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# Regularity of the one-phase problem

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# 1 Introduction

A *free boundary problem* is a special partial differential equation, where the complete solution consists of a couple $(u, \Omega)$, the solution function $u$ and a domain $\Omega \subset \mathbb{R}^n$. Prominent examples include among others the *Stefan problem*, describing the melting of ice and the *obstacle problem*. Inherent to such problems are phase transitions, in some parts of the domain $u$ is qualitatively different than in others. This gives rise to a rich theory and widespread applications in engineering, fluid dynamics, electromagnetism, financial mathematics and many more (see e.g. [BL82; Rod87; Bai74; LS09]).

The theory and study of free boundary problems established itself as its own mathematical discipline in the 80's trough the fundamental works of Caffarelli, Alt, Friedman, Spruck, Kinderlehrer and many more (e.g. [Caf77; ACF84; AC81; KNS78]). Present-day research includes classical techniques from partial differential equations, potential theory and harmonic analysis, but also from geometric measure theory and minimal surface theory.

From a purely mathematical standpoint, both properties of the solution $u$, of the domain $\Omega$ and its boundary $\partial\Omega$ are of interest and attract widespread attention in contemporary research. In this work, we focus on a particular instance, the *one-phase problem* (also known as Bernoulli problem). It is one of the most prominent examples, having motivated many regularity theories for other problems as well. As many other free boundary problems (and problems in the calculus of variations for that matter), it comes from the minimization of a certain energy functional over a suitable function space (i.e. over all non-negative $u \in W^{1,2}(D)$ satisfying Dirichlet boundary conditions, see (2.1)).

**Definition 1.1.** Given a bounded domain $D \subset \mathbb{R}^d$ and a real constant $\Lambda > 0$, the energy functional for the one-phase problem is defined as

$$F_\Lambda(u, D) := \int_D |\nabla u|^2 + \Lambda \chi_{\{u>0\}} dx = \int_D |\nabla u|^2 dx + \Lambda |\{u > 0\} \cap D|. \tag{1.1}$$

It consists of the classical Dirichlet energy and an additional measure term.

Intuitively, the minimizer $u$ finds a balance between being harmonic (minimizing the Dirichlet energy) and being zero (minimizing the measure term of the positivity set), of course depending on the ambient domain $D$ and the given boundary datum on $\partial D$. As it turns out, the solution $u$ splits into a set where it is strictly positive $\Omega_u := \{u > 0\}$ and a zero set $\{u = 0\}$. The boundary $\partial\Omega_u = \partial\{u > 0\}$ is known as the *free boundary*, since it depends on the solution $u$ itself and is not determined beforehand. It is important to differentiate the given fixed ambient domain $D$ with its boundary datum and the unknown domain $\Omega_u$ inside $D$. Here, there is only one change (a *phase transition*) in qualitative behaviour, hence the name "one-phase". There are other free boundary problems (e.g. the two-phase problem) with several phase transitions, leading to even more intricate theories.

There are many different questions in the study of the one-phase problem, we follow the modern treatments in [Vel23] and [Kri19], not the original methods in [AC81; Caf87; Caf88]. We give a self-contained exposition of basic regularity properties, combining and modifying some proofs from the literature. Additionally, we obtain some new boundary regularity results. The structure of the work is as follows:

- Already correctly setting up the problem, and showing existence are not entirely trivial tasks and are part of Chapter 2.
- In contrast to other free boundary problems, solutions are a priori not unique. However, we argue in Section 2.1 there are some positive results for uniqueness, depending on the boundary datum. As it turns out, for "almost all" boundary datum, the solution

is unique and the cases with non-unique minimizers are rare. This is known in the literature as *generic uniqueness*.

- In Chapter 3 we start studying the regularity of $u$. Optimal interior regularity results are already well established in the literature. Yet, similar to classical potential theory for harmonic functions, in Section 3.3, we are also interested in the regularity up to the boundary. In particular, we show that regularity is preserved for continuous and Hölder continuous boundary datum. These are original results of this thesis.

- In Chapter 4 we start to look at the free boundary $\partial\Omega_u$ and its structure itself. Particularly, in Section 4.2, we use the framework of viscosity solutions to define appropriate solutions of the corresponding Euler-Lagrange equation and show that minimizers of $F_\Lambda$ are in this class.

- In Chapter 5 we use a new alternative approach to obtain $C^\infty$ regularity of the free boundary at regular points. The classical result of $C^{1,\alpha}$ regularity originates in [AC81], however we follow the more recent strategy (”improvement of flatness”) from [Vel23; DeS09]. In [KN77] the result is lifted to even analyticity of the free boundary through a change of variables, the *partial hodograph transform*. Our alternative approach in Section 5.4 uses $C^{1,\alpha}$ (for any $\alpha$ arbitrarily close to 1) regularity and local energy estimates ($u$ is in $W^{2,2}(\bar{\Omega}_u)$ around regular points) together with a bootstrap argument. The argument resembles more the proof of smoothness of the free boundary for the classical obstacle problem in [FR22] and the approach in [YZ23].

- Lastly, we study the case when the blow-up is not the half-space solution, in Chapter 6. Those points are called *singular* and not much can be said about the solution there. The focus here lies in estimating the dimension of the set of singular points, linked to the dimension of the ambient space. We provide one proof and another brief sketch (from [CJK04]) of such estimates.

## 2 Existence and basic properties of solutions

After introducing the problem, we study basic properties of solutions, namely existence, (non-) uniqueness and some explicit examples. We start by showing existence.

**Proposition 2.1.** *Let $D \subset \mathbb{R}^d$ bounded and $g \in H^1(D)$ be a nonnegative function. The optimization problem,*

$$\min\{F_\Lambda(u, D) : u \in H^1(D), u - g \in H^1_0(D)\}, \tag{2.1}$$

*admits a solution $u \geq 0$.*

*Proof.* Without loss of generality assume $\Lambda = 1$. For the first point, note that for any $u \in H^1(D)$ and $u_+ = \max(u, 0)$,

$$\begin{aligned} F_\Lambda(u, D) - F_\Lambda(u_+, D) &= \int_D \left(|\nabla u|^2 - |\nabla u_+|^2\right) \\ &= \int_D \left(|\nabla u|^2 - \chi_{\{u>0\}}|\nabla u|^2\right) = \int_{D\cap\{u<0\}} |\nabla u|^2 \geq 0. \end{aligned}$$

Take a minimizing sequence $u_n$ such that $u_n - g \in H^1_0(D)$ and $u_n \geq 0$. Note that $\nabla u_n$ is bounded in $L^2(D)$ and thus by Poincaré's inequality, $u_n$ is bounded in $H^1(D)$. Thus take a subsequence (not relabeled) $u_n \rightharpoonup u$, weakly in $H^1(D)$, strongly in $L^2(D)$ and $L^2(\partial D)$ and pointwise almost everywhere in $D$ (up to another subsequence). Now almost everywhere in $D$, we have

$$u(x) > 0 \implies u_n(x) > \frac{u(x)}{2} > 0 \qquad \text{for } n \text{ sufficiently large,}$$

hence a.e. $\chi_{u>0}(x) \le \liminf_n \chi_{u_n}(x)$, which is equivalent to

$$|\{u > 0\} \cap D| \le \liminf_n |\{u_n > 0\} \cap D|.$$

Since the $H^1$-norm is weakly lower semicontinuous,

$$F_\Lambda(u, D) \le \liminf_n F_\Lambda(u_n, D),$$

which establishes the existence and non-negativity. $\square$

As a first result we show that restricting a minimizer to a subset does not change its minimality.

**Lemma 2.2.** *For any minimizer $u$ of (2.1) on $D$, the restriction to a subset $E \subset D$, $u|_E$ is a minimizer to (2.1) with datum $u|_{\partial E}$ on $E$.*

*Proof.* Suppose that there exists an admissible $v : E \to \mathbb{R}$ with $F_\Lambda(v, E) < F_\Lambda(u|_E, E)$. Then setting $u'(x) = \chi_E v(x) + (1 - \chi_E)u(x)$ gives

$$\begin{aligned} F_\Lambda(u', D) = \int_D u' + \Lambda|\Omega_{u'} \cap D| &= \int_E v + \int_{E^c} u + \Lambda(|\Omega_v \cap E| + |\Omega_u \cap E^c|) \\ &< \int_E u + \int_{E^c} u + \Lambda(|\Omega_u \cap E| + |\Omega_u \cap E^c|) = F_\Lambda(u, D), \end{aligned}$$

contradicting the minimality of $u$. $\square$

We now proceed with the basic harmonicity properties, connecting the solution $u$ to a partial differential equation. Intuitively, we expect harmonicity to show up somewhere, since we are partially minimizing the Dirichlet energy.

**Proposition 2.3.** *The solution $u$ to (2.1) satisfies:*

1. *$u$ is subharmonic, $\Delta u \ge 0$ in the sense of distributions,*
2. *the set $\Omega_u \coloneqq \{u > 0\}$ is open,*
3. *$u$ is harmonic in $\Omega_u$.*

*Proof.* Without loss of generality assume $\Lambda = 1$. We show (weak) subharmonicity of $u$. Let $\varphi \in C_c^\infty(D)$ be a non-negative test function and set $v \coloneqq u - t\varphi$ with $t$ positive. Since $F_\Lambda(v, D) - F_\Lambda(v_+, D)$ and $u \ge v_+$,

$$\begin{aligned} \int_D |\nabla u|^2 + |\{u > 0\} \cap D| = F_\Lambda(u, D) \le F_\Lambda(v_+, D) &= \int_D |\nabla v_+|^2 + |\{v_+ > 0\} \cap D| \\ &\le \int_D |\nabla v|^2 + |\{u > 0\} \cap D| \\ \implies \int_D |\nabla u|^2 \le \int_D |\nabla v|^2 &= \int_D |\nabla(u - t\varphi)|^2. \end{aligned}$$

Then letting $t \to 0^+$ in

$$0 \le \frac{1}{t} \int_D |\nabla(u - t\varphi)|^2 - |\nabla u|^2 = -2 \int_D \nabla u \cdot \nabla\varphi + t \int_D |\nabla\varphi|^2 \implies 0 \ge \int_D \nabla u \cdot \nabla\varphi.$$

Knowing that $\nabla u \geq 0$, gives for any point $x \in D$ that $r \to \fint_{\partial B_r(x)} u$ is non-decreasing. Thus the limit as $r \to 0$ exists (in particular every point is a Lebesgue point), and so $u$ and $\{u > 0\}$ are defined pointwise.

Secondly, to show that $\{u > 0\}$ is open, we show that $\{u = 0\}$ is closed. Let $\{u = 0\} \ni x_n \to x$. Note that Proposition 3.2, which is shown later, depends only on the existence of a minimizer and its subharmonicity. Hence from the first bound in Proposition 3.2,

$$\fint_{B_r(x_n)} u \leq Cr \quad \forall n \implies \fint_{B_r(x)} u \leq Cr \implies u(x) \leq \lim_{r\to 0} \fint_{B_r(x)} u = 0,$$

so $x \in \{u = 0\}$ and thereby $\{u > 0\}$ open.

Lastly, let now $\varphi \in C_c^\infty(\Omega_u)$ be a test function and $w := u + t\varphi$ with $t$ positive. Then $u|_{\partial\Omega_u} = w|_{\partial\Omega_u}$ and by Lemma 2.2,

$$\begin{aligned}
&\int_{\Omega_u} |\nabla u|^2 + |\{u > 0\}| = F_\Lambda(u, \Omega_u) \leq F_\Lambda(w, \Omega_u) = \int_{\Omega_u} |\nabla u|^2 + |\{w > 0\} \cap \{u > 0\}| \\
&\implies \int_{\Omega_u} |\nabla u|^2 \leq \int_{\Omega_u} |\nabla w|^2 = \int_{\Omega_u} |\nabla u|^2 + 2t\nabla u \cdot \nabla\varphi + t^2|\nabla\varphi|^2 \\
&\iff 0 \leq \int_{\Omega_u} 2\nabla u \cdot \nabla\varphi + t|\nabla\varphi|^2 \xrightarrow{t\to 0^+} \int_{\Omega_u} \nabla u \cdot \nabla\varphi \geq 0.
\end{aligned}$$

Thus inside $\Omega_u$, $u$ is weakly superharmonic, but since it is weakly subharmonic in the whole domain $D$, it is weakly harmonic in and hence strongly harmonic (and smooth) inside $\Omega_u$. □

Let us now give the definition of local and global minimizers, we need their properties at a later time.

**Definition 2.4.** Let $D$ be a bounded open set in $\mathbb{R}^d$, we say that the nonnegative function $u : D \to \mathbb{R}$ is a **local minimizer** of $F_\Lambda$ in $D$ if $u \in H^1_{loc}(D)$ and for any $U \subset\subset D$,

$$F_\Lambda(u, U) \leq F_\Lambda(v, U) \quad \text{for every } v \in H^1_{loc}(D) \text{ such that } u - v \in H^1_0(U). \tag{2.2}$$

If $D = \mathbb{R}^d$ we say that $u$ is a **global minimizer**. From now on, $D$ denotes always a bounded open set of $\mathbb{R}^d$, unless specified otherwise.

First, we show that any minimizer of (2.1) on $D$ is a local minimizer on $D$. We also note that a local minimizer on $D$ is a minimizer of (2.1) for $D' \subset\subset D$. It is equivalent to show regularity statements for local minimizers or minimizers to (2.1), we use the terms interchangeably, as it is also widely done in the literature.

**Lemma 2.5.** *Any minimizer in the sense of* (2.1) *is a local minimizer in the sense of* (2.2)*, i.e. there always exists a local minimizer.*

*Proof.* Let $u$ be a minimizer to (2.1), but suppose that there exists $U \subset\subset D$ and $w \in H^1_{loc}(D)$ with $w - u \in H^1_0(U)$ such that $F_\Lambda(w, U) < F_\Lambda(u, U)$. Take

$$H^1(D) \ni v = \begin{cases} w & \text{in } U, \\ u & \text{in } D\backslash U, \end{cases}$$

which clearly fulfills the boundary condition $v - g \in H^1_0(D)$. Then

$$\begin{aligned}
F_\Lambda(v, D) = F_\Lambda(v, U) + F_\Lambda(v, D\backslash U) &= F_\Lambda(w, U) + F_\Lambda(u, D\backslash U) \\
&< F_\Lambda(u, U) + F_\Lambda(u, D\backslash U) = F_\Lambda(u, D),
\end{aligned}$$

contradicting the minimality of $u$. □

## 2.1 Uniqueness of minimizers

For many free boundary problem the underlying functional is convex, giving directly uniqueness of minimizers. The prominent example is here the obstacle problem, which the author studied during a semester project [Grü22]. We refer also to [FR22, Chapter 5]. For the one-phase problem this is not the case and we show that minimizers are not unique.

**Remark 2.6.** Minimizers for the one-phase problem are not unique. Take the case where $d = 1$, $D = B_R = B_2$, $\Lambda = 1$ and boundary datum $g \equiv 1$. Since in one dimension harmonic functions are affine,

$$\min F_\Lambda(u, D) = \min_{r \leq 2} \int_{B_r^c} |\nabla u|^2 + |\{u > 0\}| = 2\left((R-r) + \frac{1}{R-r}\right),$$

which achieves its minimum at $r = R - 1$. The minimizer is then given by

$$u(x) = \begin{cases} -x-1 & \text{if } -2 \leq x \leq -1, \\ 0 & \text{if } -1 < x < 1, \\ x-1 & \text{if } 1 \leq x \leq 2, \end{cases}$$

with cost $F_\Lambda(u, B_2) = 4$. However, the constant function $\tilde{u} \equiv 1$ has cost $F_\Lambda(\tilde{u}, B_2) = 4$ and is then a minimizer different from $u$.

Yet it is possible to say that the cases with several minimizers are rare and we expect "almost everywhere" a unique minimizer. These results are also known as *generic uniqueness*, since the boundary datum $g \in H^1 D$ lies in an infinite dimensional space (and not on $\mathbb{R}^d$ with the usual Lebesgue measure), one has to carefully define what is meant by "almost everywhere". This connects closely to the so called *theory of prevalence* [OY05]. A generalized result for a wider class of functionals can be found in [FY23]. We start with a general Lemma that will also prove useful in Chapter 3.

**Lemma 2.7.** *Let $D$ be a bounded open domain of $\mathbb{R}^d$ and $g, g' : \partial D \to \mathbb{R}_{>0}$ with $g'|_{\partial D} \geq g|_{\partial D}$ continuous and $g'(x_1) > g(x_1) > 0$ at some $x_1$ in each connected component of $\partial D$. Then for the corresponding minimizers to* (2.1) *$u_g$ and $u_{g'}$ we have on $\bar{D}$, $u_{g'} \geq u_g$.*

*Proof.* Since on $\{u_g = 0\}$ the result holds trivially, consider the open set $\Omega_{u_g} = \{u_g > 0\} \cap D$. Define $u^+(x) := \max\{u_{g'}(x), u_g(x)\}$. Suppose for contradiction that there exists $x_0 \in \Omega_{u_g}$ such that $u_g(x_0) > u_{g'}(x_0)$, then $u^+(x_0) = u_g(x_0)$. Set $h(x) := u^+(x) - u_g(x) \geq 0$, then on $\Omega_{u_g}$, we have

- if $u^+(x) = u_g(x)$ then $\Delta u^+(x) = \Delta u_g(x) = 0$, since $u_g$ is harmonic where positive,
- if $u^+(x) = u_{g'}(x)$ then $\Delta u^+(x) = \Delta u_{g'}(x) = 0$, since $u_{g'}(x) \geq u_g(x) > 0$ and $u_{g'}$ is harmonic where positive.

In other words, $\Delta u^+ = 0$ in $\Omega_{u_g}$ and $h$ is harmonic there. However, $h$ attains its minimum value of 0 inside $\Omega_{u_g}$ at $x_0$, hence by the strong maximum principle it is constantly 0 on $\bar{\Omega}_{u_g}$. Thus $u^+ = u_g$ and $u_g \geq u_{g'}$ on $\bar{\Omega}_{u_g}$, in particular

$$g = u_g \geq u_{g'} = g' \qquad \text{on } \partial D \cap \bar{\Omega}_{u_g}.$$

But this is a contradiction since for $x_1$ in $\partial D \cap \bar{\Omega}_{u_g}$, we have $u_{g'}(x_1) = g'(x_1) > g(x_1) = u_g(x_1)$ on $\partial D$. □

We are now able to prove the generic uniqueness for the one-phase problem, using the argument from [FY23, Proposition 3.2].

**Theorem 2.8.** *Consider a bounded domain $D \subset \mathbb{R}^d$ and $g : \partial D \to \mathbb{R}$ a non-negative continuous function. Let $g_\lambda \coloneqq g + \lambda$, where $\lambda \in S_\lambda \coloneqq \{\lambda \in \mathbb{R} : g_\lambda \geq 0\} \subset \mathbb{R}$. Then for almost every $\lambda \in S_\lambda$, the minimizer $u_\lambda$ of $F_\Lambda(\cdot, D)$ with boundary datum $g_\lambda$ is unique.*

*Proof.* During the proof we assume the Lipschitz continuity of the minimizers, which is shown later in Proposition 3.3. Also without loss of generality take $\Lambda = 1$. By Lemma 2.7, minimizers are ordered with respect to the boundary datum, i.e. $\lambda' > \lambda > 0$ implies that $u_{\lambda'} \geq u_\lambda$. (Note that it remains open whether non-unique minimizers for a fixed $\lambda$ are ordered.)

Let $\lambda$ be such that there are at least two distinct minimizers $u_\lambda^1, u_\lambda^2$. Let $u_\lambda^+ = \max\{u_\lambda^1, u_\lambda^2\}$ and $u_\lambda^- = \min\{u_\lambda^1, u_\lambda^2\}$, then

$$\begin{aligned}
F_\Lambda(u_\lambda^+) + F_\Lambda(u_\lambda^-) &= \int_{D\cap\{u_\lambda^1 \geq u_\lambda^2\}} |\nabla u_\lambda^1|^2 + \int_{D\cap\{u_\lambda^1 < u_\lambda^2\}} |\nabla u_\lambda^2|^2 + |(\{u_\lambda^1 > 0\} \cup \{u_\lambda^2 > 0\}) \cap D| \\
&\quad + \int_{D\cap\{u_\lambda^1 \geq u_\lambda^2\}} |\nabla u_\lambda^2|^2 + \int_{D\cap\{u_\lambda^1 < u_\lambda^2\}} |\nabla u_\lambda^1|^2 + |\{u_\lambda^1 > 0\} \cap \{u_\lambda^2 > 0\} \cap D| \\
&= F_\Lambda(u_\lambda^1) + F_\Lambda(u_\lambda^2).
\end{aligned}$$

As $u_\lambda^1$ and $u_\lambda^2$ are minimizers, so are $u_\lambda^+$ and $u_\lambda^-$. Now let $x_0$ be a point where $u_\lambda^1$ and $u_\lambda^2$ differ, i.e. without loss of generality $u_\lambda^1(x_0) - u_\lambda^2(x_0) = \varepsilon$. By Lipschitz continuity, there exists $B_r(x_0)$ where $u_\lambda^1(x_0) - u_\lambda^2(x_0) > \varepsilon/2$. Thus for $\rho = \min\{\varepsilon/3, r\}$, there exists a $d+1$ dimensional ball $B_\rho$ such that $B_\rho \subset \mathrm{epi}(u_\lambda^2)\backslash\mathrm{epi}(u_\lambda^1)$.

Repeating the argument for any $\lambda$ with non-unique minimizers, gives a collection of disjoint (as minimizers are ordered with respect to the boundary datum) balls. As the open balls contain each a distinct rational number, there are only countable many $\lambda$ with non-unique minimizers, which finishes the proof. ☐

**Remark 2.9.** In the one dimensional case, the boundary datum can be identified with a point in $\mathbb{R}^2$. We show now that the set of non-negative boundary conditions that give non-unique minimizers,

$$\begin{aligned}
S \coloneqq \{(A, B) :\ &\exists u_1 \neq u_2 \text{ minimizing } F_\Lambda(u, (a, b)) \\
&\text{with } u_1(a) = u_2(a) = A \text{ and } u_1(b) = u_2(b) = B\},
\end{aligned}$$

is small. In particular, we have $\mathcal{H}^1(S) \leq 2\sqrt{\Lambda}(b-a)$, where $\mathcal{H}^1$ is the one-dimensional Hausdorff measure from Appendix A.3. Let $u_1$ be the solution connecting $(a, A)$ and $(b, B)$ in a straight line (with empty contact set),

$$u_1(x) = A + \frac{B-A}{b-a}(x-a)$$

and $u_2$ the solution with a non-empty contact set $\{u_2 = 0\} \neq \varnothing$. As a first condition, by Proposition 4.4, which is shown later, $|\nabla u_2| = \sqrt{\Lambda}$, so $u_2$ is given as

$$u_2(x) = \begin{cases} A - \sqrt{\Lambda}(x-a) & \text{if} \quad a \leq x \leq a + A/\sqrt{\Lambda}, \\ 0 & \text{if} \quad a + A/\sqrt{\Lambda} < x < b - B/\sqrt{\Lambda}, \\ B - \sqrt{\Lambda}(b-x) & \text{if} \quad b - B/\sqrt{\Lambda} \leq x \leq b \end{cases}$$

This gives directly the condition on the boundary datum $(A, B)$,

$$a + A/\sqrt{\Lambda} \leq b - B/\sqrt{\Lambda} \implies A + B \leq \sqrt{\Lambda}(b-a),$$

otherwise the contact set is not empty. We calculate the corresponding energies as

$$\begin{aligned}
E_1 &\coloneqq F_\Lambda(u_1, (a, b)) = \frac{(B-A)^2}{b-a} + \Lambda(b-a), \\
E_2 &\coloneqq F_\Lambda(u_2, (a, b)) = 2\sqrt{\Lambda}(A+B),
\end{aligned}$$

which are equal (i.e. two minimizers exist) whenever the datum $(A, B)$ lies on the implicit quadratic parabola

$$\frac{(B-A)^2}{b-a} + \Lambda(b-a) = 2\sqrt{\Lambda}(A+B).$$

By the constraint $A + B \leq \sqrt{\Lambda}(b-a)$, the arc-length and thereby also $\mathcal{H}^1(S)$ is finite. More explicitly, a change of variables, $A' = \frac{A+B}{\sqrt{2}}$ and $B' = \frac{A-B}{\sqrt{2}}$, gives

$$\mathcal{H}^1(S) = 2\int_0^{\sqrt{\Lambda/2}(b-a)} \sqrt{1 + \frac{A'^2}{8\Lambda(b-a)^2}} dA' \leq 2\sqrt{\Lambda}(b-a).$$

Finally, we show that for large boundary datum $g$ the solution does not touch the coordinate plane and the minimizer minimizes exactly the classical Dirichlet energy.

**Proposition 2.10.** *For any open domain $D$, there exists a constant $C = \operatorname{diam}(D)$ such that for the boundary datum $g > C$, the minimizer $u$ to* (2.1) *is given by the unique harmonic function on $D$ with $u|_{\partial D} = g$. Furthermore, in that case*

$$F_\Lambda(u, D) = \int_D |\nabla u|^2 dx + |D|.$$

*Proof.* By Lemma 2.7 it suffices to show that there exists $C > 0$ such that $u \equiv C$ is the minimizer to (2.1). We show that any minimizer $u$ with $u|_{\partial D} = C$ is strictly positive in $D$ and thus harmonic and constant by the maximum principle (with cost $F_\Lambda(u \equiv C, D) = \Lambda|D|$). Let $\varepsilon > 0$, without loss of generality, we suppose that $\bar{D} \subset \{x : x_d \geq \varepsilon\}$. Let now $v$ be the half-plane solution from Proposition 2.17, if $C > \operatorname{diam}(D) + \varepsilon$, then $C = u|_{\partial D} > v|_{\partial D}$. By Lemma 2.7 we have $u \geq v > 0$ on $D$ and the result follows as $\varepsilon$ was arbitrary. Since harmonic functions are minimizers of the Dirichlet energy, the proof is finished. □

## 2.2 Examples of explicit solutions

We state the explicit solutions in the ball and the annulus with radial symmetric boundary datum, for proofs see [Vel23, Chapter 2.4/2.3]. For general domains and boundary datum it is very difficult to arrive at explicit expressions for the minimizer, even testing whether a candidate solution is a minimizer is hard. First, the solution in a ball is given.

**Proposition 2.11.** [Vel23, Proposition 2.16] *For $D = B_R$ with $R > C_d > 0$ and $g|_{B_R} = 1$, the unique solution of* (2.1) *is given by*

$$u_R = \begin{cases} 1 & \text{on } \partial B_R, \\ 0 & \text{in } B_r, \\ \frac{|x|^{2-d} - r^{2-d}}{R^{2-d} - r^{2-d}} \text{ if } d \geq 3 \text{ and } \frac{\log|x| - \log r}{\log R - \log r} \text{ if } d = 2 & \text{in } B_R \backslash B_r, \end{cases}$$

*for some $r = r(R) < R$. Also $|\nabla u_R(x)| = \left(\frac{|x|}{r}\right)^{1-d}$ and for $R$ large enough $R - 2 < r < R$.*

Secondly, the solution in an annulus is given.

**Proposition 2.12.** [Vel23, Proposition 2.15] *For any $r > 0$, there exists $R = R(r, d) > r$ such that $u_r$ is the unique solution of* (2.1) *on $A_{r,R} = B_R \backslash B_r$ with $g|_{\partial B_r} = 1$ and $g|_{\partial B_R} = 0$. In particular,*

$$u_r = \begin{cases} 1 & \text{on } \partial B_r, \\ 0 & \text{on } \partial B_R, \\ \frac{|x|^{2-d} - R^{2-d}}{r^{2-d} - R^{2-d}} \text{ if } d \geq 3 \text{ and } \frac{\log|x| - \log R}{\log r - \log R} \text{ if } d = 2 & \text{in } B_R \backslash B_r. \end{cases}$$

*Also $r < R < r + 1$ and $|\nabla u_r(x)| = \left(\frac{|x|}{R}\right)^{1-d}$.*

The radial solutions are scale invariant.

**Remark 2.13.** We note that the solution from Proposition 2.12 is scale invariant, instead of considering $A_{1,R}$ with datum $g|_{\partial B_1} = 1$ and $g|_{\partial B_R} = 0$, we consider $A_{\rho,\rho R}$ with inner datum $g|_{\partial B_\rho} = \rho$ and $1 < R(d) < 2$. Then the unique solution is given by $\rho u_r(\rho x)$.

For the outer radius $R$ we have the following.

**Remark 2.14.** From [Vel23, Proposition 2.15], $R = R(1, d)$ is given by the relations

$$\begin{aligned} d = 2: &\quad R\log(R) - 1 = 0, \\ d \geq 3: &\quad R^{d-1} - R - d + 2 = 0. \end{aligned}$$

In particular $R \leq e^{W(1)} < 1.8$, for any dimension $d$. However, as $d \to \infty$, also $R \to 1$.

The restriction of the unique solution in Proposition 2.12 to a subset gives again a unique solution to (2.1) on that same subset.

**Corollary 2.15.** *Let $D$ be an open, non-empty subset $B_R \backslash B_r$ respectively and $u_r$ the solution from Proposition 2.12. Then for $g = u_r|_{\partial D}$ the unique minimizer to* (2.1) *is given by $u = u_r|_D$.*

*Proof.* Suppose that $u \neq u_R$ is a minimizer of (2.1) on $D$. Then

$$\tilde{u}_r = \begin{cases} u & \text{in } D, \\ u_r & \text{otherwise} \end{cases},$$

is a minimizer on $B_R$ or $B_R \backslash B_r$, not identically equal to $u_r$, contradicting the uniqueness from Proposition 2.12. □

We now introduce an explicit example of a global minimizer.

**Proposition 2.16.** [Vel23, Proposition 2.10] *For any $\nu \in \mathbb{S}^{d-1}$, the half-plane solution $H_\nu(x) = \sqrt{\Lambda}(x \cdot \nu)_+$ is a global minimizer.*

Analogous to Lemma 2.2, a restriction is again a minimizer.

**Proposition 2.17.** *Let $D \subset \mathbb{R}^d$ and $g = H_\nu|_{\partial D}$. Then a solution of* (2.1) *on $D$ with boundary datum $g$ is given by $H_\nu|_D$.*

*Proof.* Suppose there exists $v$ with $v|_{\partial D} = g$ and $F_\Lambda(v, D) < F_\Lambda(H_\nu|_D, D)$. Then for

$$\tilde{H}_\nu = \begin{cases} v & \text{in } D, \\ H_\nu & \text{otherwise} \end{cases},$$

we have $F_\Lambda(\tilde{H}_\nu, \mathbb{R}^d) < F_\Lambda(H_\nu, \mathbb{R}^d)$, contradicting the global minimality of $H_\nu$. □

# 3 Regularity of the solution

We now show the optimal interior regularity of local minimizers. This in turn gives an estimate for the behaviour of the solution $u$ around a free boundary point $x_0$, namely that $u(x) \sim |x - x_0|$, which indicates the natural scaling factor $r$. The strategy is to first show that the solution $u$ is locally Lipschitz continuous, giving the upper bound $u(x) \leq C|x - x_0|$ and then the non-degeneracy condition, $u(x) \geq c|x - x_0|$. Similar results are know for a variety of other free boundary problems and are normally the first step in a quantitative understanding of the solution.

### 3.1 Interior Lipschitz continuity

We aim to show that the gradient $\nabla u$ is locally bounded in $L^\infty$, which is equivalent to $u$ being Lipschitz. Without loss of generality, assume that $x_0 = 0$, otherwise translate the problem. The proof (from [Vel23, Chapter 3]) consists of two preparatory lemmas, which prove useful in more general settings as well.

**Lemma 3.1.** (Laplacian estimate) *Suppose that $u$ is a local minimizer of $F_\Lambda$ in $D \subset \mathbb{R}^d$. If $B_{2r}(x_0) \subset D$, then*

$$\Delta u(B_r(x_0)) = \int_{B_r(x_0)} \Delta u \leq C(d, \Lambda) r^{d-1}.$$

*Proof.* We have $x_0 = 0$. For $u \in H^1(D)$, the distribution $\Delta u$ acting on $\phi \in C_c^\infty(D)$ as

$$\Delta u(\phi) := -\int_D \nabla u \nabla \phi,$$

corresponds by the Riesz-Markov-Kakutani representation theorem to a unique Radon measure, also denoted $\Delta u$. We first estimate $\Delta u(\phi)$ for every $\phi \in C_c^\infty(B_r)$. By optimality of $u$, for $\psi \in C_c^\infty(B_r)$,

$$\begin{aligned}\int_{B_r} |\nabla u|^2 \leq \int_{B_r} |\nabla u|^2 + \Lambda |\{u > 0\} \cap B_r| &\leq \int_{B_r} |\nabla (u+\psi)|^2 + \Lambda |\{u + \psi > 0\} \cap B_r| \\ \leq \Lambda |B_r| + \int_{B_r} |\nabla(u+\psi)|^2 &= \Lambda |B_r| + \int_{B_r} |\nabla u|^2 + \int_{B_r} |\nabla \psi|^2 + 2\int_{B_r} \langle \nabla u, \nabla \psi \rangle.\end{aligned}$$

Take now $\psi = r^{d/2} \|\nabla \phi\|_{L^2(B_r)}^{-1} \phi$, then

$$\begin{aligned}-\int_{B_r} \langle \nabla u, \nabla \psi \rangle \leq \frac{1}{2}\int_{B_r} |\nabla \psi|^2 + \Lambda \frac{w_d}{2} r^d &\leq \frac{1}{2} r^d \|\nabla \phi\|_{L^2(B_r)}^{-2} \int_{B_r} |\nabla \phi|^2 + \Lambda \frac{w_d}{2} r^d = \frac{1 + \Lambda w_d}{2} r^d \\ \implies -\int_{B_r} \langle \nabla u, \nabla \phi \rangle = -\frac{\|\nabla \phi\|_{L^2(B_r)}}{r^{d/2}} \int_{B_r} \langle \nabla u, \nabla \psi \rangle &\leq \frac{1 + \Lambda w_d}{2} r^{d/2} \|\nabla \phi\|_{L^2(B_r)}\end{aligned}$$

Take now $\phi \in C_c^\infty(B_{2r})$ such that $\phi \geq 0$ on $B_{2r}$, $\phi|_{B_r} \equiv 1$ and $\|\nabla \phi\|_{L^\infty(B_{2r})} \leq \frac{2}{r}$. Since $\Delta u \geq 0$ in the sense of distributions and $\phi \geq \chi_{B_r}$,

$$\Delta u(B_r) = \int_D \Delta u \chi_{B_r} \leq \int_D \Delta u \phi \leq C(2r)^{d/2} \|\nabla \phi\|_{L^2(B_{2r})} \leq C(2r)^{d/2} |B_{2r}|^{1/2} \|\nabla \phi\|_{L^\infty(B_{2r})} \leq C r^{d-1},$$

which finishes the proof. □

Secondly, we estimate the growth of the mean value.

