Some Regularity Results for the Obstacle Problem

Semester project

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Outline

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Introduction to the obstacle problem

$$\text{minimize } \int_\Omega \frac{1}{2}|\nabla v(x)|^2 dx \qquad \text{s.t. } v \geq \varphi \text{ and } v\big|_{\partial\Omega} = g \tag{OP}$$

$$\text{minimize } \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + fu\right) dx \quad \text{s.t. } u \geq 0, u|_{\partial\Omega} = g - \varphi, f := -\Delta\varphi \tag{ZOP}$$

- Under reasonable assumptions existence and uniqueness of minimizer
- Domain has two parts $\{u = 0\}$ and $\{u > 0\}$
- How does $\partial\{u > 0\}$ look like? Smooth?
- Blackboard-Sketch of basic setup

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Regularity of solution

- EL equation: $\Delta u = f\chi_{\{u>0\}} \Leftrightarrow \begin{cases} \Delta u &= f \quad \text{in } \{u>0\}, \\ u &= 0 \quad \text{on } \partial\{u>0\}, \\ \nabla u &= 0 \quad \text{on } \partial\{u>0\}. \end{cases}$
- Directly $u \in C^{1,1-\epsilon}$ by Schauder estimate
- $0 < cr^2 < \sup_{B_r(x_0)} u \leq Cr^2$ for any FBP x_0 (Harnack's inequality and scaling)
- Nondegeneracy needs extra assumption: $f \geq c_0 > 0$
- Quadratic growth & Schauder estimates give $C^{1,1}$ regularity of u.
- Schauder estimate: solution is two derivatives smoother than RHS

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Free Boundary: Classification of blow-ups

- Our setting: $\begin{cases} u & \in & C^{1,1}(B_1), \\ u & \geq & 0, \\ \Delta u & = & \chi_{\{u>0\}}, \\ 0 & \text{is FBP.} & \end{cases}$

- Rescaling ("zooming in") $\quad u_r(x) := \frac{u(x_0+rx)}{r^2} \longrightarrow \mathbf{u_0} \quad$ in $C^1_{loc}(\mathbb{R}^n)$

- How does the blow-up (if it exists) look?

- Global problem: $\begin{cases} u_0 & \in & C^{1,1}_{\text{loc}}(\mathbb{R}^n), \\ u_0 & \geq & 0, \\ \Delta u_0 & = & \chi_{\{u_0>0\}}, \\ 0 & \text{is FBP.} & \end{cases}$

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- Blow-ups are 2-homogeneous and convex!
- Two types of blow-ups:
    1. $u_0(x) = \frac{1}{2}(x \cdot e)_+^2$ if $\{u_0 = 0\}$ has non-empty interior (half-space)
    2. $u_0(x) = x^T A x$ $\{u_0 = 0\}$ has empty interior (line)

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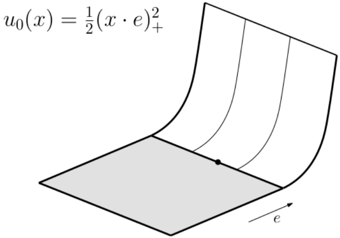

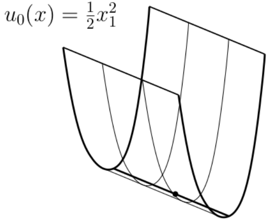

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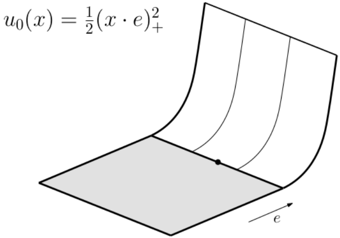

