Regularity for the one-phase problem

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Introduction

Given a bounded open domain $D \subset \mathbb{R}^d$, $\Lambda > 0$ and $g \in H^1(D)$ nonnegative, we seek out to minimize the functional

$$\mathcal{F}_\Lambda(u, D) := \underbrace{\int_D |\nabla u|^2 dx}_{\text{Dirichlet energy}} + \underbrace{\Lambda |\{u > 0\} \cap D|}_{\text{measure term}}, \tag{1}$$

over $u \in H^1 D$ with $u|_{\partial D} = g$. A nonnegative minimizer u exists, but is not necessarily unique. It satisfies

$$\begin{cases} \Delta u \geq 0 & \text{in } D, \\ \Delta u = 0 & \text{in } \{u > 0\} =: \Omega_u. \end{cases}$$

The domain Ω_u, where u is harmonic, is not determined beforehand and is part of the solution itself. Its boundary $\partial\Omega_u$ is called the **free boundary**.

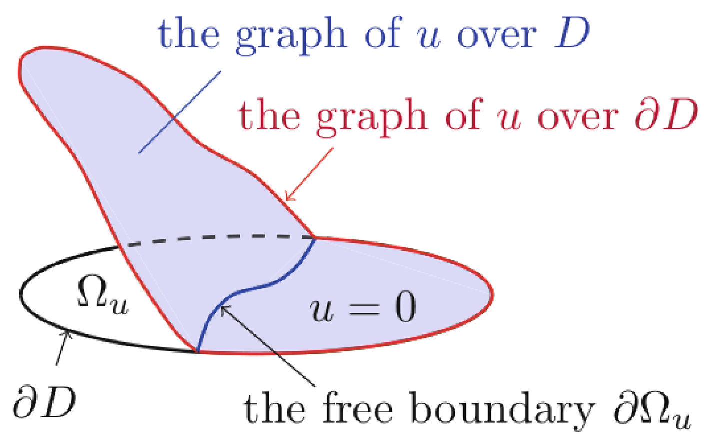

Figure 1: A possible minimizer u on $D = B_1$

Global minimizer

We say that $u \in H^1_{loc}$ is a **global minimizer** if for every $U \subset\subset \mathbb{R}^d$ and $v \in H^1(U)$ with $u - v \in H^1_0(U)$, $\mathcal{F}_\Lambda(u, U) \leq \mathcal{F}_\Lambda(v, U)$. An example is given by the half-plane solution $h_\nu(x) = \sqrt{\Lambda}(x \cdot \nu)_+$.

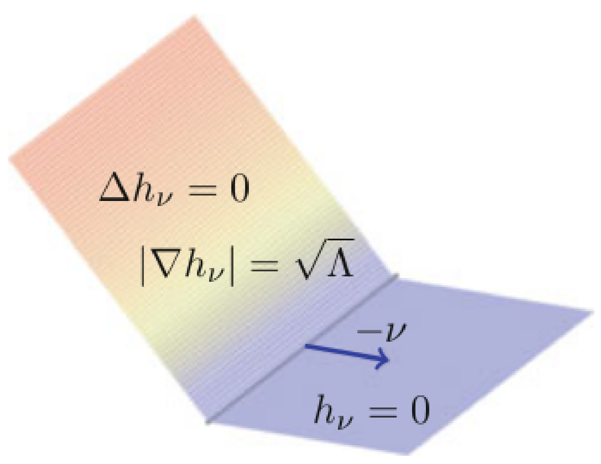

Figure 2: the half-plane solution, a global minimizer

Generic uniqueness of minimizers

Despite minimizers not being unique in general, for "almost all" boundary datum they are unique.

> **Lemma**[3]: Minimizers to $\mathcal{F}_\Lambda$ are ordered with respect to their boundary datum, i.e. $g_1 > g_2 \implies u_{g_1} \geq u_{g_2}$.

> **Theorem**[3]: Let $g : \partial D \to \mathbb{R}$ be nonnegative and continuous. Set $S_\lambda = \{\lambda \in \mathbb{R} : g + \lambda \geq 0\}$, then for almost every $\lambda \in S_\lambda$, the minimum of $\mathcal{F}_\Lambda(u, D)$ with datum g_λ is unique.

