Excitation of spikes in the monostable FitzHugh-Nagumo system

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Abstract

We consider the FHN model in the case of a single stable equilibrium and discuss sufficient conditions for the creation of spikes due to an impulse that can be extended or of Dirac-type. We first consider the definition of a spike based on non-convexity of the trajectory, whose existence a priori requires sufficiently strong scale separation. We then introduce a notion of spikes based on a voltage threshold that can be used for any parameter set. Finally, we discuss repeated impulses and the creation of periodic orbits in a hybrid impulse system.

1 Introduction and Motivation

This work was produced during two visits at Bremen University (August 2022) and Hamburg University (February 2023).

In nature, there exist many excitable media, that is a system that is reacting to an impulse with some form of wave. Yet after propagating a wave, the system needs some time to "cool down" before another impulse can be processed and a new wave emitted. Such processes occur in chemistry (Belousov–Zhabotinsky reaction) and in cardiac and neural systems. An impulse, a rapid rise and fall in the membrane potential of a cell will cause a propagating electric spike (known as *action potential*). After the spike, at entering into the refractory period, the system becomes unresponsive to further excitation, depending of course on the biological components involved.

The more realistic mathematical approach to action potentials in neurons is the 4-dimensional *Hodgkin-Huxley* model, however we focus on the 2-dimensional FitzHugh-Nagumo model. One standard formulation of the (non-spatial) *FitzHugh-Nagumo model* is given by

$$\begin{aligned} \dot{u}(t) &= f(u,w) + I(t) &&= -bu(t)(u(t)-1)(u(t)-a) - w(t) + I(t), \\ \dot{w}(t) &= \varepsilon g(u,w) &&= \varepsilon(u(t) - cw(t)), \end{aligned} \tag{FHN}$$

where $\varepsilon, a, b, c > 0, \varepsilon \ll 1$ are real parameters and $I : \mathbb{R} \to \mathbb{R}$ a control impulse, i.e. a perturbation/excitation to the dynamical system that causes the stable state $(0,0)$ to change into a nontrivial trajectory.

We investigate two different definitions to characterize spikes and give in both cases sufficient conditions on impulses. The first definition is based on a convexity condition, commonly used in the literature and only works for ε sufficiently small. Therefore, we propose an alternative condition based directly on the u-value.

Definition 1.1. *An integrable function $I : \mathbb{R} \to \mathbb{R}^+$, with support in $[0,T]$ is called an* ***impulse*** *of length T.*

Definition 1.2. *A* ***Dirac impulse*** *at time t_0 of size I_0, i.e. $I(t) = I_0\delta_{t_0}$, maps $(u(t_0), w(t_0))$ to $(u(t_0+), w(t_0+)) = (u(t_0) + I_0, w(t_0))$. By abuse of notation we say a Dirac impulse is an impulse of length $0+$.*

Definition 1.3. *The* ***epigraph*** *of a function* $\phi : \mathbb{R} \to \mathbb{R}$ *is defined as*

$$\operatorname{epi} \phi = \{(x, y) \in \mathbb{R}^2 : y \geq \phi(x)\}. \tag{1.1}$$

In the following analysis the so called *nullclines* for $I(t) = 0$, i.e. the curves where $\dot{v}$ or $\dot{w}$ are zero, will be crucial. Thus the curves

$$\begin{aligned} \mathcal{N}_u : &\quad w = -bu(u-a)(u-1) = -bu^3 + b(a+1)u^2 - abu =: h(u), \\ \mathcal{N}_w : &\quad w = \tfrac{1}{c}u \end{aligned} \tag{1.2}$$

can have one, two (degenerate) or three intersections depending on the choice of parameters. For large a and b (in fact $(1-a)^2 > 4/(bc)$) there are three intersection points and the system, becomes bistable ([DP22] Fig 2.4e). We are only interested in the case with one intersection, assume thus always that $(1-a)^2 < 4/(bc)$. Moreover as $a > 0$, the equilibrium at $(0, 0)$ is locally stable. Note also that the system is dissipative and all trajectories (even with a spike) will eventually converge to the origin.
For our examples we set the parameters as $a = 3/8$, $b = 5$, $c = 1$.

2 Spikes based on convexity criterion

The aim is to investigate under what parameters and impulses $I(t)$, spikes occur in (FHN). A common definition for spikes (cf [DJ11], [DKR12]) is based on a change in curvature of the trajectory.

Definition 2.1. *Let* $(u(t), w(t))$ *with* $(u(0), w(0)) = (0, 0)$ *be a trajectory, perturbed by a pulse* $I(t)$ *with* $\operatorname{supp} I = [0, T]$. *We call it a* ***spike*** *if there exists* $t_s \geq T > 0$, *such that it has a change in the (signed) curvature at time* t_s.

Remark 2.2. In view of the system (FHN), before t_s, the trajectory is the graph of a strictly convex function, after t_s the graph of a strictly concave function. At some other point $t_2 > t_s$, there is a second change in curvature and the trajectory returns to the origin. In other words, the region enclosed by the trajectory is not completely convex, since its boundary has a change in curvature.

We are interested in the inflection set, i.e. the set where $w''(u) = 0$. This leads to

$$\frac{d^2}{du^2} w(u) = \frac{d}{du} w'(u) = \frac{d}{du} \frac{\frac{dw}{dt}}{\frac{du}{dt}} = \frac{d}{du} \frac{\dot{w}}{\dot{u}} = \frac{d}{du} \frac{\varepsilon g(u, w(u))}{f(u, w(u))}. \tag{2.1}$$

In turn we obtain the *inflection equation*

$$i(u, w) = f(f_u g - f g_u) + \varepsilon g(f_w g - f g_y), \tag{2.2}$$

which in view of (FHN) evaluates to (Figure 1)

$$\begin{aligned} i(u, w) &= (h(u) - w)(h_u(u)(u - cw) - (h(u) - w)) + \varepsilon(u - cw)(-(u - cw) + c(h(u) - w)) \\ &= (-bu^3 + b(a+1)u^2 - abu - w)\left((-3bu^2 + 2b(a+1)u - ab)(u - cw) - (-bu^3 + b(a+1)u^2 - abu - w)\right) \\ &\quad + \varepsilon(u - cw)(-(u - cw) + c(-bu^3 + b(a+1)u^2 - abu - w)) \\ &= \left(-5u^3 + \frac{55}{8}u^2 - \frac{15}{8}u - w\right)\left(\left(-15u^2 + \frac{110}{8}u - \frac{15}{8}\right)(u - w) - \left(-5u^3 + \frac{55}{8}u^2 - \frac{15}{8}u - w\right)\right) \\ &\quad + \varepsilon u\left(-(u - w) + \left(-5u^3 + \frac{55}{8}u^2 - \frac{15}{8}u - w\right)\right) \end{aligned} \tag{2.3}$$