**Lemma 3.2.** *Suppose that $u \in H^1(B_R)$ is non-negative, subharmonic and $u(0) = 0$. Assume that the upper bound from Lemma 3.1, $\Delta u(B_r) \leq C r^{d-1}$, holds for any $0 < r < R$, then*

$$\fint_{\partial B_r} u \, dx \leq \frac{C}{d\omega_d} r \qquad \text{and} \qquad \fint_{B_r} u \, d\mathcal{H}^{d-1} \leq \frac{C}{\omega_d} r \qquad \forall r \in (0, R).$$

*Proof.* Take $u_n$ smooth such that $u_n \xrightarrow{H^1(B_r)} u$, then by Green's theorem

$$\frac{d}{dr} \fint_{\partial B_r} u_n d\mathcal{H}^{d-1} = \fint_{\partial B_r} \frac{\partial u_n}{\partial n} d\mathcal{H}^{d-1} = \frac{1}{d\omega_d r^{d-1}} \int_{B_r} \Delta u_\varepsilon(x) = \frac{1}{d\omega_d r^{d-1}} \nabla u_n(B_r).$$

Integrate in $r$ and letting $n \to \infty$ ($\Delta u_n(B_r) \to \Delta u(B_r)$, by definition of the Laplacian measure, since the convergence is in $H^1$) gives

$$\fint_{\partial B_r} u \, d\mathcal{H}^{d-1} \leq \int_0^r \frac{\Delta u(B_r)}{d\omega_d r^{d-1}} \leq \frac{C}{d\omega_d} r.$$

The second statement follows directly as

$$\fint_{B_r} u\ dx = \frac{1}{\omega_d r^d}\int_0^r \int_{\partial B_\rho} u\ d\mathcal{H}^{d-1}\, d\rho \leq \frac{1}{\omega_d r^d}\int_0^r Cr^d d\rho = \frac{C}{\omega_d} r,$$

which finishes the proof. $\square$

We are now able to completely prove the interior Lipschitz regularity.

**Proposition 3.3.** *Let $u \in H^1(D)$ be a minimizer to (2.1). Then we bound $|\nabla u|$ away from $\partial D$ on $D_\delta \coloneqq \{x \in D : \operatorname{dist}(x, \partial D) > \delta\}$ as*

$$\|\nabla u\|_{L^\infty(D_\delta)} \leq C\left(1 + \frac{\|u\|_{L^\infty(D_{\delta/2})}}{\delta}\right) \qquad \text{with } C = C(\Lambda, d),$$

*that is, $u$ is locally Lipschitz continuous in $D$.*

*Proof.* We already have:

- $u$ is harmonic in the open set $\Omega_u \coloneqq \{u > 0\}$, by Proposition 2.3.
- For the growth of the function $u$ at the free boundary, by Lemma 3.2,

$$\fint_{\partial B_r} u\ d\mathcal{H}^{d-1} \leq \frac{C}{d\omega_d} r \qquad \forall r \in (0, r_0).$$

**Step 1:** We show that for $B_{2r}(x_0) \subset D$ and $u(x_0) = 0$, then $\sup_{B_{r/2}(x_0)} u \leq C(d,\Lambda) r$. We have $x_0 = 0$. By the maximum principle for subharmonic functions, the supremum is attained at some point $x \in \partial B_{r/2}(x_0)$. Set $\delta = \operatorname{dist}(x, \partial\Omega_u) \leq r/2$. Since $u$ is harmonic on $B_\delta(x)$, by Harnack's inequality (Theorem A.18), $\inf_{B_{\delta/2}(x)} u \geq \frac{2^{n-2}}{3} u(x)$. Let $y \in \partial B_\delta(x)$ with $u(y) = 0$ and take $q : B_1 \to \mathbb{R}$ to be the unique harmonic function vanishing on $\partial B_1$ and $u|_{\partial B_{1/2}} = 1$, i.e. $q(z) = \frac{|z|^{2-d}-1}{2^{d-2}-1}$ if $d \geq 3$ and $q(z) = \frac{-\log|z|}{\log 2}$ if $d = 2$. Let

$$q_u(z) = c_d u(x) q(x + \delta z),$$

thus $q_u = 0$ on $\partial B_\delta(x)$ and $q_u \leq u$ on $\partial B_{\delta/2}(x)$, so by the maximum principle $q_u \leq u$ in $B_\delta \backslash B_{\delta/2}$. If $c_d = |\nabla q|_{\partial B_1}$, $q(z) \geq c_d(1 - |z|)$ for $|z| \in [1/2, 1]$, that is

$$u(z) \geq q_u(z) \geq c_n u(x)(1 - |z - x|/\delta) \qquad \text{for } z \in B_\delta(x)\backslash B_{\delta/2}(x).$$

Now for $s \leq \delta$, we have that $\mathcal{H}^{d-1}\left(\partial B_s(y) \cap B_{\delta - \frac{s}{2}}(x)\right) \geq c_d' s^{d-1}$, but also $\partial B_s(y) \cap B_{\delta-\frac{s}{2}}(x) \subset B_\delta(x)\backslash B_{\delta/2}(x)$, so we have

$$u(z) \geq c_d u(x)\left(1 - \frac{|z-x|}{\delta}\right) \geq c_d u(x)\left(1 - \frac{\delta - s/2}{\delta}\right) = c_d u(x)\frac{s}{\delta} \quad \forall z \in \partial B_s(y) \cap B_{\delta-\frac{s}{2}}(x).$$

Thus

$$\begin{aligned}
\fint_{\partial B_s(y)} u = \frac{1}{d\omega_d s^{d-1}}\int_{\partial B_s(y)} u &\geq \frac{1}{d\omega_d s^{d-1}}\int_{\partial B_s(y)\cap B_{\delta-\frac{s}{2}}(x)} u \\
&\geq \frac{1}{d\omega_d s^{d-1}}\,\mathcal{H}^{d-1}\left(\partial B_s(y) \cap B_{\delta-\frac{s}{2}}(x)\right) c_d u(x)\frac{s}{\delta} \\
&\geq \frac{c_d'}{d\omega_d} c_d u(x)\frac{s}{\delta} = c_d u(x)\frac{s}{\delta}.
\end{aligned}$$

By Lemma 3.1 and Lemma 3.2 we arrive at

$$\Delta u(B_s(y)) \cdot cu(x)\frac{s}{\delta} \leq \Delta u(B_s(y)) \cdot \fint_{\partial B_s(y)} u \leq C(1+\Lambda)s^{d-1}\frac{s}{d\omega_d} = Cs^d.$$

Since for any subharmonic function $\fint_{B_{2r}} u - u(0) = c_d \int_0^{2r} \frac{\Delta u(B_s)}{s^{d-1}} ds$, the fact that $u(y) = 0$ and the mean value property give

$$\frac{cu(x)}{\delta}u(x) \leq \frac{cu(x)}{\delta}\fint_{\partial B_\delta(y)} u = \frac{cu(x)}{\delta}\fint_{\partial B_\delta(y)} u - u(y) = \frac{cu(x)}{\delta}\int_0^\delta \frac{\Delta u(B_s)}{s^{d-1}} ds \leq C\delta.$$

Hence $u(x) \leq C(d,\Lambda)\delta \leq C\frac{r}{2} = Cr$, as was to be shown.

**Step 2:** We show the final bound. Let $x \in D_\delta$ and $\delta' = \delta'(x) = \operatorname{dist}(x, \partial\Omega_u)$. If $16\delta' \geq \operatorname{dist}(x, \partial D) =: \delta$, then $u$ is harmonic on $B_{\delta/16}(x)$ and by the standard elliptic estimate (Theorem A.20) (using $\delta/16$ and $\delta/32$),

$$|\nabla u(x)| \leq \sup_{B_{\delta/32}(x)} |\nabla u| \leq \frac{C}{\delta/32} \sup_{B_{\delta/16}(x)} \leq \frac{C}{\delta} \sup_{D_{\delta/2}} u \leq C\left(1 + \frac{\|u\|_{L^\infty(D_{\delta/2})}}{\delta}\right).$$

If not, then $16\delta' < \operatorname{dist}(x, \partial D) =: \delta$ and so pick $y$ on $\partial B_{\delta'}(x)$ with $u(y) = 0$. Clearly $8\delta' < \operatorname{dist}(y, \partial D)$ and $\operatorname{dist}(y, \partial D) \geq 15\delta'$, so $B_{8\delta'}(y)$ is wholly contained in $D$. Apply now the first step with $r = 4\delta'$, to get that $u(x) \leq C\delta'$ on $B_{2\delta'}(y)$ and since $B_{\delta'}(x) \subset B_{2\delta'}(y)$ (as $|x-y| = \delta'$) also on $B_{\delta'}(x)$. Applying now the standard elliptic estimate (Theorem A.20) on $B_{\delta'}(x)$ gives

$$|\nabla u(x)| \leq \sup_{B_{\delta'/2}(x)} |\nabla u| \leq \frac{C}{\delta'} \sup_{B'_\delta(x)} u \leq \frac{C}{\delta'} \sup_{B_{2\delta'}(y)} u \leq C \leq C\left(1 + \frac{\|u\|_{L^\infty(D_{\delta/2})}}{\delta}\right),$$

which is what we wanted to show. □

## 3.2 Non-degeneracy at the free boundary

We now show that the interior Lipschitz regularity is optimal in the sense that $u$ grows at least linearly at a free boundary point, in particular $u$ is not in $C^1(D)$ but only in $C^{0,1}(D)$.

**Proposition 3.4.** *Suppose that $u$ is a local minimizer on $D$, $x_0 \in \bar{\Omega}_u$ and $B_r(x_0) \subset\subset D$, then*

$$\sup_{B_r(x_0)} u = \max_{\partial B_r(x_0)} u \geq c_D\sqrt{\Lambda}r.$$

*Proof.* The idea comes from [Kri19]. Since $u$ is subharmonic and harmonic on $\{u > 0\}$, by the maximum principle

$$\sup_{B_r(x_0)} u = \max_{\bar{B}_r(x_0)\cap\{u>0\}} u = \max_{\partial B_r(x_0)\cup\partial\{u>0\}} u = \max_{\partial B_r(x_0)} u.$$

Suppose $\sup_{B_r(x_0)} u \leq \varepsilon\sqrt{\Lambda}r$ for any $\varepsilon < \varepsilon_0$. We want to show that this implies that $u(x) = 0$ for any $x \in B_{r/4}(x_0)$, in other words $x_0 \notin \bar{\Omega}_u$.

We first claim that for any $B_{2^{-k}r}(x)$ with $k \in \mathbb{N}$ and $\varepsilon$ sufficiently small,

$$\sup_{B_{2^{-k}r}(x)} u \leq \varepsilon\sqrt{\Lambda}2^{-k}r \quad \Longrightarrow \quad \sup_{B_{2^{-(k+1)}r}(x)} u \leq \varepsilon\sqrt{\Lambda}2^{-(k+1)}r.$$

Let $\eta_1$ be a smooth cutoff with $\eta_1 \equiv 1$ on $B_{\frac{9}{10}2^{-k}r}(x)$, $\operatorname{spt}\eta_1 \subset B_{2^{-k}r}(x)$ and $|\nabla\eta_1| \leq \frac{C}{2^{-k}r}$. Then test $\eta_1^2 u$ against $\Delta u \geq 0$, by Cacciopolli identities (Theorem A.19, $u \geq 0$ is a minimizer so subharmonic, i.e. $u\Delta u \geq 0$),

$$\int_{B_{\frac{9}{10}2^{-k}r}(x)} |\nabla u|^2 \leq \int_{B_{2^{-k}r}(x)} \eta_1^2|\nabla u|^2 \leq \frac{4C}{2^{-2k}r^2}\int_{B_{2^{-k}r}(x)} u^2 \leq \frac{4C}{(2^{-k}r)^2}\varepsilon^2\Lambda(2^{-k}r)^2|B_{2^{-k}r}|.$$

The assumption $u \leq \varepsilon\sqrt{\Lambda}2^{-k}r$ is used in the last inequality. Take a second cutoff $\eta_2$ supported on $\mathbb{R}^d\setminus B_{\frac{3}{4}2^{-k}r}(x)$, $\eta_2 \equiv 1$ on $\mathbb{R}^d\setminus B_{\frac{9}{10}2^{-k}r}(x)$ and $|\nabla\eta_2| \leq \frac{C'}{2^{-k}r}$. Set $v = \eta_2 u$, since $u$ was a local minimizer,

$$\Lambda|\{u>0\}\cap D| + \int_D |\nabla u|^2 = E(u,D) \leq E(v,D) = \Lambda|\{v>0\}\cap D| + \int_D |\nabla v|^2.$$

Then

$$|\{u>0\}\cap D| - |\{v>0\}\cap D| = |\{u>0\}\cap B_{\frac{3}{4}2^{-k}r}(x)^c|$$

and $|\nabla v|^2 \leq 2(\eta_2|\nabla u|)^2 + 2(|\nabla\eta_2|u)^2$ lead to

$$\begin{aligned}
\Lambda|\{u>0\}\cap B_{\frac{3}{4}2^{-k}r}(x)^c| &\leq \int_D |\nabla v|^2 - |\nabla u|^2 = \int_{B_{\frac{9}{10}2^{-k}r}(x)} |\nabla v|^2 - |\nabla u|^2 \leq \int_{B_{\frac{9}{10}2^{-k}r}(x)} |\nabla v|^2 \\
&\leq 2\int_{B_{\frac{9}{10}2^{-k}r}(x)} |\nabla u|^2 + 2C'^2\int_{B_{\frac{9}{10}2^{-k}r}(x)} \frac{u^2}{(2^{-k}r)^2} \\
&\leq \frac{C}{(2^{-k}r)^2}\varepsilon^2\Lambda(2^{-k}r)^2|B_{2^{-k}r}| \\
\implies |\{u>0\}\cap B_{\frac{3}{4}2^{-k}r}(x)| &\leq C\varepsilon^2|B_{2^{-k}r}|.
\end{aligned}$$

Pick now $y \in B_{2^{-(k+1)}r}(x)$ and use the mean value property for the subharmonic function $u$,

$$\begin{aligned}
u(y) &= \fint_{B_{\frac{1}{4}2^{-k}r}(y)} u \leq \frac{1}{|B_{\frac{1}{4}2^{-k}r}|}\int_{B_{\frac{3}{4}2^{-k}r}(x)} u \\
&\leq \frac{1}{|B_{\frac{1}{4}2^{-k}r}|}|\{u>0\}\cap B_{\frac{3}{4}2^{-k}r}(x)| \sup_{B_{2^{-k}r}(x)} u \\
&\leq \frac{1}{|B_{\frac{1}{4}2^{-k}r}|}C\varepsilon^2|B_{2^{-k}r}|\varepsilon\sqrt{\Lambda}2^{-k}r = 4^d\cdot C\varepsilon^3\sqrt{\Lambda}2^{-k}r \leq \varepsilon\sqrt{\Lambda}2^{-(k+1)}r,
\end{aligned}$$

where in the last inequality we require that $\varepsilon_0$ is sufficiently small such that $4^d\cdot C\varepsilon_0^2 \leq 1/2$. Hence the claim holds.

Using the same argument to $B_r(x_0)$(i.e. $k=0$) and supposing that $4^d\cdot C\varepsilon^2 \leq 1/4$ we can show (after adjusting $\varepsilon_0$ to $\varepsilon_0/2$ that is $4^d\cdot C\varepsilon_0^2 \leq 1/4$),

$$\text{for } \varepsilon \leq \varepsilon_0\text{:} \qquad \sup_{B_r(x_0)} u \leq \varepsilon\sqrt{\Lambda}r \quad\implies\quad \sup_{B_{r/2}(x_0)} u \leq \varepsilon\sqrt{\Lambda}\frac{r}{4}.$$

We pick now $x \in B_{r/4}(x_0)$, then $B_{2^{-2}r}(x) \subset B_{r/2}(x_0)$ and so

$$\sup_{B_{2^{-2}r}(x)} u \leq \sup_{B_{r/2}(x_0)} u \leq \varepsilon\sqrt{\Lambda}\frac{r}{4} = \varepsilon\sqrt{\Lambda}2^{-2}r.$$

In other words, the hypothesis of the claim holds for $k=2$. By applying the claim inductively for increasing $k = 2,3,\ldots$ we have $\sup_{B_{2^{-k}r}(x)} u \leq \varepsilon\sqrt{\Lambda}2^{-k}r(x)$ for any $\mathbb{N} \ni k \geq 2$. Thus

$$u(x) \leq \lim_{k\to\infty}\sup_{B_{2^{-k}r}(x)} u \leq \lim_{k\to\infty}\varepsilon\sqrt{\Lambda}2^{-k}r = 0,$$

as was to be shown. □

### 3.3 Regularity up to the boundary

Notice that the previous estimates were interior estimates, nothing is said about how the solution behaves when it meets the boundary $\partial D$: a priori it is only known that $u \in W^{1,2}(D)$. We now improve up on that by two results, first for continuous datum, secondly for Hölder continuous datum. We hope to achieve similar results as in the case for the Laplace equation with Dirichlet boundary conditions. These results are original and, to our knowledge, previously unreported in the literature. First, we have a qualitative regularity result.

**Theorem 3.5.** *For a bounded open domain $D$, suppose that at least one of*

- *$D$ is convex,*
- *$\partial D$ is locally $c$-Lipschitz for some constant $c = c(d) < R(1, d) - 1$ ($R(1, d)$ comes from Proposition 2.12), i.e. $\partial D$ is locally the graph of $g : U \to \mathbb{R}$ with*

$$|g(x) - g(y)| \le c|x - y| \qquad \forall x, y \in U.$$

*is satisfied. Let $u$ be a minimizer of* (2.1) *with $g \in C^0(\partial D) \cap H^{1/2}(\partial D)$. Then $u \in C^0(\bar{D})$.*

**Remark 3.6.** Note that the second condition holds whenever $\partial D$ is $C^1$.

*Proof.* Since $g$ is an admissible trace for $u \in W^{1,2}(D)$, the problem (2.1) is well posed and a solution $u$ exists. By contradiction, suppose that $u$ is not continuous, then there exists $x_0 \in \bar{B}_1$ and $x_k \to x_0$ with $u(x_k) \nrightarrow u(x_0)$. Since $u$ is (Lipschitz) continuous inside $D$, necessarily $x_0 \in \partial D$.
**Case $x_0 \in \partial D \cap \{u = 0\}$:** Since $u$ is subharmonic, $u \le \tilde{u}$, where

$$\begin{cases} \Delta \tilde{u} = 0 & \text{in } D, \\ \tilde{u} = g & \text{on } \partial D. \end{cases}$$

This in turn gives

$$0 \le u(x_k) \le \tilde{u}(x_k) \to g(x_0) = 0,$$

where the convergence follows from the continuity up to the boundary of harmonic functions, [GT77, Theorem 2.14].
**Case $x_0 \in \partial D \cap \Omega_u$:** Without loss of generality, after a translation, assume that $x_0 = 0$. Let us show $u > 0$ in $B_\delta(x_0) \cap D$ for some $\delta > 0$. Let $r$ be such that $g(x) \ge g(x_0)/2$ in $B_r(x_0) \cap \partial D$, by continuity of $g$. Then we have the two cases:

- $D$ **is convex:** Up to a rigid motion, we assume that $D \subset H$, where $H = \{(x', x_d) : x_d \ge 0\}$. Consider the solution $v$ from Proposition 2.12 in the annulus $A_{1,R}(x_1)$ with radii 1 and $R(1, d) > 1$ ($v|_{\partial B_1} = 1$ and $v|_{\partial B_R} = 0$). Take $\rho$ sufficiently small such that $\sqrt{R^2 - (R - \rho)^2} < r$ ($\Longleftarrow$ $2R\rho < r^2$) and on the open annulus $A_{R-\rho,R}(x_1)$, $0 < v \le g(x_0)/3$. By translation invariance, take $x_1 = (0, -(R - \rho/2))$ and $E = A_{R-\rho,R} \cap D$. It follows that $E \subset B_r(x_0)$ and that $\partial E = (\partial B_R(x_1) \cap D) \cup (\partial D \cap A_{R-\rho,R}) =: E_1 \cup E_2$. On $E_1$,

  $$v(x) \le g(x_0)/3 < g(x_0)/2 \le g(x) = u(x)$$

  and on $E_2$, $v(x) = 0 \le u(x)$.

- $\partial D$ **is $c$-Lipschitz:** Without loss of generality (up to rotation), we assume that $\partial D$ is locally around $x_0$ the graph of a $C^1$ function $f : \mathbb{R}^{d-1} \to \mathbb{R}$, with tangent plane at $x_0$ equal to $\{x_d = 0\}$ and the interior of $D$ lies below $f$. As $f$ is $c$-Lipschitz, for any $h > 0$, the graph of $f$ lies within $B_h^{d-1} \times [-ch, ch]$.

Take now $\rho = \min(r/4, g(x_0)/2)$ and $x_1 = (0', (1+c)\rho)$ and consider the solution $v$ on the annulus $A_{\rho,R\rho}(x_1)$ given by Remark 2.13. Now $\mathrm{graph}(f) \cap A_{\rho,R\rho}(x_1)$ lies within $B^{d-1}_{2\rho} \times [-2c\rho, 2c\rho]$, we set $E$ to be the connected component of $\mathrm{hyp}(f) \cap A_{\rho,R\rho}(x_1)$ containing $x_0$. The set $E$ is not empty as long as $(1+c)\rho < R\rho$, which is equivalent to $c < R-1$. The boundary $\partial E$ splits into $E_1 \subset \mathrm{graph}(f)$ with $E_1 \subset A_{\rho,R\rho}(x_1)$, so $v \le g(x_0)/2$ there; and $E_2 \subset \partial B_{R\rho}(x_1)$, so $v = 0$ there.

In both cases, we have constructed a set $E$ where the boundary consists of two parts $E_1$ and $E_2$, and $u|_{E_1} > v|_{E_1}$ and $u|_{E_2} \ge v|_{E_2}$. Since $v_E$ and $u_E$ are minimizers on $E$ for their own boundary datum (Lemma 2.2), by Lemma 2.7 we have $u \ge v$. However inside $E$, $v > 0$ and so $u > 0$.

Now, since $u$ is strictly positive, it is harmonic, i.e. $\Delta u = 0$ in $B_r(x_0) \cap D \subset E$ for some small $r$ and so $u$ is continuous on $\overline{B_{r/2}(x_0) \cap D}$. In particular, we apply [GT77, Lemma 2.13], using the continuity of $g$ at the regular (i.e. existence of barrier, see [GT77]) boundary point $x_0$, to get $u(x_k) \to u(x_0)$, contradicting the assumption. □

For the second result on the Hölder regularity up to the boundary, we follow the steps of [ESV22], where a similar result is given for Lipschitz functions: for Lipschitz continuous boundary datum, the solution is $\alpha$-Hölder continuous up to the boundary for any $\alpha < 1$.

**Theorem 3.7.** *Suppose that $D$ is a $C^{1,\alpha}$ domain. Let $u$ be a minimizer of* (2.1) *on $D$ with boundary datum $g \in C^{\gamma_0}(\partial D) \cap H^1 D$ with $1 > \gamma_0 > \frac{1}{2}$. Then $u \in C^{\gamma_0}(\bar{D})$.*

*Proof.* Let $x_0 \in \partial D$ and $\rho < 1$ sufficiently small such that $\partial D$ is the graph of a $C^{1,\alpha}$ function in $B_\rho(x_0)$. It suffices to show $\gamma_0$-Hölder continuity in a small ball $B_{\rho/8}(x_0)$. Denote the positive and negative half spaces by $H^+$ and $H^-$.

First, extend $\nabla u : B_\rho(x_0) \cap D \to \mathbb{R}^d$ to $\nabla u : B_\rho(x_0) \to \mathbb{R}^d$. Let $\Phi : B_\rho \to B_\rho(x_0)$ such that $\Phi(\{x_d = 0\}) = \partial D \cap B_\rho(x_0)$, inheriting the $C^{1,\alpha}$ regularity. Up to translation and rotation, we assume $\Phi(0) = x_0$ and $D\Phi(0) = I_{d\times d}$. Let $\pi : (y', y_d) \mapsto (y', -y_d)$, we define the extension

$$\nabla u(x) = \begin{cases} \nabla u(x) & \text{if } x \in D \cap B_\rho(x_0), \\ \nabla u(\Phi \circ \pi \circ \Phi^{-1}(x)) & \text{if } x \in D^c \cap B_\rho(x_0). \end{cases}$$

The idea is to arrive at an estimate for $u$ of the form

$$\forall \bar{x} \in B_{\rho/8}(x_0), \forall r \le \frac{\rho}{2} : \qquad \int_{B_r(\bar{x})} |\nabla u|^2 \le C r^{d+2(\gamma_0-1)}, \tag{3.1}$$

and then use the Morrey Lemma (Lemma A.5), which gives directly $\gamma_0$-Hölder regularity in $B_{\rho/8}(x_0)$ (and so also for any $\gamma < \gamma_0$ by Hölder embeddings). Since for any $\bar{x} \in B_{\rho/8}(x_0)$ and $r \le \frac{\rho}{2}$, we have $B_r(\bar{x}) \subset B_{\frac{5}{8}\rho}(x_0)$ , it suffices to consider only points on $\partial D$ and to show

$$\forall x_0 \in \partial D \cap B_{\rho/8}, \forall r \le \frac{5}{8}\rho : \qquad \int_{B_r(x_0)} |\nabla u|^2 \le C r^{d+2(\gamma_0-1)}. \tag{3.2}$$

If we show the inequality $\int_{B_r(x_0)} |\nabla u|^2 \le C r^{d+2(\gamma_0-1)}$ for any $r > 0$ small enough, we take $\rho$ possibly smaller to get the above. Denote $D_r = \Phi(H^+ \cap B_r)$ and $D_\rho = \Phi(H^+ \cap B_\rho)$.

By using the change of variable, $x = \Phi(y), dx = |D\Phi(y)| dy$, we have

$$\begin{aligned} \int_{B_r(x_0)} |\nabla u(x)|^2 dx &= \int_{\Phi(B_r)} |\nabla u(x)|^2 dx = \int_{B_r} |\nabla u(\Phi(y))|^2 |D\Phi(y)| dy \\ &= \int_{B_r \cap H^+} |\nabla u(\Phi(y))|^2 |D\Phi(y)| dy + \int_{B_r \cap H^-} |\nabla u(\Phi \circ \pi \circ \Phi^{-1}(\Phi(y)))|^2 |D\Phi(y)| dy \qquad (3.3) \\ &= 2 \int_{B_r \cap H^+} |\nabla u(\Phi(y))|^2 |D\Phi(y)| dy = 2 \int_{\Phi(H^+ \cap B_r)} |\nabla u(x)|^2 dx. \end{aligned}$$

**Step 1:** Let $h_g : H^+ \cap B_\rho \to \mathbb{R}$ such that

$$\begin{cases} \Delta h_g = 0 & \text{in } H^+ \cap B_\rho, \\ h_g = g \circ \Phi & \text{on } \partial(H^+ \cap B_\rho), \end{cases}$$

which is in $C^{\gamma_0}(\bar{H}^+ \cap B_\rho)$ with $\|h_g\|_{C^{\gamma_0}(H^+ \cap B_\rho)} \leq \|g\|_{C^{\gamma_0}(D_\rho)}$ [MS06, Proposition 2.1]. We claim that

$$|Dh_g(x)| \leq C \operatorname{dist}(x, \partial H^+).$$

Let $x_1 \in \partial H^+$ with $|x - x_1| = \operatorname{dist}(x, \partial H^+) =: d(x)$ and set $v(x) := h_g(x) - h_g(x_1)$. By $\gamma_0$-Hölder regularity of $v$, either directly from the definition of Hölder continuity or from Theorem A.4,

$$v(x) = v(x_1) + R(x) = R(x) \qquad \text{with } |R(x)| \leq C_d \|h_g\|_{C^{\gamma_0}(H^+ \cap B_R)} |x - x_1|^{\gamma_0} \leq C_{d,g} |x - x_1|^{\gamma_0}.$$

By standard harmonic estimates (Theorem A.20),

$$|Dv(x)| \leq \sup_{B_{d(x)/4}(x)} |Dv(y)| \leq \frac{C_d}{d(x)} \sup_{B_{d(x)/2}(x)} |v(y)| \leq \frac{C_{d,g}}{d(x)} \sup_{B_{d(x)/2}(x)} |y - x_1|^{\gamma_0} \leq C_{d,g} d(x)^{\gamma_0 - 1}.$$

In the co-area formula (A.1), take $f(x) = d(x)$ and $g(x) = d(x)^{2(\alpha-1)}$. Since $\partial H^+ = \{x_d = 0\}$ we have $|\nabla f(x)| = 1$. Hence, as $f^{-1}(t) = \{x \in H^+ \cap B_r : d(x) = t\}$,

$$\begin{aligned} \int_{H^+ \cap B_r} |\nabla h_g|^2 dx &\leq C_{d,g} \int_{H^+ \cap B_r} d(x)^{2(\gamma_0 - 1)} dx \\ &= C_{d,g} \int_{H^+ \cap B_r} d(x)^{2(\gamma_0 - 1)} |\nabla f| dx \\ &= C_{d,g} \int_{\mathbb{R}} \int_{\{x \in H^+ \cap B_r : d(x) = t\}} d(x)^{2(\gamma_0 - 1)} d\,\mathcal{H}^{d-1}(x) dt \\ &= C_{d,g} \int_{\mathbb{R}} t^{2(\gamma_0 - 1)}\, \mathcal{H}^{d-1}(\{x \in H^+ \cap B_r : d(x) = t\}) dt \\ &= C_{d,g} \int_0^r t^{2(\gamma_0 - 1)}\, \mathcal{H}^{d-1}(\{x \in H^+ \cap B_r : d(x) = t\}) dt \\ &\leq C_{d,g} r^{d-1} \int_0^r t^{2(\gamma_0 - 1)} dt = C_{d,g} r^{d + 2(\gamma_0 - 1)}. \end{aligned} \tag{3.4}$$

for any $r$ with $0 < r < R$. The rest of the proof, provided for completeness, follows exactly as in [ESV22].

**Step 2:** Define the symmetric matrix $A(x) : D_r \to \mathbb{R}^{d \times d}$

$$A(x) := |\det \Phi(\Phi^{-1}(x))|^{-1} D\Phi(\Phi^{-1}(x)) D\Phi(\Phi^{-1}(x))^T.$$

By $C^{1,\alpha}$ regularity,

$$\exists C_g : \quad (1 - C_g r^\alpha) I \leq A(x) \leq (1 + C_g r^\alpha) I \qquad \forall x \in B_r. \tag{3.5}$$

For $r < R$, let $h$ be the solution to

$$\begin{cases} \Delta h = 0 & \text{in } H^+ \cap B_r, \\ h = u \circ \Phi & \text{on } \partial(H^+ \cap B_r), \end{cases} \tag{3.6}$$

and $f := h \circ \Phi^{-1} \Leftrightarrow h = f \circ \Phi$. We have $\nabla h(y) = D\Phi^T(y) \nabla f(\Phi(y))$. Furthermore, $f : D_r \to \mathbb{R}$ satisfies

$$\begin{cases} \operatorname{div}(A(x) \nabla f(x)) = 0 & \text{in } D_r, \\ f(x) = u(x) & \text{on } \partial D_r. \end{cases} \tag{3.7}$$

To show this, let $\tilde{\varphi} \in C_c^\infty(D_r)$ and consider the test function $\tilde{\varphi} \circ \Phi =: \varphi \in C_c^1(H^+ \cap B_r)$. From (3.6) we get, using again the change of variable $x = \Phi(y)$, $dx = |(\det \Phi)(\Phi^{-1}(x))|dy$,

$$
\begin{aligned}
0 &= \int_{H^+\cap B_r} \nabla\varphi(y) \cdot \nabla h(y) dy \\
&= \int_{H^+\cap B_r} \nabla\varphi(y) \cdot D\Phi^T(y) \nabla f(\Phi(y)) dy \\
&= \int_{D_r} \nabla\varphi(\Phi^{-1}(x)) \cdot D\Phi^T(\Phi^{-1}(x)) \nabla f(x) |(\det \Phi)(\Phi^{-1}(x))|^{-1} dx \\
&= \int_{D_r} D\Phi^T(\Phi^{-1}(x)) \nabla(\varphi \circ \Phi^{-1})(x) \cdot D\Phi^T(\Phi^{-1}(x)) \nabla f(x) |(\det \Phi)(\Phi^{-1}(x))|^{-1} dx \\
&= \int_{D_r} \nabla(\varphi \circ \Phi^{-1})(x) \cdot |(\det \Phi)(\Phi^{-1}(x))|^{-1} D\Phi(\Phi^{-1}(x)) D\Phi^T(\Phi^{-1}(x)) \nabla f(x) dx \\
&= \int_{D_r} \nabla(\varphi \circ \Phi^{-1})(x) \cdot A(x) \nabla f(x) dx \\
&= \int_{D_r} \nabla\tilde{\varphi}(x) \cdot A(x) \nabla f(x) dx,
\end{aligned}
$$

using the fact

$$
\nabla(\varphi \circ \Phi^{-1})(x) = D\Phi^{-T}(\Phi^{-1}(x)) \nabla\varphi(\Phi^{-1}(x)) \qquad \left( \Longleftarrow \; D(\Phi^{-1})(x) = \underbrace{D\Phi^{-1}}_{(D\Phi)^{-1}}(\Phi^{-1}(x)) \right)
$$

in the fourth equality.

