Regularity for the one-phase problem

Master Project in mathematics

Florian Noah Grün
École Polytechnique Fédérale de Lausanne (EPFL), Switzerland

Supervisor Dr. Xavier Fernández-Real (EPFL)
External expert Prof. Hui Yu (National University of Singapore)

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Outline

- Introduction
- Existence and (non-) uniqueness
- Interior regularity
- Boundary regularity
  - Continuous boundary datum NEW!
  - Hölder continuous boundary datum NEW!
- The free boundary
  - First results
  - Blow-ups: regular and singular points
  - Local $C^{1,\alpha}$ regularity by improvement of flatness
  - From $C^{1,\alpha}$ to C^{∞} regularity NEW!
  - Singular set

Introduction: Motivation

- Free boundary problem: solve PDE for couple (u, Ω)
- Stefan problem: melting of ice
- One-phase (Bernoulli free boundary) problem
  $\hookrightarrow$ flame propagation, jet flows, ...

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- Bounded open domain $D \subset \mathbb{R}^d$, $\Lambda > 0$, $u \in H^1(D)$:

$$\boxed{F_\Lambda(u, D) := \underbrace{\int_D |\nabla u|^2 dx}_{\text{Dirichlet energy}} + \underbrace{\Lambda |\{u > 0\} \cap D|}_{\text{measure term}}}$$

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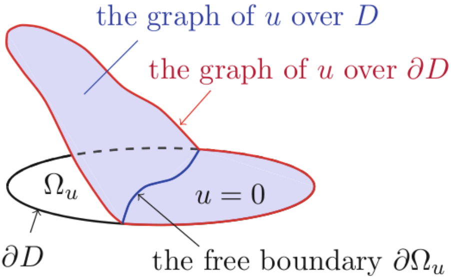
Existence: Some terminology

- $F_\Lambda(u, D) = \int_D |\nabla u|^2 dx + \Lambda|\{u > 0\} \cap D|$
- **Positivity set** $\Omega_u := \{x \in \mathbb{R}^d : u(x) > 0\}$,
  contact set $\{x : u(x) = 0\}$
- **Free Boundary** (FB) $\partial\Omega_u$
- $\begin{cases} \Delta u \geq 0 & \text{in } D, \\ \Delta u = 0 & \text{in } \Omega_u \end{cases}$



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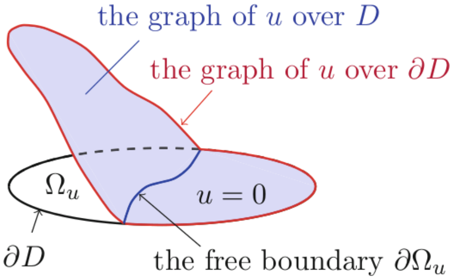

Existence: Global minimizers

- Recall $F_\Lambda(u,D) = \int_D |\nabla u|^2 dx + \Lambda|\{u>0\}\cap D|$
- Global minimizer $H^1_{loc} \ni u : \mathbb{R}^d \to \mathbb{R}$

  $\forall U \subset\subset \mathbb{R}^d$: $\forall v \in H^1(U)$ with $u - v \in H^1_0(U)$:

  $$F_\Lambda(u,U) \le F_\Lambda(v,U)$$

- Half-plane solution $h_\nu(x) = \sqrt{\Lambda}(x\cdot\nu)_+$

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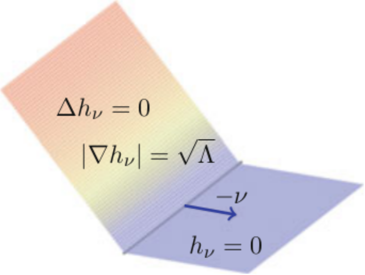
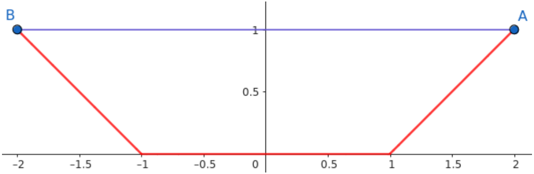


Uniqueness: Non-uniqueness

- NON uniqueness of minimizers: $d = 1$, $D = (-2, 2)$, $\Lambda = 1$



Uniqueness: Generic uniqueness

- If $g > \mathrm{diam}(D)$, then empty contact set $\implies$ uniqueness.