The zero level set W_ε of the inflection equation has 4 components (cf. [DKR12] has only 3?). Interesting for our purpose here is the middle component. When increasing ε, it wanders eventually below

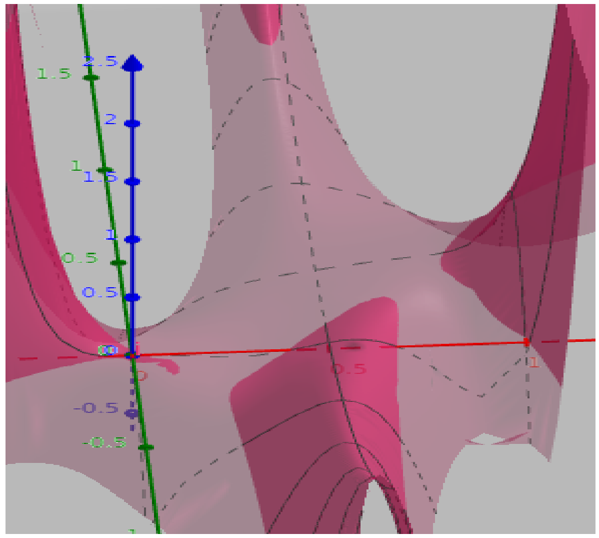


Figure 1: The inflection equation as a function $i(u, w)$ for $\varepsilon = 0.1$

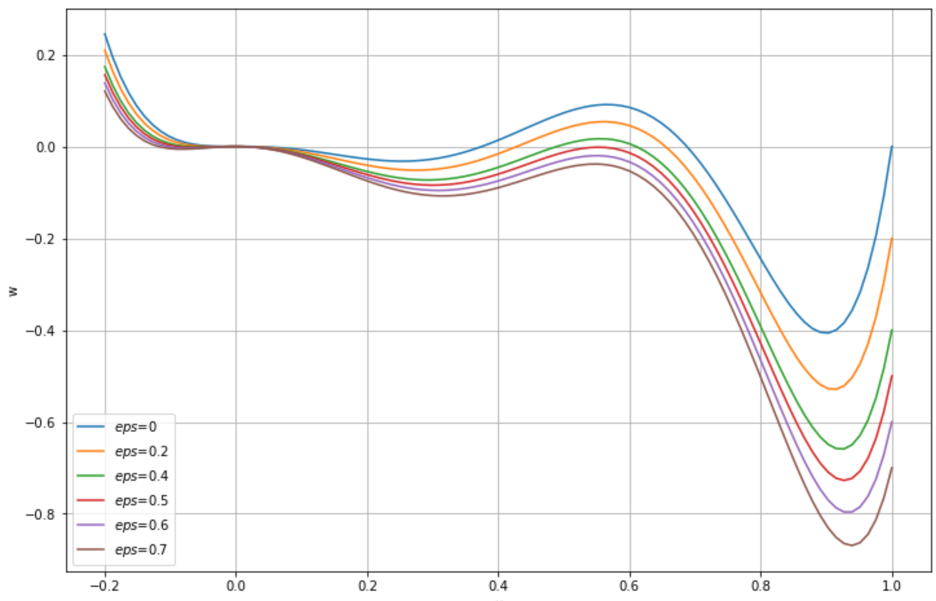


Figure 2: Inflection equation on the u-axis ($w \equiv 0$) for different ε

the u-axis (cf. [DKR12] where it becomes complex and vanishes). (When e.g. $c = 1/2$, then increasing ε will cause the middle component to move below the u-axis and eventually vanish).

This can be seen when solving the inflection equation in u for $w = 0$. For $\varepsilon > \varepsilon_0 \approx 0.5$, two real solutions become complex (Figure 2). Due to the fact that the middle component of the zero level set only becomes negative and does not vanish, makes it difficult to find an analytic expression for the threshold ε_0. For the parameter set $a = 3/8, b = 5, c = 1$, the inflection equation for $w = 0$ becomes a quartic polynomial

$$p_\varepsilon(x) = -23\varepsilon/8 + (-(825/64) + (55\varepsilon)/8)x + (4225/64 - 5\varepsilon)x^2 - (825x^3)/8 + 50x^4, \tag{2.4}$$

with discriminant

$$\Delta_\varepsilon = -\frac{15625}{1073741824}(-265869140625 - 495493750000\varepsilon + 1122641035200\varepsilon^2 + 1889203003392\varepsilon^3 + 233414852608\varepsilon^4 + 17582522368\varepsilon^5). \tag{2.5}$$

The real solution (i.e. where the discriminant changes sign and the two real roots become complex) can be calculated numerically to be $\varepsilon_0 \approx 0.491$. In [DKR12] an analytic expression for ε_0 is given, since there it suffices to show that certain solutions to (2.2) become complex, whereas in the setting here the middle component wanders below the u-axis without necessarily becoming complex. This increases the problem difficulty and appears to make an analytic expression impossible.

Furthermore, setting $w = \beta$ in (2.3) and solving for u numerically, gives the middle component of the inflection sets (Figure 3).