We calculate

$$
\begin{aligned}
\int_{D_r} \nabla(u-f) \cdot A(x) \nabla(u-f) dx &= \int_{D_r} \nabla u \cdot A\nabla u - 2\nabla(u-f) \cdot A\nabla f - \nabla f \cdot A\nabla f dx \\
&= \int_{D_r} \nabla u \cdot A\nabla u - \nabla f \cdot A\nabla f - 2 \underbrace{\int_{\partial D_r} (u-f) A\nabla f \cdot \nu}_{=0} + 2 \underbrace{\int_{D_r} \operatorname{div}(A(x)\nabla f(x))}_{=0} \\
&\leq (1 + C_g r^\alpha) \left( \int_{D_r} |\nabla u|^2 dx - \frac{1 - C_g r^\alpha}{1 + C_g r^\alpha} \int_{D_r} |\nabla f|^2 \right) \\
&\leq (1 + C_g r^\alpha) \left( \int_{D_r} |\nabla u|^2 dx - \int_{D_r} |\nabla f|^2 dx + C_g r^\alpha \int_{D_r} |\nabla f|^2 \right) \\
&\leq (1 + C_g r^\alpha) \left( |\{u > 0\}| - |\{f > 0\}| + C_g r^\alpha \int_{D_r} |\nabla f|^2 \right) \\
&\leq (1 + C_g r^\alpha) \left( |D_r| + C_g r^\alpha \int_{D_r} |\nabla f|^2 \right) \leq (1 + C_g r^\alpha) \left( r^d + C_g r^\alpha \int_{D_r} |\nabla f|^2 \right),
\end{aligned}
$$

using (3.5) and in line 5 the optimality of $u$ in $D_r$, i.e.

$$
F_\Lambda(u, D_r) = \int_{D_r} |\nabla u|^2 + |\{u > 0\}| \leq \int_{D_r} |\nabla f|^2 + |\{f > 0\}| = F_\Lambda(f, D_r).
$$

Again by (3.7) we get,

$$
\int_{D_r} \nabla f \cdot A(x) \nabla f \leq \int_{D_r} \nabla u \cdot A(x) \nabla u,
$$

combining it several times with (3.5) gives

$$\begin{aligned}\frac{1-C_g r^\alpha}{1+C_g r^\alpha}\int_{D_r}|D(u-f)|^2 &\le \frac{1}{1+C_g r^\alpha}\int_{D_r} D(u-f)\cdot A(x)D(u-f)\\ &\le r^d + C_g r^\alpha \int_{D_r}|Df|^2\\ &\le r^d + C_g r^\alpha \frac{1}{1-C_g r^\alpha}\int_{D_r} Df\cdot A(x)Df\\ &\le r^d + C_g r^\alpha \frac{1}{1-C_g r^\alpha}\int_{D_r} Du\cdot A(x)Du\\ &\le r^d + C_g r^\alpha \frac{1+C_g r^\alpha}{1-C_g r^\alpha}\int_{D_r}|Du|^2.\end{aligned}$$

Upon choosing $R > r$ small enough such that $C_g R^\alpha \le \frac{1}{2}$, we get $\frac{1-C_g r^\alpha}{1+C_g r^\alpha} \le \frac{1}{3}$ and thus arrive eventually at

$$\int_{D_r}|\nabla(u-f)|^2 dx \le C_g r^d + C_g r^\alpha \int_{D_r}|\nabla u|^2 dx. \tag{3.8}$$

**Step 3:** Let $0<\kappa<1$ (to be defined later), we estimate

$$\begin{aligned}&\int_{\Phi(H^+\cap B_{\kappa r})}|Du|^2 \le \int_{\Phi(H^+\cap B_r)}|D(u-f)|^2 + \int_{\Phi(H^+\cap B_{\kappa r})}|Df|^2\\ &\quad\le C_g r^d + C_g r^\alpha \int_{\Phi(H^+\cap B_r)}|Du|^2 + C_g\int_{H^+\cap B_{\kappa r}}|Dh|^2\\ &\quad\le C_g r^d + C_g r^\alpha \int_{\Phi(H^+\cap B_r)}|Du|^2 + C_g\int_{H^+\cap B_{\kappa r}}|Dh_g|^2 + C_g\int_{H^+\cap B_{\kappa r}}|D(h-h_g)|^2\\ &\quad\le C_{d,\varphi,g} r^{d+2(\gamma_0-1)} + C_g r^\alpha \int_{\Phi(H^+\cap B_r)}|Du|^2 + C_g\frac{|B_r|}{|B_{\kappa r}|}\int_{H^+\cap B_r}|D(h-h_g)|^2\\ &\quad\le C_{d,\varphi,g} r^{d+2(\gamma_0-1)} + C_g r^\alpha \int_{\Phi(H^+\cap B_r)}|Du|^2 + C_g\kappa^d\int_{H^+\cap B_r}|Dh|^2\\ &\quad\le C_{d,\varphi,g} r^{d+2(\gamma_0-1)} + C_g(r^\alpha+\kappa^d)\int_{\Phi(H^+\cap B_r)}|Du|^2,\end{aligned} \tag{3.9}$$

using (3.8) in the second, (3.4) and harmonicity of $h-h_g$ (also $(h-h_g)|_{\partial H^*\cap B_r}=0$) in the fourth and $\int_{H^+\cap B_r}|Dh|^2 \le \int_{\Phi(H^+\cap B_r)}|Du|^2$ (follows from harmonicity of $h$) in the last line.

**Step 4:** Define

$$M_n := \frac{1}{r_n^{d+2(\gamma_0-1)}}\int_{\Phi(H^+\cap B_{r_n})}|Du|^2 \qquad \text{with } r_n = R\kappa^n. \tag{3.10}$$

Note $r_{n+1}=\kappa r_n$ and for $n\ge d/\alpha$, we have $r_n^\alpha+\kappa^d \le \kappa^{n\alpha}+\kappa^d \le 2\kappa^d$. We estimate by (3.9),

$$\begin{aligned}M_{n+1} &= \frac{1}{(\kappa r_n)^{d+2(\gamma_0-1)}}\int_{\Phi(H^+\cap B_{\kappa r_n})}|Du|^2\\ &\le \frac{1}{(\kappa r_n)^{d+2(\gamma_0-1)}}\left(C_{d,\varphi,g} r_n^{d+2(\gamma_0-1)} + C_g(r_n^\alpha+\kappa^d)\int_{\Phi(H^+\cap B_{r_n})}|Du|^2\right)\\ &\le \underbrace{C_{d,\varphi,g}\kappa^{d+2(\gamma_0-1)}}_{=:A} + \underbrace{2C_g\frac{\kappa^d}{\kappa^{d+2(\gamma_0-1)}}}_{=:B}\frac{1}{(r_n)^{d+2(\gamma_0-1)}}\int_{\Phi(H^+\cap B_{r_n})}|Du|^2 = A + BM_n.\end{aligned}$$

However, for $\kappa$ sufficiently small, $B = 2C_g\kappa^{2(1-\gamma_0)} < 1$ and so for any $n \geq n_o = \lceil d/\alpha \rceil$,

$$M_n \leq A + BM_{n-1} \leq \cdots \leq A(1 + B + \dots B^{n-n_o}) + B^{n-n_o}M_{n_o} \leq \frac{1}{1-B}A + M_{n_o} \leq C.$$

Rewriting $M_n$ gives

$$\int_{\Phi(H^+\cap B_{r_n})} |Du|^2 \leq r_n^{d+2(\gamma_0-1)},$$

so for any $r \in (0, R)$, pick $n$ such that $r_{n+1} \leq r \leq r_n$, then

$$\begin{aligned}\int_{\Phi(H^+\cap B_r)} |Du|^2 &\leq \int_{\Phi(H^+\cap B_{r_n})} |Du|^2 \\ &\leq Cr_n^{d+2(\gamma_0-1)} = C\left(\frac{r_{n+1}}{\kappa}\right)^{d+2(\gamma_0-1)} \\ &\leq \frac{C}{\kappa^{d+2(\gamma_0-1)}} r^{d+2(\gamma_0-1)}.\end{aligned}$$

Substituting in (3.3) gives (3.2), which finishes the proof. □

**Remark 3.8.** For $\gamma_0 < \frac{1}{2}$, it is not a priori given that $g$ is the trace of a function in $H^1(D)$. For $\gamma_0 \leq \frac{1}{2}$ and $g \in C^{\gamma_0} \cap H^{1/2}(D)$, the previous proof does not work.

**Remark 3.9.** If $\gamma_0 = 1$ (i.e. the datum is Lipschitz), then using the same argument as above,

$$\forall x_0 \in \bar{\Omega} \cap B_{R/2}, r \leq R/2 : \qquad \int_{B_r(x_0)} |\nabla u|^2 \leq Cr^{d+2(\gamma-1)}, \qquad \gamma < 1.$$

This implies local $\gamma$-Hölder regularity for any $\gamma < 1$ (as in [ESV22]), but not Lipschitz regularity, in the exact same fashion as for the Laplace equation with Dirichlet boundary condition. It would be interesting to know whether this result could be improved to show e.g. log-Lipschitz continuity of the solution,

$$|u(x) - u(y)| \leq C|\log|x-y|| \cdot |x-y| \qquad \forall x, y \in \bar{D}.$$

# 4 Finer properties of the free boundary

In this chapter, we focus on the sets $\{u > 0\}$, $\{u = 0\}$ and in particular the free boundary $\partial\{u > 0\}$ and finally arrive at an Euler-Lagrange equation for the minimizer of (2.1).

## 4.1 Density estimate

As a first result, we show that all free boundary points have positive density and that the free boundary separates the positivity set and the contact set "evenly". In other words, the free boundary does not contain cusps as e.g. the obstacle problem.

**Proposition 4.1.** *Suppose that $u$ is a local minimizer for $F_\Lambda$ on $D$ and $0 \in \partial\{u > 0\}$. Then for any $B_r \subset\subset D$,*

$$0 < \delta_D < \frac{|B_r \cap \{u > 0\}|}{|B_r|} < 1 - \delta_D < 1.$$

*Proof.* First, we show the lower bound. By non-degeneracy, for a point $x \in \partial B_{r/2}$, $u(x) \geq c\frac{r}{2}$. However by the local Lipschitz continuity in $B_r$, $|\nabla u| \leq L$, and so $u > \frac{cr}{4} > 0$ on $B_{\kappa r}(x)$, where $\kappa = \min{(1/4, c/(2L))}$. Then

$$\frac{|B_r \cap \{u > 0\}|}{|B_r|} \geq \frac{|B_{\kappa r}(x)|}{|B_r|} \geq \kappa^d > 0.$$

For the upper bound, let $h$ be the harmonic replacement of $u$ in $B_{r/2}$, that is $\Delta h = 0$ in $B_{r/2}$ and $h|_{\partial B_{r/2}} = u|_{\partial B_{r/2}}$. By harmonicity and the strong maximum principle, $h$ can only be zero on $\partial B_{r/2}$ and is positive in the interior of $B_{r/2}$. By optimality,

$$\int_{B_{r/2}} |\nabla u|^2 + \Lambda|\{u > 0\} \cap B_{r/2}| = E(u, B_{r/2}) \leq E(h, B_{r/2}) = \int_{B_{r/2}} |\nabla h|^2 + \Lambda|B_{r/2}|.$$

Thus, as $h - u \in H_0^1(B_{r/2})$,

$$\begin{aligned}\int_{B_{r/2}} |\nabla(u-h)|^2 &= \int_{B_{r/2}} |\nabla u|^2 - |\nabla h|^2 - 2\langle \nabla h, \nabla(u-h)\rangle \\ &= \int_{B_{r/2}} |\nabla u|^2 - |\nabla h|^2 \leq \Lambda|\{u = 0\} \cap B_{r/2}|.\end{aligned}$$

Using Poincaré's inequality on $B_{r/2}$ and Jensen's inequality give,

$$\Lambda|\{u = 0\} \cap B_{r/2}| \geq \int_{B_{r/2}} |\nabla(u-h)|^2 \geq \frac{C_d}{r^2}\int_{B_{r/2}} |(u-h)|^2 \geq \frac{C_d}{|B_{r/2}|}\left(\frac{1}{r}\int_{B_{r/2}} |h-u|\right)^2.$$

Now we estimate, using that $\inf_{B_{\kappa r}(x)} u \geq \frac{cr}{4}$,

$$h(0) = \fint_{\partial B_{r/2}} h = \fint_{\partial B_{r/2}} u \geq \frac{1}{|\partial B_{r/2}|}\int_{\partial B_{r/2} \cap B_{\kappa r}(x)} u \geq \frac{|\partial B_{r/2} \cap B_{\kappa r}(x)|}{|\partial B_{r/2}|}\frac{cr}{4}u \geq \kappa^{d-1}c'r.$$

Since $\sup_{B_{r/4}} h \geq h(0)$, we apply Harnack's inequality (Theorem A.18) to get $h \geq C'r$ in $B_{r/4}$. But $u(0) = 0$ and $u$ Lipschitz gives $u \leq L\varepsilon r$ in $B_{\varepsilon r}$, thus taking $\varepsilon$ sufficiently small ($C' \geq 2\varepsilon L$ and $\varepsilon \leq 1/4$) gives

$$\int_{B_{r/2}} |h-u| \geq \int_{B_{\varepsilon r}} h - u \geq \frac{1}{2}C'r|B_{\varepsilon r}|.$$

To conclude

$$\Lambda|\{u = 0\} \cap B_r| \geq \Lambda|\{u = 0\} \cap B_{r/2}| \geq C\frac{|B_{\varepsilon r}|^2}{|B_r|} \geq C\varepsilon^{2d}|B_r| =: \delta|B_r|,$$

and hence

$$\frac{|B_r \cap \{u > 0\}|}{|B_r|} = \frac{|B_r| - |B_r \cap \{u = 0\}|}{|B_r|} \leq 1 - \delta_D < 1.$$

Setting $\delta_D = \min(\kappa^d, \delta)$ finishes the proof. □

We are now able to show that the free boundary has locally finite perimeter.

**Proposition 4.2.** *Let $u$ be a local minimizer of $F_\Lambda$ on $D$. Then for every compact set $K \subset D$, we have $\mathcal{H}^{d-1}(K \cap \partial\Omega_u) < \infty$.*

*Proof.* We only sketch the proof of [Vel23, Chapter 5.3].
**Step 1:** We show that

$$\Lambda|\{0 < u \leq \varepsilon\} \cap B_r(x_0)| \leq \int_{\{0<u\leq\varepsilon\}\cap B_r(x_0)} |\nabla u|^2 dx + \Lambda|\{0 < u \leq \varepsilon\} \cap B_r(x_0)| \leq C\varepsilon \quad \forall \varepsilon \in (0,1].$$

Let $\phi$ be a smooth cutoff such that $\phi|_{B_r} = 0$ and $\phi|_{\mathbb{R}^d \setminus B_{2r}} = 1$. Using the auxiliary functions $u_\varepsilon = (u-\varepsilon)_+$ and $\tilde{u}_\varepsilon = \phi u + (1-\phi)u_\varepsilon$, a careful calculation of $|\nabla \tilde{u}_\varepsilon|^2$ gives the estimate for the chosen constant $C = 2\|\nabla u\|_{L^2(B_{2r})}\|\nabla\phi\|_{L^2(B_{2r})} + \|\nabla\phi\|^2_{L^2(B_{2r})}$.

**Step 2:** We show that Step 1 implies that $\mathcal{H}^{d-1}(K \cap \partial\Omega_u) < \infty$. In particular, take a finite covering $\{B_\delta(x_j)\}_{j=1}^N$ of $K \cap \partial\Omega_u$ with $B_{\delta/5}(x_j)$ disjoint. Using the Lipschitz continuity together with the nondegeneracy, we arrive at a bound

$$N d\omega_d \delta^{d-1} \leq C = C(d, c, L),$$

with $L$ and $c$ the constants from the nondegeneracy and Lipschitz continuity. The RHS is independent of $\delta$ and taking the infimum over all partitions gives the $\mathcal{H}_\delta^{d-1}$-measure, then passing to the limit $\delta \to 0$ on the LHS yields the result. □

The finite $d-1$ dimensional Hausdorff measure implies directly the locally finite perimeter of the free boundary.

**Remark 4.3.** Since $\partial^*\Omega_u \subseteq \partial\Omega$, we have that $\mathcal{H}^{d-1}(K \cap \partial^*\Omega_u) < \infty$ and so directly by Federer's characterization theorem [Fed69, Theorem 4.5.11] $\partial\Omega_u$ has locally finite perimeter in the sense of [Mag12, Chapter 12].

## 4.2 Condition on the free boundary and viscosity solutions

We already know that $u$ is harmonic inside $\Omega_u$ and vanishes outside $\Omega_u$, but what happens exactly at the free boundary? Note that $u$ is only in $H^1(D)$, we do not expect that $u$ is harmonic or even differentiable across the free boundary $\partial\Omega_u$. (Actually $\Delta u$ behaves like a singular measure concentrated on $\partial\Omega_u$.) Nevertheless, by the Lipschitz continuity and the non-degeneracy, $u$ behaves like the positive part of a linear function near $\partial\Omega_u$. We can make this more precise: in fact, on $\partial\Omega_u$, the norm of the gradient is constant across the whole free boundary $\partial\Omega_u$. Note that even for $u$ continuous, $\nabla u$ is not continuous across $\partial\Omega_u$ (though integrable on $D$), so when writing $\nabla u(x)$ for $x \in \partial\Omega_u$ we actually mean $\lim_{n\to\infty} \nabla u(x_n)$ for any sequence $\Omega_u \ni x_n \to x$.

**Proposition 4.4.** *Assuming that the local minimizer $u$ of $F_\Lambda$ on $D$ has a (piecewise) smooth free boundary $\partial\Omega_u$, then*

$$|\nabla u| = \sqrt{\Lambda} \qquad \textit{a.e. on } \partial\Omega_u.$$

*Proof.* The idea (from [Kri19]) is again to construct a competitor $u_t$, using an inner variation, which modifies also the free boundary $\partial\Omega_u$. Then use $F_\Lambda(u_t, D) - F_\Lambda(u, D) \geq 0$ for arbitrary $t$ to arrive at the free boundary condition. Take a vector field $T \in C_c^\infty(D, \mathbb{R}^n)$ and set $\phi_t(x) = x + tT(x)$. Then we have,

$$\nabla\phi_t(x) = I + t\nabla T(x) \quad \text{invertible for small } t \text{, so } \phi \text{ is a diffeomorphism,}$$

$$\nabla(\phi_t^{-1})(x) = \big(\nabla\phi_t(\phi_t^{-1}(x)\big)^{-1} = \big(I + t\nabla T(\phi_t^{-1}(x)) + O(t^2)\big)^{-1} = (I - t\nabla T(\phi_t^{-1}(x)) + O(t^2),$$

$$|\det \nabla\phi_t(x)| = 1 + t\,\mathrm{Tr}(\nabla T(x)) + O(t^2) = 1 + t\,\mathrm{div}\,T(x) + O(t^2).$$

Define now $u_t(x) \coloneqq u(\phi_t^{-1}(x))$,

$$\nabla u_t = \nabla\phi_t^{-1}(\nabla u \circ \phi_t^{-1}) = \big(I - t\nabla T(\phi_t^{-1}(x))\big)(\nabla u \circ \phi_t^{-1}) + O(t^2),$$

$$|\nabla u_t(x)|^2 = |\nabla u(\phi^{-1}(x))|^2 - 2t\left\langle \nabla u(\phi^{-1}(x)), \nabla T(\phi^{-1}(x))\nabla u(\phi^{-1}(x))\right\rangle + O(t^2).$$

Using a change of variables, $x \mapsto \phi_t(x)$, in the energy leads to

$$\begin{aligned}
F_\Lambda(u_t, D) &= \int_D |\nabla u_t|^2 + \Lambda\chi_{\Omega_{u_t}} = \int_D \big(|\nabla u|^2 - 2t\langle\nabla u, \nabla T\nabla u\rangle\big)(\phi_t^{-1}) + \Lambda\chi_{\Omega_{u_t}} + O(t^2) \\
&= \int_D \big(|\nabla u|^2 - 2t\langle\nabla u, \nabla T\nabla u\rangle + \Lambda\chi_{\Omega_u}\big)(1 + t\,\mathrm{div}\,T) + O(t^2) \\
&= \int_D |\nabla u|^2 + \Lambda\chi_{\Omega_u} + t\underbrace{\int_D |\nabla u|^2\,\mathrm{div}\,T - 2\langle\nabla u, \nabla T\nabla u\rangle + \Lambda\,\mathrm{div}\,T\chi_{\Omega_u}}_{=:A} + O(t^2) \\
&= F_\Lambda(u, D) + tA + O(t^2).
\end{aligned}$$

Since $t$ can be either positive or negative, if $|t|$ sufficiently small, it follows that $A = 0$ and

$$\begin{aligned}0 &= \int_D |\nabla u|^2 \operatorname{div} T - 2\langle \nabla u, \nabla T \nabla u\rangle + \Lambda \operatorname{div} T \chi_{\Omega_u} \\ &= \int_{\Omega_u} |\nabla u|^2 \operatorname{div} T - 2\langle \nabla u, \nabla T \nabla u\rangle + \Lambda \operatorname{div} T \chi_{\Omega_u}.\end{aligned}$$

We proceed by integration by parts (regularity of $\Omega_u$ is needed to define the outward unit normal $\nu$; note also that $T = 0$ on $\partial D$). Let $T^i$ denote the components of $T$ and $u_j := \frac{\partial}{\partial x_j} u$. Then, using that $u_{jj} = 0$ inside $\Omega_u$ by harmonicity,

$$\begin{aligned}0 &= \int_{\Omega_u} |\nabla u|^2 \operatorname{div} T - 2\langle \nabla u, \nabla T \nabla u\rangle + \Lambda \operatorname{div} T \chi_{\Omega_u} = \int_{\Omega_u} u_i^2 T_j^j - 2T_j^i u_i u_j + \Lambda T_i^i \chi_{\Omega_u} \\ &= \int_{\partial\Omega_u} u_i^2 T^j \nu^j - 2\int_{\Omega_u} u_i u_{ij} T^j - 2\int_{\partial\Omega_u} T^i u_i u_j \nu^j + 2\int_{\Omega_u} T^i (u_{ij} u_j + u_i u_{jj}) + \int_{\partial\Omega_u} \Lambda T^j \nu^j \\ &= \int_{\partial\Omega_u} u_i^2 T^j \nu^j - 2T^i u_i u_j \nu^j + \Lambda T^j \nu^j.\end{aligned}$$

Since on $\partial\Omega_u$, $\nabla u = -\nu|\nabla u|$ (i.e. $u_i = \nu^i |\nabla u|$),

$$\begin{aligned}0 &= \int_{\partial\Omega_u} u_i^2 T^j \nu^j - 2T^i u_i u_j \nu^j + \Lambda T^j \nu^j = \int_{\partial\Omega_u} u_i^2 T^j \nu^j - 2T^i \nu^i \sqrt{u_k^2}\nu^j \sqrt{u_k^2}\nu^j + \Lambda T^j v^j \\ &= \int_{\partial\Omega_u} -u_i^2 T^j \nu^j + \Lambda T^j \nu^j = \int_{\partial\Omega_u} (\Lambda - |\nabla u|^2)\langle T, \nu\rangle.\end{aligned}$$

For any point $x_0 \in \partial\Omega_u$, suppose that $\Lambda - |\nabla u(x_0)| = a > 0$. Let $\rho$ be sufficiently small such that $B_\rho(x_0) \subset\subset D$ and $\Lambda - |\nabla u(x_0)| \geq a/2 > 0$, by continuity on $\partial\Omega_u$ such a $\rho > 0$ exists. Take now $\psi : D \to [0,1]$ be supported in $B_\rho(x_0)$ and set $T := \psi\nu$, then

$$0 = \int_{\partial\Omega_u} (\Lambda - |\nabla u|^2)\langle T, \nu\rangle = \int_{\partial\Omega_u \cap B_\rho(x_0)} (\Lambda - |\nabla u(x)|^2)\psi(x) \geq \int_{\partial\Omega_u \cap B_\rho(x_0)} \frac{a}{2}\psi(x) > 0,$$

a contradiction. In the case $a < 0$, take $T = -\psi\nu$. □

Next, we upgrade this to the case where the free boundary is not necessarily smooth. In order to do so we need to define precisely in what sense the condition $|\nabla u| = \sqrt{\Lambda}$ on $\partial\Omega_u$ is satisfied. Therefore, we introduce the concept of a viscosity solution and show that the minimizer $u$ satisfies a PDE (the Euler-Lagrange equation of the functional $F_\Lambda$) in the so called viscosity sense, which then recovers the condition $|\nabla u| = \sqrt{\Lambda}$ on $\partial\Omega_u$. Up until now the techniques were mostly variational; at present, we connect back to the original thought of having a PDE satisfied in an unknown domain and use so called viscosity methods, which are more recent than the variational approach.

**Definition 4.5.** Let $D \subseteq \mathbb{R}^d$ be open, $a > 0$ and $u : D \to \mathbb{R}_+$ be continuous. Let

$$\mathcal{A}_{x_0} := \{\phi \in C(\mathbb{R}^d) : \phi(x_0) = u(x_0) \text{ and } \exists B_r(x_0) \text{ with } \phi(x) \geq u(x)\ \forall x \in B_r(x_0)\}, \quad \text{(above)}$$
$$\mathcal{B}_{x_0} := \{\phi \in C(\mathbb{R}^d) : \phi(x_0) = u(x_0) \text{ and } \exists B_r(x_0) \text{ with } \phi(x) \leq u(x)\ \forall x \in B_r(x_0)\}. \quad \text{(below)}$$

We say that $u$ satisfies the PDE

$$\begin{cases} \Delta u = 0 & \text{in } \Omega_u, \\ |\nabla u| = a & \text{on } \partial\Omega_u \cap D, \end{cases}$$

**in the viscosity sense** (or $u$ is a viscosity solution) if for every $x_0 \in \bar{\Omega}_u \cap D$ we have:

- If $x_0 \in \Omega_u$ : $\begin{cases} \Delta\phi(x_0) \le 0 & \text{for any } \phi \in \mathcal{B}_{x_0} \cap C^\infty(\mathbb{R}^d), & \text{(subsolution)} \\ \Delta\phi(x_0) \ge 0 & \text{for any } \phi \in \mathcal{A}_{x_0} \cap C^\infty(\mathbb{R}^d). & \text{(supersolution)} \end{cases}$
- If $x_0 \in \partial\Omega_u \cap D$ : $\begin{cases} |\nabla\phi(x_0)| \le a & \text{for any } \phi \in \mathcal{B}_{x_0} \cap C^\infty(\mathbb{R}^d), \\ |\nabla\phi(x_0)| \ge a & \text{for any } \phi \in C^\infty(\mathbb{R}^d) \text{ such that } \phi_+ \in \mathcal{A}_{x_0}. \end{cases}$

Viscosity solutions connect back to classical solutions in the following remark.

**Remark 4.6.** A function $u \in C^0(D)$ that is harmonic in the viscosity sense on $D$ (that is, it satisfies the first condition above for $x_0 \in \Omega_u$), is harmonic in the classical sense on $D$, i.e. $u \in C^2(D)$ and $\Delta u = \frac{\partial^2}{\partial x_1^2}u + \cdots + \frac{\partial^2}{\partial x_d^2}u = 0$ pointwise.

Moreover, the case where $\Lambda \neq 1$ can be reduced to $\Lambda = 1$.

**Lemma 4.7.** *The continuous nonnegative function $u : D \to \mathbb{R}$ is a viscosity solution to*

$$\Delta u = 0 \ \text{ in } \Omega_u, \quad |\nabla u| = a \ \text{ on } \partial\Omega_u \cap D,$$

*if and only if the function $v := \frac{1}{a}u$ is a viscosity solution to*

$$\Delta u = 0 \ \text{ in } \Omega_u, \quad |\nabla u| = 1 \ \text{ on } \partial\Omega_u \cap D.$$

*Proof.* The condition on the Laplacian follows trivially. If $\phi \in \mathcal{B}_{x_0} \cap C^\infty(\mathbb{R}^d)$, then $|\nabla v(x_0)| = \frac{1}{a}|\nabla u(x)| \le 1$, analogous for the other case. □

We are now able to derive the Euler-Lagrange equation to (2.1), in the viscosity sense.

**Proposition 4.8.** *A minimizer $u \in H^1(D)$ of $F_\Lambda$ in $D$ is a viscosity solution of*

$$\Delta u = 0 \quad \text{in } \Omega_u, \qquad |\nabla u| = \sqrt{\Lambda} \quad \text{on } \partial\Omega_u \cap D.$$

*Proof.* Without loss of generality, assume $\Lambda = 1$. For $x_0$ in $\Omega_u$, take $\phi \in \mathcal{B}_{x_0} \cap C^\infty(\mathbb{R}^d)$, since $\Omega_u$ is open, $u$ is harmonic on $B_r(x_0) \subset \mathcal{N}(x_0)$. Then for some small $r$, by definition of subharmonicity,

$$\phi(x_0) = u(x_0) = \int_{B_r(x_0)} u(x) \le \int_{B_r(x_0)} \phi(x) \quad \Longrightarrow \quad \Delta\phi(x_0) \ge 0.$$

For $\phi \in \mathcal{A}_{x_0}$, by the same argument, for some small $r$,

$$\phi(x_0) = u(x_0) = \int_{B_r(x_0)} u(x) \ge \int_{B_r(x_0)} \phi(x) \quad \Longrightarrow \quad \Delta\phi(x_0) \le 0.$$

It remains to check the boundary condition, so let $x_0 \in \partial\Omega_u \cap D$. First, let $\phi \in \mathcal{B}_{x_0}$ and suppose for the sake of contradiction that $|\nabla\phi(x_0)| = a > 1$. Set $\nu := \frac{1}{a}\nabla\phi(x_0) \in \mathbb{S}^{d-1}$, assume for simplicity that $x_0 = 0$. By continuity of $\nabla\phi$, non-negativity of $u$ and the definition of $\mathcal{B}_{x_0}$, for $\rho > 0$ sufficiently small

$$u(x) \ge \phi_+(x) \ge \frac{1+a}{2}(x \cdot \nu)_+ \qquad \forall x \in B_\rho \subset D.$$

Take now $r > 1$ large enough such that $(1 + 1/r)^{d-1} \le \frac{2+a}{3}$, then for the unique solution $u_r$ from Proposition 2.12 on $A_{r,R}$ we have

$$|\nabla u_r| \le \left(\frac{R}{|x|}\right)^{d-1} \le \left(\frac{r+1}{r}\right)^{d-1} \le \frac{2+a}{3}.$$

Define $\tilde{u}_\varepsilon(x) := u_r(x - (R-\varepsilon)\nu)$ on $A_{r,R}((R-\varepsilon)\nu) =: \tilde{A}_{r,R}$, clearly $\tilde{u}_\varepsilon(0) > 0$. For $\varepsilon$ sufficiently small, by the bound on $|\nabla u_r|$

$$\tilde{u}_\varepsilon(x) \leq \frac{1+a}{2}(x \cdot \nu)_+ \qquad \forall x \in \partial B_\rho \cap \tilde{A}_{r,R}.$$

Hence combining the two inequalities gives

$$\tilde{u}_\varepsilon(x) \leq u(x) \implies \min(u, \tilde{u}_\varepsilon) = \tilde{u}_\varepsilon \quad \text{and} \quad \max(u, \tilde{u}_\varepsilon) = u \quad \text{on } \partial B_\rho \cap \tilde{A}_{r,R}.$$

By Remark 2.2, $u$ and $\tilde{u}_\varepsilon$ are minimizers in $B_\rho \cap \tilde{A}_{r,R}$ with respect to their own boundary datum, i.e.

$$\begin{aligned} F_\Lambda(\tilde{u}_\varepsilon, B_\rho \cap \tilde{A}_{r,R}) &\leq F_\Lambda(\min(\tilde{u}_\varepsilon, u), B_\rho \cap \tilde{A}_{r,R}), \\ F_\Lambda(u, B_\rho \cap \tilde{A}_{r,R}) &\leq F_\Lambda(\max(\tilde{u}_\varepsilon, u), B_\rho \cap \tilde{A}_{r,R}). \end{aligned}$$

Combining it with

$$F_\Lambda(\tilde{u}_\varepsilon, B_\rho \cap \tilde{A}_{r,R}) + F_\Lambda(u, B_\rho \cap \tilde{A}_{r,R}) = F_\Lambda(\min(\tilde{u}_\varepsilon, u), B_\rho \cap \tilde{A}_{r,R}) + F_\Lambda(\max(\tilde{u}_\varepsilon, u), B_\rho \cap \tilde{A}_{r,R}),$$

we get actually

$$\begin{aligned} F_\Lambda(\tilde{u}_\varepsilon, B_\rho \cap \tilde{A}_{r,R}) &= F_\Lambda(\min(\tilde{u}_\varepsilon, u), B_\rho \cap \tilde{A}_{r,R}), \\ F_\Lambda(u, B_\rho \cap \tilde{A}_{r,R}) &= F_\Lambda(\max(\tilde{u}_\varepsilon, u), B_\rho \cap \tilde{A}_{r,R}). \end{aligned}$$

Set now

$$\tilde{v}_\varepsilon = \begin{cases} \tilde{u}_\varepsilon & \text{in } \tilde{A}_{r,R} \backslash B_\rho, \\ \min(\tilde{u}_\varepsilon, u) & \text{in } B_\rho \cap \tilde{A}_{r,R}, \end{cases}$$

and $v_r(x) := \tilde{v}_\varepsilon(x + (R-\varepsilon)\nu)$, translating back the shifted version. We calculate

$$\begin{aligned} F_\Lambda(v_r, A_{r,R}) &= F_\Lambda(\tilde{v}_\varepsilon, \tilde{A}_{r,R}) \\ &= F_\Lambda(\min(\tilde{u}_\varepsilon, u), B_\rho \cap \tilde{A}_{r,R}) + F_\Lambda(\tilde{u}_\varepsilon, \tilde{A}_{r,R} \backslash B_\rho) \\ &= F_\Lambda(\tilde{u}_\varepsilon, B_\rho \cap \tilde{A}_{r,R}) + F_\Lambda(\tilde{u}_\varepsilon, \tilde{A}_{r,R} \backslash B_\rho) \\ &= F_\Lambda(\tilde{u}_\varepsilon, \tilde{A}_{r,R}) = F_\Lambda(u_r, A_{r,R}). \end{aligned}$$

But also

$$v_r(-(R-\varepsilon)\nu) = \tilde{v}_\varepsilon(0) = \min(u(0), \tilde{u}_\varepsilon(0)) = 0 < \tilde{u}_\varepsilon(0) = u_r(-(R-\varepsilon)\nu) \implies u_r \neq v_r,$$

a contradiction, since $u_r$ was the unique minimizer and so $|\nabla \phi(x_0)| \leq 1$.

