Optimality gaps and regularity for one-dimensional variational problems

Bachelor thesis MATK11

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Introduction

$$\min\ J(x) = \int_a^b \Lambda(t, x, x')dt, \quad x \in \mathcal{S}, x(a) = A, x(b) = B \quad \text{(P)}$$

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- weak minimizer
  $(|x_*(t) - x(t)| + |x_*'(t) - x'(t)| < \delta, \qquad \forall t \in [a, b])$
- strong minimizer $(|x_*(t) - x(t)| < \delta, \qquad \forall t \in [a, b])$
- <u>global</u> minimizer $(J(x_*) \leq J(x), \quad \forall x \in \mathcal{S})$

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Theorem

If $x_* \in C^2[a,b]$ and $\Lambda(t,x,v) \in C^2$, the Euler-Lagrange (EL) equation is satisfied

$$\Lambda_x(t, x_*(t), x_*'(t)) - \frac{d}{dt}\Lambda_v(t, x_*(t), x_*'(t)) = 0, \quad x_*(a) = A, x_*(b) = B$$

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- Integral EL: $x_* \in Lip[a,b]$ instead,
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If $\Lambda(t,x,v)$ convex in (x,v) then any solution of EL is the global minimum.

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A function $x : [a, b] \to \mathbb{R}$ is called **absolutely continuous** if there exists a Lebesgue-integrable function y s.t. $y = x'$ almost everywhere and

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$$C^2[a, b] \subsetneq C^1[a, b] \subsetneq Lip[a, b] \subsetneq AC[a, b] \subsetneq C[a, b]$$

Existence of solutions

Theorem (Tonelli 1915)

- $\Lambda(t, x, v)$, $\Lambda_v(t, x, v)$ *continuous*
- $\Lambda(t, x, v)$ *convex in* v
- $\Lambda(t, x, v) \geq \alpha |v|^r + \beta \quad \forall (t, x, v) \in [a, b] \times \mathbb{R} \times \mathbb{R}$ *where* $\alpha > 0$ *and* $r > 1$

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Extension: $\Lambda(t, x, v) \geq \alpha|v|^r - \gamma|x|^s + \beta$ where
$r > 1, r > s \geq 0, \gamma > 0$

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4. Show lower semicontinuity of functional J, i.e. $\liminf_{i\to\infty} J(x_i) \geq J(x_*)$

Tonelli's conditions (growth and convexity) almost "optimal".

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Counter-intuitive, $x \in AC$ can be approximated uniformly by $x \in Lip$, yet the infima do not become arbitrary close.
Bad for numerical solvers.

Lavrentiev phenomenon

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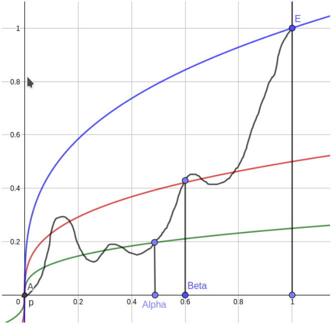
Idea:

1. choose subinterval $[\alpha, \beta]$ where function is squeezed between $cx^{1/3}$.
2. use upper bound and variable substitution to arrive at $J(x) \geq C \int_{\alpha'}^{\beta'} \bar{x}(s)^{2m} ds$
3. move power to the outside (Jensen's ineq.)
4. integrate and simplify $J(x) \geq \frac{7^2(2m-3)^{2m-1}}{2^{4m+6}(2m-1)^{2m-1}} > 0$



Lavrentiev phenomenon

Gap between *Lip* and C^1 using $g(t,x) = \frac{x^4 - t^4}{x^4 + t^4}$,

$J(x) = \int_{-1}^{1} \Lambda(t, x, x')dt = \int_{-1}^{1} |\frac{d}{dt} g(t, x(t))|dt, \;\; x(-1) = x(1) = 0$

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$$g(\epsilon, x(\epsilon)) = \frac{(x(0) + x'(0)\epsilon + h(\epsilon)\epsilon)^4 - \epsilon^4}{(x(0) + x'(0)\epsilon + h(\epsilon)\epsilon)^4 + \epsilon^4}$$

