

# Action potentials in action: Sufficient conditions for spikes in the FitzHugh-Nagumo system

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# Outline

- Introduction
- The FitzHugh-Nagumo model
- Problem statement
- Definition of spikes
- Conditions for spikes
  - Dirac impulse
  - general impulse
- Periodic hybrid system
- Outlook

# Introduction

- Excitable media: impulse  $\rightarrow$  wave  $\rightarrow$  cool-down
- Examples: wildfire, chemical oscillators and cardiac or neuronal tissue
- Hodkin-Huxley model:  $4D$  non-linear dynamical system
- FitzHugh-Nagumo model:  $2D$  non-linear dynamical system
- Reference: George Datseris and Ulrich Parlitz. *Nonlinear Dynamics*. Springer 2022

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# The FitzHugh-Nagumo model

- FitzHugh-Nagumo equations:

$$\begin{aligned}\dot{u}(t) &= -bu(t)(u(t)-1)(u(t)-a) - w(t) + I(t), \\ \dot{w}(t) &= \varepsilon(u(t) - cw(t)),\end{aligned}\tag{0.1}$$

with  $a, b, c, \varepsilon \in \mathbb{R}$ .

- Impulse: A measurable function  $I : [t_0, t_0 + T] \rightarrow \mathbb{R}^+$
- Dirac Impulse:  $I_0 \delta_{t_0}, (u(t_0), w(t_0)) \mapsto (u(t_0) + I_0, w(t_0))$

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- No impulse:

$$\begin{aligned}\dot{u} &= -bu(u-1)(u-a) - w, \\ \dot{w} &= \varepsilon(u - cw),\end{aligned}\tag{0.2}$$

- Nullclines determine global behaviour

$$\begin{aligned}\mathcal{N}_{\dot{u}} : \quad w &= -bu(u-a)(u-1) =: f(u), \\ \mathcal{N}_{\dot{w}} : \quad w &= \frac{1}{c}u\end{aligned}\tag{0.3}$$

- if  $(1-a)^2 < \frac{4}{bc}$  unique globally stable fixpoint at origin
- if  $(1-a)^2 > \frac{4}{bc}$  bistable; if  $a, c > 0, b < 0$  periodic cycle



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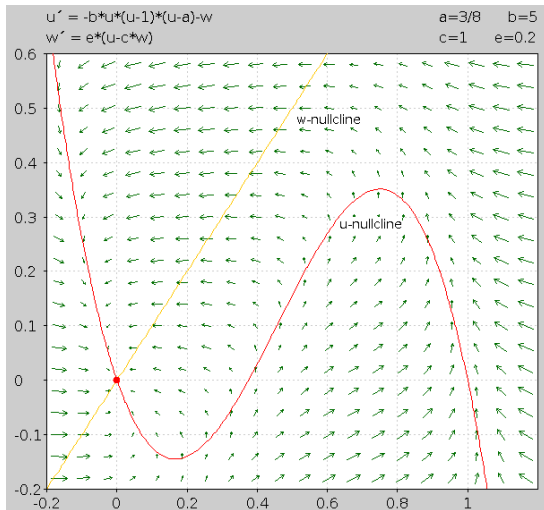


Figure:  $u$ -vs- $w$  phase plot for  $a = 3/8$ ,  $b = 5$ ,  $c = 1$ ,  $\varepsilon = 0.2$

# Problem statement

What kind of impulses to the fixpoint  $(0,0)$  will cause the system to spike, i.e. make an "excursion in phase space" with "large"  $u$ -value?

- How to define a spike rigorously?
- Calculate the exact value of a Dirac impulse leading to a spike for a fixed set of parameters
- What can be said about a general impulse?