Secondly, suppose that $\phi \in \cap C^\infty(\mathbb{R}^d)$ with $\phi_+ \in \mathcal{A}_{x_0}$ and $|\nabla \phi(x_0)| = a < 1$. Again set $\nu := \frac{1}{a}\nabla \phi(x_0) \in \mathbb{S}^{d-1}$, for simplicity assume $x_0 = 0$. By continuity of $\nabla \phi$, non-negativity of $u$ and the definition of $\mathcal{A}_{x_0}$, for $\rho > 0$ sufficiently small

$$u(x) \leq \phi_+(x) \leq \frac{1+a}{2}(x \cdot \nu)_+ \qquad \forall x \in B_\rho \subset D.$$

Take now $R$ sufficiently large such that $\left(1 - \frac{2}{R}\right)^{d-1} \geq \frac{2+a}{3}$ and let $u_R$ be the unique radial solution from Proposition 2.11, with $u_R = 1$ on $\partial B_R$, $u_R = 0$ in $B_r$ and

$$|\nabla u_R(x)| = \left(\frac{r}{|x|}\right)^{d-1} \geq \left(\frac{R-2}{R}\right)^{d-1} \geq \frac{2+a}{3}.$$

Define $\tilde{u}_\varepsilon(x) \coloneqq u_R(x + (r-\varepsilon)\nu)$ on $B_R((r-\varepsilon)\nu) =: \tilde{B}_R$, clearly $\tilde{u}_\varepsilon = 0$ in $B_\varepsilon$. As $x_0$ is a free boundary point, there exists $\bar{x} \in B_\varepsilon$ with $u(\bar{x}) > \tilde{u}_\varepsilon(\bar{x}) = 0$. If $\varepsilon$ sufficiently small, then by the bound on $|\nabla u_R|$ and if necessary taking $\rho$ smaller (ensuring that $B_\rho \subset \tilde{B}_R$)

$$\tilde{u}_\varepsilon(x) \geq \frac{1+a}{2}(x \cdot \nu)_+ \qquad \text{on } \partial B_\rho.$$

Hence combining the two inequalities gives

$$\tilde{u}_\varepsilon(x) \geq u(x) \implies \max(u, \tilde{u}_\varepsilon) = \tilde{u}_\varepsilon \quad \text{and} \quad \min(u, \tilde{u}_\varepsilon) = u \quad \text{on } \partial B_\rho.$$

By Remark 2.2, $u$ and $\tilde{u}_\varepsilon$ are minimizers in $B_\rho$ with respect to their own boundary datum, i.e.

$$\begin{aligned} F_\Lambda(\tilde{u}_\varepsilon, B_\rho) &\leq F_\Lambda(\max(\tilde{u}_\varepsilon, u), B_\rho), \\ F_\Lambda(u, B_\rho) &\leq F_\Lambda(\min(\tilde{u}_\varepsilon, u), B_\rho). \end{aligned}$$

Combining it with

$$F_\Lambda(\tilde{u}_\varepsilon, B_\rho) + F_\Lambda(u, B_\rho) = F_\Lambda(\min(\tilde{u}_\varepsilon, u), B_\rho) + F_\Lambda(\max(\tilde{u}_\varepsilon, u), B_\rho),$$

we get

$$\begin{aligned} F_\Lambda(\tilde{u}_\varepsilon, B_\rho) &= F_\Lambda(\max(\tilde{u}_\varepsilon, u), B_\rho), \\ F_\Lambda(u, B_\rho) &= F_\Lambda(\min(\tilde{u}_\varepsilon, u), B_\rho). \end{aligned}$$

Set now

$$\tilde{v}_\varepsilon = \begin{cases} \tilde{u}_\varepsilon & \text{in } \tilde{B}_R \backslash B_\rho, \\ \max(\tilde{u}_\varepsilon, u) & \text{in } B_\rho, \end{cases}$$

and $v_R(x) \coloneqq \tilde{v}_\varepsilon(x - (r-\varepsilon)\nu)$, translating back the shifted version. We calculate

$$\begin{aligned} F_\Lambda(v_R, B_R) &= F_\Lambda(\tilde{v}_\varepsilon, \tilde{B}_R) \\ &= F_\Lambda(\max(\tilde{u}_\varepsilon, u), B_\rho) + F_\Lambda(\tilde{u}_\varepsilon, \tilde{B}_R \backslash B_\rho) \\ &= F_\Lambda(\tilde{u}_\varepsilon, B_\rho) + F_\Lambda(\tilde{u}_\varepsilon, \tilde{B}_R \backslash B_\rho) \\ &= F_\Lambda(\tilde{u}_\varepsilon, \tilde{B}_R) = F_\Lambda(u_R, B_R). \end{aligned}$$

But also

$$v_R(\bar{x} + (r-\varepsilon)\nu) = \tilde{v}_\varepsilon(\bar{x}) = \min(u(\bar{x}), \tilde{u}_\varepsilon(\bar{x})) = u(\bar{x}) > 0 = \tilde{u}_\varepsilon(\bar{x}) = u_R(\bar{x} + (r-\varepsilon)\nu),$$

which implies $v_R \neq u_R$, contradicting the uniqueness of $u_R$. Thus $|\nabla \phi(x_0)| \geq 1$. □

Having an explicit value of $|\nabla u|$ on $\partial \Omega_u$ leads to the following corollary.

**Corollary 4.9.** *For a global minimizer $u$, in $\Omega_u$ we have $|\nabla u| \leq \sqrt{\Lambda}$.*

*Proof.* Since the function $|\nabla u|^2$ is subharmonic on $\Omega_u$ and vanishes outside of it, the result follows from the maximum principle. □

# 5 Regularity of the free boundary

Despite the results in Chapter 4, to study the free boundary in more detail, we need to employ more technical tools from the theory of partial differential equations and geometric measure theory. The aim is to derive regularity properties of $\partial \Omega_u$: a priori it is not known whether the free boundary is even differentiable, it could have e.g. a fractal like structure.

## 5.1 Blow-Ups

The first technique to study the free boundary is to "zoom in" infinitely on a free boundary point and relate the limit back to the original solution $u$. The method is inspired from the theory of minimal surfaces and has been a standard tool in the study of free boundary problems for over 40 years.

**Definition 5.1.** Given $D$, an open set of $\mathbb{R}^d$, $u : D \to \mathbb{R}$ a local minimizer of $F_\Lambda$ in D and a free boundary point $x_0 \in \partial\Omega_u \cap D$, we define the **rescaling**

$$u_{x_0,r}(x) := \frac{1}{r}u(x_0 + rx), \qquad r > 0.$$

Note directly that by translation and scaling invariance of $F_\Lambda$, $u_{x_0,r}$ is a local minimizer in $rD + x_0$.

**Definition 5.2.** The function $u_0 : \mathbb{R}^d \to \mathbb{R}$ is called a **blow-up** limit of $u$ at $x_0$ for the sequence $r_n \to 0$ if

$$\lim_{n\to\infty} \|u_{x_0,r_n} - u_0\|_{L^\infty(B_R)} = 0 \qquad \forall R > 0. \tag{5.1}$$

First, we show that the blow-up limit always exists.

**Lemma 5.3.** *The blow-up limit $u_0$ of $u$ at $x_0 \in \partial\Omega_u$ for a sequence $r_n \to 0$ exists and is non-negative, Lipschitz continuous (in $\mathbb{R}^d$) and vanishes at zero.*

*Proof.* For a fixed $R > 0$, take $N$ large enough such that for any $n > N$, $B_R \subset \frac{1}{r_n}(-x_0 + D)$, i.e. $u_{x_0,r}$ is defined on $B_R$. Then since

$$\nabla u_{x_0,r_n}(x) = \nabla u(x_0 + r_n x) = \nabla\left(\frac{1}{r}u(x_0 + r_n x)\right) \qquad \text{in } B_R,$$

$$\|\nabla u_{x_0,r_n}\|_{L^\infty(B_R)} = \|\nabla u\|_{L^\infty(B_{Rr_n}(x_0))}.$$

Since $u$ vanishes at $x_0$ and is locally Lipschitz, the sequence $u_{x_0,r_n}$ is uniformly bounded and equicontinuous on $B_R$, so by the Arzelà-Ascoli theorem, there is a uniformly convergent (in $B_R$) subsequence. Repeat the argument for $R + k$ with $k \in \mathbb{N}$ and extract each time a subsubsequence of the previous subsequence, then extract a diagonal sequence we arrive at a sequence satisfying (5.1). The non-negativity and the fact that $u_0(x_0) = 0$ follows from the uniform convergence and the equicontinuity gives that $u_0$ is Lipschitz. □

**Remark 5.4.** We note directly that the blow-up limit usually depends on the chosen subsequence of $r_n$ and is not unique.

We now characterize the properties of the blow-up limit, following [Vel23, Chapter 6] and [Kri19, Section 6].

**Proposition 5.5.** *For an open set $D \subseteq \mathbb{R}^d$, let $u \in H^1_{loc}(D)$ be a local minimizer of $F_\Lambda$ in $D$. For a free boundary point $x_0$ and $r_n \to 0$, suppose that the blow-up sequence $u_{x_0,r_n}$ converges locally uniformly to $u_0 : \mathbb{R}^d \to \mathbb{R}$, then up to a subsequence*

- *$u_{x_0,r_n}$ converges to $u_0$ strongly in $H^1(B_R)$ for all $R > 0$;*
- *$\chi_{\Omega_{u_{x_0,r_n}}} = \chi_{\{u_{x_0,r_n}>0\}}$ converges to $\chi_{\Omega_{u_0}} = \chi_{\{u_0>0\}}$ in $L^1(B_R)$ for all $R > 0$;*
- *$u_0$ is a non-trivial local minimizer of $F_\Lambda$ in $\mathbb{R}^d$.*

*Proof.* Note that by Lemma 5.3 we already have local uniform convergence and so $u_{x_0,r_n} =: u_n \to u_0$ strongly in $L^2(B_R)$ for any $R > 0$. Note that from

$$\|\nabla u_n\|_{L^\infty(B_R)} = \|\nabla u\|_{L^\infty(B_{Rr_n}(x_0))},$$

we get that (for a fixed $R$) $u_{x_0,r_n}$ is uniformly bounded in $H^1(B_R)$. Thus up to a subsequence, $u_n \to u_0$ weakly in $H^1(B_R)$ and by lower semicontinuity of the norm with respect to weak convergence,

$$\|\nabla u_0\|_{L^2(B_r)} \le \liminf_n \|\nabla u_n\|_{L^2(B_r)} \qquad \forall r \in (0, R]. \tag{5.2}$$

Also, by the uniform convergence,

$$x \in \Omega_{u_0} \implies u_0(x) > 0 \implies u_d(x) > 0 \quad \forall n \text{ suff. large} \implies x \in \Omega_{u_d} \quad \forall n \text{ suff. large},$$

so $\chi_{\Omega_{u_0}} \le \liminf_n \chi_{\Omega_{u_d}}$, giving for $0 < r \le R$

$$|\Omega_{u_0} \cap B_r| \le \int_{B_r} \chi_{\Omega_{u_0}} \le \int_{B_r} \liminf_n \chi_{\Omega_{u_n}} \le \liminf_n \int_{B_r} \chi_{\Omega_{u_n}} = \liminf_n |\Omega_{u_n} \cap B_r|. \tag{5.3}$$

In particular,

$$\int_{B_R} |\nabla u_0|^2 dx + |\Omega_{u_0} \cap B_R| \le \liminf_n \int_{B_R} |\nabla u_n|^2 dx + |\Omega_{u_n} \cap B_R|.$$

**Step 1:** Both inequalities (5.2) and (5.3) are equalities if and only if the convergence (in $H^1(B_r)$ and $L^2(B_r)$) is strong. Hence we want to show that for each $r \in (0, R)$,

$$\|\nabla u_0\|_{L^2(B_r)} = \liminf_n \|\nabla u_n\|_{L^2(B_r)} \quad \text{ and } \quad |\Omega_0 \cap B_r| = \liminf_n |\Omega_{u_n} \cap B_r|,$$

which follows by (5.2) and (5.3) once we show the claim

$$\int_{B_R} |\nabla u_0|^2 dx + |\Omega_{u_0} \cap B_R| \ge \liminf_n \int_{B_R} |\nabla u_n|^2 dx + |\Omega_{u_n} \cap B_R|.$$

Since $R$ was arbitrary and we have strong convergence (in $H^1(B_r)$ and $L^2(B_r)$ for $0 < r < R$) the points (i) and (ii) follow.
To show the claim, let $C^\infty(B_R) \ni \eta : B_R \to [0, 1]$ be a smooth cutoff with $\eta \equiv 1$ in $B_r$ and $\eta \equiv 0$ on $\partial B_R$ and let $v_n = \eta u_0 + (1 - \eta) u_n$. First, we have

$$\begin{aligned}|\Omega_{v_n} \cap B_R| - |\Omega_{u_n} \cap B_R| &= |\Omega_{v_n} \cap \{\eta = 1\}| - |\Omega_{u_n} \cap \{\eta = 1\}| + |\Omega_{v_n} \cap \{\eta < 1\}| - |\Omega_{u_n} \cap \{\eta < 1\}| \\ &\le |\Omega_{u_0} \cap B_r| - |\Omega_{u_n} \cap \{\eta = 1\}| + |\{\eta < 1\}|.\end{aligned}$$

Since by Fatou's Lemma,

$$|\Omega_{u_0} \cap \{\eta = 1\} \backslash B_r| \le \liminf_n |\Omega_{u_n} \cap \{\eta = 1\} \backslash B_r|,$$

we get for the measure term,

$$\begin{aligned}
&\limsup_n |\Omega_{v_n} \cap B_R| - |\Omega_{u_n} \cap B_R| \\
&\le |\Omega_{u_0} \cap \{\eta = 1\}| + \limsup_n (-|\Omega_{u_n} \cap \{\eta = 1\}|) + |\{\eta < 1\}| \\
&= |\Omega_{u_0} \cap B_r| + |\Omega_{u_0} \cap \{\eta = 1\} \backslash B_r| \\
&\qquad + \limsup_n (-|\Omega_{u_n} \cap \{\eta = 1\} \backslash B_r| - |\Omega_{u_n} \cap B_r|) + |\{\eta < 1\}| \\
&= |\Omega_{u_0} \cap B_r| + \limsup_n (-|\Omega_{u_n} \cap B_r|) + |\Omega_{u_0} \cap \{\eta = 1\} \backslash B_r| \\
&\qquad - \liminf_n (|\Omega_{u_n} \cap \{\eta = 1\} \backslash B_r|) + |\{\eta < 1\}| \\
&\le |\Omega_{u_0} \cap B_r| + \limsup_n (-|\Omega_{u_n} \cap B_r|) + |\{\eta < 1\}| \\
&= \limsup_n (|\Omega_{u_0} \cap B_r| - |\Omega_{u_n} \cap B_r|) + |\{\eta < 1\}|.
\end{aligned} \tag{5.4}$$

For the Dirichlet energy term, we first calculate

$$\begin{aligned}|\nabla v_n|^2 - |\nabla u_n|^2 &= |\nabla(\eta u_0 + (1-\eta)u_n)|^2 - |\nabla u_n|^2 \\ &= |(u_0 - u_n)\nabla\eta + \eta\nabla u_0 + (1-\eta)\nabla u_n|^2 - |\nabla u_n|^2 \\ &= (u_0-u_n)^2|\nabla\eta|^2 + \eta^2|\nabla u_0|^2 + (1-\eta)^2|\nabla u_n|^2 - |\nabla u_n|^2 \\ &\quad + (u_0-u_n)\eta\nabla\eta\cdot\nabla u_0 + (u_0-u_n)(1-\eta)\nabla\eta\cdot\nabla u_n + \eta(1-\eta)\nabla u_0\cdot\nabla u_n.\end{aligned}$$

Since $u_n \to u_0$ strongly in $L^2(B_R)$ (for 1st, 5th and 6th term) and weakly in $H^1(B_R)$ (last term),

$$\begin{aligned}\limsup_n \int_{B_R} |\nabla v_n|^2 - |\nabla u_n|^2 &= \limsup_n \int_{B_R} \eta^2|\nabla u_0|^2 + (1-\eta)^2|\nabla u_n|^2 - |\nabla u_n|^2 + \eta(1-\eta)|\nabla u_0|^2 \\ &= \limsup_n \int_{B_R} \eta|\nabla u_0|^2 + (\eta^2 - 2\eta)|\nabla u_n|^2 \\ &\le \limsup_n \int_{B_R\cap\{\eta=1\}} |\nabla u_0|^2 - |\nabla u_n|^2 + \int_{B_R\cap\{\eta<1\}} |\nabla u_0|^2 \\ &\le \limsup_n \int_{B_R\cap\{\eta=1\}} |\nabla u_0|^2 - |\nabla u_n|^2 + \int_{B_R\cap\{\eta<1\}} |\nabla u_0|^2 \\ &\le \limsup_n \int_{B_r} |\nabla u_0|^2 - |\nabla u_n|^2 + \int_{B_R\cap\{\eta<1\}} |\nabla u_0|^2,\end{aligned} \tag{5.5}$$

using in the last inequality the weak $H^1$ convergence of $u_n$ to $u_0$ on $\{\eta=1\}\backslash B_r$, i.e.

$$\limsup_n \int_{\{\eta=1\}\backslash B_r} |\nabla u_0|^2 - |\nabla u_n|^2 = \int_{\{\eta=1\}\backslash B_r} |\nabla u_0|^2 - \liminf_n \int_{\{\eta=1\}\backslash B_r} |\nabla u_n|^2 \le 0.$$

By combining (5.5) and (5.4) and using the optimality of $u_n$,

$$\begin{aligned}0 &\le \limsup_n \left(F_\Lambda(v_n, B_R) - F_\Lambda(u_n, B_R)\right) \\ &\le \limsup_n \left(F_\Lambda(u_0, B_r) - F_\Lambda(u_n, B_r)\right) + |\{\eta<1\}| + \int_{B_R\cap\{\eta<1\}} |\nabla u_0|^2 \\ &\iff \liminf_n F_\Lambda(u_n, B_r) \le F_\Lambda(u_0, B_r) + |\{\eta<1\}| + \int_{B_R\cap\{\eta<1\}} |\nabla u_0|^2.\end{aligned}$$

However since $\eta$ can be taken such that $\eta = 1$ in $B_{R-\varepsilon}$ for any $\varepsilon$, i.e. the last two terms can be made arbitrarily small, we get

$$\liminf_n F_\Lambda(u_n, B_r) \le F_\Lambda(u_0, B_r),$$

proving the claim and thereby (i) and (ii).

**Step 2:** For (iii), let $0 < r < R$ and $\varphi \in H_0^1(B_r)$ Take a cutoff $\eta \in C_c^\infty(B_R, [0,1])$ with $\eta \equiv 1$ in $B_r$, and define $v_n \coloneqq u_n + \varphi + \eta(u_0 - u_n)$ and $v_0 \coloneqq u_0 + \varphi$. We show that

$$F_\Lambda(u_0, \operatorname{spt}\eta) \le F_\Lambda(v_0, \operatorname{spt}\eta) \iff F_\Lambda(u_0, B_r) \le F_\Lambda(v_0, B_r) = F_\Lambda(u_0 + \varphi, B_r).$$

By Step 1, the optimality of $u_n$, and $u_n \to u_0$ strongly in $\operatorname{spt}\eta$,

$$\begin{aligned}F_\Lambda(u_0, \operatorname{spt}\eta) &= \lim_n F_\Lambda(u_n, \operatorname{spt}\eta) \le \liminf F_\Lambda(v_n, \operatorname{spt}\eta) \\ &= \int_{\operatorname{spt}\eta} |\nabla v_0|^2 + \Lambda \liminf |\{v_n > 0\} \cap \operatorname{spt}\eta|.\end{aligned}$$

Concerning the last term, note that $v_n = v_0$ on $\{\eta = 1\}$,

$$\begin{aligned}|\{v_n > 0\} \cap \operatorname{spt} \eta| &= |\{v_n > 0\} \cap \operatorname{spt}\{\eta = 1\}| + |\{v_n > 0\} \cap \{0 < \eta < 1\}| \\ &\leq |\{v_0 > 0\} \cap \operatorname{spt} \eta| + |\{0 < \eta < 1\}|,\end{aligned}$$

resulting in

$$F_\Lambda(u_0, \operatorname{spt} \eta) \leq F_\Lambda(v_0, \operatorname{spt} \eta) + |\{0 < \eta < 1\}|.$$

However letting now $\operatorname{spt} \eta \to B_r$ gives (iii). □

## 5.2 Regular and singular parts

We now study, with help of the blow-ups, the local structure of the free boundary. Depending on the exact form of the blow-up, the boundary point might be regular (here $C^\infty$) or singular, but the latter case does not happen often, so to speak.

**Definition 5.6.** For $D \subset \mathbb{R}^d$ open, $u$ a (local) minimizer of $F_\Lambda$ and $x_0$ a free boundary point, we say that $x_0$ is a **regular point** if there exists a blow-up limit $u_0$ to $u$ at $x_0$ with

$$u_0(x) = \sqrt{\Lambda}(x \cdot v)_+ \quad \text{for some } v \in \mathbb{S}^{d-1} .$$

The set of all regular points is denoted by $Reg(\partial\Omega_u)$ and the set of singular points as $Sing(\partial\Omega_u) = (\partial\Omega_u \cap D)\backslash Reg(\partial\Omega_u)$.

The fact that at regular points the free boundary is actually smooth will be proven in the next sections. Note that this is highly non-trivial, since the half space solution blow-up function itself is result of a limiting process, but for regularity of the original free boundary something has to be said for a finite rescaling. Roughly speaking, for a sufficiently large finite rescaling the free boundary is "close" enough to the half space solution, which gives local$C^{1,\alpha}$ regularity and then also local $C^\infty$ regularity.

To estimate the size of the singular part (see also Chapter 6) is topic of current research, the best known estimate is that the dimension of the singular set is less than $d-5$. By standard results from geometric measure theory without deeper analysis it is possible to show that the singular set has at most dimension $d-1$, something much weaker.

**Proposition 5.7.** *For $D \subset \mathbb{R}^d$ open and bounded and $u$ a minimizer of $F_\Lambda$ on $D$, let $x_0$ be a free boundary point such that for some $r_n \to 0$,*

$$\chi_{\Omega_n} \to \chi_{H_v} \qquad \textit{for some } v \in \mathbb{R}^d .$$

*Then $x_0$ is a regular free boundary point.*

*Proof.* We sketch the proof of [Vel23, Lemma 6.11]. Since $\chi_{\Omega_n} \xrightarrow{L^1(B_R)} \chi_{\Omega_{u_0}}$, we have directly $\Omega_{u_0} = H_v$. Next we show that $\nabla u_0 = \sqrt{\Lambda} v$ on $\partial H_v$, by a contradiction argument, using the radial solution to construct a competitor $\tilde{h}$ that is not harmonic in its positivity set. However there is only one way to extend $\tilde{u}_0$ with $\tilde{u}_0|_{\partial H_v} = 0$ and $|\nabla \tilde{u}_0|_{\partial H_v} = \sqrt{\Lambda} v$ harmonically into $H_v$, namely $\tilde{u}_0(x) = \sqrt{\Lambda}(x \cdot v)$, thus $u_0(x) = \sqrt{\Lambda}(x \cdot v)_+$. □

**Corollary 5.8.** *Let $u \in H^1(D)$ be a minimizer of $F_\Lambda$ in $D$. Then $\mathcal{H}^{d-1}(Sing(\partial\Omega_u)) = 0$.*

*Proof.* We sketch the proof of [Vel23, Proposition 6.12]. By [Mag12, Theorem 5.15] the condition $\chi_{\Omega_n} \to \chi_{H_v}$ holds for any point in $\partial^*\Omega_u$, so $\partial^*\Omega_u \subseteq Reg(\partial\Omega_u)$. Now,

$$\mathcal{H}^{d-1}(Sing(\partial\Omega_u)) \leq \mathcal{H}^{d-1}(\partial\Omega_u \backslash Reg(\partial\Omega_u)) = \mathcal{H}^{d-1}(\partial\Omega_u \backslash (\Omega_u^{(0)} \cup \Omega_u^{(1)} \cup \partial^*\Omega_u)) = 0$$

using that there are no points of density 1 or 0 in the first and Federer's Second theorem [Mag12, Theorem 16.2] in the second equality. □

## 5.3 $C^{1,\alpha}$-regularity for regular points by improvement of flatness

We now introduce a collection of technical propositions, in order to analyze the free boundary.

**Definition 5.9.** Let $u : B_1 \to \mathbb{R}$ be a given function, we say it is $\varepsilon$ flat in direction $v \in \mathbb{S}^{d-1}$ in $B_1$ if

$$(x \cdot v - \varepsilon)_+ \leq u(x) \leq (x \cdot v + \varepsilon)_+ \qquad \text{in } B_1,$$

or equivalently

$$|u - x \cdot v| \leq \varepsilon \qquad \text{in } \{u > 0\} \cap B_1.$$

We can assume without loss of generality that $D = B_1$ and that the origin is a free boundary point and, by rotating, that $v = e_d$. We prove first the so called "improving of flatness" in the same direction on a smaller ball using an argument appearing first in [DeS09]. Secondly, we show that upon changing the direction slightly (i.e. rotate a small amount) the flatness can be improved even more.

**Proposition 5.10.** *Let $u$ be a minimizer of* (2.1)*, in particular $u$ is a viscosity solution of* (4.8) *on $B_1$ with $0 \in \partial\Omega_u$. There are dimensional constants $\bar{\varepsilon} > 0$, $c \in (0,1)$ such that for real numbers $a_0 < 0 < b_0$ and $|a_0| < 1/10$ with*

$$|b_0 - a_0| \leq \bar{\varepsilon} \qquad \text{and} \qquad (x_d + a_0)_+ \leq u(x) \leq (x_d + b_0)_+ \quad \text{in } B_1,$$

*there exists $a_1, b_1 \in \mathbb{R}$ with $a_0 \leq a_1 < 0 < b_1 \leq b_0$ and*

$$|b_1 - a_1| \leq (1-c)|a_0 - b_0| \qquad \text{and} \qquad (x_d + a_1)_+ \leq u(x) \leq (x_d + b_1)_+ \quad \text{in } B_{1/20}.$$

*Proof.* Start by defining $\bar{x} = (0, ..., 1/5)$, $\bar{c} = (20^d - (4/3)^d)^{-1}$ and the auxiliary function

$$w(x) = \begin{cases} 1 & \text{in } \bar{B}_{1/20}(\bar{x}), \\ 0 & \text{in } (B_{3/4}(\bar{x}))^c, \\ \bar{c}\left(|x - \bar{x}|^{-d} - (3/4)^{-d}\right) & \text{in } B_{3/4}(\bar{x}) \backslash \bar{B}_{1/20}(\bar{x}). \end{cases}$$

Let also $p(x) \coloneqq x_d + a_0$ and $\varepsilon = |b_0 - a_0|$. We note directly that

- in $B_{3/4}(\bar{x}) \backslash \bar{B}_{1/20}(\bar{x})$, $\Delta w(x) = 2d\bar{c}|x - \bar{x}|^{-d-2} \geq 2d\bar{c}(4/3)^{d+2} > 0$,
- in $\left(B_{3/4}(\bar{x}) \backslash \bar{B}_{1/20}(\bar{x})\right) \cap \{x_d < 1/10\}$: $|\nabla w| \geq \partial_{x_d} w = \frac{d}{10}|4/3|^{d+2} =: c_w > 0$.

**Case $u(\bar{x}) \geq p(\bar{x}) + \frac{\varepsilon}{2}$:** As $p(x) \leq (x + a_0)_+ \leq u(x)$ in $B_1$ and $\Delta(u - p) = \Delta u = 0$ in $B_{1/10}(\bar{x}) \subset \Omega_u$, we apply Harnack's inequality (Theorem A.18), to get

$$\frac{\varepsilon}{2} = u(\bar{x}) - p(\bar{x}) \leq \sup_{B_{1/20}(\bar{x})} u - p \leq C_H \inf_{B_{1/20}(\bar{x})} u - p \leq C_H(u(x) - p(x)) \quad \forall x \in B_{1/20}(\bar{x})$$

$$\Leftrightarrow \text{in } B_{1/20}(\bar{x}) : \quad c\varepsilon \leq u - p \quad \text{for } c \coloneqq (2C_H)^{-1}.$$

We show now that for $c_d > 0$, we have

$$u(x) \geq p(x) + c_d\varepsilon \quad \text{in } B_{1/20} \quad \left(\Leftrightarrow u(x) \geq (p(x) + c_d\varepsilon)_+ \quad \text{in } B_{1/20}\right).$$

Consider therefore

$$v(x) = p(x) + c\varepsilon w(x),$$

and suppose that $u - v$ is minimized at some point $x \in B_1$ with $u(x) - v(x) < 0$. Since $u - v = u - p \geq 0$ on $B_1 \backslash B_{3/4}(\bar{x})$ and $u - v = u - p - c\varepsilon \geq 0$ in $\bar{B}_{1/20}(\bar{x})$, $x$ has to be in $B_{3/4}(\bar{x}) \backslash \bar{B}_{1/20}(\bar{x})$. However, in $\left(B_{3/4}(\bar{x}) \backslash \bar{B}_{1/20}(\bar{x})\right) \cap \Omega_u$, $\Delta(u - v) = -\Delta v = -c\varepsilon\Delta w < 0$, so $u - v$ is superharmonic and has its minimum on the boundary. In $\left(B_{3/4}(\bar{x}) \backslash \bar{B}_{1/20}(\bar{x})\right) \cap \{x_d <$

$1/10\}$, we have $|\nabla v| \geq \partial_{x_d} v(x) = 1 + c\varepsilon c_w > 1$. Thus if $x \notin \bar{\Omega}_u$, then $|\nabla u| = u = 0$ and so $|\nabla(u-v)| \neq 0$ and $u - v$ does not have a minimum here. In the case $x \in \partial\Omega_u$, the same argument works as $|\nabla(u-v)| = |\nabla v| - |\nabla u| = c\varepsilon c_w > 0$. We conclude that $x \notin B_{3/4}(\bar{x}) \backslash \bar{B}_{1/20}(\bar{x})$ and the minimum of $u - v$ is positive, i.e.

$$u(x) \geq v(x) = p(x) + c\varepsilon w(x) \quad \forall x \in B_1.$$

Since $w(x) \geq \bar{c}\left((5/20)^{-d} - (3/4)^{-d}\right) = c' > 0$ in $B_{1/20}$, we have for $c_d := \min(cc', 1/2)$

$$u(x) \geq p(x) + c_d\varepsilon \quad \text{in } B_{1/20} \quad \iff \quad u(x) \geq (x_d + a_0 + c_d\varepsilon)_+ \quad \text{in } B_{1/20}.$$

Setting now $a_1 = a_0 + c_d\varepsilon$ and $b_1 = b_0$ gives the result.

**Case $u(\bar{x}) < p(\bar{x}) + \frac{\varepsilon}{2}$:** Again from $u(x) \leq (x_d + b_0)_+ = (p(x)+\varepsilon)_+$, we get that $p + \varepsilon - u$ is a non-negative, harmonic function inside $B_{1/10}(\bar{x})$, and Harnack's inequality gives $p+\varepsilon-u \geq c\varepsilon$ in $B_{1/20}(\bar{x})$. Let now

$$v(x) = p(x) + \varepsilon - c\varepsilon w(x),$$

as previously $v_+ \geq u$ in $B_1 \backslash B_{3/4}(\bar{x}) \cup \bar{B}_{1/20}(\bar{x})$. We show that actually $v_+ \geq u$ everywhere in $B_1$. Suppose there were $y \in B_{3/4}(\bar{x}) \backslash \bar{B}_{1/20}(\bar{x})$ with $v_+(y) - u(y) < 0$. Since $v_+ \geq 0$, $y$ has to be strictly inside $\Omega_u$. However this implies

$$0 > v_+(y) - u(y) \geq v(y) - u(y) \geq \inf_{\left(B_{3/4}(\bar{x}) \backslash \bar{B}_{1/20}(\bar{x})\right) \cap \Omega_u} v - u.$$

Since $v - u$ is superharmonic,

$$\Delta(v-u) = \Delta v = -c\varepsilon \Delta w \leq 0 \quad \text{in } \left(B_{3/4}(\bar{x}) \backslash \bar{B}_{1/20}(\bar{x})\right) \cap \Omega_u,$$

the infimum is actually a minimum attained at a point $z$ on

$$\partial\left(\left(B_{3/4}(\bar{x}) \backslash \bar{B}_{1/20}(\bar{x})\right) \cap \Omega_u\right) = \left(\partial B_{3/4}(\bar{x}) \cap \Omega_u\right) \cup \partial B_{1/20}(\bar{x}) \cup \left(\partial\Omega_u \cap B_{3/4}(\bar{x})\right).$$

Clearly $x \notin \partial B_{1/20}(\bar{x})$, as by continuity $v = v_+ \geq u$ there.

Secondly, suppose $x \in \partial\Omega_u \cap B_{3/4}(\bar{x})$. We show that $|\nabla v|^2 < 1$, which together with $|\nabla u| = 1$ on $\partial\Omega_u$ implies that $|\nabla(v-u)| \geq |\nabla u| - |\nabla v| > 0$, so $x$ can not be a minimum. In fact, since $\partial\Omega_u$ lies in the strip $\{x \in B_{3/4}(\bar{x}) : |x_d| < 1/10\}$, we have

$$-\frac{3}{10} < x_d - \bar{x}_d < -\frac{1}{10} \quad \iff \quad \frac{1}{10} < \bar{x}_d - x_d < \frac{3}{10} \qquad \text{and} \qquad \frac{1}{10} < |x - \bar{x}| \leq \frac{3}{4}.$$

We estimate

$$\begin{aligned}
|\nabla v|^2 &= \langle e_d - c_H\varepsilon\nabla w, e_d - c_H\varepsilon\nabla w\rangle = 1 - 2c_H\varepsilon\langle\nabla w, e_d\rangle + c_H^2\varepsilon^2|\nabla w|^2 \\
&= 1 - 2c_h\varepsilon\left(-\bar{c}d|x-\bar{x}|^{-d-2}(x_d - \bar{x}_d)\right) + c_H^2\varepsilon^2\bar{c}d|x-\bar{x}|^{-d-1} \\
&= 1 - 2c_h\varepsilon\left(\bar{c}d|x-\bar{x}|^{-d-2}(\bar{x}_d - x_d)\right) + c_H^2\varepsilon^2\bar{c}d|x-\bar{x}|^{-d-1} \\
&\leq 1 - \frac{1}{5}c_h\varepsilon\left(\bar{c}d|x-\bar{x}|^{-d-2}\right) + c_H^2\varepsilon^2\bar{c}d|x-\bar{x}|^{-d-1} \\
&\leq 1 - \frac{1}{5}c_h\varepsilon\left(\bar{c}d(3/4)^{-d-2}\right) + c_H^2\varepsilon^2\bar{c}d(1/10)^{-d-1} < 1,
\end{aligned}$$

if $\bar{\varepsilon} \geq \varepsilon$ is taken sufficiently small as then

$$\frac{1}{5}c_h\varepsilon\left(\bar{c}d(3/4)^{-d-2}\right) \geq c_H^2\varepsilon^2\bar{c}d(1/10)^{-d-1} \quad \Leftrightarrow \quad (4/3)^{d+2} \geq c_H\varepsilon 5^{d+2}.$$

Lastly, suppose $z \in \partial B_{3/4}(\bar{x}) \cap \Omega_u$. Then $w(z) = 0$, $u(z) > 0$ and so by using the assumption of the theorem

$$0 > v(z) - u(z) = z_d + b_0 \implies z_d < -b_0 \implies 0 = (z_d + b_0)_+ \geq u(z) > 0,$$

a contradiction. Thus the initial assumption, the existence of $y$ with $v_+(y) - u(y)$ is wrong and so $v_+ \geq u$ in $B_1$. Since $w(x) \geq \bar{c}\left((5/20)^{-d} - (3/4)^{-d}\right) = c' > 0$ in $B_{1/20}$, we have for $c_d \coloneqq \min(cc', 1/2)$

$$u(x) \leq v(x) = (p(x) + \varepsilon - c_d\varepsilon)_+ = (x_d + a_0 + \varepsilon - c_d\varepsilon)_+ = (x_d + b_0 - c_d\varepsilon)_+ \quad \text{in } B_{1/20}.$$

Setting now $a_1 = a_0$ and $b_1 = b_0 - c_d\varepsilon$ gives the result. The proof is finished. □

**Remark 5.11.** In [Vel23, Lemma 7.10] and [Kri19] a whole family of functions, $v_t$ with $t < 1$, is considered. Then passing to the limit as $t \to 1$, gives the same inequalities.