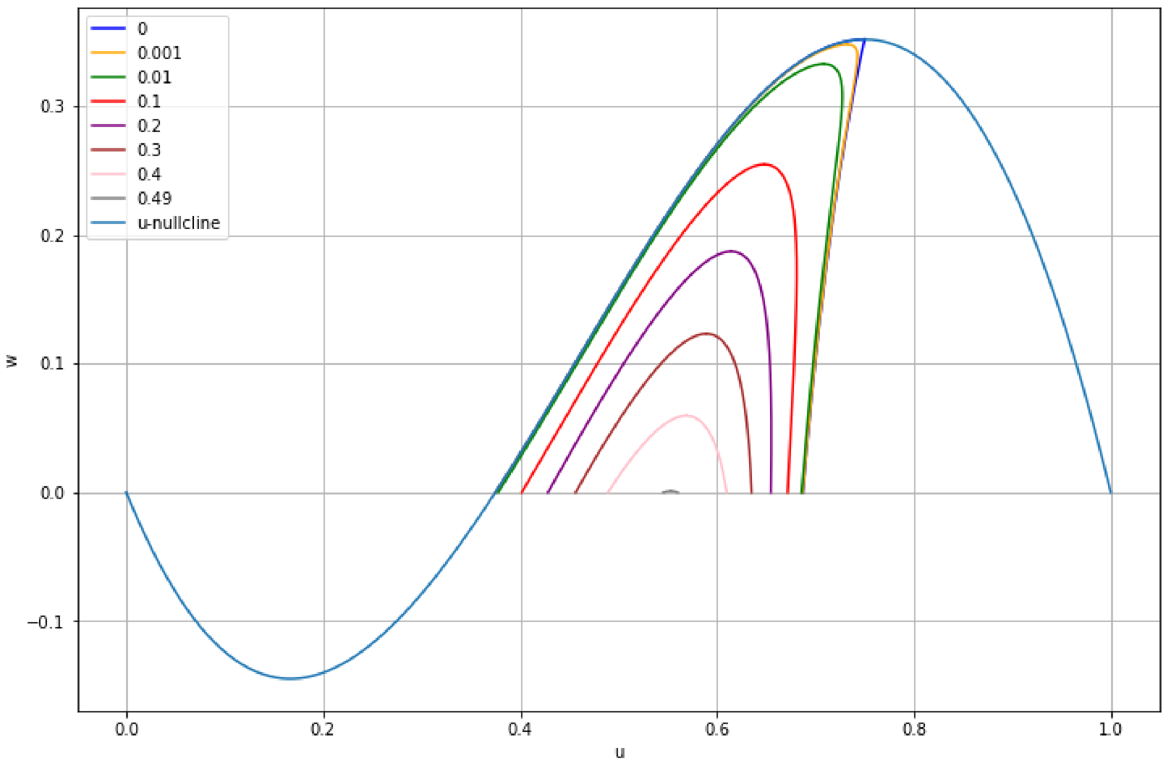


Figure 3: Inflection sets for different ε

We note that as $\varepsilon \to 0$, the inflection equation becomes

$$f(f_u g - f g_u) = 0, \tag{2.6}$$

with solutions $w_1(u) = h(u)$, the u-nullcline, and $w_2(u) = \frac{h - h_u(u)}{1 - h_u(u)} = \frac{-55x^2+80x^3}{23-110x+120x^2}$. In other words the inflection sets W_ε converge to the area bounded by the u-axis, w_1 and w_2. Interestingly to note, for some ε_1 with $\varepsilon_0 > \varepsilon_1 > 0$, W_ε changes from "leaning right" to the proper graph of a function. To conclude, inside W_ε, the trajectory is the graph of a concave function $w(u)$, outside it is convex and on ∂W_ε the signed curvature changes. In other words, any spike lies partially in W_ε

We want now a theorem giving a sufficient condition on arbitrary impulses of length T. Clearly, if $(u(T), w(T)) \in W_\varepsilon$, the trajectory is a spike, by definition. To estimate arbitrary impulses, we need to fit a rectangular box under W_ε. (We loose a lot of the area of W_ε, maybe some other shapes like triangles could make for finer estimates) Let $0 < \varepsilon < \varepsilon_0$ be fixed, we propose the following:

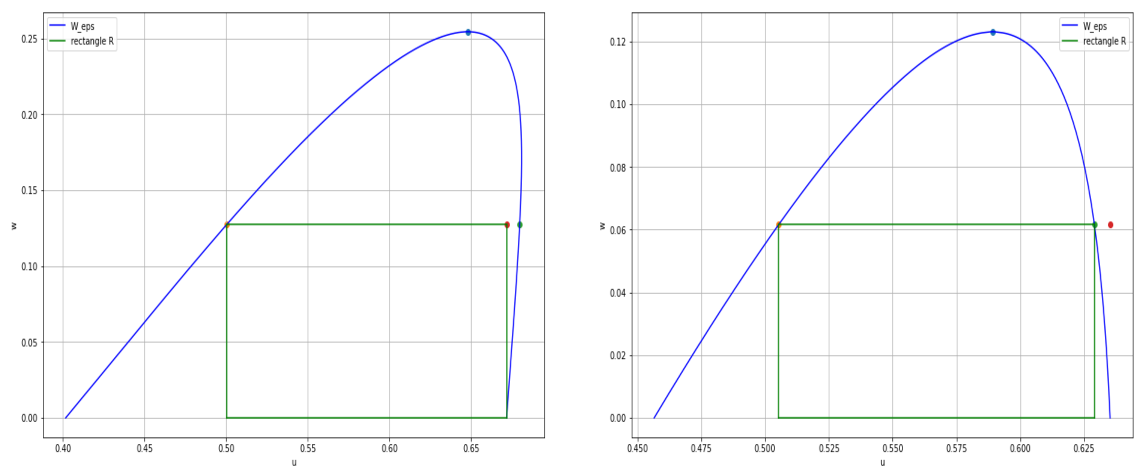
1. Calculate β_{max}, the maximum of W_ε, as the second real positive root of the discriminant of $x \mapsto i(x, \beta)$. The discriminant is a high order polynomial (here degree 14!) and can only be solved numerically. For $\beta = 0$, the resulting polynomial $p_\varepsilon(x)$ is of degree 4, but for $\beta > 0$, $p_\varepsilon(x)$ is of degree 6, thus the discriminant is of higher degree. Note $u \mapsto i(u, \beta_{max})$ has a double root.

2. Calculate the 4th and 5th root of $i(u, \beta_{max}/2)$, and denote as $r_{4,\beta/2}$ and $r_{5,\beta/2}$.

3. Calculate the 5th root of $i(u, 0)$, denoted as $r_{5,0}$.

4. Set $W_{max} := \beta_{max}/2$ and $U_{min} := r_{4,\beta_{max}/2}$, $U_{max} := \min\{r_{5,\beta_{max}/2}, r_{5,0}\}$. The last part is needed to make sure that the rectangle R with vertices $(U_{min}, 0)$, $(U_{min}, \beta_{max}/2)$, $(U_{max}, 0)$, $(U_{max}, \beta_{max}/2)$ lies completely inside W_ε, even for "rightward leaning" W_ε ($\varepsilon < \varepsilon_1$). Compare to Figure 4.

Remark 2.3. We should actually check that the rectangle R is not empty, i.e. $U_{min} < U_{max} \leq r_{5,0}$. This follows from the fact that the limiting case $\varepsilon \to 0$ has the furthest rightward leaning W_ε and there $U_{min} < r_{5,0} = \frac{11}{16}$, hence W_ε does not lean enough to the right, such that $(U_{min}, \beta_{max}/2$ were to the right of $(U_{max}, 0)$.