Lemma 2.7: Minimizers are ordered wrt data,

$$g_1 > g_2 \implies u_{g_1} \geq u_{g_2}.$$

Theorem 2.8: $g \in C(\partial D)$, $S_\lambda = \{\lambda \in \mathbb{R} : g + \lambda \geq 0\}$
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Interior regularity

- Let u be a minimizer of $F_\Lambda(u, D)$ with $u|_{\partial D} = g$.

Prop 3.3: u locally Lipschitz cont. in $D_\delta = \{\mathrm{dist}(x, \partial D) > \delta\}$:

$$\|\nabla u\|_{L^\infty(D_\delta)} \leq C(\Lambda, d) \left(1 + \frac{\|u\|_{L^\infty(D_{\delta/2})}}{\delta}\right)$$

$\hookrightarrow$ Lipschitz constant depends on distance to ∂D!

Prop 3.4: $x_0 \in \partial\Omega_u$, $B_r(x_0) \Subset D$:

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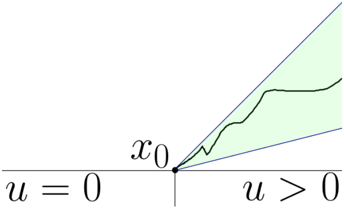

Boundary regularity: Motivation

- Classical potential theory: $D \subset \mathbb{R}^d$ sufficiently "nice",

$$\begin{cases} \Delta u = 0 & \text{in } D, \\ u = g & \text{on } \partial D, \end{cases} \quad \text{then} \quad \begin{array}{l} g \text{ cont.} \implies u \text{ cont. in } \bar{D}, \\ g\ \alpha\text{-Hölder} \implies u\ \alpha\text{-Hölder in } \bar{D}, \\ g \text{ Lip.} \implies u\ \alpha\text{-Hölder } \forall \alpha < 1 \text{ in } \bar{D}. \end{array}$$

- Similar results for minimizers to F_Λ?
  ↪ Useful for generic uniqueness
  ↪ Edelen, Spolaor, Velichkov [ESV22]: g is Lipschitz **YES**

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Boundary regularity: Continuous datum

Theorem 3.5: continuous datum g, boundary point $x_0 \in \partial D$
Then $x_k \to x_0 \implies u(x_k) \to u(x_0)$.

- Idea 1: Show $u > 0$ in $B_r(x_0) \cap D \implies \Delta u = 0$,
  then use harmonic regularity.

- Idea 2: solution on annulus and ordering lemma
  $\hookrightarrow$ works for convex domain D
  $\hookrightarrow$ works for C^1 domain D
  $\hookrightarrow$ works for c-Lipschitz domain D ($c \to 0$ when $d \to \infty$)

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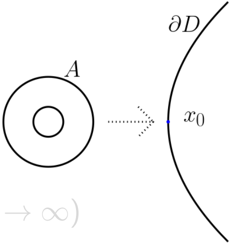

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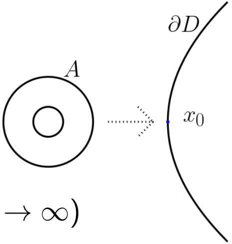

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Theorem 3.7: $g \in C^{\gamma_0}(\partial D), \quad \gamma_0 \in \left(\frac{1}{2}, 1\right) \implies u \in C^{\gamma_0}(\bar{D}).$

- Similar iteration argument from [ESV22]
- Morrey Lemma: show $\int_{B_r(x_0)} |\nabla u|^2 \leq C r^{d+2(\gamma_0 - 1)}, \quad x_0 \in \partial D$
- Change of variables Φ: straighten $B_r(x_0) \cap D$ to $B_r \cap H^+$
- Harmonic extension h_g of $g \circ \Phi$
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Free boundary: First results