We get now a Hölder condition, but only for pairs of points sufficiently far apart.

**Corollary 5.12.** *Let $u$ be a viscosity solution of (4.8) on $B_1$ with $0 \in \partial\Omega_u$. Then if $u$ is $\varepsilon$ flat in $x_d$ direction,*

$$|u - x_d^+| \leq \varepsilon < \frac{\bar{\varepsilon}}{2} \qquad \text{in } B_1 \cap \Omega_u,$$

*the rescaling $v(x) = \frac{u(x) - x_d}{\varepsilon}$ is "almost" Hölder continuous i.e.*

$$|v(x) - v(y)| \leq C|x - y|^\alpha, \quad \alpha < 1 \qquad \text{for all } x, y \in B_{1/2} \cap \Omega_u \text{ with } |x - y| \geq C'\frac{\varepsilon}{\bar{\varepsilon}}.$$

*Proof.* Take $n$ big enough such that

$$\frac{1}{2}\frac{1}{20^{n+1}} \leq \frac{\varepsilon}{\bar{\varepsilon}} < \frac{1}{2}\frac{1}{20^n},$$

and set $r_j \coloneqq \frac{1}{2}\frac{1}{20^j}$, clearly $\varepsilon \leq \bar{\varepsilon} r_j$ for any $j = 0, \dots, n$. We set $a_0 = -\varepsilon$, $b_0 = \varepsilon$, then $|a_0 - b_0| = 2\varepsilon \leq \bar{\varepsilon}$ and take $x_0 \in B_{1/2} \cap \bar{\Omega}_u$. Apply the theorem iteratively $n$ times around $x_0$ to obtain a sequence

$$a_0 \leq a_1 \leq \dots \leq a_n < b_n \leq \dots \leq b_1 \leq b_0,$$

with $|a_j - b_j| \leq (1-c)^j |a_0 - b_0|$ for some $c > 0$ and

$$(x_d + a_j)_+ \leq u(x) \leq (x_d + b_j)_+ \quad \text{in } B_{r_j}(x_0) \iff x_d + a_j \leq u(x) \leq x_d + b_j \quad \text{on } B_{r_j}(x_0) \cap \bar{\Omega}_u.$$

This gives

$$|u(x) - x_d - a_j| \leq |b_j - a_j| \leq 2(1-c)^j\varepsilon \qquad \text{in } B_{r_j}(x_0) \cap \bar{\Omega}_u,$$

so in turn

$$|v(x) - v(x_0)| = \frac{1}{\varepsilon}|u(x) - x_d - a_j - (u(x_0) - x_d - a_j)| \leq \frac{2}{\varepsilon} 2(1-c)^j \varepsilon = 4(1-c)^j.$$

Let now $x \in B_{1/2}(x_0) \backslash B_{\varepsilon/\bar{\varepsilon}}(x_0)$ and $j$ be such that $r_{j+1} \leq |x - x_0| \leq r_j$. Take $\alpha$ sufficiently small such that $\frac{1}{20^\alpha} = 1 - c$, then

$$|x - x_0|^\alpha \geq r_{j+1}^\alpha = \frac{1}{2^\alpha}\frac{1}{20^{(j+1)\alpha}} = \frac{1}{2^\alpha}(1-c)^j \frac{1}{20^\alpha} \geq \frac{1}{4}\frac{1}{40^\alpha}|v(x) - v(x_0)|,$$

which finishes the proof. □

We also need a general result, permitting us to smoothly extend a viscosity solution.

**Lemma 5.13.** *Let $u \in C^0(B_1 \cap \{x_d \geq 0\})$ be harmonic on $B_1 \cap \{x_d > 0\}$ satisfying the Neumann boundary condition in the viscosity sense, i.e.*

1. *for every $\phi \in C^\infty(B_1 \cap \{x_d \geq 0\})$ with $\phi \geq u$ and equality $\phi(x) = u(x)$ only at $x$, where $x_d = 0$, we have $\partial_d \phi(x) \geq 0$.*
2. *for every $\phi \in C^\infty(B_1 \cap \{x_d \geq 0\})$ with $\phi \leq u$ and equality $\phi(x) = u(x)$ only at $x$, where $x_d = 0$, we have $\partial_d \phi(x) \leq 0$.*

*Then, the even extension of $u$ on $B_1$ given by*

$$\bar{u}(x) = \begin{cases} u(x) & \text{if } x_d \geq 0 \\ u(x', -x_d) & \text{if } x_d < 0 \end{cases},$$

*is harmonic, thereby smooth on $B_1$ and $u$ satisfies the Neumann boundary condition in the classical sense.*

*Proof.* Take $\phi \in C^\infty(B_1)$ with $\phi(x) = u(x)$ for some $x$ such that $x_d = 0$ and $\phi > \bar{u}$ otherwise. (To assure that such a $\phi$ exists, we have to show that there exists $\phi \geq 0$ touching $u$ at $x$ with $\partial_d \phi(x) = 0$. But if $\partial_d \phi(x) > 0$ for any $\phi \geq u$, then the second condition can not hold.) Suppose now that $\Delta \phi(x) < 0$, for $x$ with $x_d = 0$. Set $\psi(x) = \frac{\phi(x', x_d) + \phi(x', -x_d)}{2}$, then $\Delta \psi(x) = \Delta \phi(x) < 0$ and by continuity for $\tau, c > 0$ sufficiently small,

$$\Delta \psi < 0 \quad \text{in } B_\tau(x) \qquad \text{and} \qquad \psi > \bar{u} + c \quad \text{on } \partial B_\tau(x).$$

Define $\psi : B_1 \to \mathbb{R}$ as

$$\psi_\varepsilon(z) \coloneqq \psi(z) - \varepsilon z_d - \inf_{\xi \in B_\tau(x)} (\psi(\xi) - \varepsilon \xi_d - \bar{u}(\xi)),$$

clearly $\psi_\varepsilon \geq u$ on $B_\tau(x)$ and so there exists $y \in \bar{B}_\tau(x)$ such that $\psi_\varepsilon(y) = u(y)$. Firstly, we show that $y \in B_\tau(x)$ when $\varepsilon < c/2$. If not (i.e. $y \in \partial B_\tau(x)$), from $\psi_\varepsilon > \bar{u}$ on $B_\tau(x)$, we get

$$\psi(z) - \varepsilon z_n - \bar{u}(z) > \inf_{B_\tau(x)} \psi(\xi) - \varepsilon \xi_d - \bar{u}(\xi) = \min_{\partial B_\tau(x)} \psi(\xi) - \varepsilon \xi_d - \bar{u}(\xi) > \min_{\partial B_\tau(x)} c - \varepsilon \xi_d = c - \varepsilon \tau.$$

Then $\phi(z) - \bar{u}(z) > c - 2\varepsilon\tau > 0$ in $B_\tau(x)$, which contradicts $\psi(x) = \phi(x) = u(x) = \bar{u}(x)$.

Secondly, we show that $y_d = 0$. Since $y$ is a minimum of $\psi_\varepsilon - u$,

$$0 \leq \Delta \psi_\varepsilon(y) - \Delta \bar{u}(y) = \Delta \psi(y) - \Delta \bar{u}(y).$$

However, $\Delta \psi < 0$ in $B_\tau(x)$, which implies that $\Delta \bar{u}(y) < 0$ which cannot happen in $\{y_d \neq 0\}$ as $\bar{u}$ is harmonic there. Since $\psi_\varepsilon$ then touches $u$ above at a point in $\{x_d = 0\}$, by the first viscosity Neumann condition,

$$0 \leq \partial_d \psi_\varepsilon(y) = \partial_d \psi(y) - \varepsilon = -\varepsilon,$$

using that $\partial_d \psi(y) = 0$ by symmetry, a contradiction, thus $\nabla \phi(x) \geq 0$. Another similar argument gives $\nabla \phi(x) \leq 0$, so $\bar{u}$ is harmonic in the viscosity sense and thus also classically harmonic and smooth. ☐

We can finally prove the complete improvement of flatness, based on [Kri19].

**Proposition 5.14.** (Improvement of flatness) *Let $\tau > 0$ and $u$ a viscosity solution of (4.8) on $B_1$ with $0 \in \partial \Omega_u$. Assume that it is $\varepsilon$-flat in $e_d$ direction, i.e.*

$$\sup_{\Omega_u \cap B_1} |u(x) - x_d| \leq \varepsilon \leq \bar{\varepsilon}(d, \tau).$$

*Then there exists $e \in \mathbb{S}^{d-1}$ with $|e - e_d| \leq C' \varepsilon$ such that*

$$\sup_{B_\tau \cap \Omega_u} |u(x) - x \cdot e| \leq C_d \tau^2 \varepsilon.$$

*Proof.* By contradiction, assume it does not hold. Then there is a sequence $\varepsilon_k \to 0$ and $u_k$ viscosity solutions of (4.8) (on $B_1$ with $0 \in \partial\Omega_{u_k}$ for any $k$) with

$$|u_k(x) - x_d| \leq \varepsilon_k \qquad \text{in } B_1 \cap \{u_k > 0\},$$

but

$$\sup_{B_\tau \cap \Omega_{u_k}} |u_k(x) - x \cdot e| > C_d \tau^2 \varepsilon_k \qquad \forall e \in \mathbb{S}^{d-1} : |e - e_d| \leq C' \varepsilon_k.$$

Let the rescalings $v_k$ be given as

$$v_k(x) = \frac{u_k(x) - x_d}{\varepsilon_k},$$

we proceed as follows:

1. Show that $v_k$ converges to a limit function $v : B_1 \cap \{x_d > 0\} \to \mathbb{R}$.
2. Show that $v$ is (Hölder-) continuous up to the boundary.
3. Show that $v$ satisfies the viscosity Neumann condition in Lemma 5.13 and is thereby $C^2$ up to the boundary.
4. Show that this entails a contradiction.

**Step 1:** From $|u_k(x) - x_d| \leq \varepsilon_k$ we get that $x_d - \varepsilon_k \leq \varepsilon_k$, thus $\Omega_{u_k} := \{u_k > 0\} \supseteq \{x_d > \varepsilon_k\}$ and clearly $v_k$ is harmonic on $\Omega_{u_k > 0}$. Take now $\Omega := B_1 \cap \{x_d > 0\}$ and let $K$ be a compact subset of $\Omega$, that is $K \subset\subset B_1 \cap \{x_d > \delta\}$ for some $\delta > 0$. Thus for large enough $k$, $|v_k| \leq 1$ in $B_1 \cap \{x_d > \delta\}$, using again $|u_k(x) - x_d| \leq \varepsilon_k$. By Theorem A.20, this gives

$$\begin{aligned}
\sup_K |\nabla v_k| \leq C \frac{1}{\operatorname{dist}(K, B_1 \cap \{x_d > \delta\})} \sup_{B_1 \cap \{x_d > \delta\}} |v_k| \leq C \frac{1}{\operatorname{dist}(K, B_1 \cap \{x_d > \delta\})} \\
\implies |v_k(x) - v_k(y)| \leq \sup_K |\nabla v_k| |x - y|,
\end{aligned}$$

hence the sequence $v_k$ is equi-continuous (and uniformly bounded). Applying the Arzelà-Ascoli theorem gives a uniform converging subsequence on $K$. Denote its limit by $v : \Omega \to \mathbb{R}$ (note $|v| \leq 1$), as the uniform limit of harmonic functions it is harmonic itself on $\Omega$.

**Step 2a:** By Corollary 5.12 we have $|v_k(x) - v_k(y)| \leq C|x - y|^\alpha$ in $B_{1/2} \cap \Omega_{u_k}$ as long as $|x - y| \geq C' \frac{\varepsilon_k}{\bar{\varepsilon}}$. If $|x - y| < C' \frac{\varepsilon_k}{\bar{\varepsilon}}$, take $z \in B_{1/2} \cap \Omega_{u_k}$ with

$$C' \frac{\varepsilon_k}{\bar{\varepsilon}} \leq |x - z|, |y - z| \leq 3C' \frac{\varepsilon_k}{\bar{\varepsilon}},$$

then

$$|v_k(x) - v_k(y)| \leq |v_k(x) - v_k(z)| + |v_k(y) - v_k(z)| \leq C|x - z|^\alpha + C|y - z|^\alpha \leq 6CC' \varepsilon_k^\alpha.$$

In other words,

$$|v_k(x) - v_k(y)| \leq C(|x - y| + \varepsilon_k)^\alpha \qquad \text{in } B_{1/2} \cap \Omega_{u_k} \tag{5.6}$$

and passing to the limit gives

$$|v(x) - v(y)| \leq C|x - y|^\alpha \qquad \text{in } B_{1/2} \cap \{x_d > 0\},$$

that is $v$ extends to a continuous function on $B_{1/2} \cap \{x_d \geq 0\}$.

**Step 2b:** We claim that for $x \in \bar{\Omega}_{u_k} \cap B_{1/2}$ and $y \in \bar{\Omega} \cap B_{1/2}$, with $|x-y| < b\varepsilon_k$ ($b \geq 1$) the estimate $|v_k(x) - v(y)| \leq o_k(1)$ holds. In particular, as $0 \in \bar{\Omega}_{u_k}$, $0 \in \bar{\Omega}$ and $v_k(0) = 0$, we get $v(0) = 0$.

Pick $z \in B_{1/2} \cap \Omega_{u_k} \cap \Omega$, since $\{x_d > \varepsilon_k\}$ is contained in both $\Omega_{u_k}$ and $\Omega$, we can assume that $|x-z|, |y-z| < 3b\varepsilon_k$.

$$\begin{aligned}|v_k(x) - v(y)| &\leq |v_k(x) + v_k(z)| + |v_k(z) - v(z)| + |v(y) - v(z)| \\ &\leq C(|x-y| + \varepsilon_k)^\alpha + C|y-z|^\alpha + |v_k(y) - v(y)| \\ &< C\varepsilon_k + |v_k(y) - v(y)| = o_k(1).\end{aligned}$$

In the last step we used the local uniform convergence of $v_k$ to $v$ on $\Omega$.

**Step 3a:** In order to use Lemma 5.13 we show that for a smooth function $\phi \in C^\infty(B_{1/2})$ with $\phi \leq v$ on $B_{1/2} \cap \bar{U}$ and we have equality only at $x$ with $x_d = 0$, we have $\partial_d \phi(x) \leq 0$.

Suppose for the sake of contradiction that there exists some $\phi$ with $\partial_d \phi(x) = c_1 > 0$. Since we are interested only in local properties, replacing $\phi$ locally (that is on $B_\rho(x)$ for some $\rho > 0$ small) by a quadratic polynomial of the form

$$\phi(y) = a + \frac{2}{3} c_1 y_d + \nu \cdot (y' - x') + (y-x)^T A (y-x) \qquad \text{for some } \nu \in R^{d-1},$$

where $A$ has strictly positive trace on $B_\rho(x)$. Adjust $\rho$, if necessary, such that

$$\text{for some } c_2 > 0,\ \phi < v - c_2 \text{ on } \partial B_\rho(x) \cap \bar{\Omega} \qquad \text{and} \qquad \partial_d \phi \geq c_1/2 \text{ on } B_\rho(x) \tag{5.7}$$

using the compactness of $\partial B_\rho(x) \cap \bar{\Omega}$, the fact that $v > \phi$ unless at $x$ and also the smoothness of $\phi$.

The idea is to construct a function $w_t$ touching for some large $k$, $u_k$, a viscosity solution to (5.13), from below at a point $z$ (specified later) and derive a contradiction. Note that as $\varepsilon_k \to 0$, for large enough $k$, so eventually $B_\rho(0) \cap \Omega_{u_k} \neq \varnothing$.

1. $v_k \geq \phi - o_k(1)$ on $\bar{\Omega}_{u_k} \cap B_\rho(x)$, which is equivalent to $u_k \geq x_d + \varepsilon_k \phi - \varepsilon_k o_k(1)$: Take $\xi \in \bar{\Omega}_{u_k} \cap B_\rho(x)$ and $y \in \bar{\Omega} \cap B_\rho(x)$ with $|\xi - y| < \varepsilon_k$ and by Step 2b,

$$-v_k(\xi) + v(y) \leq |v_k(\xi) - v(y)| \leq o_k(1) \Longleftrightarrow v(y) \leq v_k(\xi) + o_k(1).$$

Since $y \in \bar{\Omega} \cap B_{1/2}$, $v(y) \geq \phi(y)$ and since $\phi$ Lipschitz, $|\phi(\xi) - \phi(y)| \leq L\varepsilon_k = o_k(1)$, giving

$$\phi(\xi) + o_k(1) \leq \phi(y) \leq v(y) \leq v_k(\xi) + o_k(1) \implies v_k(\xi) \geq \phi(\xi) - o_k(1).$$

2. $v_k \geq \phi + c_2 - o_k(1)$ on $\bar{\Omega}_{u_k} \cap \partial B_\rho(x)$, which is equivalent to $u_k \geq x_d + \varepsilon_k \phi + \varepsilon_k c_2 - \varepsilon_k o_k(1)$: Similar to above, take $\xi \in \bar{\Omega}_{u_k} \cap \partial B_\rho(x)$ and $y \in \bar{\Omega} \cap \partial B_\rho(x)$ with $|x - y| \leq 2\varepsilon_k$, then again applying Step 2b gives

$$-v_k(\xi) + v(y) \leq |v_k(\xi) - v(y)| \leq o_k(1) \Longleftrightarrow v(y) \leq v_k(\xi) + o_k(1).$$

Since $y \in \bar{\Omega}\partial \cap B_{1/2}$, $v(y) \geq \phi(y) + c_2$ and regularity of $\phi$ as above,

$$\phi(\xi) + c_2 + o_k(1) \leq \phi(y) + c_2 \leq v(y) \leq v_k(\xi) + o_k(1) \implies v_k(\xi) \geq \phi(\xi) + c_2 - o_k(1).$$

3. For some point $\bar{x} \in \bar{\Omega}_{u_k} \cap B_\rho(x)$ we have $v_k(\bar{x}) \leq \phi(\bar{x}) + o_k(1) \Leftrightarrow u_k(\bar{x}) \leq \bar{x}_d + \varepsilon_k \phi(\bar{x}) + \varepsilon_k o_k(1)$: Note that we have $v(x) = \phi(x)$ by definition, hence if $x \in \bar{\Omega}_{u_k}$, take $\bar{x} = x$. If $x \notin \bar{\Omega}_{u_k}$, take $\bar{x} \in \bar{\Omega}_{u_k} \cap B_\rho(x)$ such that $x_d < \bar{x}_d$ but $x' = \bar{x}'$ and $|x - \bar{x}| < \varepsilon_k$. Since by assumption $\partial_d \phi \geq c_1/2 > 0$ on $B_\rho(x)$, $\phi(\bar{x}) \geq \phi(x)$, so again by Step 2b we get

$$|v_k(\bar{x}) - v(x)| \leq o_k(1) \implies v_k(\bar{x}) \leq v(x) + o_k(1) = \phi(x) + o_k(1) \leq \phi(\bar{x}) + o_k(1). \tag{5.8}$$

**Step 3b:** Let now be $w_t(x) \coloneqq (x_d + \varepsilon_k \phi(x) + \varepsilon_k t)_+$, then we have:

1. $t \le -\frac{c_2}{2}$: Then $w_t \le u_k$ on $B_\rho(x) \cap \bar{\Omega}_{u_k}$ as
$$x_d + \varepsilon_k + \varepsilon_k t \le x_d + \varepsilon_k - \varepsilon_k \frac{c_2}{2} \le x_d + \varepsilon_k - \varepsilon_k o_k(1),$$
for $k$ large enough. If $w_t = 0$, it follows trivially.

2. $t \le \frac{c_2}{2}$: Then on $\partial B_\rho(x) \cap \bar{\Omega}_{u_k}$,
$$w_t \le x_d + \varepsilon_k \phi + \varepsilon_k \frac{c_2}{2} \le x_d + \varepsilon_k \phi + \varepsilon_k c_2 - \varepsilon_k o_k(1) \le u_k.$$

3. For any $t \in (0, \frac{c_2}{2})$ fixed, we have for $k$ large enough
$$w_t(\bar{x}) = \bar{x}_d + \varepsilon_k \phi(\bar{x}) + \varepsilon_k t \ge \bar{x}_d + \varepsilon_k \phi(\bar{x}) + \varepsilon_k o_k(1) \ge u_k(\bar{x}).$$
Note that $\bar{x}_d + \varepsilon_k \phi(\bar{x}) + \varepsilon_k t > 0$, otherwise $x_d + \varepsilon_k \phi(\bar{x}) < -\varepsilon_k t$, which then gives $u(\bar{x}) < 0$ in (5.8), a contradiction.

Fix now $k$ large enough so that the first two statements hold and the third holds for some $t = \frac{c_2}{3}$. Let now $-\frac{c_2}{2} \le t^* \le t$ be the smallest $t$ such that all three statement hold (i.e. for $t < t^*$, the third statement does not hold anymore) and $z$ a point in $B_\rho(x) \cap \Omega_{u_k}$ with $w_{t^*}(z) = u_k(z)$.

**Step 3c:** Now there are two cases:

(i) $u_k(z) > 0$: Then $w_{t^*}(z) > 0$ and so $\Delta w_{t^*}(z) = \Delta \phi(z) = \mathrm{Tr}(A) > 0$. However this is a contradiction, since by the definition of viscosity solution to $u_k$, $\Delta w_{t^*}(z) \le 0$ for any function touching from below.

(ii) $u_k(z) = 0$: Then, since $\partial_d \phi(z) \ge c_1/2 > 0$,
$$|\nabla w_{t^*}(z)|^2 = 1 + 2\varepsilon_k \partial_d \phi(z) + \varepsilon_k^2 |\nabla \phi(z)|^2 > 1,$$
again a contradiction to the boundary condition of the viscosity solution $u_k$.

A similar argument treats the case $\phi \ge v$ and $\partial_d \phi(x) \ge 0$. By Lemma 5.13, the symmetric extension $\bar{v}$ is harmonic and so $C^2$, then $v$ is $C^2$ up to the boundary and $\partial_d v(x) = 0$ on $\{x_d = 0\}$ by symmetry.

**Step 4:** By the previous steps, $\bar{v} \in C^2(B_{1/4})$ therefore $v \in C^2(B_{1/4} \cap \bar{\Omega})$ and $\partial_d v(0) = 0$, hence $[v]_{C^2(B_\tau \cap \{x_d > 0\})}, |\nabla v(0)| \le C_d$.

Set $\tilde{e}_k = \frac{e_d + \varepsilon_k \nabla v(0)}{|e_d + \varepsilon_k \nabla v(0)|}$, then
$$\begin{aligned} |e_d - \tilde{e}_k| &= \left| e_d - \frac{e_d + \varepsilon_k \nabla v(0)}{|e_d + \varepsilon_k \nabla v(0)|} \right| = \left| e_d \left( 1 - \frac{1}{\sqrt{1 + \varepsilon_k^2 |\nabla v(0)|^2}} \right) + \frac{\varepsilon_k \nabla v(0)}{\sqrt{1 + \varepsilon_k^2 |\nabla v(0)|^2}} \right| \\ &\le 2\varepsilon_k^2 |\nabla v(0)|^2 + \varepsilon_k |\nabla v(0)| \le C' \varepsilon_k, \end{aligned}$$
using that $\partial_d v(0) = 0$ and $1 - \frac{1}{\sqrt{1+x}} \le 2x$ for $|x| < 1/2$, which holds for large $k$. Hence $\tilde{e}_k$ is admissible. By Taylor expansion of the even extension $\bar{v}$ across $\{x_d = 0\}$ (which is in $C^2(B_{1/4})$, we get
$$\begin{aligned} &\bar{v}(x) = \bar{v}(0) + \nabla \bar{v}(0) \cdot x + \xi^T \cdot D^2 \bar{v}(0) \cdot \xi \quad \text{for some } \xi \in B_\tau \\ &\implies |\bar{v}(x) - \nabla\, v(0) \cdot x| \le [\bar{v}]_{C^2(B_{1/4})} = [v]_{C^2(B_{1/4} \cap \bar{\Omega})} \quad \text{on } B_\tau. \end{aligned}$$

This in turn gives

$$\sup_{B_\tau\cap\bar\Omega_{u_k}} |v_k(x)-\nabla v(0)\cdot x| \le \sup_{B_\tau\cap\bar\Omega_{u_k}} |v_k(x)-\bar v(x)| + \sup_{B_\tau\cap\bar\Omega_{u_k}} |\bar v(x)-\nabla\bar v(0)\cdot x|$$
$$\le o_k(1) + [v]_{C^2(B_{1/4}\cap\bar\Omega)}\tau^2 \le 2[v]_{C^2(B_{1/4}\cap\bar\Omega)}\tau^2,$$

using the local uniform convergence if $x\in\{x_d>0\}$ or Step 2 if $x\in\{x_d=0\}$. Note also that from $\partial_d v(0)=0$ we have,

$$|e_d+\varepsilon_k\nabla v(0)-\tilde e_k| = |e_d+\varepsilon_k\nabla v(0)|-1 = \sqrt{1+\varepsilon_k^2|\nabla v(0)|^2} \le \varepsilon_k^2|\nabla v(0)|^2.$$

Finally, we estimate using that $\varepsilon_k\to 0$,

$$\sup_{B_\tau\cap\bar\Omega_{u_k}}\left|\frac{u_k(x)-e_k\cdot x}{\varepsilon_k}\right| \le \sup_{B_\tau\cap\bar\Omega_{u_k}}\left(\left|\frac{u_k(x)-(e_d+\varepsilon_k\nabla v(0))\cdot x}{\varepsilon_k}\right|+\left|\frac{(e_d+\varepsilon_k\nabla v(0)-\tilde e_k)\cdot x}{\varepsilon_k}\right|\right)$$
$$\le \sup_{B_\tau\cap\bar\Omega_{u_k}}|v_k(x)-\nabla v(0)\cdot x| + \frac{1}{\varepsilon_k}|e_d+\varepsilon_k\nabla v(0)-\tilde e_k|\tau$$
$$\le 2[v]_{C^2(B_{1/4}\cap\bar\Omega)}\tau^2 + \frac{1}{\varepsilon_k}\varepsilon_k^2|\nabla v(0)|^2\tau \le 3[v]_{C^2(B_{1/4}\cap\bar\Omega)}\tau^2.$$

However this contradicts the assumption, since $|\nabla v(0)|$ and $[v]_{C^2(B_{1/4}\cap\bar\Omega)}$ are bounded by a constant, which finishes the proof. □

We use now the technical tools to show the regularity of the free boundary at regular points. The strategy [Kri19, Chapter 11] is to show that the rescalings before the blowup (i.e. at some finite scaling radius) are $\varepsilon$-flat, and then that $\varepsilon$-flatness implies the $C^{1,\alpha}$ regularity.

**Theorem 5.15.** *The set of regular free boundary points, $Reg(\partial\Omega_u)$ is locally a $C^{1,\alpha}$ manifold in $\mathbb{R}^d$ for any $\alpha\in(0,1)$. To be more precise, for any $\alpha\in(0,1)$ and any regular free boundary point $x_0$, there exists $r=r(\alpha,x_0)$ such that $\nabla u\in C^{1,\alpha}(B_r(x_0),\mathbb{R}^d)$.*

*Proof.* Let $x_0$ be a regular free boundary point, without loss of generality $x_0=0$, and $u_{r_n}$ a corresponding blow-up sequence. We use "improvement of flatness", Proposition 5.14, on $u_{r_n}$.

**Step 1:** For $0<\alpha<1$ fixed, take $\tau<1/2$ such that $C_d\tau\le\tau^\alpha$, where $C_d$ is the constant from Proposition 5.14. Fix now $\varepsilon=\bar\varepsilon(d,\tau)/3$. By the strong $H^1$ convergence of the blow-up sequence to the blow up limit $u_0(x)=(x\cdot\nu)_+$ (as $x_0$ is regular) in e.g. $B_1$, we have $\|u_{r_n}-u_0\|_{L^\infty(B_1)}\to 0$. In particular $|u_{r_n}-(x\cdot\nu)_+|<2\varepsilon$ in $B_1$, i.e. $u_{r_n}$ is $2\varepsilon$-flat in $B_1$ for any $n$ sufficiently large (depending on $\bar\varepsilon$). Again without loss of generality, up to a rotation, we assume that $\nu=\bar e_d=(0,\dots,0,1)\in\mathbb{R}^d$. It remains to show that the free boundary of $u_{r_n}$ is $C^{1,\alpha}$ in some small ball $B_r$, then by rescaling, the $C^{1,\alpha}$ regularity also holds for the free boundary of the original $u$ in $B_{rr_n}(x_0)$. We thus can fairly assume that $|u-(x\cdot v_d)_+|<2\varepsilon$ in $B_1$, renaming $u_{r_n}$ to $u$ for simplification.

**Step 2:** We now apply the improvement of flatness, Proposition 5.14, to $u$. This gives $e_1\in\mathbb{S}^{d-1}$ with $|e_1-\bar e_d|\le C'\varepsilon$ with

$$\sup_{B_\tau\cap\{u>0\}}|u(x)-x\cdot e_1|\le C_d\tau^2\varepsilon\le\tau^{1+\alpha}\varepsilon,$$

so $\tau^{1+\alpha}\varepsilon$ flat in $e_1$ direction. Repeating now the same argument to get $e_2\in\mathbb{S}^{d-1}$ with $|e_2-e_1|\le C'\tau^{1+\alpha}\varepsilon$ and

$$\sup_{B_{\tau^2}\cap\{u>0\}}|u(x)-x\cdot e_1|\le C_d\tau^2\tau^{1+\alpha}\varepsilon\le(\tau^{1+\alpha})^2\varepsilon.$$

Iterating gives a sequence $e_k \in \mathbb{S}^{d-1}$ with $|e_k - e_{k-1}| \le C'(\tau^{1+\alpha})^k \varepsilon$ and

$$\sup_{B_{\tau^k} \cap \{u>0\}} |u(x) - x \cdot e_k| \le \tau^{k(1+\alpha)} \varepsilon, \qquad \text{for any } \alpha \in (0,1).$$

However, $\{e_k\}$ is a Cauchy sequence, which by completeness of $\mathbb{S}^{d-1}$, converges to a limit vector $e \in \mathbb{S}^{d-1}$. For each $k$ we have

$$|e_{k-1} - e| \le \sum_{i=k-1}^{\infty} |e_i - e_{i+1}| \le C'\varepsilon \sum_{i=k-1}^{\infty} (\tau^{1+\alpha})^i \le C'\varepsilon \frac{\tau^{k(1+\alpha)}}{1-\tau^{1+\alpha}} \le 2C'\varepsilon\tau^{k(1+\alpha)}.$$

The triangle inequality gives now

$$\begin{aligned} \sup_{B_{\tau^k} \cap \{u>0\}} |u(x) - x \cdot e| &\le |e_k - e| + \sup_{B_{\tau^k} \cap \{u>0\}} |u(x) - x \cdot e_k| \\ &\le 2C'\varepsilon\tau^{(k+1)(1+\alpha)} + \tau^{k(1+\alpha)}\varepsilon \le C\tau^{k(1+\alpha)}\varepsilon. \end{aligned}$$

This estimate works for any regular free boundary point $x_0$ in $B_1$, we proceed now by showing that actually for the limiting vector $e_{x_0}$

$$\forall x_0 \in \partial\Omega_u \cap B_1 : \quad |u(z) - e_{x_0} \cdot (z - x_0)| \le C|z - x_0|^{(1+\alpha)} \qquad \forall z \in \{u > 0\} \cap B_1.$$

If $|z - x_0| \ge \tau$, then as $z \mapsto |u(z) - e_{x_0} \cdot (z - x_0)|$ is uniformly (independent of $x_0$) bounded by $M := \sup_D u(z) + 1$, setting $C = M/\tau$ suffices. If $|z - x_0| < \tau$, then there exists $k$ such that $\tau^{k+1} \le |z - x_0| \le \tau^k$ and so

$$|u(z) - e_{x_0} \cdot (z - x_0)| \le \sup_{B_{\tau^k} \cap \{u>0\}} |u(z) - (z - x_0) \cdot e_{x_0}| \le C(\tau^k)^{1+\alpha} \le C|z - x_0|^{(1+\alpha)}. \tag{5.9}$$

Since the $\varepsilon$-flatness is an open condition, the estimate 5.3 holds for any free boundary point $x_1$ in $B_{1/2}(x_0)$ and $z \in B_{1/2}(x_1)$, with the constant $C = C(d, \alpha, x_0)$ independent of $x_1$.