(a) $a = \frac{3}{8}$, $b = 5$, $c = 1$, $\varepsilon = 0.1$ (b) $a = \frac{3}{8}$, $b = 5$, $c = 1$, $\varepsilon = 0.3$

Figure 4: The set W_ε and the rectangle R for different values of ε, illustrating the definition of U_{max}

We can now characterize impulses leading to a spike, i.e. sufficient conditions such that $(u(T), w(T)) \in R \subseteq W_\varepsilon$.

Theorem 2.4. *For ε fixed, let $I(t)$ be an impulse of finite length T for* (FHN) *to the steady state* $(0, 0)$. *Calculate W_{max}, U_{max} and U_{min} as proposed. If the following conditions*

(i) $\int_0^T I(t) \leq (1 - T^2\varepsilon)U_{max} - Th(u_s)$ (upper bound 1),

(ii) $T\varepsilon U_{max} \leq W_{max}$ (upper bound 2),

(iii) $\int_0^T I(t) \geq \left(1 - \frac{T^2\varepsilon}{1-T^2\varepsilon}\right)^{-1} \left(U_{min} + \frac{T^3\varepsilon h(u_s)}{1-T^2\varepsilon} - T \min_{u\in[0,u_s]} h(u)\right)$ (lower bound),

are satisfied, then the trajectory is a spike.

Proof. To assure that the impulse does not send the trajectory too far, start by estimating u and w. We use that here $w(t) \geq 0$ for all $t \in [0,T]$ and $t_{m_w}, t_{m_u} \leq T$, where w and u respectively are maximized over $[0,T]$.

$$
\begin{aligned}
w_m &:= \max_{t\in[0,T]} w(t) = w(t_{m_w}) = \int_0^{t_{m_w}} \varepsilon u(t) - \varepsilon c w(t) \leq \int_0^{t_{m_w}} \varepsilon u(t) \leq T\varepsilon u_m, \\
u_m &:= \max_{t\in[0,T]} u(t) = u(t_{m_u}) = \int_0^{t_{m_u}} h(u(t)) - w(t) + I(t) \\
&\leq T \max_{t\in[0,T]} h(u(t)) + \int_0^T I(t) \\
&\leq T \max_{t\in[0,T]} \{w_m, h(u_s)\} + \int_0^T I(t) \\
&\leq T^2 \varepsilon u_m + T h(u_s) + \int_0^T I(t) \\
&\implies u_m \leq \frac{1}{1-T^2\varepsilon}\left(Th(u_s) + \int_0^T I(t)\right) \leq U_{max} \qquad \text{from upper bound 1,} \\
&\implies w_m \leq T\varepsilon u_m \leq T\varepsilon U_{max} \leq W_{max} \qquad \text{from upper bound 2.}
\end{aligned}
\tag{2.7}
$$

Hence $u(T) \leq U_{max} \leq u_s$ and $w(T) \leq W_{max}$, so it remains to check that $(u(T), w(T))$ is inside R, i.e. $u(T) \geq U_{min}$. Remember $W_{max} = \beta_{max}/2$, then

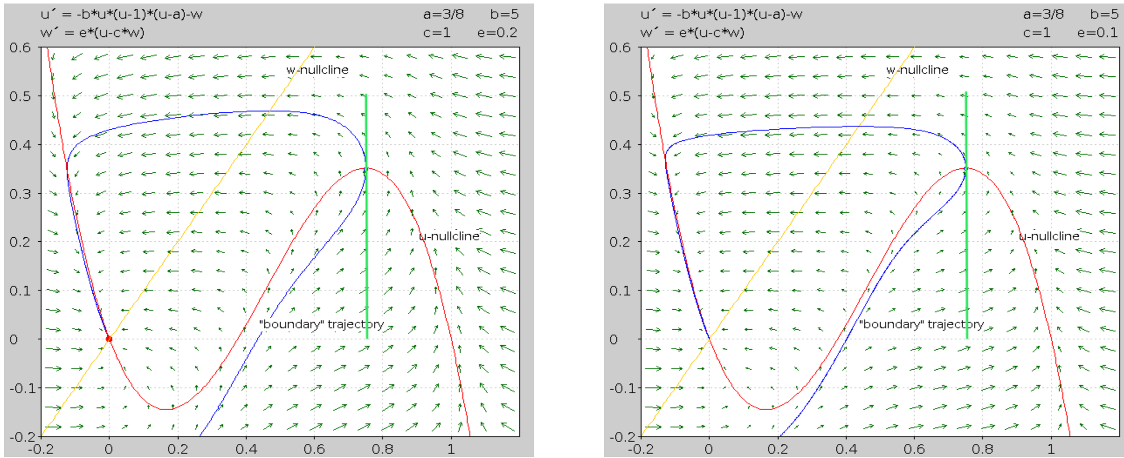
$$
\begin{aligned}
u(T) &= \int_0^T h(u(t)) - \int_0^T w(t) + \int_0^T I(t) \\
&\geq T \min_{u\in[0,u_s]} h(u) - T w_m + \int_0^T I(t) \\
&\geq T \min_{u\in[0,u_s]} h(u) - T^2 \varepsilon u_m + \int_0^T I(t) \\
&\geq T \min_{u\in[0,u_s]} h(u) - \frac{T^3 \varepsilon h(u_s)}{1-T^2\varepsilon} + \left(1 - \frac{T^2\varepsilon}{1-T^2\varepsilon}\right) \int_0^T I(t) \\
&\geq U_{min} \qquad \text{from the lower bound.}
\end{aligned}
\tag{2.8}
$$

It is already known that $0 > \min_{u\in[0,u_s]} h(u) = h\left(\frac{a+1-\sqrt{a^2-a+1}}{3}\right) < -\infty$. We conclude that $U_{min} \leq u(T) \leq U_{max}$ and $0 \leq w(T) \leq W_{max}$ and thereby $(u(T), w(T))$ lies in R so the trajectory is a spike. □

3 Spikes based on u-elongation

As an alternative to the previous definition of spikes (which only works for $\varepsilon < \varepsilon_0$), we propose an alternative definition. The motivation stems from the idea that the u-coordinate represents the voltage of a neuronal system, whereas the w-variable is an abstract variable representing the refractory process, i.e. u has a greater physical meaning, it is natural to ask whether u exceeds a certain threshold u_s. The threshold u_s exactly crossing the local maximum of $\mathcal{N}_u$ is motivated by the fact that left of u_s, $\dot{u}$ is increasing, but decreasing to the right, i.e. $\ddot{u}$ vanishes. There exists a "boundary trajectory" barely touching the u-threshold, that separates spike trajectories from non-spike trajectories (Figure 5).

