} \frac{1\{C_s = 0\}}{f_{C_s|\bar{C}_{s-1},\bar{\boldsymbol{X}}_{s-1},\bar{A}_{s-1}}(0 \mid \bar{0}, \bar{\boldsymbol{X}}_{s-1}, \bar{A}_{s-1})}
\sum_{a_0=0}^{1} \cdots \sum_{a_{k'}=0}^{1} \prod_{s=0}^{k'} \frac{f_{A_s|\bar{C}_s,\bar{\boldsymbol{X}}_s,\bar{A}_{s-1}}(a_s \mid \bar{0}, \bar{\boldsymbol{X}}_s, \bar{A}_{s-1})1\{A_s = g_j(Y_s, a_s)\}}{f_{A_s|\bar{C}_s,\bar{\boldsymbol{X}}_s,\bar{A}_{s-1}}(A_s \mid \bar{0}, \bar{\boldsymbol{X}}_{s-1}, \bar{A}_{s-1})}\Bigg].
$$

By the indicator functions $1\{C_0 = 0\}, \ldots, 1\{C_k = 0\}$ and the definition of censoring (Definition 9 of the main article), this is equal to

$$
E\Bigg[\tilde{Y}_k \prod_{s=0}^{k} \frac{1\{C_s = 0\}}{f_{C_s|\bar{C}_{s-1},\bar{\tilde{\boldsymbol{X}}}_{s-1},\bar{\tilde{A}}_{s-1}}(0 \mid \bar{0}, \bar{\tilde{\boldsymbol{X}}}_{s-1}, \bar{\tilde{A}}_{s-1})}
\sum_{a_0=0}^{1} \cdots \sum_{a_{k'}=0}^{1} \prod_{s=0}^{k'} \frac{f_{\tilde{A}_s|\bar{C}_s,\bar{\tilde{\boldsymbol{X}}}_s,\bar{\tilde{A}}_{s-1}}(a_s \mid \bar{0}, \bar{\tilde{\boldsymbol{X}}}_s, \bar{\tilde{A}}_{s-1})1\{\tilde{A}_s = g_j(\tilde{Y}_s, a_s)\}}{f_{\tilde{A}_s|\bar{C}_s,\bar{\tilde{\boldsymbol{X}}}_s,\bar{\tilde{A}}_{s-1}}(\tilde{A}_s \mid \bar{0}, \bar{\tilde{\boldsymbol{X}}}_{s-1}, \bar{\tilde{A}}_{s-1})}\Bigg].
$$

□

C. Sequentially doubly robust identification formula

Denote

$$
m_{j,k,k'}(\bar{l}_{k'}, \bar{y}_{k'}, \bar{a}_{k'}) = E[Y_k \mid \bar{L}_{k'} = \bar{l}_{k'}, \bar{Y}_{k'} = \bar{y}_{k'}, A_0 = g_j(y_0, a_0), \ldots, A_{k'} = g_j(y_{k'}, a_{k'})],
$$

and, recursively, for $t = k' - 1, \dots, 0$, denote

$$m_{j,k,t}(\bar{l}_t, \bar{y}_t, \bar{a}_t) = \int \dots \int m_{j,k,t+1}(\bar{l}_{t+1}, \bar{y}_{t+1}, \bar{a}_{t+1})$$
$$f_{L_{t+1},Y_{t+1},A_{t+1}|\bar{L}_t,\bar{Y}_t,\bar{A}_t}(l_{t+1}, y_{t+1}, a_{t+1} \mid \bar{l}_t, \bar{y}_t, g_j(y_0, a_0), \dots, g_j(y_t, a_t))$$
$$\mathrm{d}\,l_{t+1}\,\mathrm{d}\,y_{t+1}\,\mathrm{d}\,a_{t+1},$$

for $j \in \{0,1\}$, $k \geq 1$, $\bar{l}_{k'} \in \mathcal{L}^{k'+1}$, $\bar{y}_{k'} \in \mathcal{Y}^{k'+1}$, and $\bar{a}_{k'} \in \{0,1\}^{k'+1}$, where $k' = \min\{k-1, \kappa\}$.

Using this notation, we present a convenient reformulation of the g-formula in Equation 12 of the main article. The notation matches that of Theorem 1 in Díaz et al. (2023). Accordingly, the following identification formula is a special case of the formula in Theorem 1 of Díaz et al. (2023), though it is derived under different assumptions.