**Step 3:** We show now the $\alpha$-Hölder continuity of the free boundary. It suffices to assume that $\alpha \ge \frac{1}{2}$, then regularity for smaller $\alpha$ is implied. Note that by definition, for $x_0$ regular, $\nabla u(x_0) = e_{x_0}$ is the unit normal of $\Omega_u$ at $x_0$. Let $x_1, x_2 \in \partial\Omega_u \cap B_{1/2}$, set $\delta := |x_1 - x_2|$, we show

$$|e_{x_1} - e_{x_2}| \le C|x_1 - x_2|^\alpha. \tag{5.10}$$

For $\delta \ge \frac{1}{100}$, the result holds by choosing a large constant, so assume that $\delta < \frac{1}{100}$. Moreover, without loss of generality, we assume that $x_1 = 0$ and $e_{x_1} = \bar{e}_d$. First, by triangle inequality and (5.3), we estimate for any $z \in B_\delta \cap \Omega_u$

$$\begin{aligned} f(z) := |e_{x_1} \cdot (z - x_1) - e_{x_2} \cdot (z - x_2)| &\le |u(z) - e_{x_1} \cdot (z - x_1)| + |u(z) - e_{x_2} \cdot (z - x_2)| \\ &\le |z - x_1|^{1+\alpha} + |z - x_2|^{1+\alpha} \\ &\le \delta^{1+\alpha} + (2\delta)^{1+\alpha} \le C\delta|x_1 - x_2|^\alpha. \end{aligned}$$

To prove (5.10), it remains to show that

$$\sup_{z \in B_\delta \cap \Omega_u} f(z) \ge c\delta|e_{x_1} - e_{x_2}|.$$

Since $x_1 = 0$ and $e_{x_1} = e_d$ by assumption, from (5.3),

$$|u(z) - e_{x_1} \cdot z| = |u(z) - z_d| \le |z|^{1+\alpha},$$

and so $u(z) > 0$ in $B_\delta \cap \text{epi}(z_d = 4|z'|^{1+\alpha}) =: B_\delta \cap E$, where $z = (z', z_d)$. If this were not the case, i.e. $u(z) = 0$ for some $z \in B_\delta$ with $z_d \geq 4|z'|^{1+\alpha}$, then (using $(a+b)^p \leq 2a^p + 2b^p$ for $a, b \geq 0, p \leq 2$)

$$|z_d| = |u(z) - z_d| \leq |z|^{1+\alpha} \leq 2|z_d|^{1+\alpha} + 2|z'|^{1+\alpha} \leq 2|z_d|^{1+\alpha} + \frac{|z_d|}{4} \implies |z_d| \leq 4|z_d|^{1+\alpha} \leq 4|z_d|^{\frac{3}{2}}.$$

However since $|z_d| \leq |z| \leq \frac{1}{100}$ and $\frac{1}{100} \geq 4\frac{1}{100^{3/2}} \Leftrightarrow 1 \geq \frac{2}{5}$, it is not possible that $|z_d| \leq 4|z_d|^{1+\alpha}$. We continue by showing

$$\sup_{z \in B_\delta \cap E} f(z) = \sup_{z \in B_\delta \cap E} |(e_{x_1} - e_{x_2}) \cdot z + e_{x_2} \cdot x_2| = \sup_{z \in B_\delta \cap E} |a \cdot z + c_2| \geq \frac{\delta}{4}|a|,$$

where we set $a := e_{x_1} - e_{x_2}$ and $c_2 := e_{x_2} \cdot x_2$. If $|c_2| \geq \frac{\delta}{4}|a|$, then by taking $z = 0$,

$$\sup_{z \in B_\delta \cap E} |a \cdot z + c_2| \geq |a \cdot 0 + c_2| = |c_2| \geq \frac{\delta}{4}|a|.$$

If $|c_2| \leq \frac{\delta}{4}|a|$, by the reverse triangle inequality,

$$\sup_{z \in B_\delta \cap E} |a \cdot z + c_2| = \sup_{z \in B_\delta \cap E} |a \cdot z - (-c_2)| \geq \sup_{z \in B_\delta \cap E} ||a \cdot z| - |c_2|| \geq \left(\frac{\delta}{2} - \frac{\delta}{4}\right)|a| = \frac{\delta}{4}|a|,$$

using that

$$\text{if } \bar{e}_d \cdot a \geq 0\text{: } z := \frac{\delta}{2}\left(e_d + \frac{a}{|a|}\right) \implies |a \cdot z| = \frac{\delta}{2}(e_d \cdot a + |a|) \geq \frac{\delta}{2}|a|,$$
$$\text{if } \bar{e}_d \cdot a < 0\text{: } z := \frac{\delta}{2}\left(e_d + \frac{-a}{|a|}\right) \implies |a \cdot z| = |e_d \cdot a - |a|| = \frac{\delta}{2}(-e_d \cdot a + |a|) \geq \frac{\delta}{2}|a|.$$

In both cases, as $\delta < \frac{1}{100}$,

$$z_d = z \cdot e_d \geq \frac{\delta}{2} \geq 4\delta^{1+\alpha} \geq 4|z|^{1+\alpha} \geq 4|z'|^{1+\alpha} \quad \text{and} \quad |z| \leq \delta \implies z \in B_\delta \cap E.$$

To conclude, by continuity of $f$,

$$c\delta|e_{x_1} - e_{x_2}| \leq \sup_{z \in B_\delta \cap E} f(z) \leq \sup_{z \in B_\delta \cap \bar{\Omega}_u} f(z) = \sup_{z \in B_\delta \cap \Omega_u} f(z) \leq C\delta|x_1 - x_2|^\alpha,$$

leading directly to (5.10). Thus $\partial\Omega_u$ is locally $\alpha$- Hölder continuous. □

### 5.4 From $C^{1,\alpha}$ to smoothness

We now proceed with improving the $C^{1,\alpha}$ regularity from the previous subsection to smoothness ($C^\infty$) of the regular part of the free boundary. Compared to the obstacle problem, the boundary Harnack inequality can not be used, the gradient $\nabla u$ does not vanish on the free boundary as $|\nabla u|\big|_{\partial\Omega_u} = \sqrt{\Lambda}$. Without loss of generality, we assume that for the rest of the chapter $\Lambda = 1$.

In [KN77], a change of variables, the *partial hodograph transform*, is used to show something stronger, namely analyticity of $Reg(\partial\Omega_u)$. We will not proceed in this direction and instead have the following approach:

1. Using the fact that $Reg(\partial\Omega_u)$ is locally $C^{1,\alpha}$ for any $\alpha > 1/2$, show that locally $u \in W^{2,p}(\Omega_u)$ for any $1 \leq p < \infty$.

2. Show that locally around $x_0 \in \partial\Omega_u$, the normal to $\partial\Omega_u$ satisfies a certain divergence form equation with Neumann boundary condition weakly.
3. Show that this implies local $C^{2,\alpha}$ regularity of $Reg(\partial\Omega_u)$.
4. Bootstrap the argument to get local $C^\infty$ regularity of $Reg(\partial\Omega_u)$.

We will not be able to prove the third point rigorously, but believe strongly that the result needed holds nevertheless. We refer also to [YZ23] for a similar method, also without the use of the partial hodograph transform. Let us start with the first point.

**Proposition 5.16.** *Let $u$ be a minimizer to* (2.1), *in particular $u$ is a viscosity solution to* (4.8) *in $D$. For a regular free boundary point $x_0 \in Reg(\partial\Omega_u)$ and for any $p \in [1,\infty)$, there exists an $r > 0$ such that $u \in W^{2,p}(B_r(x_0) \cap \Omega_u)$.*

*Proof.* Without loss of generality we assume that $x_0 = 0$. Since $Reg(\partial\Omega_u)$ is locally a $C^{1,\alpha}$ manifold, take $r$ small enough such that $B_r$ is contained in one connected component, i.e. $B_r \cap \partial\Omega_u$ contains only regular points.

**Step 1:** We claim that for any multi-index $\beta$ with $|\beta| = 2$ and any $x \in B_r \cap \Omega_u$, we have $|D^\beta u(x)| \le \operatorname{dist}(x, \partial\Omega)^{\alpha-1}$. First, let $x_1 \in \partial\Omega_u$ such that $|x - x_1| = \operatorname{dist}(x, \partial\Omega_u) =: d(x)$, note that $[x, x_1] \in \bar{\Omega}_u$. Set now

$$v(x) = u(x) - \nabla u(x_1) \cdot (x - x_1),$$

clearly $v$ is harmonic wherever $u$ is and $D^\beta u(x) = D^\beta v(x)$. We note also that $v(x_1) = u(x_1) = 0$, $\nabla v(x_1) = 0$ and $[u]_{\alpha;B_r\cap\Omega_u} = [v]_{\alpha;B_r\cap\Omega_u}$, for the seminorm

$$[u]_{\alpha;B_r\cap\Omega_u} := \max_{|\gamma|=1} \sup_{x,y\in B_r\cap\Omega_u} \frac{|D^\gamma u(x) - D^\gamma u(y)|}{|x-y|^\alpha}.$$

In particular, by Theorem A.20 and Theorem A.4,

$$\begin{aligned}
|D^\beta u(x)| = |D^\beta v(x)| &\le \sup_{B_{d(x)/4}(x)} |D^\beta v(y)| \le \frac{C_d}{d(x)^2} \sup_{B_{d(x)/2}(x)} |v(y)| \\
&\le \frac{C_d}{d(x)^2} \sup_{B_{d(x)/2}(x)} |v(x_1) + \nabla v(x_1)\cdot(y - x_1) + R(y)| = \frac{C_d}{d(x)^2} \sup_{B_{d(x)/2}(x)} |R(y)| \\
&\le \frac{C'_d}{d(x)^2} [u]_{\alpha;B_r\cap\Omega_u} \sup_{B_{d(x)/2}(x)} |y - x_1|^{1+\alpha} \le \frac{C'_d}{d(x)^2} \|u\|_{C^{1,\alpha}(B_r\cap\Omega_u)} d(x)^{1+\alpha} \\
&= C'_d \|u\|_{C^{1,\alpha}(B_r\cap\Omega_u)} d(x)^{\alpha-1}.
\end{aligned}$$

**Step 2:** We now integrate the distance to the free boundary in the previous inequality with the co-area formula. So far,

$$\|D^\beta u\|^p_{L^p(B_r\cap\Omega_u)} = \int_{B_r\cap\Omega_u} |D^\beta u|^p \le C_d^p \|u\|^p_{C^{1,\alpha}(B_r\cap\Omega_u)} \int_{B_r\cap\Omega_u} d(x)^{p(\alpha-1)} dx.$$

In (A.1), take $f(x) = d(x)$ and $g(x) = d(x)^{p(\alpha-1)}$, note that $|\nabla f(x)| = 1$. Hence, as $f^{-1}(t) = \{x \in B_r \cap \Omega_u : d(x) = t\}$, we get

$$\begin{aligned}
\int_{B_r\cap\Omega_u} d(x)^{p(\alpha-1)} dx &= \int_{B_r\cap\Omega_u} d(x)^{p(\alpha-1)} |\nabla f| dx \\
&= \int_{\mathbb{R}} \int_{\{x\in B_r\cap\Omega_u : d(x)=t\}} d(x)^{p(\alpha-1)} d\,\mathcal{H}^{d-1}(x) dt \\
&= \int_{\mathbb{R}} t^{p(\alpha-1)}\, \mathcal{H}^{d-1}(\{x \in B_r \cap \Omega_u : d(x) = t\}) dt \\
&= \int_0^r t^{p(\alpha-1)}\, \mathcal{H}^{d-1}(\{x \in B_r \cap \Omega_u : d(x) = t\}) dt \\
&\le C \int_0^r t^{p(\alpha-1)} dt.
\end{aligned}$$

We used the locally finite perimeter (Proposition 4.2) of $\partial\Omega_u$ in the last step. Now for a fixed $p$, take $\alpha$ such that $p(1-\alpha)$ sufficiently small, i.e. $p(1-\alpha)<1$, then

$$\int_{B_r\cap\Omega_u} d(x)^{p(\alpha-1)}dx \le C\int_0^r t^{p(\alpha-1)}dt = C\frac{1}{1-(1-\alpha)p}r^{1-p(1-\alpha)} < \infty.$$

To conclude $\|D^\beta u\|_{L^p(B_r\cap\Omega_u)} \le C_d\|u\|_{C^{1,\alpha}(B_r\cap\Omega_u)}$ for any $\beta$ with $|\beta|=2$ and hence $u\in W^{2,p}(B_r\cap\Omega_u)$. □

We can even control the size of the local $W^{2,p}$ estimate.

**Remark 5.17.** Let $\varepsilon<1/2$ be fixed and take $\alpha$ such that $p(1-\alpha)<\varepsilon$. For $r=r(\alpha)$ sufficiently small,

$$\int_{B_r\cap\Omega_u} d(x)^{p(\alpha-1)}dx \le C\int_0^r t^{p(\alpha-1)}dt < 2Cr^{1-\varepsilon} \implies \|D^\beta u\|_{L^p(B_r\cap\Omega_u)} \le C_d\|u\|_{C^{1,\alpha}(B_r\cap\Omega_u)}r,$$

which, using Poincaré's inequality, gives the linear scaling estimate $\|u\|_{W^{2,p}(B_r\cap\Omega_u)}\le Cr$.

We proceed now with the second step. Without loss of generality assume that $x_0$ is the origin and up to a rotation $\nabla u(x_0)=e_d$. Thus by local $C^{1,\alpha}$ continuity of $\nabla u$, in a small enough ball $B_r\cap\bar\Omega_u$, $\frac{1}{2}\le u_d\le\frac{3}{2}$. The inward unit normal to the free boundary $\frac{\nabla u}{|\nabla u|}$ has the components

$$\left(\frac{\nabla u}{|\nabla u|}\right)_i = \frac{u_i}{\sqrt{u_1^2+\cdots+u_d^2}} = \frac{u_i/u_d}{\sqrt{1+\frac{u_1^2}{u_d^2}+\cdots+\frac{u_{d-1}^2}{u_d^2}}},$$

hence in order to show smoothness of the free boundary, it is enough to show that locally the function $w=\frac{u_i}{u_d}$ for $i=1,\dots,d-1$ is smooth. We already know that $\partial\Omega_u$ is $C^{1,\alpha}$ in $B_r\cap\bar\Omega_u$ thus $w\in C^\alpha(B_r\cap\bar\Omega_u)$. We aim to show that $w\in C^{1,\alpha}(B_r\cap\bar\Omega_u)$.

**Remark 5.18.** The level sets $B_r\cap\{u=t\}$ with $t>0$ are smooth $d-1$ dimensional manifolds. This follows from the fact that $|\nabla u|=\sqrt{\Lambda}\neq 0$, thus $|\nabla u|\neq 0$ in $B_r\cap\Omega_u$, the smoothness of $u$ inside $\Omega_u$ and the implicit function theorem.

First, we prove a general lemma, that for a sequence of $\alpha$-Hölder continuous functions converging to a constant, the derivatives converge somehow weakly to zero. Note that this is a nontrivial result, since the functions being only $\alpha$-Hölder does not give direct control of the derivatives and we have to perform a "fractional integration by parts".

**Lemma 5.19.** *Let $\Omega\subset\mathbb{R}^d$ be a bounded domain. Suppose that for $g_t\in W^{1,2}(\Omega)$ we have $\|g_t\|_{C^\alpha(\Omega)}\to 0$ and $v\in C_c^\alpha(\Omega,\mathbb{R}^d)$, where $\frac{3}{4}\le\alpha<1$. Then*

$$\int_\Omega \nabla g_t\cdot v\to 0.$$

We provide two proofs, dealing differently with the fractional Sobolev spaces (see Appendix A.2 for the relevant definitions of the fractional Sobolev norms used).

*Proof.* (Using the fractional Laplacian) Suppose $\operatorname{spt} g_t\cup\operatorname{spt} v\subset K$, for $K$ compact (extend outside of $\Omega$ if necessary). By Plancherel, Cauchy-Schwarz and [DPV12, Proposition 3.4],

$$\int_\Omega \nabla g_t\cdot v \le \int_{\mathbb{R}^d}|\xi||\hat g_t||\hat v| \le \left(\int_{\mathbb{R}^d}|\xi||\hat g_t|^2\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^d}|\xi||\hat v|^2\right)^{\frac{1}{2}} = C_d\|g_t\|_{\dot H^{1/2}(\mathbb{R}^d)}\|v\|_{\dot H^{1/2}(\mathbb{R}^d)}.$$

Moreover, by [DPV12, Proposition 3.6] ($s=1/2$),

$$\int_\Omega \nabla g_t\cdot v \le C_d\|g_t\|_{\dot H^{1/2}(\mathbb{R}^d)}\|v\|_{\dot H^{1/2}(\mathbb{R}^d)} = 4\|(-\Delta)^{1/4}g_t\|_{L^2(\mathbb{R}^d)}\|(-\Delta)^{1/4}v\|_{L^2(\mathbb{R}^d)}. \tag{5.11}$$

We note that both $g_t$ and $v$ are compactly supported, from [FR24, Lemma 1.10.2],

$$\|(-\Delta)^{1/4} g_t\|_{L^\infty(B_{\frac{1}{2}})} \le \|(-\Delta)^{1/4} g_t\|_{C^\beta(B_{\frac{1}{2}})} \le C(d,\beta)\left(\|g_t\|_{C^{1/2+\beta}(B_1)} + \int_{\mathbb{R}^d} \frac{|g_t(y)|}{1+|y|^{d+1/2}} dy\right)$$
$$\|(-\Delta)^{1/4} v\|_{L^\infty(B_{\frac{1}{2}})} \le \|(-\Delta)^{1/4} v\|_{C^\beta(B_{\frac{1}{2}})} \le C(d,\beta)\left(\|v\|_{C^{1/2+\beta}(B_1)} + \int_{\mathbb{R}^d} \frac{|v(y)|}{1+|y|^{d+1/2}} dy\right).$$

Let $\{x_k\}_{k=1}^\infty$ be a countable collection of points in $\mathbb{R}^d$ such that $\bigcup B_{1/2}(x_k) \supset \mathbb{R}^d$ and $\{x_k\}_{k=1}^\infty \cap K \le N_K < \infty$ for the compact set $K$ (e.g. all points with coordinates in the lattice $(\frac{1}{2}\mathbb{Z})^d$). We estimate, taking $\beta = 1/4$ and the translation invariance [FR24, Lemma 1.10.3],

$$\begin{aligned}
\|(-\Delta)^{1/4} g_t\|_{L^2(\mathbb{R}^d)} &\le \sum_{k=1}^\infty \|(-\Delta)^{1/4} g_t\|_{L^2(B_{1/2}(x_k))} \\
&\le C_d \sum_{k=1}^\infty \|(-\Delta)^{1/4} g_t\|^2_{L^\infty(B_{1/2}(x_k))} \\
&\le C_d \sum_{k=1}^\infty \left(\|g_t\|_{C^{3/4}(B_1(x_k))} + \int_{\mathbb{R}^d} \frac{|g_t(y+x_k)|}{1+|y|^{d+1/2}} dy\right).
\end{aligned}$$

Thanks to the compact support of $g_t$, $\|g_t\|_{C^{3/4}(B_1(x_k))}$ vanishes for all but finitely many $k$, thus

$$\sum_{k=1}^\infty \|g_t\|_{C^{3/4}(B_1(x_k))} \le N_K |B_1| \|g_t\|_{C^{3/4}(\Omega)}.$$

For the integral term, we note that its integrand is only supported in $K - x_k$, giving

$$\begin{aligned}
\sum_{k=1}^\infty \int_{\mathbb{R}^d} \frac{|g_t(y+x_k)|}{1+|y|^{d+1/2}} dy &= \sum_{k=1}^\infty \int_{K-x_k} \frac{|g_t(y+x_k)|}{1+|y|^{d+1/2}} dy \\
&= \|g_t\|_{L^\infty(\Omega)} \sum_{k=1}^\infty \int_{K-x_k} \frac{1}{1+|y|^{d+1/2}} dy \\
&\le \|g_t\|_{L^\infty(\Omega)} N_K \int_{\mathbb{R}^d} \frac{1}{1+|y|^{d+1/2}} dy \le 2N_K \|g_t\|_{L^\infty(\Omega)},
\end{aligned}$$

using that at most $N_K$ of the sets $K - x_k$ overlap and

$$\int_{\mathbb{R}^d} \frac{1}{1+|y|^{d+1/2}} dy = \int_0^\infty \frac{1}{1+r^{d+1/2}} r^{d-1} \le \int_0^1 \frac{1}{2} + \int_1^\infty \frac{r^{d-1}}{2r^{d+1/2}} = \frac{1}{2} + \int_1^\infty r^{-3/2} \le 2.$$

To summarize, $\|(-\Delta)^{1/4} g_t\|_{L^2(\mathbb{R}^d)} \le C\|g_t\|_{C^{3/4}(\Omega)}$. The same estimate holds for $v$, thus continuing from (5.11),

$$\int_\Omega \nabla g_t \cdot v \le 4\|(-\Delta)^{1/4} g_t\|_{L^2(\mathbb{R}^d)} \|(-\Delta)^{1/4} v\|_{L^2(\mathbb{R}^d)} \le C^2 \|g_t\|_{C^{3/4}(\Omega)} \|v\|_{C^{3/4}(\Omega)} \to 0,$$

whenever $t \to 0$ by the uniform $C^\alpha(\Omega)$ convergence of $g_t$ to 0, where $\alpha \in (0,1)$ arbitrary, finishing the proof. $\square$

For the second proof, we use embedding properties of fractional Sobolev spaces from [DPV12]. There are many more general embedding results involving generalizations of Sobolev spaces, known as Besov spaces and Lorentz spaces. For an in detail treatment, we refer to [Saw18].

*Proof.* (Using fractional Sobolev spaces) Suppose $\operatorname{spt} g_t \cup \operatorname{spt} v \subset K$, for $K$ compact ($g_t$ outside of $\operatorname{spt} v$ does not contribute to the integral). We set the extensions $\tilde{g}_t, \tilde{v} : \mathbb{R}^d \to \mathbb{R}$ to be

$$\tilde{g}_t(x) = \begin{cases} f(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^d \backslash \Omega, \end{cases} \qquad \tilde{v}(x) = \begin{cases} v(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^d \backslash \Omega. \end{cases}$$

Then, as before, by Plancherel and Cauchy-Schwarz,

$$\begin{aligned} \int_\Omega \nabla g_t \cdot v = \int_\Omega \nabla \tilde{g}_t \cdot \tilde{v} &\le \int_{\mathbb{R}^d} |\xi||\hat{\tilde{f}}_t||\hat{\tilde{v}}| \le \left(\int_{\mathbb{R}^d} |\xi||\hat{\tilde{f}}_t|^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} |\xi||\hat{\tilde{v}}|^2\right)^{\frac{1}{2}} \\ &= C_d \|\tilde{g}_t\|_{\dot{H}^{1/2}(\mathbb{R}^d)} \|\tilde{v}\|_{\dot{H}^{1/2}(\mathbb{R}^d)}. \end{aligned}$$

Now using first [DPV12, Proposition 3.4] (equivalence between the fractional Sobolev norm $\dot{H}^{1/2}$ defined through the Fourier transform and the one defined through the Gagliardo seminorms $W^{1/2,2}$) and then [DPV12, Lemma 5.1] (upper bounding the fractional Sobolev norm of the extension) gives

$$\begin{aligned} \|\tilde{g}_t\|_{\dot{H}^{1/2}(\mathbb{R}^d)} &\simeq [\tilde{g}_t]_{W^{1/2,2}(\mathbb{R}^d)} \le \|\tilde{g}_t\|_{W^{1/2,2}(\mathbb{R}^d)} \le C_d \|g_t\|_{W^{1/2,2}(\Omega)}, \\ \|\tilde{v}\|_{\dot{H}^{1/2}(\mathbb{R}^d)} &\simeq [\tilde{v}]_{W^{1/2,2}(\mathbb{R}^d)} \le \|\tilde{v}\|_{W^{1/2,2}(\mathbb{R}^d)} \le C_d \|v\|_{W^{1/2,2}(\Omega)}. \end{aligned}$$

Finally, take $\varepsilon < \alpha - \frac{1}{2}$, then for a $C^{1/2+\varepsilon}(\Omega)$ function $u$,

$$\begin{aligned} \|u\|_{W^{1/2,2}(\Omega)} &= \left(\int_\Omega |u|^2 + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{d+1}} dy dx\right)^{1/2} \\ &\le |\Omega| \|u\|_{L^\infty(\Omega)} + \left(\int_\Omega \int_\Omega \left(\frac{|u(x) - u(y)|}{|x-y|^{1/2+\varepsilon}}\right)^2 \frac{1}{|x-y|^{d-2\varepsilon}} dy dx\right)^{1/2} \\ &\le |\Omega| \|u\|_{L^\infty(\Omega)} + [u]_{C^{1/2+\varepsilon}(\Omega)} \left(\int_\Omega \int_\Omega \frac{1}{|x-y|^{d-2\varepsilon}} dy dx\right)^{1/2} \\ &\le C_{\Omega,d} \|u\|_{C^{1/2+\varepsilon}(\Omega)}, \end{aligned}$$

using that by a change of variable ($z = y - x$) and boundedness of $\Omega$

$$\begin{aligned} \int_\Omega \int_\Omega \frac{1}{|x-y|^{d-2\varepsilon}} dy dx = \int_\Omega \int_{\Omega - x} \frac{1}{|z|^{d-2\varepsilon}} dz dx &\le \int_\Omega \int_{B_R} \frac{1}{|z|^{d-2\varepsilon}} dz dx \\ &= \int_\Omega \int_0^R \frac{1}{r^{d-2\varepsilon}} r^{d-1} dr dx = |\Omega| R^{2\varepsilon} < \infty. \end{aligned}$$

Thus

$$\int_\Omega \nabla g_t \cdot v \le C_{\Omega,d} \|g_t\|_{C^{1/2+\varepsilon}(\Omega)} \|v\|_{C^{1/2+\varepsilon}(\Omega)} \to 0,$$

since $\|g_t\|_{C^{1/2+\varepsilon}(\Omega)} \le \|g_t\|_{C^\alpha(\Omega)} \to 0$ and $\|v\|_{C^{1/2+\varepsilon}(\Omega)} \le \|v\|_{C^\alpha(\Omega)} < \infty$ by elementary Hölder space embeddings, Theorem A.3. □

We are now able to complete the second step.

**Proposition 5.20.** *Let $u \in W^{1,2}(D)$ be a minimizer to (2.1), in particular a viscosity solution to (4.8) in $D$, $0 \in \partial\Omega_u$, $\nabla u(0) = e_d$ and $w = \frac{u_i}{u_d}$ for some $1 \le i \le d-1$. Then there exists $B_r$ such that for $\Omega = B_r \cap \Omega_u$*

$$\begin{cases} \operatorname{div}(u_d^2 \nabla w) &= 0 \qquad \text{in } \Omega, \\ \partial_\nu w &= 0 \qquad \text{on } \partial\Omega_u \cap B_r, \end{cases}$$

*is satisfied in the weak sense, i.e.*

$$\int_\Omega u_d^2 \nabla w \cdot \nabla \varphi = \int_{B_r \cap \partial\Omega_u} u_d^2 \nabla w \cdot \nu \varphi = 0 \qquad \forall \varphi \in C_c^\infty(B_r \cap \bar{\Omega}_u). \tag{5.12}$$

*Proof.* By Proposition 5.16, $u \in W^{2,2}(B_r)$ and thereby $w \in W^{1,2}(B_r)$ for some $r$ sufficiently small, the integral $\int_\Omega u_d^2 \nabla w \cdot \nabla \varphi$ is finite. Note also that we can assume $\frac{9}{10} \leq u_d \leq \frac{10}{9}$ in $\Omega$.

Fix $\varphi \in C_c^\infty(B_r \cap \bar{\Omega}_u)$. We show that for $\Omega_t = B_r \cap \{u \geq t\}$,

$$\int_{\Omega_t} u_d^2 \nabla w \cdot \nabla \varphi \to 0, \tag{5.13}$$

which combined with $u \in W^{2,2}(\Omega)$ gives the result, since the remaining integral over $\Omega \backslash \Omega_t$ is eventually bounded by $C|\Omega \backslash \Omega_t|$.

For (5.13), from harmonicity inside $\Omega_u$,

$$0 = \Delta u_i = \Delta(u_d w) = \Delta u_d w + \frac{1}{u_d} \operatorname{div}(u_d^2 \nabla w) \implies \operatorname{div}(u_d^2 \nabla w) = 0,$$

thus by the bounds on $u_d$

$$\left| \int_{\Omega_t} u_d^2 \nabla w \cdot \nabla \varphi \right| = \left| \int_{\partial\Omega_t \cap \operatorname{spt}\varphi} u_d^2 \varphi \nabla w \cdot \nu - \int_{\Omega_t} \varphi \operatorname{div}(u_d^2 \nabla w) \right| \leq C \left| \int_{\partial\Omega_t \cap \operatorname{spt}\varphi} \varphi \nabla w \cdot \nabla u \right|,$$

where $\nu = \frac{\nabla u}{|\nabla u|} \simeq \nabla u$, since $|\nabla u| \approx 1$ close to the free boundary $\partial\Omega$. Also since

$$\nabla w = \frac{1}{u_d^2} (u_d \nabla u_i - u_i \nabla u_d) = \frac{1}{u_d^2} D^2 u \cdot (0, \dots, 0, \overbrace{u_d}^{i\text{'th}}, 0, \dots, -u_i) \quad \text{and} \quad \nabla |\nabla u|^2 = Du^2 \cdot \nabla u,$$

$$\begin{aligned}
\int_{\partial\Omega_t \cap \operatorname{spt}\varphi} \varphi \nabla w \cdot \nabla u &= \int_{\partial\Omega_t \cap \operatorname{spt}\varphi} \frac{\varphi}{u_d^2} \nabla u^T \cdot D^2 u \cdot (0, \dots, 0, u_d, 0, \dots, -u_i) \\
&= \frac{1}{2} \int_{\partial\Omega_t \cap \operatorname{spt}\varphi} \varphi \nabla |\nabla u|^2 \cdot \tau,
\end{aligned}$$

where $\tau := \frac{1}{u_d^2}(0, \dots, 0, u_d, 0, \dots, -u_i)$ is a tangent vector to $\partial\Omega_t$. Since $\partial\Omega_t$ is a smooth $d-1$ dimensional manifold (for simplicity we assume here that the domain is contained in one chart) for each $t > 0$ we have a smooth chart $\phi_t : \partial\Omega_t \to \mathbb{R}^{d-1}$ and $D\phi_t : \mathbb{R}^d \to \mathbb{R}^{d-1}$. Moreover, we have a smooth parametrization $\phi_t^{-1} : \phi_t(\partial\Omega_t) \subset \mathbb{R}^{d-1} \to \partial\Omega_t$, where we note that $\operatorname{spt}\varphi \subset \bigcup_{t>0} \phi_t(\partial\Omega_t)$. Let

$$g_t : \mathbb{R}^{d-1} \to \mathbb{R} \quad \text{with} \qquad g_t(y) = |\nabla u|^2(\phi_t^{-1}(y)),$$

then

$$|\nabla u(x)|^2 = g_t \circ \phi_t(x) \quad \text{and} \quad \nabla |\nabla u(x)|^2 = D\phi_t(x) \nabla g_t(\phi_t(x)).$$

Thus, by $\langle Au, v \rangle = \langle u, A^T v \rangle$ and the change of variable $x = \phi^{-1}(y)$,

$$\begin{aligned}
\int_{\partial\Omega_t} \varphi(x) \nabla |\nabla u|^2(x) \cdot \tau(x) &= \int_{\partial\Omega_t} \varphi(x) \langle D\phi_t(x) \nabla g_t(\phi_t(x)), \tau(x) \rangle_{\mathbb{R}^d} \, dx \\
&= \int_{\partial\Omega_t} \varphi(x) \left\langle \nabla g_t(\phi_t(x)), D\phi_t(x)^T \tau(x) \right\rangle_{\mathbb{R}^{d-1}} dx \\
&= \int_{\phi_t(\partial\Omega_t)} \varphi(\phi_t^{-1}(y)) \left\langle \nabla g_t(y), D\phi_t(\phi_t^{-1}(y))^T \tau(\phi_t^{-1}(y)) \right\rangle_{\mathbb{R}^{d-1}} dy \\
&= \int_{\mathbb{R}^{d-1}} \tilde{\varphi}_t(y) \langle \nabla g_t(y), \tilde{\tau}_t(y) \rangle_{\mathbb{R}^{d-1}} \, dy,
\end{aligned}$$

where we set $\tilde{\varphi}_t(y) := \varphi(\phi_t^{-1}(y))$ and $\tilde{\tau}_t(y) := D\phi_t(\phi_t^{-1}(y))^T \tau(\phi_t^{-1}(y))$ in $\phi_t(\Omega_t)$ and 0 outside. To simplify notation, we define

$$I_t := \int_{\mathbb{R}^{d-1}} \tilde{\varphi}_t(y) \langle \nabla g_t(y), \tilde{\tau}_t(y) \rangle_{\mathbb{R}^{d-1}} \, dy.$$

We note that for $t \to 0$, we have uniformly in $C^\alpha(\mathbb{R}^{d-1})$ for any fixed $\alpha < 1$ (so in particular for $\alpha = 3/4$),

$$\tilde{\varphi}_t \tilde{\tau}_t \to \tilde{\varphi}_0 \tilde{\tau}_0 \quad \text{and} \quad g_t - 1 \to 0, \tag{5.14}$$

where $\tilde{\varphi}_0 = \varphi \circ \phi_0^{-1}$ $\tilde{\tau}_0 = \tau \circ \phi_0^{-1}$, a tangent vector to the free boundary. This follows from the $C^{1,\alpha}$ convergence of $\phi_t \to \phi_0$, where $\phi_0$ is a $C^{1,\alpha}$ chart of the free boundary, and $u \in C^{1,\alpha}(\bar{\Omega})$.