Regularity of the minimizer u

A minimizer u of $\mathcal{F}_\Lambda$ is locally Lipschitz continuous in D. This regularity is optimal in the sense that for a FBP x_0 and $B_r(x_0) \subset\subset D$,

$$cr \leq \sup_{B_r(x_0)} u \leq Cr.$$

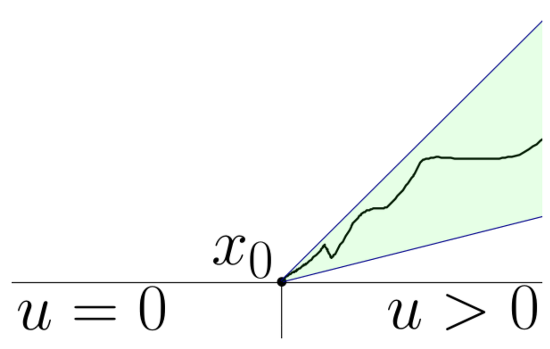

Figure 3: u locally at the FB

Similar to the classical Laplace equation with Dirichlet boundary condition, we are not only interested in interior regularity, but also in the regularity up to the boundary.

> **Theorem**[4, Theorem 3.5]: Let $g : \partial D \to \mathbb{R}$ be nonnegative and continuous. Then the minimizer u of $\mathcal{F}_\Lambda$ is in $C(\bar{D})$, i.e. for $x_0 \in \partial D$ we have $x_k \to x_0 \implies u(x_k) \to u(x_0)$.

> **Theorem**[4, Theorem 3.7]: Let $g : \partial D \to \mathbb{R}$ be nonnegative and γ_0-Hölder continuous with $\gamma_0 \in (1/2, 1)$. Then the minimizer u of $\mathcal{F}_\Lambda$ is in $C^\gamma(\bar{D})$ for any $\gamma \leq \gamma_0$.

The free boundary: First results

In the one-phase problem, the free boundary satisfies a density estimate,

$$0 < \delta_D < \frac{|B_r \cap \{u > 0\}|}{|B_r|} < 1 - \delta_D < 1,$$

and does not contain cusps. Moreover, u satisfies, in the viscosity sense,

$$\begin{cases} \Delta u = 0 & \text{in } \Omega_u, \\ |\nabla u| = \sqrt{\Lambda} & \text{on } \partial\Omega_u \cap D. \end{cases} \qquad \text{(Euler-Lagrange eq. of } \mathcal{F}_\Lambda\text{)}$$

The free boundary: Blow-ups

To study the finer regularity properties of the free boundary, we "zoom-in infinitely" around a FBP x_0. The limit of the rescalings

$$u_{x_0, r_n}(x) := \frac{1}{r_n} u(x_0 + r_n x) \xrightarrow{r_n \to 0} u_0 \in Lip_{loc}(\mathbb{R}^d),$$

is called a **blow-up**.

If $u_0(x) = \sqrt{\Lambda}(x \cdot v)_+$ for some $v \in \mathbb{S}^{d-1}$, we call it a **regular point** ($\in Reg(\partial\Omega_u)$) and a **singular point** ($\in Sing(\partial\Omega_u)$) otherwise.

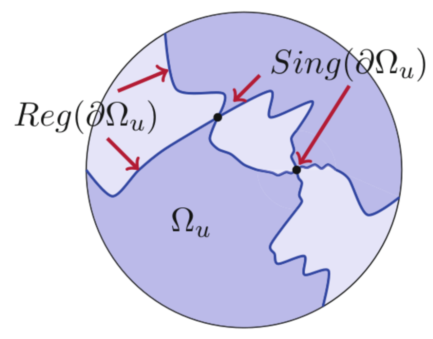

Figure 4: Free boundary points

The free boundary: local $C^{1,\alpha}$ regularity at regular points

If r_n is sufficiently small, then u_{x_0, r_n} is ε-**flat** in some direction $v \in \mathbb{S}^{d-1}$,

$$|u_{x_0, r_n} - x \cdot v| \leq \varepsilon \text{ in } \{u_{x_0, r_n} > 0\} \cap B_1.$$

By changing the direction slightly to $\tilde{v}$ and a rescaling, there is an "improvement of flatness" [9, 7]. Iterating this argument gives

> **Theorem**[1]: For any $\alpha \in (0, 1)$, the free boundary is a $C^{1,\alpha}$ manifold locally around a regular FBP x_0.