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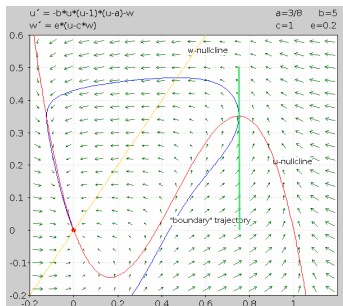
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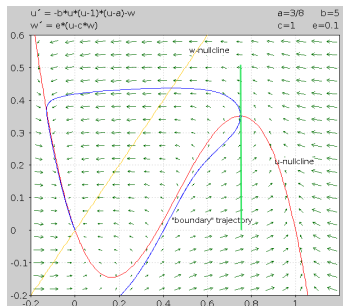
# Definition of spikes

## Definition

A trajectory  $(u, w) : t \mapsto (u(t), w(t)) \ni \mathbb{R}^2$  with  $(u(0), w(0)) = (0, 0)$  has a **spike** at time  $t_s$  if  $\dot{u}(t_s) > 0$  and  $u(t_s) = u_s$  with the spike threshold  $u_s := (\sqrt{a^2 - a + 1} + a + 1)/3$ .



(a)  $a = \frac{3}{8}, b = 5, c = 1, \varepsilon = 0.2$



(b)  $a = \frac{3}{8}, b = 5, c = 1, \varepsilon = 0.1$

# Conditions for spikes

- Problem: boundary trajectory can only be found numerically
- Goal: find region  $A_\varepsilon$  analytically s.t.  $(u, w) \in A_\varepsilon$  guarantees a spike

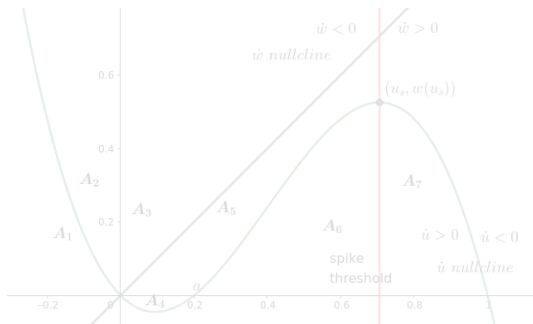


Figure:  $a = 0.2$ ,  $b = 5$ ,  $c = 1$



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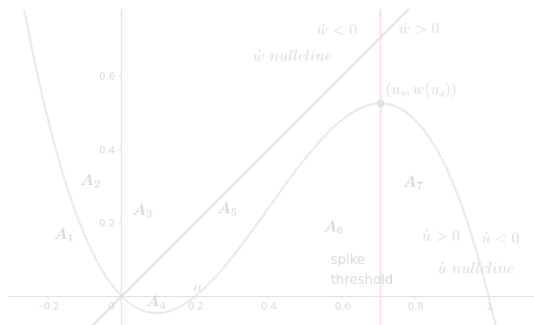


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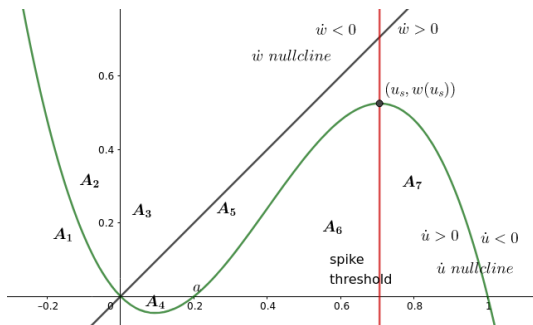


Figure:  $a = 0.2$ ,  $b = 5$ ,  $c = 1$

## Condition for spikes

Idea: estimate direction of flow

- if below  $z_\lambda(u) = 1/(1 - c\epsilon\lambda)(f(u) - \epsilon\lambda u)$  we have  $\dot{u} \geq \lambda\dot{w}$
- if below  $l_\lambda(u) := \frac{1}{\lambda}u + z(u_s) - \frac{1}{\lambda}u_s$ , flow crosses spike threshold
- Let  $A_\epsilon = \bigcup_\lambda A_\lambda$