Definition 3.1. *A trajectory $(u(t), w(t))$ with $(u(0), w(0)) = (0,0)$ is a **u-spike** if there exists t_s such that $\dot{u}(t_s) > 0$ and $u(t_s) = u_s$ with the u-spike threshold $u_s := (\sqrt{a^2-a+1} + a + 1)/3$. In view of the FitzHugh-Nagumo system* (FHN)*, the closed trajectory crosses the vertical line $u = u_s$, which intersects the local maximum of the u-nullcline.*



(a) $a = \frac{3}{8}$, $b = 5$, $c = 1$, $\varepsilon = 0.2$

(b) $a = \frac{3}{8}$, $b = 5$, $c = 1$, $\varepsilon = 0.1$

Figure 5: The boundary trajectory of $\tilde{A}_\varepsilon$ for different values of ε

Let the initial condition $(u_0, w_0) = (0, 0)$, the stable equilibrium, and give an impulse $I(t)$ of length T. It depends on the position of $(u(T), w(T))$ (or $(u(t_0+), w(t_0+))$ in case of a Dirac impulse) whether the trajectory is a u-spike or not. Of course, if $u(T) \geq u_s$, the trajectory is trivially a u-spike, so we restrict ourselves to $u(T) < u_s$. Due to the positivity of the impulse, $w(T)$ is always non-negative. Also if $(u(T), w(T))$ lies above $\mathcal{N}_u$, then the flow is leftwards and no u-spike can occur. However it is NOT the case (as could be interpreted in [DP22], Section 2.2.4) that $(u(T), w(T))$ below $\mathcal{N}_u$ implies a u-spike. If $(u(T), w(T))$ were close enough to $\mathcal{N}_u$, the trajectory just returns back over $\mathcal{N}_u$ and does not cross the threshold.

Example 3.2. Take $a = \frac{3}{8}$, $b = 5$, $c = 1$, $\varepsilon = 0.2$ and consider $I(t) = I_0 \delta_0(t)$ with $I_0 = \frac{2}{5}$. We show that despite $I_0 > a$, the trajectory is not a u-spike. At $(u, w) = \left(\frac{2}{5}, 0\right)$ we have $(u'(t), w'(t)) = \left(\frac{3}{100}, \frac{2}{25}\right)$, a slope of $\frac{8}{3}$. Define now from $\dot{w} = \frac{8}{3}\dot{u}$ (i.e. $-5u^3 + \frac{55}{8}u^2 - \frac{15}{8}u - w = \frac{3}{40}u - \frac{3}{40}w$),

$$z(u) := \left(\frac{1}{1 - 3/40}\right)\left(-5u^3 + \frac{55}{8}u^2 - \frac{78}{40}u\right). \tag{3.1}$$

Hence for all points $(u(t), w(t)) \in \operatorname{epi} z(u)$ we have $\dot{w} \geq \frac{8}{3}\dot{u}$, the flow is directed steeper upwards than $(\frac{8}{3}, 1)$. Let now $l(u)$ be the line going through the point $(\frac{2}{5}, 0)$, so

$$w(t) \geq \frac{8}{3}u - \frac{16}{15} =: l(u). \tag{3.2}$$

Therefore the intersection between the trajectory (with slope steeper than $\frac{8}{3}$) and $\mathcal{N}_u$ happens before the intersection of l (with exact slope of $\frac{8}{3}$) and $\mathcal{N}_u$, which is at $u = 0.421 \ll 0.75 = u_s$ and so the trajectory never crosses the u-spike threshold.

However, the area $\tilde{A}_\varepsilon$ enclosed by the boundary trajectory, the u-axis and the vertical threshold $u = u_s$ can only be found by solving the ODE-system numerically. The next theorem basically uses a first order approximation of the vector field, to find a closed set $A_\varepsilon \subseteq \tilde{A}_\varepsilon$ (or the curve describing its left boundary ∂A_ε) where a u-spike is guaranteed.

Theorem 3.3. *Let u_s be the u-spike threshold, ε fixed and define for $\lambda \in \mathbb{R}^+$:*

- *the cubic curve $z_\lambda(u) := \left(\frac{1}{1-c\varepsilon\lambda}\right)(f(u) - \varepsilon\lambda u)$;*
- *the point of intersection between $z_\lambda(u)$ and the vertical line at u_s, i.e. $(u_s, z_\lambda(u_s))$;*

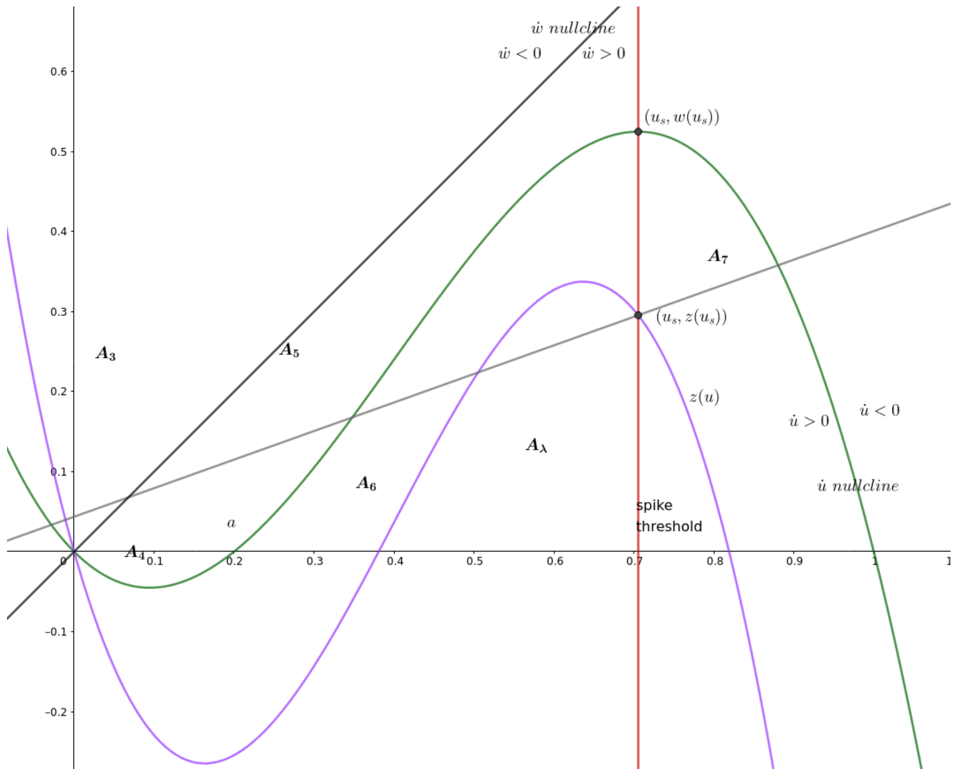


Figure 6: The idea of the proof for Theorem 3.3

- $l_\lambda(u) := \frac{1}{\lambda}u + z(u_s) - \frac{1}{\lambda}u_s$ *as the line going through* $(u_s, z_\lambda(u_s))$ *with slope* $1/\lambda$.