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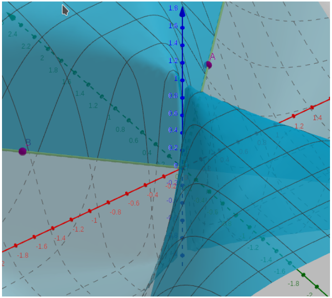
- split up integral
- if $x(0) \neq 0$ then $g(\epsilon, x(\epsilon)) \to 1$
- if $x(0) = 0$ and $x'(0) = 0$ then $g(\epsilon, x(\epsilon)) \to -1$
- if $x(0) = 0$ and $x'(0) \neq 0$ then $\exists \tau \neq 0$ s.t. $x(\tau) = 0$. $g(\tau, x(\tau)) = -1 \implies J(x) \geq \left| \int_{-1}^{\tau} g(t, x(t)) \right| = 1$



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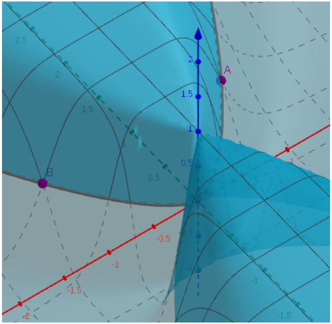
- define integrant as absolute value of derivative of $g(t, x(t)) = \frac{x(t)^4 - t^6}{x(t)^4 + t^6}$
- different limit at the origin
- use Taylor expansion and consider different cases



Regularity

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Theorem

If $\Lambda(t, x, v)$ is locally Lipschitz in v for every $(t, x) \in [a, b] \times \mathbb{R}$ and continuous in x for fixed (t, v), then there is no gap between Lip and $\mathcal{P}$, i.e. for all x admissible

$$\inf_{x \in Lip[a,b]} J(x) = \inf_{x \in \mathcal{P}[a,b]} J(x)$$

(locally Lipschitz: Lipschitz condition $|f(x) - f(y)| \leq L\|x - y\|$ holds in any compact interval)

Regularity

Ingredients of proof:

- Lusin's theorem for finding a continuous approximation to x'
- Weierstrass approx. gives approximation polynomial y
- $x - y$ bounded uniformly, $x' - y'$ bounded only in L^1
- splitting up the integral and estimating using continuity and locally Lipschitz property.

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Open question: if Λ only continuous, can there be gaps or not?

Regularity

Definition (Nagumo-growth along x_*)

$\lim_{t\to\infty} \frac{h(t)}{t} = \infty$ and $\Lambda(t, x_*(t), v) \geq h(|v|), \quad \forall t \in [a, b], v \in \mathbb{R}$

Theorem (Clarke-Vinter)

If x_* is a global minimizer for the functional $J(x)$ in the basic problem (P) over the class AC and the Lagrangian is autonomous, continuous, convex in v and has Nagumo growth along x_*, then x_* is Lipschitz.

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Theorem

If $\Lambda(t, x_*(t), v)$ strictly convex in v for almost all $t \in [a, b]$, then $x_* \in Lip[a, b]$ implies $x_* \in C^1[a, b]$.

(can be extended even to more classes C^k, C^∞)

Regularity

Sketch of proof for Clarke-Vinter:

- Set up contraint minimization problem
  $f(\alpha) = \int_a^b \Lambda(x_*(t), \frac{x_*(t)}{\alpha(t)})\alpha(t)dt, \qquad \text{s.t. } \int_a^b \alpha(t)dt = b - a$
- Lagrange multipliers (Hahn-Banach separation)
- measurable selection theorem
  $\int_a^b \inf_{p \in S} \phi(t,p)dt = \inf_{p \in \Sigma} \int_a^b \phi(t, p(t))$
- show x_*' is bounded (using Nagumo growth)

Application to boundary value problems

Idea: given ODE, find Lagrangian Λ, convex in (x, v) such that ODE is EL

Use existing theory to show existence ,uniqueness and smoothness of solution

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Example

$$x''(t) = \frac{1 - x'(t)^2}{1 + x'(t)^2} \qquad x(0) = 0, x(1) = 1$$

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1. Tonelli's theorem applies, existence is guaranteed
2. Show strict convexity of $J \implies$ minimum is unique
3. solution is Lipschitz by Clarke-Vinter theorem
4. apply higher regularity result to show smoothness.

Time for Questions!