Let the Banach spaces $X = C_c^\alpha(\mathbb{R}^{d-1})$ and $Y = \mathbb{R}$ and the family of linear bounded operators $\{T_t\}_{1 \geq t > 0} : X \to Y$ with

$$T_t(v) := \int_{\mathbb{R}^{d-1}} \nabla g_t \cdot v.$$

Note that for $t$ fixed, $|T_t(v)| \leq \|v\|_{C^\alpha} \int_{\mathbb{R}^{d-1}} |\nabla g_t| \leq C_t \|v\|_{C^\alpha}$. Applying now Lemma 5.19 with $g_t$ gives

$$\forall v \in X: \qquad T_t(v) \to 0. \tag{5.15}$$

Thus for a fixed $v \in X$, $\sup_{t \in (0,1]} |T_t(v)| < \infty$. By Theorem A.1,

$$\sup_{t \in (0,1]} \|T_t\|_{X \to Y} \leq C < \infty.$$

Finally,

$$\begin{aligned}
I_t &\leq \left| \int_{\mathbb{R}^{d-1}} \langle \nabla g_t(y), (\tilde{\varphi}_t(y)\tilde{\tau}_t(y) - \tilde{\varphi}_0(y)\tilde{\tau}_0(y)) \rangle \, dy \right| + \left| \int_{\mathbb{R}^{d-1}} \langle \nabla g_t(y), \tilde{\varphi}_0(y)\tilde{\tau}_0(y) \rangle \, dy \right| \\
&\leq |T_t(\tilde{\varphi}_t\tilde{\tau}_t - \tilde{\varphi}_0\tilde{\tau}_0)| + |T_t(\tilde{\varphi}_0\tilde{\tau}_0)| \\
&\leq C \|\tilde{\varphi}_t\tilde{\tau}_t - \tilde{\varphi}_0\tilde{\tau}_0\|_{C^\alpha(\mathbb{R}^{d-1})} + |T_t(\tilde{\varphi}_0\tilde{\tau}_0)| \to 0,
\end{aligned}$$

by (5.14) and (5.15). We conclude that for $t$ sufficiently small,

$$\begin{aligned}
\int_\Omega u_d^2 \nabla w \cdot \nabla \varphi &= \int_{\Omega_t} u_d^2 \nabla w \cdot \nabla \varphi + \int_{\Omega \setminus \Omega_t} u_d^2 \nabla w \cdot \nabla \varphi \\
&\leq \int_{\Omega_t} u_d^2 \nabla w \cdot \nabla \varphi + \left(\frac{10}{9}\right)^2 \left(\int_{\Omega \setminus \Omega_t} |\nabla w|^2\right)^{1/2} \left(\int_{\Omega \setminus \Omega_t} \nabla \varphi\right)^{1/2} \\
&\leq \int_{\Omega_t} u_d^2 \nabla w \cdot \nabla \varphi + C_{u,\varphi} |\Omega \setminus \Omega_t| \to 0 \quad \text{as } t \to 0.
\end{aligned}$$

Hence (5.12) is satisfied as $\varphi$ was arbitrary. $\square$

We are now in the position to apply the following theorem to $w = \frac{u_i}{u_d}$, to arrive at the local $C^{k,\alpha}$ for any $k \in \mathbb{N}$ (i.e. $C^\infty$) regularity of

$$\nu = \frac{\nabla u}{|\nabla u|} = \frac{1}{\sqrt{1 + \frac{u_1^2}{u_d^2} + \cdots + \frac{u_{d-1}^2}{u_d^2}}} \left(\frac{u_1}{u_d}, \ldots, \frac{u_{d-1}}{u_d}, 1\right).$$

**Theorem 5.21.** (Blackbox) *Let $\Omega \subset \mathbb{R}^d$ be a $C^{k,\alpha}$ ($k \geq 1$) domain. Suppose $w \in W^{1,2}(\Omega) \cap C^{k-1,\alpha}(\Omega)$, $A \in C^{k-1,\alpha}(\Omega, \mathbb{R}^{d \times d})$, $\frac{1}{2}I \leq A \leq 2I$ and*

$$\begin{cases} \operatorname{div}(A \nabla w) &= 0 \qquad \text{in } \Omega, \\ \partial_\nu w &= 0 \qquad \text{in } \partial\Omega, \end{cases}$$

*holds weakly (in the case $k \geq 3$ strongly), that is*

$$\int_\Omega A \nabla w \cdot \nabla \varphi = 0 \qquad \forall \varphi \in C_c^\infty(\bar{\Omega}). \tag{5.16}$$

*Then $w \in C^{k,\alpha}(\bar{\Omega})$.*

It is currently unknown to us if Theorem 5.21 holds, but the results in [TTV22] and [YZ23] strongly indicate that it does. We now conclude the chapter with the final theorem.

**Theorem 5.22.** *Let $u$ be a minimizer to* (2.1) *and suppose that $x_0 \in \partial\Omega_u$ is a regular point in the sense of Definition 5.6. Then locally, in a neighborhood of $x_0$, the free boundary is a $C^\infty$ manifold.*

*Proof.* Let $\alpha \geq \frac{3}{4}$ be fixed. Take $r > 0$ such that $\partial\Omega_u \cap B_r(x_0)$ is a $C^{1,\alpha}$ manifold and $\frac{9}{10} \leq u_d \leq \frac{10}{9}$. Then for any $1 \leq i \leq d-1$ and $w = \frac{u_i}{u_d}$, by Proposition 5.20, the condition in Theorem 5.21 with $A = u_d^2 I$, $\Omega = \Omega_u \cap B_r(x_0)$ and $k = 1$ holds and so $w \in C^{1,\alpha}(B_r(x_0) \cap \bar{\Omega}_u)$. This in turn implies that $u \in C^{2,\alpha}(B_r(x_0) \cap \bar{\Omega}_u)$ and so $\Omega_u \cap B_r(x_0)$ is a $C^{2,\alpha}$ domain. We now compute again for $\varphi \in C_c^\infty(\bar{\Omega}_u \cap B_r(x_0))$ directly (note $\nu = -\nabla u$),

$$\begin{aligned}
\int_{\Omega_u \cap B_r(x_0)} u_d^2 \nabla w \cdot \nabla\varphi &= \int_{\partial\Omega_u \cap \operatorname{spt}\varphi} u_d^2 \varphi \nabla w \cdot \nu - \int_{\Omega_u} \varphi \operatorname{div}(u_d^2 \nabla w) \\
&= -\int_{\partial\Omega_u \cap \operatorname{spt}\varphi} u_d^2 \varphi \nabla w \cdot \nabla u \\
&= -\int_{\partial\Omega_u \cap \operatorname{spt}\varphi} u_d^2 \varphi \nabla u^T D^2 u \cdot \tau \\
&= -\frac{1}{2} \int_{\partial\Omega_u \cap \operatorname{spt}\varphi} u_d^2 \varphi \partial_\tau(|\nabla u|) = 0,
\end{aligned}$$

as $|\nabla u|$ remains constant on the free boundary $\Omega_u$. Thus for the tangent vector $\tau = \frac{1}{u_d^2}(0, \ldots, 0, u_d, 0, \ldots, -u_i)$, the tangential derivative vanishes. Applying once again Theorem 5.21, gives $w \in C^{2,\alpha}(\bar{\Omega}_u \cap B_r(x_0))$ and thereby $\Omega_u \cap B_r(x_0)$ is a $C^{3,\alpha}$ domain. Iterating this argument gives that $\Omega_u \cap B_r(x_0)$ is a $C^{k,\alpha}$ domain for any $k \in \mathbb{N}$ and hence $\partial\Omega_u$ is smooth in $B_r(x_0)$. ☐

# 6 The set of singular points

So far, the estimate on the size of the singular set only gives $\mathcal{H}^{d-1}(Sing(\partial\Omega_u)) = 0$, but using finer analytical techniques it is possible to derive much stronger estimates. The exact size (or more precisely its dimension) of the singular set is topic of current research, here we will only prove that its dimension is less than $d-3$. Morally speaking, the higher the dimension, the more chance for singular points to appear.

To say something about the free boundary of general minimizers $u$, naturally we need to look at blow-ups of $u$. The main result first appeared in [Wei99], and is powerful in the sense that knowing a certain *critical dimension* $d^*$ is sufficient to characterize the size of the singular set for any minimizer $u$ for $F_\Lambda$ on $D$. We begin by showing that all blow-ups are one-homogeneous, they are already global minimizers (recall a global minimizer is a local minimizer on $\mathbb{R}^d$) by Proposition 5.5. Then, using basic properties of Hausdorff measures and Federer's dimension reduction argument we derive a relation between the singular set of the free boundary of $u$ and the critical dimension $d^*$.

**Definition 6.1.** A function $u : \mathbb{R}^d \to \mathbb{R}$ is 1-homogeneous if

$$u(tx) = tu(x) \qquad \forall x \in \mathbb{R}^d, \forall t > 0.$$

**Lemma 6.2.** *For a homogeneous local minimizer of $F_\Lambda$ in $\mathbb{R}^d$ the regular set $Reg(\partial\Omega_u)$ and the singular set $Sing(\partial\Omega_u)$ are cones (in $\mathbb{R}^d$).*

*Proof.* Since $Sing(\partial\Omega_u) = \partial\Omega_u \backslash Reg(\partial\Omega_u)$, it suffices to show that $Reg(\partial\Omega_u)$ is a cone. Let $x_0 \in Reg(\partial\Omega_u)$, then $\partial\Omega_u$ is $C^{1,\alpha}$ in $B_r(x_0)$ for some $r > 0$. By homogeneity $u(tx_0) = tu(x_0)$, so $u$ is also $C^{1,\alpha}$ in $B_{r'}(tx_0)$. ☐

**Definition 6.3.** We define the **critical dimension** $d^* \in \mathbb{N}$ as the dimension where singular points appear for the first time, i.e. $d^*$ is the smallest dimension such that there exists a function $\bar{u} : \mathbb{R}^{d^*} \to \mathbb{R}$ with

- $\bar{u}$ is a non-negative, 1-homogeneous global minimizer of $F_\Lambda$,
- the free boundary $\partial\Omega_{\bar{u}}$ is NOT a $(d-1)$ dimensional $C^{1,\alpha}$ manifold in $\mathbb{R}^{d^*}$.

We start with a first elementary estimate of the critical dimension.

**Proposition 6.4.** *Let $w : \mathbb{R}^2 \to \mathbb{R}$ be a 1-homogeneous global minimizer of $F_\Lambda$ in $\mathbb{R}^2$. Then*

$$w(x) = \sqrt{\Lambda}(x \cdot \nu)_+ \quad \textit{for some } \nu \in \mathbb{S}^1 .$$

*In particular $\partial\Omega_w = \nu^\perp \mathbb{R}$, which is smooth and thereby $d^* \geq 3$.*

*Proof.* Since $w$ is 1-homogeneous, $w(x) = w(r,\theta) = rw(\theta)|_{\partial B_1}$, set $h(\theta) := w(\theta)|_{\partial B_1}$. By continuity of $w$ and connectedness/smoothness of $\mathbb{S}^1$, $h : \partial B_1 \to \mathbb{R}$ is also continuous and so $\Omega_h = \{\theta \in \mathbb{S}^1 : h(\theta) > 0\} = \Omega_w \cap \mathbb{S}^1$ open and a countable union of disjoint arcs $\{I_i\}$. Since $w(0) = 0$ and by Proposition 4.1 the zero at the origin can not be isolated, $w(r,\theta) = 0$ for some $r > 0$, and so $\Omega_h \neq \mathbb{S}^1$. As $w$ is a minimizer, $\Delta w = 0$ in $\Omega_w$, which gives

$$0 = \Delta w(r,\theta) = \partial_{rr}(rh(\theta)) + \frac{1}{r}\partial_r w(r,\theta) + \frac{1}{r^2}\partial_{\theta\theta} rh(\theta) = \frac{1}{r}\big(h(\theta) + h''(\theta)\big) \qquad \text{in } \Omega_w$$

$$\implies \begin{cases} -h''(\theta) = h(\theta), \\ h(\theta) > 0 \quad \text{ in } I_i, \\ h(\theta) = 0 \text{ on } \partial I_i. \end{cases}$$

However this in turn implies that $I_i = (0,\pi)$ and $h(\theta) = a\sin(\theta)$ for some $a > 0$, i.e. there are at most two arcs. Again by Proposition 4.1, $|\Omega_w \cap B_1| < |B_1| = \pi$, so there is only one arc of length $\pi$. Hence $w(x) = a(x \cdot \nu)_+$ and since $\partial\Omega_w$ is smooth, by Proposition 4.4, we have $a = \sqrt{\Lambda}$. $\square$

## 6.1 Homogeneity of the blow-up

To show the homogeneity, we make use of Weiss' monotonicity formula, appearing first also in [Wei99]. In the same spirit as with the corresponding Weiss energy for the obstacle problem, we use the fact that from the monotonicity a certain energy limit exists when the radius of the rescalings converges to zero.

**Definition 6.5.** (Weiss energy) We define for $u \in H^1(B_1)$,

$$W_\Lambda(u) := \int_{B_1} |\nabla u|^2 dx - \int_{\partial B_1} u^2 d\,\mathcal{H}^{d-1} + \Lambda|\Omega_u \cap B_1|.$$

For $u \in H^1(B_r(x_0))$ and the rescalings $u_{x_0,r} = \frac{1}{r}u(rx + x_0)$ directly,

$$W_\Lambda(u_{x_0,r}) = \frac{1}{r^d}\int_{B_r(x_0)} |\nabla u|^2 dx - \frac{1}{r^{d+1}}\int_{\partial B_r(x_0)} u^2 d\,\mathcal{H}^{d-1} + \frac{\Lambda}{r^d}|\Omega_u \cap B_r(x_0)|.$$

We show the monotonicity of the Weiss energy.

**Lemma 6.6.** (Weiss' monotonicity formula) *Let $u$ be a local minimizer of $F_\Lambda$ on $D$. Let $x_0 \in D$ and $\delta = \operatorname{dist}(x_0, \partial D)$, i.e. the rescaling $u_{x_0,r}$ is a local minimizer on $B_1$ as long as $r \in (0,\delta)$. Then for almost every $r$ in $(0,\delta)$,*

$$\frac{\partial}{\partial r} W_\Lambda(u_{x_0,r}) = \frac{1}{r}\int_{\partial B_1} |x \cdot \nabla u_{x_0,r} - u_{x_0,r}|^2 d\,\mathcal{H}^{d-1} \geq 0,$$

*in particular $W_\Lambda(u_{x_0,r})$ is non-decreasing i.e.*

$$W_\Lambda(u_{x_0,s}) \leq W_\Lambda(u_{x_0,r}) \qquad \textit{for } 0 \leq s < r < \delta.$$

*Proof.* **Step 1:** Without loss of generality, assume that $x_0 = 0$. Let $z_{x_0,r}(x) := |x|u_{x_0,r}(x/|x|)$, note that $z_{x_0,r} = u_{x_0,r}$ on $\partial B_1$. We differentiate each term:

$$\begin{aligned}
\frac{\partial}{\partial r}\left(\frac{1}{r^d}\int_{B_r(x_0)} |\nabla u|^2 dx\right) &= \frac{-d}{r^{d+1}}\int_{B_r} |\nabla u|^2 dx + \frac{1}{r^d}\frac{\partial}{\partial r}\int_0^r \int_{\partial B_t} |\nabla u|^2 d\,\mathcal{H}^{d-1}\, dt \\
&= \frac{-d}{r^{d+1}}\int_{B_r} |\nabla u|^2 dx + \frac{1}{r^d}\int_{\partial B_r} |\nabla u|^2 d\,\mathcal{H}^{d-1}\, dt \\
&= \frac{-d}{r^{d+1}}\int_{B_r} |\nabla u|^2 dx + \frac{1}{r}\int_{\partial B_1} |\nabla u_r|^2 d\,\mathcal{H}^{d-1}\, dt,
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial r}\left(\frac{1}{r^{d+1}}\int_{\partial B_r} u^2 d\,\mathcal{H}^{d-1}\right) &= \frac{\partial}{\partial r}\left(\frac{1}{r^2}\cdot\frac{1}{r^{d-1}}\int_{\partial B_r} u^2 d\,\mathcal{H}^{d-1}\right) \\
&= \frac{-2}{r^3}\frac{1}{r^{d-1}}\int_{\partial B_r} u^2 d\,\mathcal{H}^{d-1} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{1}{r^{d-1}}\int_{\partial B_r} u(x)^2 d\,\mathcal{H}^{d-1}\right) \\
&= \frac{-2}{r^{d+2}}\int_{\partial B_r} u^2 d\,\mathcal{H}^{d-1} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(\int_{\partial B_1} u(ry)^2 d\,\mathcal{H}^{d-1}\right) \\
&= \frac{-2}{r^{d+2}}\int_{\partial B_r} u^2 d\,\mathcal{H}^{d-1} + \frac{1}{r^2}\int_{\partial B_1} u(ry)y\cdot\nabla u(ry) d\,\mathcal{H}^{d-1} \\
&= \frac{-2}{r^{d+2}}\int_{\partial B_r} u(x)^2 d\,\mathcal{H}^{d-1}(x) + 2r\int_{\partial B_1} u_r(x)[x\cdot\nabla u_r(x)] d\,\mathcal{H}^{d-1},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial r}\left(\frac{1}{r^d}|\Omega_u \cap B_r|\right) &= \frac{-d}{r^{d+1}}|\Omega_u \cap B_r| + \frac{1}{r^d}\,\mathcal{H}^{d-1}(\Omega_u \cap \partial B_r) \\
&= \frac{-d}{r}|\Omega_{u_r} \cap B_1| + \frac{1}{r}\,\mathcal{H}^{d-1}(\Omega_{u_r} \cap \partial B_1) \\
&= \frac{-d}{r}|\Omega_{u_r} \cap B_1| + \frac{d}{r}\,\mathcal{H}^{d-1}(\Omega_{z_r} \cap \partial B_1) = 0.
\end{aligned}$$

In particular, we can assume that $\Lambda = 0$. Now by a change of variables, write $z_{x_0,r}$ in polar coordinates as $z_{x_0,r}(\rho,\theta) = \rho z_{x_0,r}(1,\theta)$, we calculate

$$\begin{aligned}
W_0(z_{x_0,r}) &= \int_{B_1} |\nabla z_{x_0,r}|^2 dx - \int_{\partial B_1} z_{x_0,r}^2 d\,\mathcal{H}^{d-1} \\
&= \int_0^1 r^{d-1} dr \int_{\partial B_1} z_{x_0,r}(1,\theta)^2 + |\nabla_\theta z_{x_0,r}(1,\theta)|^2 d\theta - \int_{\partial B_1} z_{x_0,r}^2 d\,\mathcal{H}^{d-1} \\
&= \frac{1}{d}\int_{\partial B_1} |\nabla_\theta z_{x_0,r}| d\theta - \frac{d-1}{d}\int_{\partial B_1} z_{x_0,r}(1,\theta)^2 d\theta \\
&= \frac{1}{d}\int_{\partial B_1} \left(|\nabla u_r(x)|^2 - (x\cdot\nabla u_r(x))^2\right) d\,\mathcal{H}^{d-1} - \frac{d-1}{d}\int_{\partial B_1} u_r^2 d\,\mathcal{H}^{d-1}.
\end{aligned}$$

Thus,

$$
\begin{aligned}
&\frac{d}{r}\left(W_0(z_{x_0,r})-W_0(u_{x_0,r})\right)+\frac{1}{r}\int_{\partial B_1}|x\cdot\nabla u_{x_0,r}(x)-u_{x_0,r}(x)|^2d\,\mathcal{H}^{d-1}\\
&\quad=\frac{1}{r}\int_{\partial B_1}|\nabla u_r|^2-(x\cdot\nabla u_r)^2d\,\mathcal{H}^{d-1}-\frac{d-1}{r}\int_{\partial B_1}u_r^2d\,\mathcal{H}^{d-1}-\frac{d}{r^{d+1}}\int_{B_r}|\nabla u|^2\\
&\quad+\frac{d}{r^{d+2}}\int_{\partial B_r}u^2d\,\mathcal{H}^{d-1}+\frac{1}{r}\int_{\partial B_1}(x\cdot\nabla u_{x_0,r})^2-u_r2(x\cdot\nabla u_r)+u_r^2d\,\mathcal{H}^{d-1}\\
&\quad=\frac{-d}{r^{d+1}}\int_{B_r}|\nabla u|^2+\frac{1}{r}\int_{\partial B_1}|\nabla u_r|^2+\frac{2}{r^{d+2}}\int_{\partial B_r}u^2-\frac{2}{r}\int_{\partial B_1}u_r(x\cdot\nabla u_r)\\
&\quad=\frac{\partial}{\partial r}\left(\frac{1}{r^d}\int_{B_r}|\nabla u|^2dx-\frac{1}{r^{d+1}}\int_{\partial B_r}u^2d\,\mathcal{H}^{d-1}\right)=\frac{\partial}{\partial r}W_0(u_{x_0,r}).
\end{aligned}
$$

Note that $W_\Lambda(z_{x_0,r})-W_\Lambda(u_{x_0,r})\geq 0$, since $u_{x_0,r}$ is a minimizer of $F_\Lambda$ in $B_1$ (first and third term) and $z_{x_0,r}=u_{x_0,r}$ on $\partial B_1$ (second term). Hence

$$
\frac{\partial}{\partial r}\geq 0+\frac{1}{r}\int_{\partial B_1}|x\cdot\nabla u_{x_0,r}-u_{x_0,r}|^2d\,\mathcal{H}^{d-1}\geq 0,
$$

as was to be shown.
**Step 2:** To show that $W_\Lambda(u_{x_0,r})$ is non-decreasing, it suffices to show local absolute continuity of $\phi: r\mapsto W_\Lambda(u_{x_0,r})$ on $(0,\delta)$, since $\phi$ is differentiable a.e. with non-negative derivative. The local absolute continuity of

$$
r\mapsto\frac{1}{r^d}\int_{B_r(x_0)}|\nabla u|^2\quad\text{and}\quad r\mapsto\frac{1}{r^d}|\{u>0\}\cap B_r(x_0)|=\frac{1}{r^d}\int_{B_r(x_0)}\chi_{\{u>0\}}
$$

follows from the absolute continuity of the Lebesgue integral and that the product of absolutely continuous functions on a compact interval stays absolutely continuous. For

$$
\psi: r\mapsto\frac{1}{r^{d+1}}\int_{\partial B_r(x_0)}u^2d\,\mathcal{H}^{d-1},
$$

we estimate by the Lipschitz continuity and boundedness of $u$

$$
\begin{aligned}
\left|\int_{\partial B_r(x_0)}u^2d\,\mathcal{H}^{d-1}-\int_{\partial B_s(x_0)}u^2d\,\mathcal{H}^{d-1}\right|&=\left|\int_{\mathbb{S}^{d-1}}u(r\theta+x_0)^2-u(s\theta+x_0)^2d\,\mathcal{H}^{d-1}\right|\\
&\leq\int_{\mathbb{S}^{d-1}}2M|u(r\theta+x_0)-u(s\theta+x_0)|\leq 2ML\int_{\mathbb{S}^{d-1}}|r-s|\leq C_d|r-s|.
\end{aligned}
$$

Since Lipschitz functions are absolutely continuous, the result follows. □

Next, we use Weiss' monotonicity formula to show that blow-ups are 1-homogeneous.

**Proposition 6.7.** *Let $u$ be a local minimizer of $F_\Lambda$ on $D$ and $x_0\in\partial\Omega_u\cap D$. Then every blow-up limit $u_0:\mathbb{R}^d\to\mathbb{R}$ is a 1-homogeneous local minimizer of $F_\Lambda$ in $\mathbb{R}^d$.*

*Proof.* Write the blow-up as $u_0=\lim u_{x_0,r_n}$ for some sequence $r_n\to 0$, by Proposition 5.5. Note that then

$$
\lim_{n\to\infty}W_\Lambda(u_{r_n,x_0})=W_\Lambda(u_0)=:L<\infty.
$$

We now lift this convergence, i.e.

$$
L=\lim_{n\to\infty}W_\Lambda(u_{sr_n,x_0})\qquad\forall 0<s\leq 1.
$$

This follows essentially from the monotonicity in Lemma 6.6. For any $sr_n$, find $m$ such that $r_{m+1} \le sr_n \le r_m$, then

$$L \le W(u_{r_{m+1},x_0}) \le W(u_{sr_n,x_0}) \le W(u_{r_m,x_0}),$$

the RHS converges also to $L$. However, $u_{sr_n,x_0}$ converges also strongly in $H^1(B_1)$ to its blow-up $u_s := \frac{1}{s}u_0(s\cdot)$ and $\chi_{\Omega_{u_{sr_n,x_0}}}$ strongly in $L^2(B_1)$ to $\chi_{\Omega_{u_s}}$, hence

$$\lim_{n\to\infty} W_\Lambda(u_{sr_n,x_0}) = W_\Lambda(u_s).$$

In other words, $W_\Lambda(u_s) = L$ for any $s \in (0,1]$ is constant. Therefore, as $u_s$ is a local minimizer on $B_1$,

$$0 = \frac{\partial}{\partial s} W_\Lambda(u_s) \ge \frac{1}{s}\int_{\partial B_1} |x\cdot\nabla u_s - u_s|^2 d\mathcal{H}^{d-1},$$

which can be seen as an alternative definition of homogeneity. To show the standard definition $u_0(\lambda x) = \lambda u_0(x)$, we show that $\frac{u_0(\lambda x)}{\lambda} = u_0(x)$ on $\partial B_1$ a.e. (and everywhere by continuity). We rewrite (suppose that $\lambda > 1$, if not switch roles of 1 and $\lambda$)

$$\frac{u_0(\lambda x)}{\lambda} - u_0(x) = \int_1^\lambda \frac{\partial}{\partial s}\frac{u_0(sx)}{s} ds = \int_1^\lambda \frac{x\cdot\nabla u_0(sx)}{s} - \frac{u_0(sx)}{s^2} ds = \int_1^\lambda \frac{1}{s}(x\cdot\nabla u_s(x) - u_s(x))ds,$$

then

$$\begin{aligned}\int_{\partial B_1}\left|\frac{u_0(\lambda x)}{\lambda} - u_0(x)\right|^2 &\le \int_{\partial B_1}\left(\int_1^\lambda \frac{1}{s}(x\cdot\nabla u_s(x) - u_s(x))ds\right)^2 d\mathcal{H}^{d-1} \\ &\le \int_{\partial B_1}\left(\int_1^\lambda \frac{1}{s^2}ds\right)\left(\int_1^\lambda |x\cdot\nabla u_s(x) - u_s(x)|^2 ds\right) d\mathcal{H}^{d-1} = 0.\end{aligned}$$

Thus $u_0$ is 1-homogeneous. □

## 6.2 The dimension of the singular set

We first prove some preliminary lemmas linking global minimizers in different dimensions, before using the dimension reduction argument to arrive at the final dichotomy.

**Lemma 6.8.** *For a bounded set $D \subset \mathbb{R}^d$, let $u$ be a local minimizer of $F_\Lambda$ on $D$. For a free boundary point $x_0 \in \partial\Omega_u$, let $u_n(x) := \frac{u(r_n x + x_0)}{r_n}$ be a blow-up sequence, where $r_n$ is such that $u_n$ converges locally uniformly to a blow-up limit $u_0$.*
*Then for any compact set $K \subset D$ and open set $U \subset D$ we have:*

$$Sing(\partial\Omega_{u_0}) \cap K \subset U \implies \exists n_0 \in \mathbb{N} \text{ such that } Sing(\partial\Omega_{u_n}) \cap K \subset U \quad \forall n \ge n_0.$$

*Moreover, for every $s > 0$,*

$$\mathcal{H}^s_\infty(Sing(\partial\Omega_{u_0}) \cap K) \ge \limsup_{n\to\infty} \mathcal{H}^s_\infty(Sing(\partial\Omega_{u_n}) \cap K).$$

*Proof.* Note first that the local Lipschitz continuity (say in $B_r(x_0)$) and non-degeneracy of the minimizer $u$ give uniform bounds for $u_n$, namely for $r$ sufficiently small ($\bar{B}_r(x_0) \subset D$),

$$\|\nabla u\|_{L^\infty(B_r(x_0))} \le C \implies \|\nabla u_n\|_{L^\infty(B_1)} \le \|\nabla u_n\|_{L^\infty(B_{r/r_n})} \le C \quad \text{for } n \text{ sufficiently large},$$

$$\begin{aligned}&\|u\|_{L^\infty(B_r(x_0))} \ge cr \quad \text{for any } x_0 \in \partial\Omega_u \\ &\implies \|u_n\|_{L^\infty(B_r(x_0))} = \frac{1}{r_n}\|u\|_{L^\infty(B_{rr_n}(x_0))} \le \frac{1}{r_n} c r_n r = cr.\end{aligned}$$

Suppose by contradiction that there exists $K$ and $U$ such that

$$Sing(\partial\Omega_{u_0}) \cap K \subset U \quad \text{ but } \quad Sing(\partial\Omega_{u_{n_k}}) \cap K \ni x_{n_k} \notin U \quad \forall n_k.$$

Since $K$ is compact, up to a subsequence $x_{n_k} \to \bar{x}$, we show now that $\bar{x} \in \partial\Omega_{u_0}$. By the local uniform convergence of $u_n$ to $u_0$, and the uniform Lipschitz continuity,

$$|u_n(x_n)-u_0(x_0)| \leq |u_n(x_n)-u_n(x_0)|+|u_n(x_0)-u_0(x_0)| \leq L|x_n-x_0|+|u_n(x_0)-u_0(x_0)| \to 0,$$

thus $u_0(x_0) = 0$. But also for $r > 0$ sufficiently small,

$$\begin{aligned}\|u_0\|_{L^\infty(B_r(\bar{x}))} &= \|u_n - (u_n - u_0)\|_{L^\infty(B_r(\bar{x}))} \geq \|u_n\|_{L^\infty(B_r(\bar{x}))} - \|u_n - u_0\|_{L^\infty(B_r(\bar{x}))} \\ &\geq \|u_n\|_{L^\infty(B_{r/2}(x_n))} - \|u_n - u_0\|_{L^\infty(B_r(\bar{x}))} \geq c\frac{r}{2} - o(n),\end{aligned}$$

hence $\bar{x} \in \partial\Omega_{u_0}$. However as $\bar{x} \in K$, $\bar{x}$ has to be in $Reg(\partial\Omega_{u_0})$. That means for some $r_n \to 0$

$$\left\|\frac{1}{r_n}u_0(r_n x + \bar{x}) - \sqrt{\Lambda}(\nu \cdot x)_+\right\|_{L^\infty_x(B_1)} \to 0 \implies \frac{1}{r_n}\|u_0(x)-\sqrt{\Lambda}(\nu\cdot(x-\bar{x}))_+\|_{L^\infty_x(B_{r_n}(\bar{x}))} \to 0,$$

so for $n$ sufficiently big,

$$\|u_0(x) - \sqrt{\Lambda}(\nu \cdot (x - \bar{x}))_+\|_{L^\infty_x(B_{r_n}(\bar{x}))} \leq \frac{\bar{\varepsilon}}{3}r_n.$$

The continuity of $u_0$ and the local uniform convergence of $u_n$ to $u_0$ give

$$\|u_0(x)-\sqrt{\Lambda}(\nu\cdot(x-\bar{x}))_+\|_{L^\infty_x(B_{r_n}(\bar{x}))} \leq \frac{2\bar{\varepsilon}}{3}r_n \quad \text{and} \quad \|u_n(x)-\sqrt{\Lambda}(\nu\cdot(x-\bar{x}))_+\|_{L^\infty_x(B_{r_n}(\bar{x}))} \leq \bar{\varepsilon}r_n.$$

However, this means that the $u_n$ is $\bar{\varepsilon}$-flat in $\nu$ direction and thus by the proof of Theorem 5.15 locally $C^{1,\alpha}$ around $x_n$. This yields that $x_n \in Reg(\partial\Omega_{u_n})$, a contradiction. The upper semi-continuity follows from the definition of the Hausdorff measure. □

Secondly, we get an invariance property of an 1-homogeneous function.

**Lemma 6.9.** *Let $w : \mathbb{R}^d \to \mathbb{R}$ be a 1-homogeneous locally Lipschitz continuous function and $0 \neq x_0 \in \partial\Omega_w$. For a blow-up sequence $w_{r_n,x_0}$ converging locally uniformly to $w_0$, $w_0$ is invariant in the $x_0$ direction, i.e.*

$$w_0(x + tx_0) = w_0(x) \quad \forall x \in \mathbb{R}^d \quad \forall t \in \mathbb{R}\,.$$

*Proof.* Fix $t \in \mathbb{R}$, then by the local uniform convergence and homogeneity

$$\begin{aligned}w_0(x + tx_0) &= \lim_n w_{r_n,x_0}(x + tx_0) = \lim_n \frac{1}{r_n}w(x_0 + r_n(x + tx_0)) \\ &= \lim_n \frac{1}{r_n}w(r_n x + (1 + r_n t)x_0) \\ &= \lim_n \frac{1}{r_n}w\left((1 + r_n t)\left(x_0 + \frac{r_n}{1 + r_n t}x\right)\right) \\ &= \lim_n \frac{1 + r_n t}{r_n}w\left(x_0 + \frac{r_n}{1 + r_n t}x\right) = w_0(x).\end{aligned}$$

The last inequality follows from the local Lipschitz continuity and using $w(x_0) = 0$,

$$\begin{aligned}&\left|\frac{1 + r_n t}{r_n}w\left(x_0 + \frac{r_n}{1 + r_n t}x\right) - \frac{1}{r_n}w(x_0 + r_n x)\right| \\ &\quad\leq \frac{1}{r_n}\left|w\left(x_0 + \frac{r_n}{1 + r_n t}x\right) - w(x_0 + r_n x)\right| + t\left|w\left(x_0 + \frac{r_n}{1 + r_n t}x\right) - w(x_0)\right| \\ &\quad\leq L\frac{1}{r_n}\left|x_0 + \frac{r_n}{1 + r_n t}x - (x_0 + r_n x)\right| + Lt\left|x_0 + \frac{r_n}{1 + r_n t}x - x_0\right| \xrightarrow{r_n \to 0} 0,\end{aligned}$$

as we wanted to show. □

Next, we link global minimizers in $\mathbb{R}^d$ and $\mathbb{R}^{d-1}$.

**Lemma 6.10.** *For $u \in H^1_{loc}(\mathbb{R}^{d-1})$ let $\tilde{u} : \mathbb{R}^d \to \mathbb{R}$ be given as $\tilde{u}(x) = u(x')$ where $x = (x', x_d) \in \mathbb{R}^d$. Then $u$ is a local minimizer of $F_\Lambda$ in $\mathbb{R}^{d-1}$ if and only if $\tilde{u}$ is a local minimizer of $F_\Lambda$ in $\mathbb{R}^d$.*

*Proof.* We only provide a sketch, for the complete proof see [Vel23, Lemma 10.10]
**Step 1:** Suppose $u$ is a local minimizer, but not $\tilde{u}$. Then there exists $C_R := B'_R \times (-R, R) \subset \mathbb{R}^{d-1} \times \mathbb{R}$ and $\tilde{v} : \mathbb{R}^d \to \mathbb{R}$ with $\tilde{v}|_{\partial C_R} = \tilde{u}|_{\partial C_R}$ but $F_\Lambda(\tilde{u}, C_R) > F_\Lambda(\tilde{v}, C_R)$. Starting from $F_\Lambda(u, B'_R)$, using Fubini and the mean value theorem for integrals gives $v = \tilde{v}(\cdot, t), t \in (-R, R)$ with lower energy.
**Step 2:** Suppose $\tilde{u}$ is a local minimizer, but not $u$. Then there exists $R > 0$ and $v : \mathbb{R}^{d-1} \to \mathbb{R}^d$ with $u|_{\partial B'_R} = v|_{\partial B'_R}$ such that $F_\Lambda(u, B'_R) > F_\Lambda(v, B'_R)$. Define now $\tilde{v}(x', x_d) = v(x')\phi_t(x_d)$ with

$$\phi_t(x_d) = \begin{cases} 1 & \text{if } |x_d| < 1, \\ 0 & \text{if } |x_d| > t + 1, \\ x_d + t + 1 & \text{if } -t - 1 \le x_d \le -t, \\ x_d - t - 1 & \text{if } t \le x_d \le t + 1. \end{cases}$$

Choosing $t$ sufficiently large, $\tilde{v}$ has lower energy than $\tilde{u}$ on $B'_R \times (-t, t)$, a contradiction. □

Lastly, we show that the singular set is empty except for one point.