Let now (hyp $f := \{(u,r) \in \mathbb{R} \times \mathbb{R} : r \leq f(u)\}$)

$$A_\lambda := \{u : a \leq u \leq u_s\} \cap \operatorname{hyp} l_\lambda \cap \operatorname{hyp} z_\lambda \qquad \textit{and} \qquad \tilde{A}_\varepsilon := \bigcup_\lambda A_\lambda. \tag{3.3}$$

Then an impulse $I(t)$ *of length* T *at time* t_0 *gives a u-spike at whenever* $(u(t_0+T), w(t_0+T)) \in A_\varepsilon$.

Proof. Without loss of generality assume that $t_0 = 0$ and let I(t) be an impulse such that $P := (u(T), w(T)) \in A_\varepsilon$. Then for some $\lambda \in \mathbb{R}^+$, $P \in A_\lambda$ and so $P \in \operatorname{hyp} z_\lambda = \{(u,w) : \dot u \geq \lambda \dot w\}$, which follows directly from the state equations similar to Example 3.2.

Now the trajectory starting from P always lies below $l_\lambda(u)$, as the flow inside A_λ is directed more rightward than the vector $\left(1, \frac{1}{\lambda}\right)$ (i.e. $(u(t), w(t)) \cdot \left(1, \frac{1}{\lambda}\right) \geq 0$ for any trajectory starting in A_λ) by definition of hyp z_λ. Moreover as $|f(u) - z_\lambda(u)| \geq> 0$ (λ was fixed) for $u \in [a, u_s]$, the trajectory moves at positive speed in the u-direction (i.e. there is no point where the flow vanishes) and so reaches at some finite time t_s the point $(u(t_s), w(t_s)) = (u_s, w(t))$, and is thereby a u-spike. Moreover as $|f(u) - z_\lambda(u)| \geq> 0$ (λ was fixed) for $u \in [a, u_s]$, the trajectory moves at positive speed in the u-direction (i.e. there is no point where the flow vanishes) and so reaches at some finite time t_s the point $(u(t_s), w(t_s)) = (u_s, w(t))$, and is thereby a u-spike. See also Figure 6. □

Remark 3.4. The converse does not hold. There is positive flux into A_ε across ∂A_ε, e.g. from points in hyp $z_\lambda \backslash$ hyp l_λ close to l_λ. Thus $A_\varepsilon \subsetneq \tilde{A}_\varepsilon$.

To exactly find the area A_ε from the preceding abstract theorem involves solving a quartic equation, which is considerably easier than the ODE-system.

Remark 3.5. The left boundary of A_ε, ∂A_ε is an increasing function in u.

Corollary 3.6. *Let $I(t) = I_0\delta(t-t_0)$ be a Dirac impulse to the state $(0,\beta)$, with $\beta \le f(u_s)$. Then there exists a constant $C_\varepsilon(\beta)$ with $u_s > C_\varepsilon(\beta) > a$ such that any I_0 satisfying $u_s \ge I_0 \ge C_\varepsilon(\beta)$ causes a u-spike.*

Proof. Take $C_\varepsilon(\beta) = \min_u \{A_\varepsilon \cap \{(u,w) : w = \beta\}\}$. □

Example 3.7. Let us compute the exact value of $C_\varepsilon(0)$ for the specific set of parameters $a = \frac{3}{8}$, $b = 5$, $c = 1$, $\varepsilon = 0.2$. Now

$$u_s = \frac{3}{4} \quad , \quad z_\lambda(u) = \frac{1}{1-\lambda\varepsilon}\left(-5u^3 + \frac{55}{8}u^2 - \left(\frac{15}{8} + \lambda\varepsilon\right)u\right) \quad , \quad z_\lambda(u_s) = \frac{1}{1-\lambda\varepsilon}\left(\frac{45}{128} - \frac{3\lambda\varepsilon}{4}\right), \tag{3.4}$$

which gives the line

$$l_\lambda(u) = \frac{1}{\lambda}u + \frac{1}{1-\lambda\varepsilon}\left(\frac{45}{128} - \frac{3\lambda\varepsilon}{4}\right) - \frac{3}{4\lambda}. \tag{3.5}$$

Calculate now the roots of $z_\lambda(u)$ (here we only need the second root, $0 = r_1 < r_2 < r_3$) and $l_\lambda(u)$,

$$r_l(\lambda) = \frac{3}{4} - \frac{\lambda}{1-\lambda\varepsilon}\left(\frac{45}{128} - \frac{3\lambda\varepsilon}{4}\right) \quad \text{and} \quad r_z(\lambda) = \frac{11}{16} - \sqrt{\frac{25}{256} - \frac{\lambda\varepsilon}{5}}. \tag{3.6}$$

To find $C_\varepsilon(0)$ solve

$$C_\varepsilon(0) = \min_{\lambda \ge 0} \max\{r_l(\lambda), r_z(\lambda)\}, \tag{3.7}$$

numerically. Note that due to the shape of r_l and r_z, their rightmost intersection point is the minimum. Theoretically, one solves only a quartic polynomial in λ, so there is a closed (but complicated) form for $C_\varepsilon(0)$. Take now $\varepsilon = 0.2$ and look for solutions to

$$r_l(\lambda) = \frac{3}{4} - \frac{5\lambda}{5-\lambda}\left(\frac{45}{128} - \frac{3\lambda}{20}\right) = \frac{11}{16} - \sqrt{\frac{25}{256} - \frac{\lambda}{25}} = r_z(\lambda) \implies \lambda_* \approx 1.3099, \tag{3.8}$$

which gives

$$C_{\varepsilon=0.2}(0) = r_z(\lambda_*) \approx r_z(1.3099) \approx 0.4748. \tag{3.9}$$

Note that for $\varepsilon \to 0$ we have $C_\varepsilon(0) \to a = \frac{3}{8}$ and for $\varepsilon \to \infty$ we have $C_\varepsilon(0) \to \frac{3}{4}$ (see Figure 7). This makes sense intuitively; if ε is large $\dot{w} \gg 0$ and the upward flow dominates, only points already close to the u-spike threshold "flow over" it.