- Density estimate: $0 < \delta_D < \frac{|B_r \cap \{u>0\}|}{|B_r|} < 1 - \delta_D < 1$
- If $\partial\Omega_u$ smooth: $|\nabla u| = \sqrt{\Lambda}$ a.e. on $\partial\Omega_u$
- u is viscosity solution to $\begin{cases} \Delta u = 0 & \text{in } \Omega_u, \\ |\nabla u| = \sqrt{\Lambda} & \text{on } \partial\Omega_u \cap D \end{cases}$

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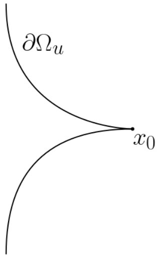

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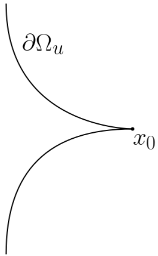

- u is viscosity solution to $\begin{cases} \Delta u = 0 & \text{in } \Omega_u, \\ |\nabla u| = \sqrt{\Lambda} & \text{on } \partial\Omega_u \cap D \end{cases}$

$\hookrightarrow x_0 \in \Omega_u : \Delta\phi(x_0) \leq 0,\ x_0 \in \partial\Omega_u : |\nabla\phi(x_0)| \leq \sqrt{\Lambda} \quad \forall\phi$ subsol.
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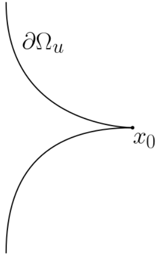
Free boundary: First results



- Density estimate: $0 < \delta_D < \frac{|B_r \cap \{u>0\}|}{|B_r|} < 1 - \delta_D < 1$
- If $\partial\Omega_u$ smooth: $|\nabla u| = \sqrt{\Lambda}$ a.e. on $\partial\Omega_u$
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Free boundary: Blow-ups

- "zoom-in infinitely" on FBP x_0

- $r_n \to 0$: $u_{x_0,r_n}(x) := \frac{1}{r_n}u(x_0 + r_n x) \xrightarrow{unif.} u_0 \in Lip_{loc}(\mathbb{R}^d)$
  $\hookrightarrow$ by Arzelà-Ascoli
  $\hookrightarrow$ dependent on r_n

- x_0 is **regular point** $(Reg(\partial\Omega_u))$:
  $u_0(x) = \sqrt{\Lambda}(x \cdot v)_+$ for $v \in \mathbb{S}^{d-1}$

- If not: x_0 is **singular point** $(Sing(\partial\Omega_u))$

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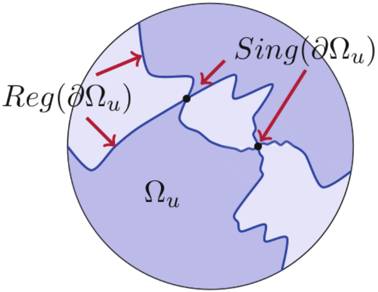

Free boundary: $C^{1,\alpha}$ regularity

- Wts: x_0 is regular FBP, then $\partial\Omega_u$ locally $C^{1,\alpha}$
- ε-**flat** for $v \in \mathbb{S}^{d-1}$: $|u - x \cdot v| \leq \varepsilon$ in $\Omega_u \cap B_1$
  $\hookrightarrow$ x_0 regular $\implies$ u_{x_0,r_n} ε-flat for large n
  $\hookrightarrow$ WLOG $v = e_d$, $x_0 = 0$, $\Lambda = 1$

Prop 5.14: improvement à la De Silva & Kriventsov
For $\tau > 0$, $\varepsilon \leq \bar{\varepsilon}(d, \tau)$, there is $e \in \mathbb{S}^{d-1}$ with $|e - e_d| \leq C'\varepsilon$ s.t.