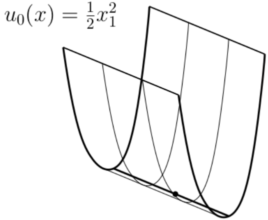

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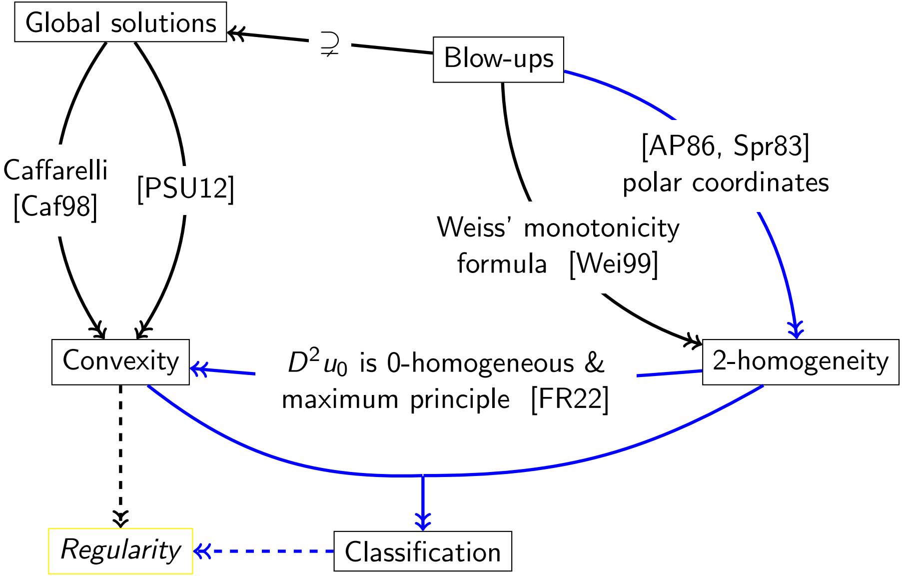

Free Boundary: Regularity

- Regular FBP: $\limsup_{r\to 0} \frac{|\{u=0\}\cap B_r(x_0)|}{|B_r|} > 0$
- u has a blow-up $u_0 = \frac{1}{2}(x\cdot e)_+^2$ for some $e \in \mathbb{S}^{n-1}$.
- u_0 and its derivative are "close" to u_{r_0} for some r_0.
- The free boundary of the scaled version u_{r_0} around 0 is contained in a strip.
- There exists a cone (not a full half space) of directions τ, in where u_{r_0} is non-decreasing. (transferring monotonicity from u_0 to u_{r_0})
- Free boundary of u_{r_0} is Lipschitz (interior and exterior cone condition) $\implies$ free boundary of u is also locally Lipschitz.

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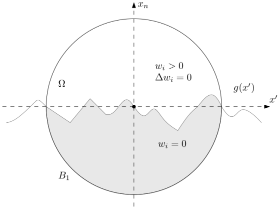
Free Boundary: Regularity

- Boundary Harnack:
  Ω is Lip, w_1, w_2 harmonic $\implies \left\| \frac{w_1}{w_2} \right\|_{C^{0,\alpha}} \leq C$.

- Higher order Boundary Harnack:
  Ω is $C^{k,\alpha}$, w_1, w_2 harmonic $\implies \left\| \frac{w_1}{w_2} \right\|_{C^{k,\alpha}} \leq C$



- Boundary Harnack gives $C^{1,\alpha}$ regularity around the origin.
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- FB has no corners!

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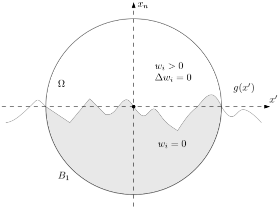

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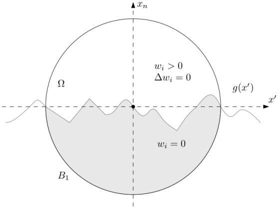

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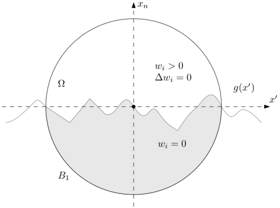

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Free Boundary: Singular points

- Singular FBP: $\limsup_{r \to 0} \frac{|\{u=0\} \cap B_r(x_0)|}{|B_r|} = 0$
- Caffarelli's dichotomy (1977):
  1. regular points: $u_0(x) = \frac{1}{2}(x \cdot e)_+^2$
  2. singular points: $u_0(x) = \frac{1}{2}x^T Ax$
- Blow-up at singular points is unique (Monneau monotonicity formula)
- Continuous dependence of blow-ups
- Caffarelli [Caf98]: The singular set is locally contained in a C^1 $(n-1)$ dimensional manifold.
- Monneau [Mon03]: Generically (for "almost every" boundary data) the free boundary has no singular points in $\mathbb{R}^2$.

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Thank you for listening!

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