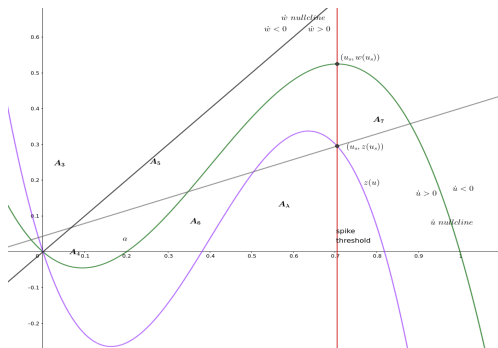


Figure:  $a = 0.2, b = 5, c = 1$

# Conditions for spikes - Dirac impulse

- $(u, w) \in A_\varepsilon$  is sufficient but not necessary condition!
- Why is  $A_\varepsilon$  easier to handle than the boundary trajectory?
- Q: For a fixed parameter set and fixed  $\varepsilon$ , determine  $C_\varepsilon > a$   
s.t.  $I_0 \delta_0$  with  $I_0 \geq C_\varepsilon$  causes a spike i.e.  $(u(0+), w(0+)) \in A_\varepsilon$   
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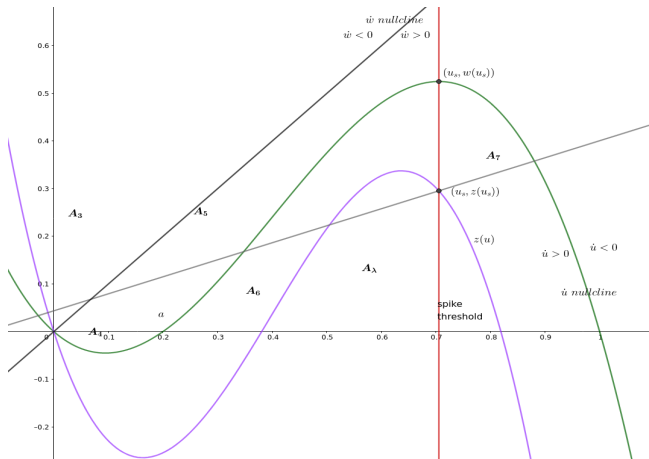


Figure:  $a = 0.2$ ,  $b = 5$ ,  $c = 1$

# Conditions for spikes - Dirac impulse

- find  $\min_u \{A_\varepsilon \cap \{(u, w) : w = 0\}\}$
- solve for roots  $r_z(\lambda)$  of  $z_\lambda(u)$  (quadratic) and  $r_l(\lambda)$  of  $l_\lambda(u)$  (linear)
  - $\hookrightarrow$  algebraic expressions involving all parameters,  $\varepsilon$  and  $\lambda$
  - $\hookrightarrow r_z(\lambda) = A - \sqrt{B - C\lambda\varepsilon}, \quad r_l(\lambda) = D - \frac{\lambda}{1-\lambda\varepsilon} (E - F\lambda\varepsilon)$
- $C_\varepsilon = \min_\lambda \max\{r_z(\lambda), r_l(\lambda)\}, \lambda_* = \operatorname{argmin}_\lambda \max\{r_z(\lambda), r_l(\lambda)\}$ 
  - $\hookrightarrow$  easier to solve than ODE
- either  $r_l(\lambda_*) = r_z(\lambda_*)$  or  $\lambda_* = \operatorname{argmin}_\lambda r_l(\lambda)$ 
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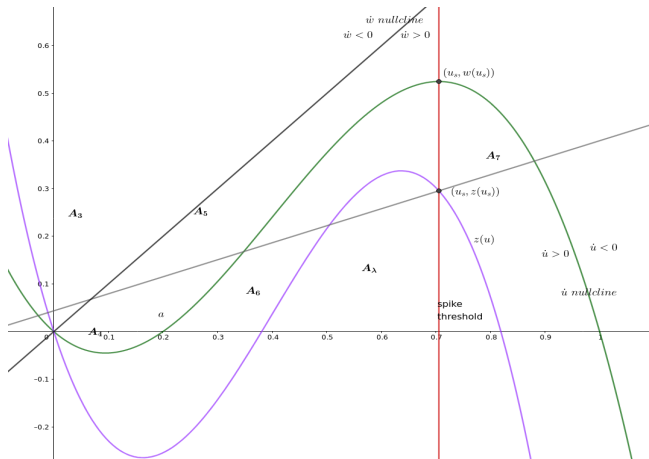
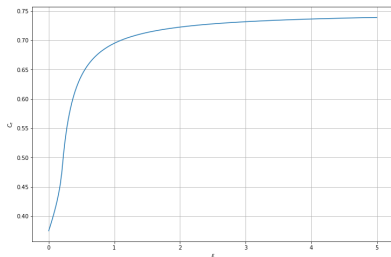