**Lemma 6.11.** *Let $w : \mathbb{R}^{d^*} \to \mathbb{R}$ be a 1-homogeneous local minimizer of $F_\Lambda$ then we have $Sing(\partial\Omega_w)\backslash\{0\} = \varnothing$ and so $\dim_{\mathcal{H}} Sing(\partial\Omega_w) = 0$.*

*Proof.* Suppose there exists a singular point $x_0 \neq 0$. Then, as $Reg(\Omega_w)$ and also $Sing(\Omega_w)$ are cones by Lemma 6.2, $tx_0$ for any $t \in \mathbb{R}$ is also a singular point. Up to a rotation, we can assume that $x_0 = e_d$. The blow-up $w_0$ at $x_0$ is also a homogeneous local minimizer. By Lemma 6.9, $w_0(x', t) = w_0(x', 0)$ and by Lemma 6.10 $w'_0 := w_0(\cdot, 0) : \mathbb{R}^{d^*-1} \to \mathbb{R}$ is a local minimizer of $F_\Lambda$. However $0' \in \mathbb{R}^{d^*-1}$ is a singular point for $\partial\Omega_{w'_0}$, contradicting the definition of $d^*$. If it were regular, the blow-up of $w'_0$ at $0'$, which is just $w'_0$ itself (this follows from the homogeneity), is a half space solution, i.e. $w'_0(x') = (x' \cdot \nu)_+$ for $\nu \in \mathbb{S}^{d^*-2}$. But then also $w_0(x', t) = (x' \cdot \nu)_+$ is a half space solution, contradicting the fact that $w_0$ was the blow-up of $w$ at the singular point $x_0$. □

We now use the general dimension reduction argument from Proposition A.16 in the context of the singular set.

**Proposition 6.12.** (Dimension reduction) *If $d \ge d^*$ and $w : \mathbb{R}^d \to \mathbb{R}$ a 1-homogeneous global minimizer of $F_\Lambda$ then $\mathcal{H}^{d-d^*+\varepsilon}(Sing(\partial\Omega_w)) = 0$ for any $\varepsilon \in (0, 1)$.*

*Proof.* Fix $s > 0$. If $d = d^*$, then $\mathcal{H}^s(Sing(\partial\Omega_w)) \le \mathcal{H}^s(\{0\}) = 0$ by Lemma 6.11. Suppose that the result holds for $d-1 \ge d^*$, we show it holds for $d$. If not, then $\mathcal{H}^{d-d^*+s}(Sing(\partial\Omega_w)) > 0$ and so by Lemma A.15 for some $\varepsilon > 0$, $0 \neq x_0 \in Sing(\partial\Omega_w)$ and some sequence $r_n \to 0$

$$\forall n \in \mathbb{N} : \frac{\mathcal{H}^{d-d^*+s}(Sing(\partial\Omega_w) \cap B_r(x_0))}{r^{d-d^*+s}} \ge \varepsilon \implies \forall n \in \mathbb{N} : \mathcal{H}^{d-d^*+s}(Sing(\partial\Omega_{w_{r_n}}) \cap B_1) \ge \varepsilon,$$

using the translation invariance and scaling properties of the Hausdorff measure. Up to a rotation and by homogeneity, we assume that $x_0 = e_d$. Up to a subsequence the rescalings $w_{r_n}(x) := \frac{1}{r_n} w(x_0 + r_n x)$ converge to a blow-up limit (of $w$) $w_0$ locally uniformly. Now

- by Lemma 6.9 we have $w_0(x', x_d) = w_0(x', 0)$,
- by Lemma 6.10 we have that $w'_0 := w_0(\cdot, 0)$ is a homogeneous local minimizer of $F_\Lambda$ in $\mathbb{R}^{d-1}$.

We now show that $Sing(\partial\Omega_{w_0}) \subset Sing(\partial\Omega_{w_0'})\times\mathbb{R}$ (in fact they are equal). By contraposition, it suffices to show that for any $\bar{x} = (\bar{x}', \bar{x}_d)$, where the blow-up of $w_0'$ at $\bar{x}'$ converges to a half-plane solution in $\mathbb{R}^{d-1}$, also the blow-up of $w_0$ converges to a half-plane solution in $\mathbb{R}^d$. This follows from

$$\frac{1}{r_n} w_0(\bar{x}' + r_n x', \bar{x}_d + r_n x_d) = \frac{1}{r_n} w_0(\bar{x}' + r_n x', 0) = \frac{1}{r_n} w_0'(\bar{x}' + r_n x') \to (x' \cdot \nu)_+ = ((x', x_d) \cdot (\nu, 0))_+ .$$

Conclude now by Proposition A.16 that

$$0 = \mathcal{H}^{d-1-d^*+s}(Sing(\partial\Omega_{w_0'})) \implies 0 = \mathcal{H}^{d-d^*+s}(Sing(\partial\Omega_{w_0'}) \times \mathbb{R}) \geq \mathcal{H}^{d-d^*+s}(Sing(\partial\Omega_{w_0})),$$

as we wanted to show. □

Using the homogeneity of the blow up and the lemmas above, we are finally able to show the complete dichotomy characterizing exactly the size of the singular set going back to [Wei99].

**Theorem 6.13.** *Let $u$ be a local minimizer for $F_\Lambda$ on $D \subset \mathbb{R}^d$ attaining the boundary datum $g \in H^1(\partial D)$ with the regular set $Reg(\partial\Omega_u)$ and the singular set $Sing(\partial\Omega_u)$. Then,*

- *if $d < d^*$, the singular set is empty.*
- *if $d = d^*$, the singular is a discrete, locally finite set.*
- *if $d > d^*$, the singular set is of Hausdorff dimension at most $d - d^*$, i.e.*

$$\mathcal{H}^{d-d^*+\varepsilon}(Sing(\partial\Omega_u)) = 0 \qquad \forall \varepsilon \in (0,1).$$

*Proof.* **Case $d < d^*$:** Take $x_0 \in \partial\Omega_u \cap D$ and let $u_{r_n,x_0}, r_n \to 0$ be rescalings converging locally uniformly to a homogeneous blow-up $u_0$. (see Section 5.1, Proposition 6.7) Since $d < d^*$, $Sing(\partial\Omega_{u_0}) = \varnothing$ and so 0 is a regular free boundary point of $u_0$. In particular its blow-up there is equal to $u_0$ (by homogeneity), giving $u_0(x) = \sqrt{\Lambda}(x \cdot \nu)_+$ for $\nu \in \mathbb{S}^{d-1}$. By the local uniform convergence,

$$\|u_{r_n,x_0} - \sqrt{\Lambda}\langle \cdot, \nu\rangle_+\|_{L^\infty(B_1)} \leq \varepsilon < \sqrt{\Lambda}\bar{\varepsilon}/3,$$

that is the rescaling $u_{r_n,x_0}$ is $\bar{\varepsilon}/3$ flat in $\nu$ direction. Proceeding as the proof of Theorem 5.15, this implies that $x_0$ is a regular free boundary point, hence $Sing(\partial\Omega_u)$ is empty.
**Case $d = d^*$:** Suppose $Sing(\partial\Omega_u)$ is not locally finite, then there exists a sequence with $Sing(\partial\Omega_u) \ni x_n \to x_0 \in Sing(\partial\Omega_u)$. Set $r_n \coloneqq |x_n - x_0| \to 0$, up to a subsequence $u_{r_n,x_0} \to u_0$, with $u_0$ a homogeneous local minimizer of $F_\Lambda$ in $\mathbb{R}^{d^*}$. Note that $\xi_n \coloneqq \frac{x_n - x_0}{r_n} \in \mathbb{S}^{d-1}$ are singular points for $u_{r_n,x_0}$, which up to a subsequence converge to $\xi_0 \in \mathbb{S}^{d-1}$. By the first step in the proof of Lemma 6.8, $\xi_0 \in Sing(\partial\Omega_{u_0})$, yet by Lemma 6.11, the only possible point in $Sing(\partial\Omega_{u_0})$ is 0, a contradiction.
**Case $d > d^*$:** We now use the dimension reduction argument. Suppose that for $\varepsilon > 0$,

$$\mathcal{H}^{d-d^*+\varepsilon}(Sing(\partial\Omega_u)) \geq \varepsilon > 0,$$

then by scaling

$$\mathcal{H}^{d-d^*+\varepsilon}(Sing(\partial\Omega_u) \cap B_{r_n}(x_0)) \geq \varepsilon r_n^{d-d^*+\varepsilon} \implies \mathcal{H}^{d-d^*+\varepsilon}(Sing(\partial\Omega_{u_{r_n}}) \cap B_1) \geq \varepsilon.$$

Again Lemma 6.8 gives $\mathcal{H}^{d-d^*+\varepsilon}(Sing(\partial\Omega_{u_0}) \cap B_1) \geq \varepsilon$, contradicting Proposition 6.12. □

We summarize, by Chapter 5 and 6.

**Corollary 6.14.** *Since we have already shown that $d^* \geq 3$, in dimension 2 and below, the complete free boundary (so $\partial\Omega_u = Reg(\partial\Omega_u) \cup Sing(\partial\Omega_u)$) of the minimizer $u$ is a $C^\infty$ manifold and there do not exist singular free boundary points.*

### 6.3 Sharper estimate on the critical dimension

It is possible to strengthen the result by one more dimension [CJK04] ($d^* \geq 4$), or even two [JS14], i.e. $d^* \geq 5$. This is currently the best known estimate.

**Theorem 6.15.** *Let $w : \mathbb{R}^3 \to \mathbb{R}$ be a 1-homogeneous global minimizer of $F_\Lambda$ in $\mathbb{R}^3$. Then up to a rotation*

$$w(x) = \sqrt{\Lambda}(x \cdot e_3)_+,$$

*in particular $\partial\Omega_w = e_3^\perp$ and so $d^* \geq 4$.*

We first note that by Theorem 6.13, the singular set is a discrete set of points, and by Lemma 6.2 it is a cone, implying that the only possible singular point is the origin. By Theorem 5.22, $\partial\Omega_w$ is smooth except at the origin. At each regular point $p$, up to a rotation, the free boundary is given by the graph of a smooth function $\phi$ with $\nabla w(P) = -e_d = \nu$, where $\nu$ is the outer normal. We turn now to the study of the mean curvature of $\partial\Omega_w$.

**Definition 6.16.** The **mean curvature** $H$ at $p$ is given by the sum of the principal curvatures, namely $H(p) = \sum_{i=1}^{d-1} \frac{\partial^2\phi}{\partial x_i^2}(p)$.

**Lemma 6.17.** *At any regular point $p$ (i.e. every point on $\partial\Omega_w$ except the origin), we have $u_{\nu\nu}(p) = -H(p)$.*

**Lemma 6.18.** *Let $\phi : U \subset \mathbb{R}^{d-1} \to \mathbb{R}^d$ be a local parametrization of the free boundary $\partial\Omega_w$. For $\nu(s)$, the outward pointing unit normal of $\partial\Omega_w$ at $\phi(s)$, set up the local coordinate system $x(s,t) = \phi(s) - t\nu(s)$. Then for the area element $d\sigma(s)$ on $\partial\Omega_w$, the volume element $dV$ is given as $dV = (1 + tH + O(t^2))d\sigma(s)dt$.*

The proofs are standard, but needed to prove a comparison inequality.

**Proposition 6.19.** *Let $w$ be a global, 1-homogeneous minimizer in $\mathbb{R}^3$. Then for every non-negative $f \in C_c^\infty(\mathbb{R}^3 \setminus \{0\})$,*

$$\int_{\partial\Omega_w} Hf^2 d\sigma \leq \int_{\Omega_w} |\nabla f|^2 dx.$$

The idea is to take $B_R \supset \operatorname{spt} f$ and $F_R$ the harmonic function on $\Omega = B_R \cap \Omega_w$ with boundary datum $f|_\Omega$. Let also $\Omega_\varepsilon = \{x \in \Omega : w(x) > \varepsilon F(x)\}$ and a competitor $v_\varepsilon = (w - \varepsilon F)\chi_{\Omega_\varepsilon}$. From the two lemmas and second order Taylor expansions, we get the so called second variation of $F_\Lambda$,

$$0 \leq F_\Lambda(w, B_R) - F_\Lambda(v_\varepsilon, B_R) = \varepsilon^2 \int_{\partial\Omega_w \cap B_R} F^2 H - FF_\nu d\sigma + O(\varepsilon^3),$$

which by optimality of $u$ and an integration by parts implies the desired inequality.

**Lemma 6.20.** *The mean curvature is non-negative, i.e. $H \geq 0$. Since $\partial\Omega_w$ is a cone, by homogeneity, it has one principal curvature vanishing. In dimension 3 (and below, but not above!) this implies that the other principal curvature is also non-negative. Hence $\mathbb{R}^3 \setminus \Omega_w$ is a finite union of convex cones.*

**Lemma 6.21.** *Let $C \subset \mathbb{R}^3$ be an open cone such that $\gamma = \partial C \cap \mathbb{S}^2$ is a finite intersection of smooth curves. Then for a point $p \in \partial C$,*

$$H(p) = \frac{1}{|p|}\kappa\left(\frac{p}{|p|}\right) = \frac{1}{|p|}\left|\gamma''\left(\frac{p}{|p|}\right)\right|.$$

The proof of the two lemmas above uses the maximum principle for $|\nabla w|^2$, the definition of $H$ and some geometric arguments inherent to dimension 3 (cf. Lemma 6.20). Then an application of Proposition 6.19 and Lemma 6.21 to a radial symmetric compactly supported function gives $\int_{\mathbb{S}^2 \cap \partial\Omega_w} \kappa ds \leq c|\mathbb{S}^2 \cap \Omega_w|$. By the Gauss-Bonnet theorem, $\mathbb{S}^2 \cap \Omega_w$ contains a hemisphere $A$ and the homogeneity gives $\mathbb{S}^2 \cap \Omega_w = A$. We conclude that the homogeneous function $w$ is the positive part of a linear function.

# 7 Conclusion and discussion

In this master thesis, we investigated regularity properties of the one-phase problem. First, we provided a brief overview on the classical existence and interior regularity estimates. Secondly, we derived new boundary regularity estimates under some additional regularity assumption on the ambient domain $D$. Furthermore, we classified the free boundary $\partial\Omega_u$ into singular and regular points and derived the local $C^{1,\alpha}$ regularity as in the literature. We then improved the regularity to $C^\infty$ using a new method (albeit one small gap remains), avoiding the use of the partial hodograph transform. Finally, we provided a short overview on singular pints and some estimates on their Hausdorff dimension.

The one-phase problem is just one of many so called free boundary problems. Out of them, the author studied a similar problem, the obstacle problem in [Grü22] based on [FR22, Chapter 5]. There are similarities and differences between them, for instance:

- For the obstacle problem, existence of minimizers is established similarly, but the minimizers of the obstacle problem are unique, compared to Section 2.1.
- The solution $u$ in both problems separates into a positivity set $\{u > 0\}$, where $u$ satisfies second order elliptic PDE and a contact set $\{u = 0\}$.
- In the obstacle problem $\Delta u = \chi_{\{u>0\}}$, so essentially a step function. Yet in the one-phase problem $\Delta u$ is a singular measure, concentrated on the free boundary $\partial\{u > 0\}$.
- Another difference is the fact that despite a very similar structure, for the obstacle, but not the one-phase problem, the gradient $\nabla u$ vanishes on the free boundary.
- Note also that instead of $cr \leq \sup_{x\in B_r(x_0)} u \leq Cr$ around a free boundary point $x_0$ (see Proposition 3.4 and Proposition 3.3), for the obstacle problem the natural scaling is $cr^2 \leq \sup_{x\in B_r(x_0)} u \leq Cr^2$. This also leads to its blow-ups being 2-homogeneous and essentially quadratic instead of linear.
- In the proof of the smoothness for the obstacle problem [Grü22, Chapter 4],[FR22, Chapter 5.6], the main step is to show Lipschitz regularity of the free boundary and then use a boundary Harnack inequality to arrive $C^{1,\alpha}$ regularity, a completely different approach to Section 5.4.
- The set of singular points for the obstacle problem can be of dimension $d - 1$ and there is no notion of the critical dimension, in that sense. Also singular points are of zero density, whereas in the one-phase problem all free boundary points are of positive density (Proposition 4.1).

There are many more things that can be said about the one-phase problem, especially concerning the dimension of the singular set and other regularity properties. Much of the research finding the critical dimension $d^*$ has connections to the more developed theory of minimal surfaces. Techniques for constructing conical counterexamples [DJ09] and establishing their minimality takes origin in the classical Simons cone [Sim68].

Another aspect is the non-uniqueness of minimizers. As shown in Chapter 2, there are even one-dimensional examples with several minimizers. One widely open question is to ask whether there is more structure to them, e.g. are several minimizers to the same boundary datum ordered or are they intersecting each other. It is widely acknowledged that the latter case holds, however explicit examples are missing and hard to construct. Besides very basic domains and boundary datum (see Section2.2) there are no general techniques to construct explicit minimizers rigorously.

Also we did not treat stationary solutions. Those are solutions that are not necessarily minimizers of the energy, but nevertheless have vanishing first variation and are critical points of the energy functional $F_\Lambda$ (compare to local minimizers versus critical points for a function $f : \mathbb{R}^d \to \mathbb{R}$). Much of the theory done in Chapter 6 translates to stationary solutions [Vel23, Chapter 9]. Also, additional constraints [Vel23, Chapter 11], their Lagrange multipliers and connections to (2.1) are of interest.

Lastly, we would like to mention a generalization, the *Alt-Phillips* problem, first appearing in [AP86]. The energy functional is now given by

$$F_\Lambda(u, D) = \int_{B_1} \left(|\nabla u|^2 + \Lambda \max(u, 0)^\gamma\right),$$

with $\gamma \in (0, 1)$. For $\gamma = 0$, we recover the one-phase problem, for $\gamma = 1$ the obstacle problem (see [FR22, Chapter 5]). When $\gamma \geq 1$, the energy becomes convex, leading to qualitatively different behaviour, e.g. uniqueness of minimizers. Recently, also negative exponents are considered, see [DS23b] and [DS23a]. Compared to the one-phase and obstacle problem, the generalized theory is less developed with a steady influx of new results.

# A Appendix

We summarize some standard results about Hölder and Sobolev functions, Hausdorff measures and harmonic functions, used throughout the work, to keep it self-contained. First, we have one of the main results from functional analysis.

**Theorem A.1.** (Uniform Boundedness Principle) *Let $X, Y$ be Banach spaces and suppose that $F \subset B(X, Y)$. Then*

$$\sup_{T \in F} \|T(x)\| < \infty \quad \forall x \in X \qquad \Longrightarrow \qquad \sup_{T \in F} \|T\|_{B(X,Y)} < \infty.$$

*Thus if the operators $T \in F$ are bounded pointwise, then they are bounded in operator norm.*

## A.1 Hölder functions

**Definition A.2.** Let $\Omega \subset \mathbb{R}^d$ and $0 < \alpha \leq 1$. The space of $\alpha$-Hölder functions on $\Omega$ is denoted by $C^{0,\alpha}(\Omega)$ and has the norm

$$\|f\|_{C^{0,\alpha}(\Omega)} := \|f\|_{L^\infty(\Omega)} + \sup_{x,y \in \Omega, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

For $\alpha = 1$, $C^{0,1}(\Omega)$ is the space of Lipschitz functions on $\Omega$. For a non-negative integer $m$, the space $C^{m,\alpha}(\Omega)$ is defined as all $C^m(\Omega)$ functions where the $m$'th derivative is $\alpha$-Hölder continuous. Its norm is

$$\|f\|_{C^{0,\alpha}(\Omega)} := \max_{|\gamma| \leq m} \sup_{x \in \Omega} |D^\gamma f| + \max_{|\gamma| = k} \sup_{x,y \in \Omega, x \neq y} \frac{|D^\gamma f(x) - D^\gamma f(y)|}{|x - y|^\alpha}.$$

The spaces of Hölder continuous functions have also good embedding properties.

**Theorem A.3.** *Let $\Omega \subset \mathbb{R}^d$ be bounded and $0 < \alpha < \beta \leq 1$. Then $C^{0,\alpha}(\Omega) \hookrightarrow C^{0,\beta}(\Omega)$.*

Secondly, we have a generalization of Taylor's theorem to $C^{k,\alpha}$ functions.

**Theorem A.4.** (Taylor's theorem for Hölder functions) *Let $A \subset \mathbb{R}^d$ be closed, $B \subset \mathbb{R}^d$ open such that $A \subset B$. Let $f \in C^{m,\alpha}(A) \cap C(B)$ with $\alpha \in [0, 1]$, then for any fixed point $a \in A$ and $x \in A$ such that $[a, x] \subset A$, we have*

$$f(x) = \sum_{|\beta| \leq m} \frac{D^\beta f(a)}{\beta!} + \sum_{|\beta| = m} R_\beta(x) = \sum_{|\beta| \leq m} \frac{D^\beta f(a)}{\beta!} + R(x),$$

*where*

$$
\begin{aligned}
|R_\beta(x)| &\le \frac{1}{\beta!}[D^\beta f]_{\alpha;A}|x-a|^{m+\alpha} \le \|f\|_{C^{m,\alpha}(A)}|x-a|^{m+\alpha},\\
|R(x)| &\le C_{d,m}[D^\beta f]_{\alpha;A}|x-a|^{m+\alpha} \le C_{d,m}\|f\|_{C^{m,\alpha}(A)}|x-a|^{m+\alpha}.
\end{aligned}
$$

To show Hölder continuity in practice, we have the following useful result.

**Lemma A.5.** (Morrey Lemma [Vel23, Lemma 3.12]) *Let $\Omega \subset \mathbb{R}^d$, $u \in H^1(\Omega)$ and for $C > 0$, $\alpha \in (0,1)$*

$$
⨍_{B_r(x_0)} |\nabla u|^2 dx \le C r^{2(\alpha-1)} \qquad \forall x_0 \in B_{R/8} \quad \forall r \le \frac{R}{2}.
$$

*Then $u \in C^{0,\alpha}(B_{R/8})$ with $\|u\|_{C^{0,\alpha}(B_{R/8})} \le C_{d,\alpha}$.*

## A.2 Fractional Sobolev norms

We introduce two different, but equivalent, definitions of fractional Sobolev norms, either defined through the Fourier transform, $\hat{f}$, or directly through the $L^p$ norm. For a short, concise introduction to fractional Sobolev spaces we refer to [DPV12].

**Definition A.6.** For $f \in L^2(\mathbb{R}^d)$ and $s \in (0,1)$ we define the $H^s$ norm, $\|\cdot\|_{H^s}(\mathbb{R}^d)$ as

$$
\|f\|_{H^s(R^d)} := \|(1+|\xi|^2)^{s/2}|\hat{f}(\xi)|\|_{L^2(\mathbb{R}^d)}.
$$

**Definition A.7.** For $f \in L^2(\mathbb{R}^d)$ and $s \in (0,1)$ we define the **homogeneous** $H^s$ norm, $\|\cdot\|_{\dot{H}^s}(\mathbb{R}^d)$ (in [DPV12] it is denoted by $[u]_{H^s(\mathbb{R}^d)}$) as

$$
\|f\|_{\dot{H}^s(\mathbb{R}^d)} := \||\xi|^s|\hat{f}(\xi)|\|_{L^2(\mathbb{R}^d)}.
$$

**Lemma A.8.** *If follows from the definition that $\|f\|_{\dot{H}^s(\mathbb{R}^d)} \le \|f\|_{H^s(\mathbb{R}^d)}$.*

**Definition A.9.** As in [DPV12, Chapter 2], we define the **fractional Sobolev norm** $\|\cdot\|_{W^{s,p}(\Omega)}$ through the *Gagliardo seminorms* as

$$
\|f\|_{W^{s,p}(\Omega)} := \left( \int_\Omega |u|^p + \underbrace{\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}} dx dy}_{[f]^p_{W^{s,p}(\Omega)}} \right)^{1/p}.
$$

**Proposition A.10.** *The two different definitions of fractional Sobolev norms are in fact equivalent* ([DPV12, Proposition 3.4]). *We have $[f]^2_{W^{s,2}(\mathbb{R}^d)} \simeq \|f\|_{\dot{H}^s(\mathbb{R}^d)}$.*

## A.3 Hausdorff measures

We now introduce the notion of the Hausdorff measure, which enables us to measure the size of lower dimensional sets. Note that the Lebesgue measure can not do that; take e.g. $A = [0,1]$ and $B = [0,2]$ as subsets of $\mathbb{R}^2$, $|A| = |B| = 0$, but clearly $B$ should give a larger measure than $A$. An in detail treatment can be found in e.g. [Mag12].

**Definition A.11.** We define for $s > 0$, $\delta \in (0,\infty]$ and $E \subseteq \mathbb{R}^d$

$$
\mathcal{H}^s_\delta(E) := \frac{\pi^{s/2}}{2^s\Gamma(s/2+1)} \inf \left\{ \sum_{j=1}^\infty (\operatorname{diam} U_j)^s : \{U_j\} \text{ with } E \subset \bigcup_{j=1}^\infty \text{ and } \operatorname{diam} U_j \le \delta \quad \forall j \right\}.
$$

**Definition A.12.** For $s > 0$, we define the **$s$-dimensional Hausdorff measure** of $E \subset \mathbb{R}^d$ as

$$\mathcal{H}^s(E) \coloneqq \lim_{\delta \to 0+} \mathcal{H}^s_\delta(E) = \sup_{\delta > 0} \mathcal{H}^s_\delta(E).$$

**Definition A.13.** We define the **Hausdorff dimension** of $E \subset \mathbb{R}^d$ as

$$\dim_{\mathcal{H}}(E) \coloneqq \inf\{s > 0 : \mathcal{H}^s(E) = 0\}.$$

From the definitions, we have several useful properties.

**Proposition A.14.** *We have the following properties:*

1. *$\mathcal{H}^s$ and $\mathcal{H}^s_\delta$ are translation invariant and increasing with respect to set inclusion.*
2. *$\mathcal{H}^d(B_r) = |B_r| = \omega_d r^d$ and $\mathcal{H}^{d-1}(\partial B_r) = d\omega_d r^{d-1}$.*
3. *$\mathcal{H}^s_\infty(E) \geq \mathcal{H}^s_\delta(E) \geq \mathcal{H}^s(E)$ for any $E \subset \mathbb{R}^d$ and $\delta > 0$.*
4. *$\mathcal{H}^s(E) = 0$ if and only if $\mathcal{H}^s_\infty(E) = 0$.*
5. *For a collection $\{E_j\}$ and $E = \cup_{j=1}^\infty E_j$ we have $\mathcal{H}^s_\delta(E) \leq \sum_{j=1}^\infty \mathcal{H}^s_\delta(E_j)$, so in particular $\mathcal{H}^s(E) = 0$ if and only if $\mathcal{H}^s(E_j) = 0$ for all $j$.*

The next lemma gives a point of positive density for a set of positive Hausdorff measure.

**Lemma A.15.** *If $\mathcal{H}^s(E) > 0$ then there is a point $x_0 \in E$ such that*

$$\limsup_{r \to 0} \frac{\mathcal{H}^s(K \cap B_r(x_0))}{r^s} = 0.$$

*Proof.* Suppose that

$$\limsup_{r \to 0} \frac{H^s(E \cap B_r(x_0))}{r^s} = 0 \qquad \forall x_0 \in E.$$

We set $E_{\delta,\varepsilon} = \{x \in E : H^s(E \cap B_r(x)) \leq \varepsilon r^s \quad \forall r \leq \delta\}$, clearly $E = \bigcup_{\delta > 0} E_{\delta,\varepsilon} = \bigcup_{n \in \mathbb{N}} E_{1/n,\varepsilon}$. Take now the countable collection $\{U_i\}_{i \geq 1}$ with $\operatorname{diam} U_i \leq \delta$ such that $E_{\delta,\varepsilon} \subset \cup_i U_i$. Then

$$\mathcal{H}^s_\delta(E_{\delta,\varepsilon}) \leq \sum_{i=1}^\infty \mathcal{H}^s_\delta(E_{\delta,\varepsilon} \cap U_i) \leq \sum_{i=1}^\infty \mathcal{H}^s(E \cap U_i) \leq \sum_{i=1}^\infty \varepsilon \operatorname{diam}(U_i)^s.$$

This in turn implies that for $\varepsilon < \frac{\omega_s}{2^s}$,

$$\mathcal{H}^s_\delta(E_{\delta,\varepsilon}) \leq \varepsilon \frac{2^s}{\omega_s} \mathcal{H}^s_\delta(E_{\delta,\varepsilon}) < a\, \mathcal{H}^s_\delta(E_{\delta,\varepsilon}) \implies \mathcal{H}^s_\delta(E_{\delta,\varepsilon}) = 0 \implies \mathcal{H}^s(E_{\delta,\varepsilon}) = 0 \implies H^s(E) = 0,$$

a contradiction. □

We now show that the Hausdorff measure behaves nicely under dimensional extension, a technique known as Federer's dimensional reduction argument.

**Proposition A.16.** (General dimension reduction) *For a set $E \subset \mathbb{R}^{d-1}$, set $\tilde{E} = E \times \mathbb{R} \subset \mathbb{R}^d$. Then, if $\mathcal{H}^s(E) = 0$, also $\mathcal{H}^{s+1}(\tilde{E}) = 0$.*

*Proof.* Write $\tilde{E} = \bigcup_{T > 0}(E \times [0,T]) \cup \bigcup_{T > 1}(E \times [-T, 0])$, by translation invariance it suffices to show $\mathcal{H}^{s+1}(E \times [0,T]) = 0$ or in fact even $\mathcal{H}^{s+1}_\infty(E \times [0,T]) = 0$. Fix $T > 1$ and $\varepsilon > 0$. Since $\mathcal{H}^s(E) = 0$ take a countable family of balls in $\mathbb{R}^{d-1}$ such that $E \subset \bigcup B_{r_i}(x_i)$ with $\sum r_i^s \leq \varepsilon$. For each $i$ set $K_i \coloneqq \lfloor T/r_i \rfloor$ and $x_{i,k} \coloneqq (x_i, kT_i)$, where $k = 0, ..., K_i + 1$.

Take now $B_{2r_i}(x_{i,k}) \subset \mathbb{R}^d$, we get directly $x' \times [0,T] \subset \bigcup_k B_{2r_i}(x_{i,k})$ for any $x' \in B_{r_i}(x_i)$, that is $\{B_{2r_i}(x_{i,k})\}_{i,k}$ is a countable covering of $E \times [0,T]$. In turn,

$$\begin{aligned}\mathcal{H}^{s+1}_\infty(E\times[0,T]) &\le \sum_{i=1}^\infty \sum_{k=0}^{K_i}(2r_i)^{s+1} = 2^{s+1}\sum_{i=1}^\infty\sum_{k=0}^{K_i} r_i^{s+1} = 2^{s+1}\sum_{i=1}^\infty (K_i+1) r_i^{s+1} \\ &\le 2^{s+1}\sum_{i=1}^\infty \frac{T}{r_i} r_i^{s+1} = 2^{s+2}T\sum_{i=1}^\infty r_i^s \le 2^{s+2}T\varepsilon,\end{aligned}$$

which can be made arbitrarily small choosing $\varepsilon = \varepsilon(T)$ accordingly. Thus $H^{s+1}_\infty(E\times[0,T]) = 0$. □

Lastly, we have one of the cornerstones of geometric measure theory.

**Theorem A.17.** (Co-area formula, [Fed69, Theorem 3.2.22]) *Let $f : \Omega \to \mathbb{R}$ be a Lipschitz function, $\Omega \subset \mathbb{R}^d$ open and $g \in L^1(\Omega)$. Then*

$$\int_\Omega g(x)|\nabla f(x)|dx = \int_{\mathbb{R}}\left(\int_{f^{-1}(t)} g(x)d\,\mathcal{H}^{n-1}(x)\right)dt, \tag{A.1}$$

*where $f^{-1}(t) = \{x \in \Omega : f(x) = t\}$.*

## A.4 Estimates for harmonic functions

First, for non-negative harmonic functions the infimum and supremum are comparable.

**Theorem A.18.** (Harnack's inequality) *There exists a dimensional constant $C$ such that for every non-negative harmonic function $u : B_{2r} \to \mathbb{R}_{\ge 0}$,*

$$\sup_{B_r} u \le C \inf_{B_r} u.$$

For a harmonic function $u$, we also have relations between the integrals of $\nabla u$ and $u$.

**Theorem A.19.** (Cacciopolli identity) *Let $u : B_{2r} \to \mathbb{R}$ satisfy $u\Delta u \ge 0$ and $\phi : B_{2r} \to \mathbb{R}$ non-negative and $\phi|_{\partial B_{2r}} = 0$. Then*

$$\int_{B_{2r}} \phi^2|\nabla u|^2 \le 4\int_{B_{2r}} |u|^2|\nabla\phi|^2,$$

$$\int_{B_{2r}} |\nabla u|^2 \le \frac{4}{r^2}\int_{B_{2r}\setminus B_r} |u|^2.$$

Eventually, we have another powerful interior estimate on the derivatives of a harmonic function $u$, that is used extensively in this work.

**Theorem A.20.** (standard elliptic estimate for harmonic functions [GT77, Theorem 2.10]) *Let $u$ be harmonic in a domain $D$ and $\Omega \subset\subset D$. Then for any multi-index $\alpha$, we have*

$$\sup_\Omega |D^\alpha u| \le \left(\frac{d|\alpha-\beta|}{\mathrm{dist}(\Omega,\partial D)}\right)^{|\alpha-\beta|}\sup_D |D^\beta u| \qquad \forall \beta : |\beta| < |\alpha|,$$

*in particular*

$$\sup_\Omega |D^\alpha u| \le \left(\frac{d|\alpha|}{\mathrm{dist}(\Omega,\partial D)}\right)^{|\alpha|}\sup_D |u|.$$

*Suppose that $\Delta u = 0$ in $B_r$, then*

$$\|\nabla u\|_{L^\infty(B_{r/2})} \le \frac{C}{r}\|u\|_{L^\infty(B_r)}.$$

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