Remark 3.8. We summarize the general algorithm to find $C_\varepsilon(\beta)$ (Figure 7) starting from $(u_0, w_0) = (0, \beta)$ as follows:

1. Calculate the u-spike threshold u_s.
2. Set up the cubic curve $z_\lambda(u)$.
3. Calculate $z_\lambda(u_s)$ to set up the line $l_\lambda(u) := \frac{1}{\lambda}u + z(u_s) - \frac{1}{\lambda}u_s$.
4. Calculate the $r_l(\lambda)$ as solution of $l_\lambda(u) = \beta$ and $r_u(\lambda)$ as solution of $z_\lambda(u) - \beta$ (take the second root, for $\beta \ne 0$ that is a cubic polynomial).
5. Solve $\min_\lambda \max\{r_l(\lambda), r_z(\lambda)\}$ numerically.

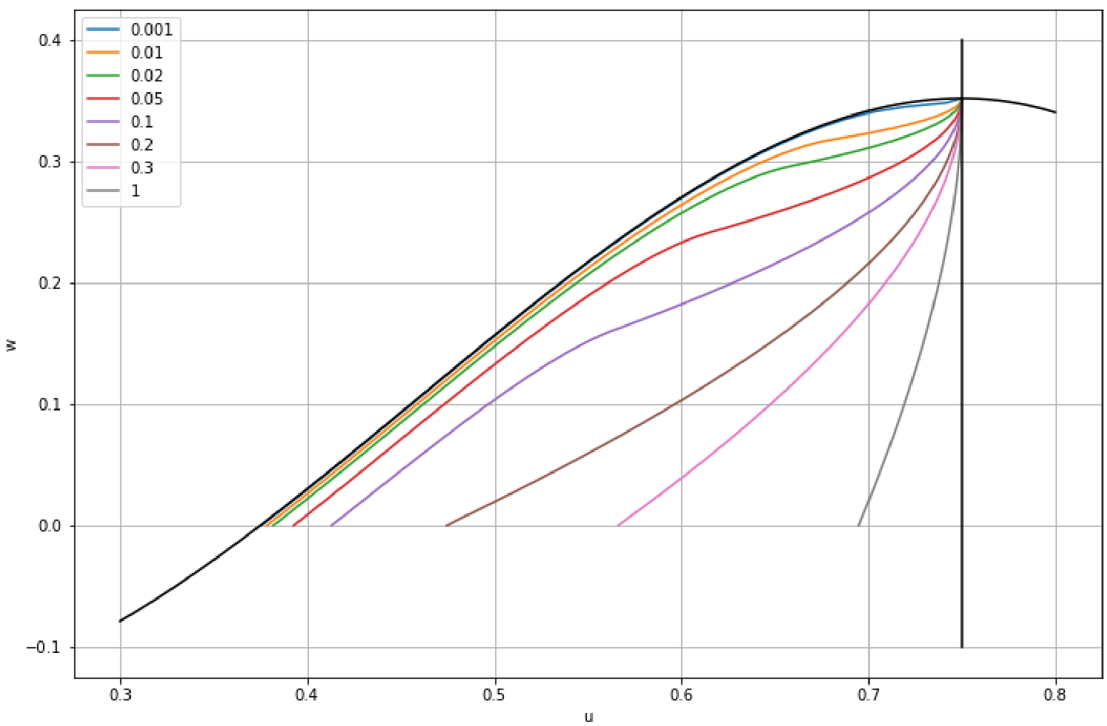
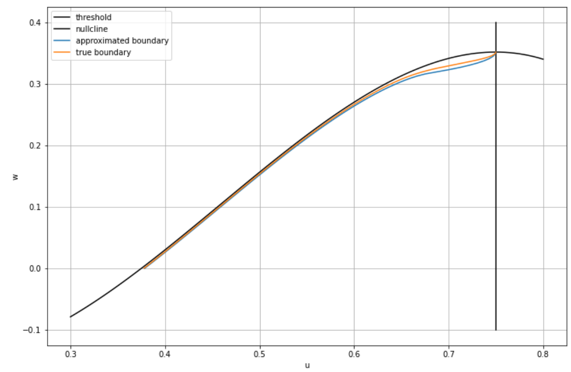
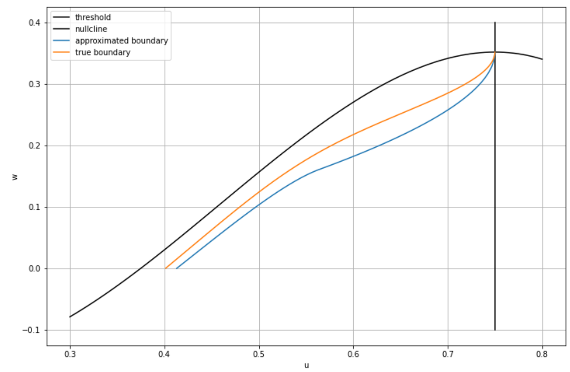


Figure 7: The boundary ∂A for different ε, computed numerically

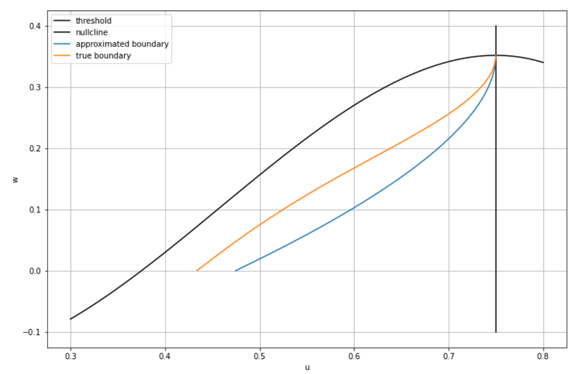
Moreover, we can compare the boundary curves of the true area $\tilde{A}$ and the estimate A (Figure 8). Finally, in case of Dirac impulses, we are interested for which ε the estimate $C_\varepsilon(0)$ of $\tilde{C}_\varepsilon(0)$ is precise (Figure 9).