Corollary 7.2. *Let g_j be an add-on-j regime and assume that consistency (Equation (6) of the main article), exchangeability (Equation (8) of the main article), and positivity (Equation (9) of the main article) hold. Then,*

$$E\left[Y_k^{g_j}\right] = E\left[m_{j,k,0}(L_0, Y_0, A_0)\right] \quad \forall j \in \{0,1\}, k \geq 1. \tag{28}$$

Proof. Let g_j be an add-on-j regime, let $k \in \{1, \dots, K\}$ and denote $k' = \min\{k-1, \kappa\}$. We prove Corollary 7.2 by showing that $E\left[m_{j,k,0}(L_0, Y_0, A_0)\right]$ is equal to the right-hand side of (6). By definition of expectations, it holds that

$$E\left[m_{j,k,0}(L_0, Y_0, A_0)\right] = \int \dots \int m_{j,k,0}(l_0, y_0, a_0) f_{L_0,Y_0,A_0}(l_0, y_0, a_0)\,\mathrm{d}\,l_0\,\mathrm{d}\,y_0\,\mathrm{d}\,a_0.$$

By definition of $m_{j,k,0}$, the above is equal to

$$\int \dots \int m_{j,k,1}(\bar{l}_1, \bar{y}_1, \bar{a}_1) f_{L_1,Y_1,A_1|L_0,Y_0,A_0}(l_1, y_1, a_1 \mid l_0, y_0, g_j(y_0, a_0))\,\mathrm{d}\,l_1\,\mathrm{d}\,y_1\,\mathrm{d}\,a_1$$
$$f_{L_0,Y_0,A_0}(l_0, y_0, a_0)\,\mathrm{d}\,l_0\,\mathrm{d}\,y_0\,\mathrm{d}\,a_0.$$

By definition of $m_{j,k,1}$, the above is equal to

$$\int \dots \int m_{j,k,2}(\bar{l}_2, \bar{y}_2, \bar{a}_2) f_{L_2,Y_2,A_2|\bar{L}_1,\bar{Y}_1,\bar{A}_1}(l_2, y_2, a_2 \mid \bar{l}_1, \bar{y}_1, g_j(y_0, a_0), g_j(y_1, a_1))\,\mathrm{d}\,l_1\,\mathrm{d}\,y_1\,\mathrm{d}\,a_1$$
$$f_{L_1,Y_1,A_1|L_0,Y_0,A_0}(l_1, y_1, a_1 \mid l_0, y_0, g_j(y_0, a_0))\,\mathrm{d}\,l_1\,\mathrm{d}\,y_1\,\mathrm{d}\,a_1$$
$$f_{L_0,Y_0,A_0}(l_0, y_0, a_0)\,\mathrm{d}\,l_0\,\mathrm{d}\,y_0\,\mathrm{d}\,a_0.$$

By recursively using the definition of $m_{j,k,t}$ for $t = 2, \dots, k' - 1$, we obtain that the above is equal to

$$\int \dots \int m_{j,k,k'}(\bar{l}_{k'}, \bar{y}_{k'}, \bar{a}_{k'})$$
$$\prod_{t=1}^{k'} f_{\boldsymbol{L}_t,Y_t,A_t|\bar{\boldsymbol{L}}_{t-1},\bar{Y}_{t-1},\bar{A}_{t-1}}(\boldsymbol{l}_t, y_t, a_t \mid \bar{\boldsymbol{l}}_{t-1}, \bar{y}_{t-1}, g_j(y_0, a_0), \dots, g_j(y_{t-1}, a_{t-1}))$$
$$f_{\boldsymbol{L}_0,Y_0,A_0}(\boldsymbol{l}_0, y_0, a_0)\,\mathrm{d}\boldsymbol{l}_{k'}\,\mathrm{d}\,y_{k'}\,\mathrm{d}\,a_{k'} \dots \mathrm{d}\boldsymbol{l}_0\,\mathrm{d}\,y_0\,\mathrm{d}\,a_0.$$

Finally, by the definition of $m_{j,k,k'}$, the above is equal to

$$E\left[Y_k^{g_j}\right] = \int \dots \int E[Y_k \mid \bar{\boldsymbol{L}}_{k'} = \bar{\boldsymbol{l}}_{k'}, \bar{Y}_{k'} = \bar{y}_{k'}, A_0 = g_j(y_0, a_0), \dots, A_{k'} = g_j(y_{k'}, a_{k'})]$$
$$\prod_{t=1}^{k'} f_{\boldsymbol{L}_t,Y_t,A_t|\bar{\boldsymbol{L}}_{t-1},\bar{Y}_{t-1},\bar{A}_{t-1}}(\boldsymbol{l}_t, y_t, a_t \mid \bar{\boldsymbol{l}}_{t-1}, \bar{y}_{t-1}, g_j(y_0, a_0), \dots, g_j(y_{t-1}, a_{t-1}))$$
$$f_{\boldsymbol{L}_0,Y_0,A_0}(\boldsymbol{l}_0, y_0, a_0)\,\mathrm{d}\boldsymbol{l}_{k'}\,\mathrm{d}\,y_{k'}\,\mathrm{d}\,a_{k'} \dots \mathrm{d}\boldsymbol{l}_0\,\mathrm{d}\,y_0\,\mathrm{d}\,a_0.$$

