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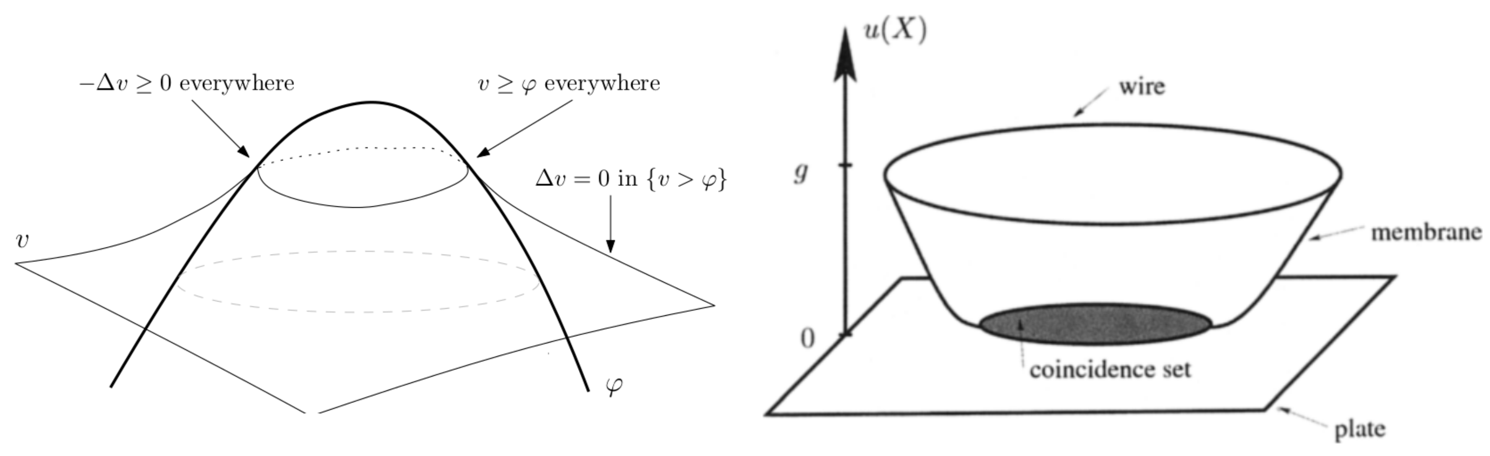
Introduction

Let us minimize the Dirichlet energy with an extra constraint, i.e.

$$\text{minimize} \quad \int_\Omega \frac{1}{2}|\nabla v(x)|^2 dx \qquad \text{s.t. } v \geq \varphi \text{ and } v\big|_{\partial\Omega} = g.$$

From now on, set $u := v - \varphi$, to get the *zero obstacle problem*

$$\text{minimize} \quad \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + fu\right) dx \quad \text{s.t. } u \geq 0, u|_{\partial\Omega} = g - \varphi, f := -\Delta\varphi.$$



(a) The normal obstacle problem

(b) The zero obstacle problem

Figure 1: The basic setup: Intuition

The domain has two parts, the *contact set* $\{x : u(x) = 0\}$ and the set $\{x : u(x) > 0\}$; the *free boundary* denotes the set $\partial\{u > 0\}$. Right now, nothing is known about the free boundary, it might be of infinite perimeter, a fractal set or discontinuous.

Regularity of Solution

Firstly, the Euler-Lagrange equation is given by

$$\Delta u = f\chi_{\{u>0\}} \Leftrightarrow \begin{cases} \Delta u = f \text{ in } \{u > 0\}, \\ u = 0 \text{ on } \partial\{u > 0\}, \\ \nabla u = 0 \text{ on } \partial\{u > 0\}. \end{cases} \tag{1}$$

The RHS is bounded, so by elementary Schauder estimates (usethe zero obstacle formalism), $u \in C^{1,1-\epsilon}$, for any $\epsilon > 0$.
Under the weak assumption, $f \geq c_0 > 0$, using Harnack's inequality, maximum principle and the continuity of u,

$$0 < cr^2 < \sup_{B_r(x_0)} u \leq Cr^2 \quad \text{for any } x_0 \in \partial\{u > 0\}. \tag{2}$$

The quadratic growth and another Schauder estimate ($\Delta u = f$ with $f \in C^{0,1} \implies u \in C^{2,1}$) gives $C^{1,1}$ interior regularity , i.e.

$$\|u\|_{C^{1,1}(B_{1/2})} \leq C_n(\|u\|_{L^\infty(B_1)} + \|f\|_{Lip(B_1)}). \tag{3}$$

As Δu is discontinuous across $\partial\{u > 0\}$, the result is "optimal".