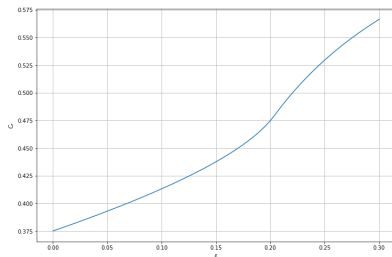
Figure:  $a = 0.2$ ,  $b = 5$ ,  $c = 1$

# Conditions for spikes - Dirac impulse

- when  $\epsilon \rightarrow 0$  then  $C_\epsilon \rightarrow a = \frac{3}{8}$  linearly
- when  $\epsilon \rightarrow \infty$  then  $C_\epsilon \rightarrow u_s = \frac{3}{4}$  asymptotically



(a)  $\epsilon$  vs  $C_\epsilon$



(b)  $\epsilon$  vs  $C_\epsilon$

Figure:  $a = \frac{3}{8}$ ,  $b = 5$ ,  $c = 1$

# Conditions for spikes - Dirac impulse

- similar constant  $C_{\varepsilon,\beta}$  for Dirac impulse to  $(u, w) = (0, \beta)$   
 $\hookrightarrow$  boundary of  $A_\varepsilon$

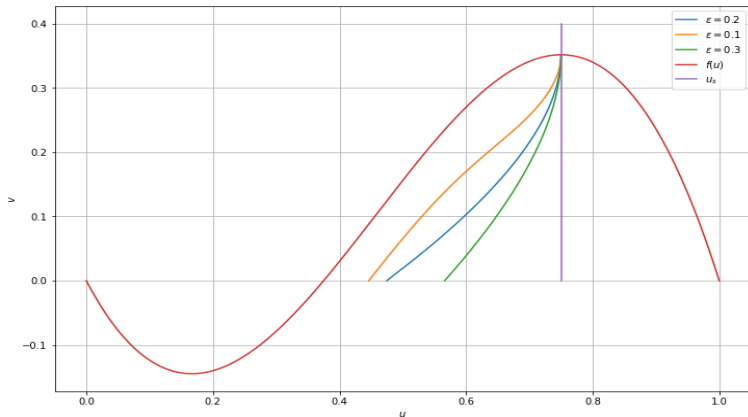
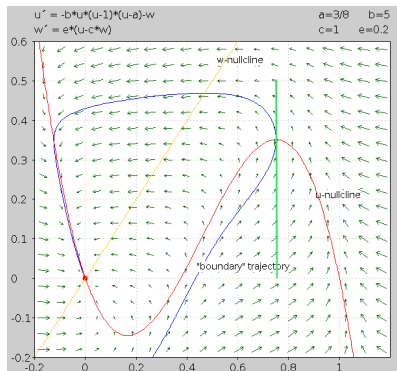