$\square$

D. Application: Variables, Target Trial Protocol, and Observational Emulation

Table 1: A summary of the target trial protocol and its corresponding observational emulation, as presented in Section 7 of the main article and implemented in the R code provided in the supplementary material.

Target Trial Specification	**Target Trial Emulation**
Eligibility criteria	
1. Registered date of discharge following the first recorded traumatic injury. This time point is hereafter referred to as the time of discharge. 2. Eligible to receive NSAIDs at discharge. 3. At least one opioid dispensation within the first month after discharge. 4. Survival through the first month after the initial opioid dispensation after discharge.	Same as in the target trial.
Time 0	
Time 0 corresponds to the first month after the initial opioid dispensation, following discharge. This time point is also referred to as month 0.	Same as in the target trial.
Treatment period	
The treatment period is measured in months, begins at month 0, and continues through month 1 for the first analysis, and through month 20 for the second analysis.	Same as in the target trial.
Follow-up	
Follow-up is measured in months, begins at month 0, and continues through month 21.	Same as in the target trial.
Outcome	
1. Monthly opioid dose over follow-up. 2. Total opioid dose during follow-up.	Same as in the target trial.

Target Trial Specification (cont.)	**Target Trial Emulation (cont.)**
Treatment strategies	
1. Add-on-0 regime: Never dispense NSAIDs when opioids are dispensed during the treatment period. When opioids are not dispensed, NSAIDs are dispensed according to usual care, without intervention. 2. Add-on-1 regime: Always dispense NSAIDs when opioids are dispensed during the treatment period. When opioids are not dispensed, NSAIDs are dispensed according to usual care, without intervention.	Same as in the target trial.
Assignment	
Participants are randomly assigned to a treatment strategy at baseline and are aware of their assignment.	Treatment assignment at each time point during the treatment period is assumed to satisfy sequential exchangeability, conditional on the observed baseline and time-varying covariates listed in Table 2. See Equation (10) of the main article for a formal definition.
Causal contrasts	
1. Add-on effect (Equation (4) of the main article). 2. Opioid-sparing effect (Equation (5) of the main article).	Same as in the target trial.
Data analysis	
Per-protocol analysis.	The g-formula estimator, corresponding to the right-hand side of Equation (10) in the main article, was implemented using the `gfoRmula` package in `R`. See the `R` code provided in the supplementary material.

Table 2: Description of variables used in the analysis presented in Section 7 of the main article and in the `R` code provided in the supplementary material.

Name	Type	Description
Participant identifier		
id	Numerical	Unique participant ID
Time		
time	Integer	Months since start of follow-up
Intervention		
nsaid_ind	Binary	Indicator of NSAID dispensation during follow-up
Outcome		
opioid_omeq	Continuous	Monthly opioid dose in oral morphine equivalents
Baseline covariates		
age	Continuous	Age at discharge
sex	Binary	Sex (1: female, 0: male)
inntektGroup	Factor	Income group at start of follow-up
nsaid_pre_dis	Binary	NSAID dispensation within 6 months before discharge
opioid_pre_dis	Binary	Opioid dispensation within 6 months before discharge
RHF	Factor	Regional Health Authority
trm_cent	Binary	Treated at a trauma center
kommuneindex	Factor	Municipality index
pt_asa_preinjury	Ordinal	ASA score before injury
acc_transport	Binary	Transport-related accident
acc_fall	Binary	Fall-related accident
acc_work	Binary	Work-related accident
inj_mechanism	Factor	External cause of injury
hosp_care_level	Factor	Highest level of hospital care received
hosp_icu_days	Continuous	Number of ICU bed days
hosp_los_days	Continuous	Total hospital length of stay (days)
res_gos_dischg	Factor	Glasgow Outcome Scale score at discharge
ais_group	Ordinal	Abbreviated Injury Scale group
Time-varying covariates		
healthcare_use	Continuous	Monthly count of KUHR-registered health care visits
B01A_ind	Binary	Indicator of antithrombotic drug dispensation
hosp_days	Continuous	Hospital bed days in a given month
opioid_omeq	Continuous	Monthly opioid dose in oral morphine equivalents

References

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