Free Boundary: Classification of blow-ups

Let $f \equiv 1$, so the problem becomes

$$\begin{cases} u & \in & C^{1,1}(B_1), \\ u & \geq & 0, \\ \Delta u & = & \chi_{\{u>0\}}, \\ & 0 \text{ is FBP.} & \end{cases} \tag{4}$$

Define a blow-up as $u_r(x) := \frac{u(x_0+rx)}{r^2}$, by Arzelà-Ascoli, $u_r(x)$ converges in C^1_{loc} to a *blow up* $\mathbf{u}_0$, satisfying a *global problem*

$$\begin{cases} u_0 & \in & C^{1,1}_{\text{loc}}(\mathbb{R}^n), \\ u_0 & \geq & 0, \\ \Delta u_0 & = & \chi_{\{u_0>0\}}, \\ & 0 \text{ is FBP.} & \end{cases} \tag{5}$$

Surprisingly, blow-ups are 2-homogeneous and convex and can thereby classified into two cases. Showing 2-homogeneity is classically done by monotonicity formulas, yet it can also be achieved by a change to spherical coordinates and a rescaling.

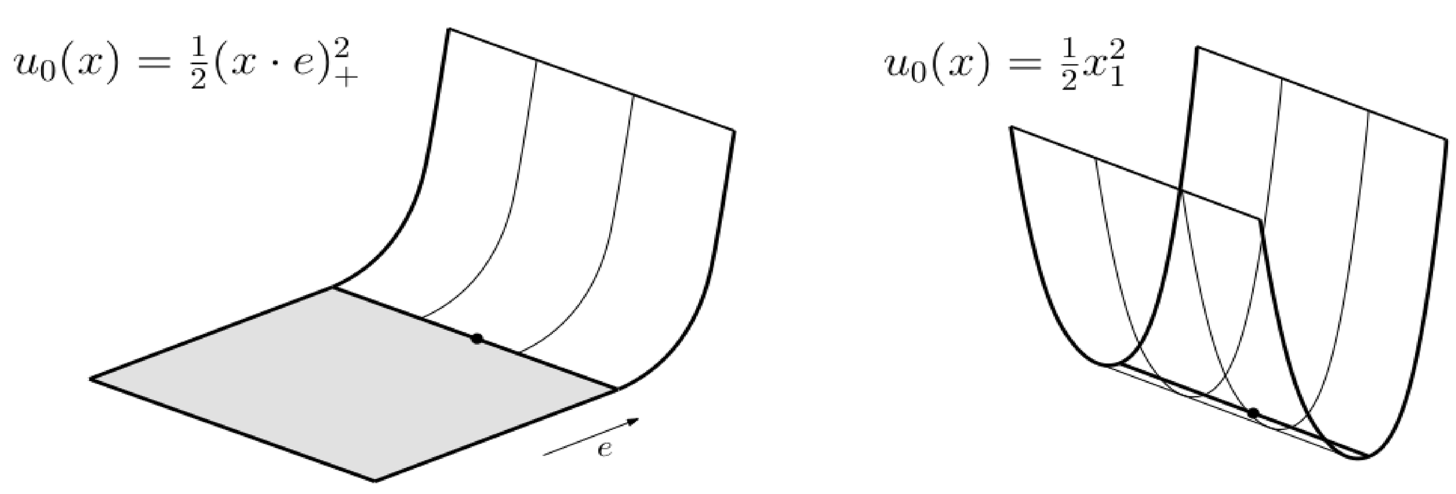
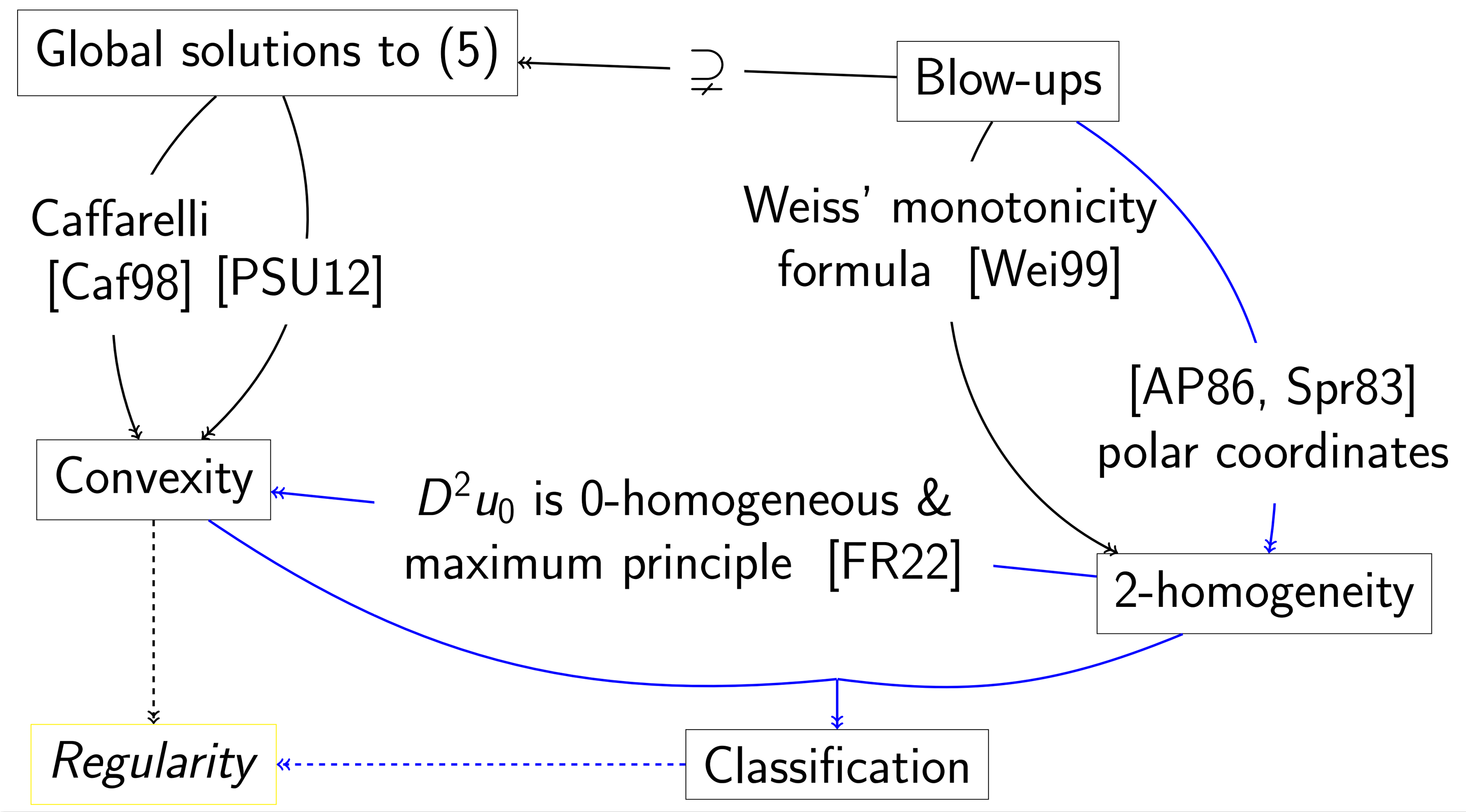