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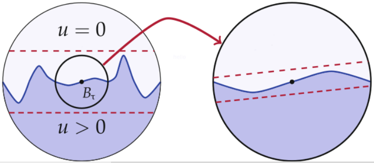

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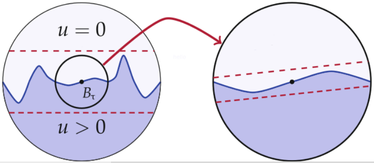

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- x_0 regular FBP, work locally
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Prop 5.20: u minimizer, 0 regular FBP, $\nabla u(0) = e_d$, $w := \frac{u_i}{u_d}$

Then $\begin{cases} \operatorname{div}(u_d^2 \nabla w) = 0 & \text{in } \Omega, \\ \partial_\nu w = 0 & \text{on } \partial\Omega_u \cap B_r, \end{cases}$ weakly **(1)**

- Terracini, Tortone, Vita [TTV22]: $w \in C^{k,\alpha}$ and **(1)** $\implies w \in C^{k+1,\alpha}$
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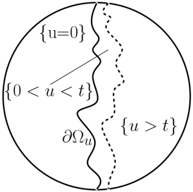

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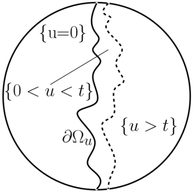

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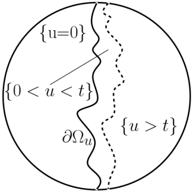

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- $u : \mathbb{R}^d \to \mathbb{R}$ is 1-*homogeneous* if $u(tx) = tu(x)\ \forall x, \forall t > 0$

Prop 6.7: Every blow-up is a 1-homogeneous global minimizer.

$$\hookrightarrow W_\Lambda(u) := \int_{B_1} |\nabla u|^2 dx - \int_{\partial B_1} u^2 d\,\mathcal{H}^{d-1} + \Lambda|\Omega_u \cap B_1|$$

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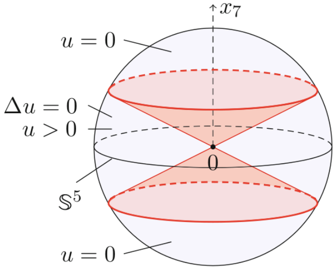

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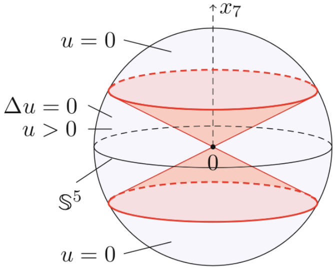

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Weiss [Wei99]: u minimizer of F_Λ on $D \subset \mathbb{R}^d$, then

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- Caffarelli, Jerison, Kenig [CJK04]: $d^* \geq 4$
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Thank you for listening!

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References

[CJK04] Luis Caffarelli, David Jerison, and Carlos Kenig. "Global energy minimizers for free boundary problems and full regularity in three dimensions". *Contemp. Math.* 350 (2004), pp. 83–97.

[DJ09] Daniela De Silva and David Jerison. "A singular energy minimizing free boundary". *Journal für die reine und angewandte Mathematik* 635 (2009), pp. 1–21.

[ESV22] Nick Edelen, Luca Spolaor, and Bozhidar Velichkov. "A strong maximum principle for minimizers of the one-phase Bernoulli problem". *arXiv:2205.00401* (2022).

[JS14] David Jerison and Ovidiu Savin. "Some remarks on stability of cones for the one-phase free boundary problem". *Geometric and Functional Analysis* 25 (2014), pp. 1240–1257.

[KN77] David Kinderlehrer and Louis Nirenberg. "Regularity in free boundary problems". *Annali Della Scuola Normale Superiore Di Pisa-classe Di Scienze* 4 (1977), pp. 373–391.

[TTV22] Susanna Terracini, Giorgio Tortone, and Stefano Vita. "Higher order boundary Harnack principle via degenerate equations". *arXiv:2301.00227* (2022).

[Wei99] Georg Weiss. "Partial regularity for a minimum problem with free boundary". *The Journal of Geometric Analysis* 9 (1999), pp. 317–326.