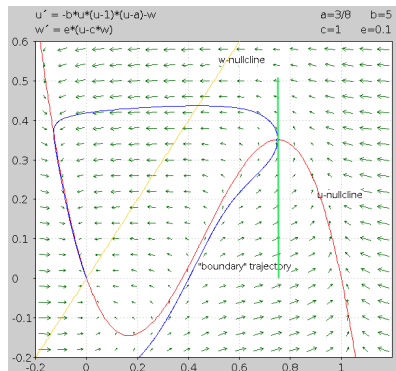
Figure: Boundaries of  $A_\varepsilon$  for different  $\varepsilon$

# Conditions for spikes - Dirac impulse

- significantly worse than "exact" boundary trajectory



(a)  $\varepsilon = 0.2$



(b)  $\varepsilon = 0.1$

Figure:  $a = \frac{3}{8}$ ,  $b = 5$ ,  $c = 1$

# Conditions for spikes - general impulse

## Theorem

Let  $I(t)$  be an impulse of finite length  $T$  to the steady state  $(0, 0)$  and  $\beta := T\varepsilon u_s$ . If the following conditions

1.  $\int_0^T I(t) \leq (1 - T^2\varepsilon)u_s - Tf(u_s)$  (upper bound 1),
2.  $T\varepsilon u_s \leq f(u_s)$  (upper bound 2),
3.  $\int_0^T I(t) \geq \left(1 - \frac{T^2\varepsilon}{1-T^2\varepsilon}\right)^{-1} \left(C_{\varepsilon,\beta} + \frac{T^3\varepsilon f(u_s)}{1-T^2\varepsilon} - T \min_{u \in [0, u_s]} f(u)\right)$  (lower bound),

are satisfied, then the trajectory has a spike for some  $t_s > T$ .

- upper bounds give  $u(T) \leq u_s$ ,  $w(T) \leq \beta \leq f(u_s)$ ,  
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# Periodic hybrid system

- What if there is a second impulse? When?
- send second Dirac impulse when returning to height  $\beta$ ;  $I(\beta, u)$ .

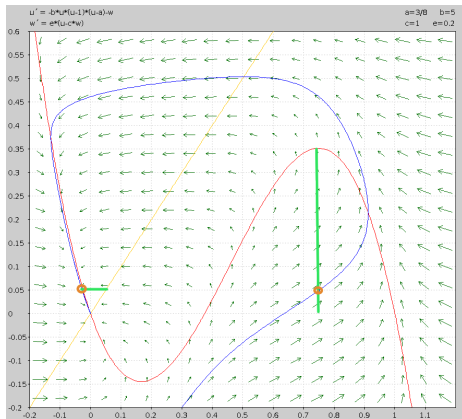


Figure: periodic hybrid system

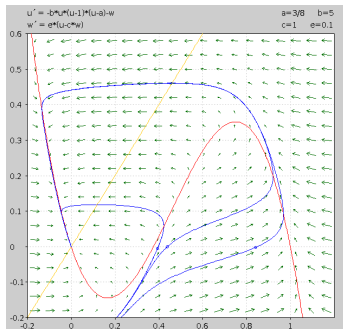
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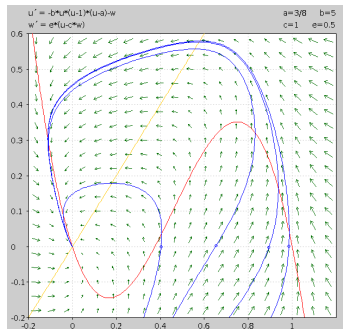
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# Outlook

- for "large"  $\varepsilon$  only convex trajectories (threshold  $\varepsilon_0$  ?)  
 $\hookrightarrow$  Mathieu Desroches and Mike Jeffrey. *Canards and curvature: the smallness of  $\varepsilon$  in slow-fast dynamics*



(a)  $\varepsilon = 0.1$



(b)  $\varepsilon = 0.5$

Figure:  $a = \frac{3}{8}$ ,  $b = 5$ ,  $c = 1$