Figure 2: The two possible blow-ups

Free Boundary: Classification of blow-ups strategy



Free Boundary: Regularity

- At *regular* points (positive density: $\limsup_{r\to 0} \frac{|\{u=0\}\cap B_r(x_0)|}{|B_r|} > 0$) u has a blow-up of the form $u_0 = \frac{1}{2}(x \cdot e)_+^2$ with $e \in \mathbb{S}^{n-1}$. By non-degeneracy, we can show that the free boundary of some scaled version u_{r_0} is contained in a strip and furthermore we can partially transfer monotonicity from u_0 to u_{r_0}: there exists a cone of directions τ, in where u_{r_0} is non-decreasing.
It follows that the free boundary is Lipschitz and by the boundary Harnack inequality $C^{0,\alpha}$. Using the higher order boundary Harnack iteratively the free boundary is C^∞ at regular points inside B_1.
- At *singular* points ($\limsup_{r\to 0} \frac{|\{u=0\}\cap B_r(x_0)|}{|B_r|} = 0$) the unique blow-up is of the form $u_0(x) = \frac{1}{2}x^TAx$. The classical way of showing uniqueness uses Monneau's monotonicity formula. By Whitney's Extension theorem, it can be shown that the singular points are contained in a $n-1$ dimensional C^1 manifold.
- In dimension $2, 3$ and 4 *generic regularity* of the free boundary.

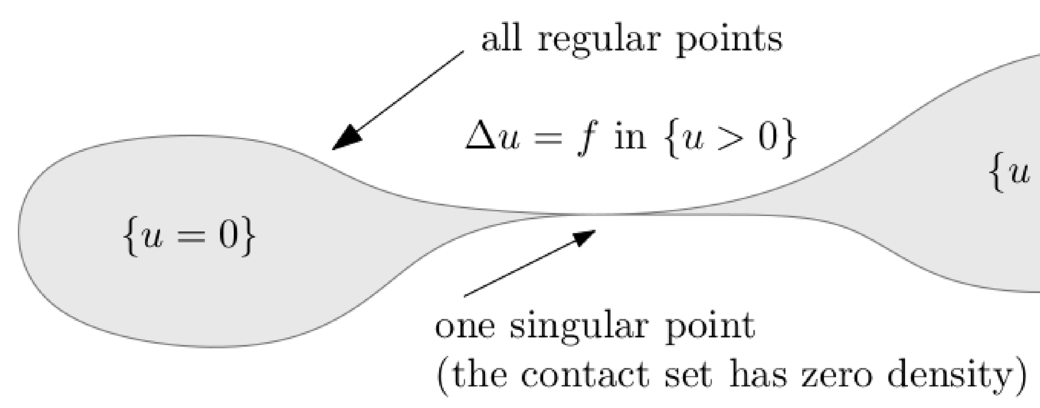


Figure 3: An example of regular and singular points

Application: Optimal stopping in Mathematical Finance

Consider a stochastic process (here standard Brownian motion starting at $x \in \mathbb{R}^n$), $x + X_t$, and a payoff function $\varphi : \mathbb{R}^n \to \mathbb{R}$. What is the optimal strategy to decide when to stop and take the payoff and when to wait for a better payoff later? We want to solve

$$u(x) := \max_\theta\{\mathbb{E}[\varphi(x + X_\theta)]\}, \quad \text{maxim. over all stopping times } \theta. \tag{6}$$

- Clearly $u(x) \geq \varphi(x)$ (just take $\theta = 0$).
- For standard Brownian motion the infinitesimal generator is

$$\frac{1}{2}\Delta u(x) = \lim_{t\to 0^+} \frac{\mathbb{E}[u(x + X_t)] - u(x)}{t}. \tag{7}$$

Furthermore for any $t > 0$, by the following,

$$\begin{aligned} u(x) &= \max_\theta \mathbb{E}[\varphi(x + X_\theta)] \geq \max_{\theta:\theta\geq t>0} \mathbb{E}[\varphi(x + X_\theta)] \\ &= \max_\theta \mathbb{E}[\varphi(x + X_{t+\theta})] = \mathbb{E}[u(x + X_t)] \\ &\implies \Delta u(x) \leq 0. \end{aligned} \tag{8}$$

- If $u(x) > \varphi(x)$, then $u(x) = \mathbb{E}[u(x + X_t)] + o(t)$ and $\Delta u(x) = 0$.

This is the obstacle problem and if the sets $\{u = \varphi\}$ and $\{u > \varphi\}$ are known, so is the optimal strategy.

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