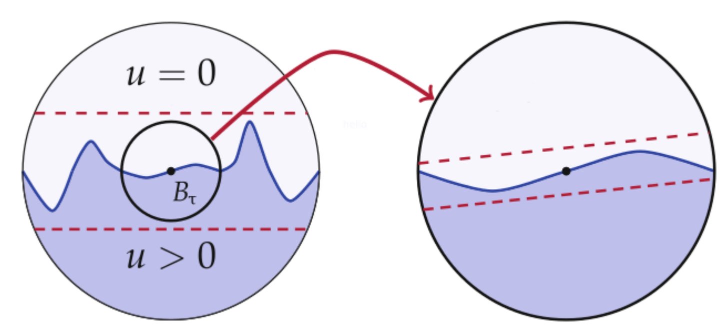

Figure 5: The idea behind the improvement of flatness

The free boundary: local C^∞ regularity at regular points

In [6], using the *hodograph transform*, local analyticity of the free boundary is shown. We follow an alternative approach using an energy estimate.

> **Proposition**[4, Prop 5.16]: Let u be a minimizer of $\mathcal{F}_\Lambda$ and x_0 a regular FBP, i.e. for any $\alpha < 1$, there exists $r > 0$ such that $u \in C^{1,\alpha}(B_r(x_0) \cap \bar{\Omega}_u)$. Then $u \in W^{2,p}(B_r(x_0) \cap \Omega_u)$ for $p < \frac{1}{1-\alpha}$.

A bootstrap argument involving [8, Theorem 2.1] gives finally

> **Theorem**[4, Theorem 5.22]: The free boundary, $\partial\Omega_u$ is a C^∞ manifold locally around any regular FBP x_0.

The free boundary: Singular points

The **critical dimension** d^* is defined as the smallest d, such that a 1-homogeneous global minimizer $u : \mathbb{R}^d \to \mathbb{R}$ with at least one singular point exists (i.e. at least one blow-up is not the half-plane solution). Currently only $5 \leq d^* \leq 7$ is known, see [5, 2]. For the size of the singular set we have the following trichotomy.

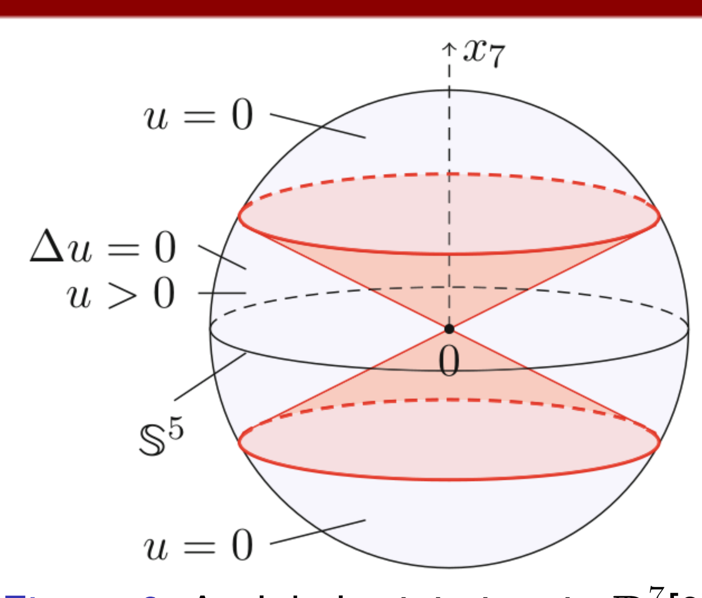

Figure 6: A global minimizer in $\mathbb{R}^7$[2]

> **Theorem**[10]: Let u be a minimizer of $\mathcal{F}_\Lambda$ on $D \subset \mathbb{R}^d$, then
> - if $d < d^*$: $Sing(\partial\Omega_u) = \varnothing$.
> - if $d = d^*$: $Sing(\partial\Omega_u)$ is discrete and locally finite.
> - if $d > d^*$: $\dim_{\mathcal{H}}(Sing(\partial\Omega_u)) = d - d^*$.

References

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