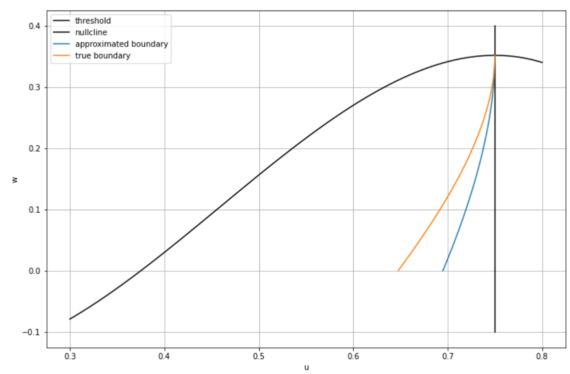
(a) $\varepsilon = 0.01$



(b) $\varepsilon = 0.1$



(c) $\varepsilon = 0.2$



(d) $\varepsilon = 1$

Figure 8: The boundaries of $\tilde{A}_\varepsilon$ and A_ε for different ε

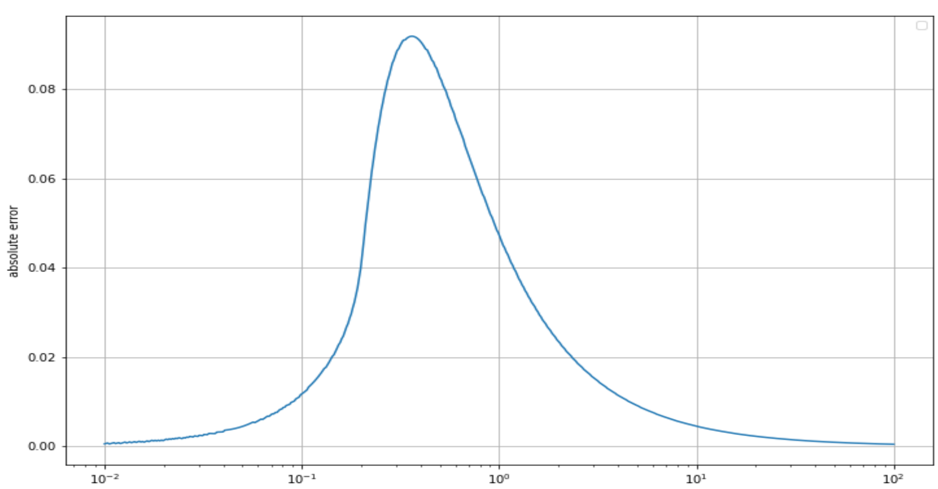


Figure 9: The absolute error of $\tilde{C}_\varepsilon(0) - C_\varepsilon(0)$ for different ε

The next theorem deals again with an arbitrary impulse, similar to Theorem 2.4. General results here are difficult to obtain, due to the coupling and absence of an analytical solution. The impulse needs to be bounded from above to ensure that the trajectory does not travel too far and bounded from below to ensure that it reaches region A_ε. As in Theorem 2.4, we construct a rectangle R, but the bounds are easier to obtain, since the right side is the vertical line $u = u_s$ and the maximum of A_ε is $(u_s, f(u_s))$ independent of ε.

Theorem 3.9. *Let $I(t)$ be an impulse of finite length T for* (FHN) *to the steady state* $(0,0)$ *and* $\beta := T\varepsilon u_s$. *If the following conditions*

(i) $\int_0^T I(t) \leq (1 - T^2\varepsilon)u_s - Tf(u_s)$ (upper bound 1),

(ii) $T\varepsilon u_s \leq f(u_s)$ (upper bound 2),

(iii) $\int_0^T I(t) \geq \left(1 - \frac{T^2\varepsilon}{1-T^2\varepsilon}\right)^{-1} \left(C_\beta(\varepsilon) + \frac{T^3\varepsilon f(u_s)}{1-T^2\varepsilon} - T\min_{u\in[0,u_s]} f(u)\right)$ (lower bound),

are satisfied, then the trajectory is a u-spike.

Proof. To assure that the impulse does not send the trajectory to far, start by estimating u and w. We use that here $w(t) \geq 0$ for all $t \in [0,T]$ and $t_{m_w}, t_{m_u} \leq T$, where w and u respectively are maximized over $[0,T]$.

$$w_m := \max_{t\in[0,T]} w(t) = w(t_{m_w}) = \int_0^{t_{m_w}} \varepsilon u(t) - \varepsilon c w(t) \leq \int_0^{t_{m_w}} \varepsilon u(t) \leq T\varepsilon u_m,$$

$$\begin{aligned} u_m := \max_{t\in[0,T]} u(t) = u(t_{m_u}) &= \int_0^{t_{m_u}} f(u(t)) - w(t) + I(t) \\ &\leq T \max_{t\in[0,T]} f(u(t)) + \int_0^T I(t) \\ &\leq T \max_{t\in[0,T]} \{w_m, f(u_s)\} + \int_0^T I(t) \\ &\leq T^2\varepsilon u_m + Tf(u_s) + \int_0^T I(t) \\ \implies u_m &\leq \frac{1}{1-T^2\varepsilon}\left(Tf(u_s) + \int_0^T I(t)\right) \leq u_s \qquad \text{from upper bound 1,} \\ \implies w_m &\leq T\varepsilon u_m \leq T\varepsilon u_s \leq f(u_s) \qquad \text{from upper bound 2.} \end{aligned} \tag{3.10}$$

Hence $u(T) \leq u_s$ and $w(T) \leq f(u_s)$, so it remains to check that $(u(T), w(T))$ is really inside A_ε. Set now $\beta = T\varepsilon u_s$ and compute $C_\beta(\varepsilon)$, then

$$\begin{aligned} u(T) &= \int_0^T f(u(t)) - \int_0^T w(t) + \int_0^T I(t) \\ &\geq T \min_{u\in[0,u_s]} f(u) - Tw_m + \int_0^T I(t) \\ &\geq T \min_{u\in[0,u_s]} f(u) - T^2\varepsilon u_m + \int_0^T I(t) \\ &\geq T \min_{u\in[0,u_s]} f(u) - \frac{T^3\varepsilon f(u_s)}{1-T^2\varepsilon} + \left(1 - \frac{T^2\varepsilon}{1-T^2\varepsilon}\right)\int_0^T I(t) \\ &\geq C_\beta(\varepsilon) \qquad \text{from the lower bound.} \end{aligned} \tag{3.11}$$

It is already known that $0 > \min_{u\in[0,u_s]} f(u) = f\left(\frac{a+1-\sqrt{a^2-a+1}}{3}\right) < -\infty$. We conclude that $C_\beta(\varepsilon) \leq u(T) \leq u_s$ and $0 \leq w(T) \leq \beta$ and thereby $(u(T), w(T))$ lies in A_ε so the trajectory is a u-spike. $\square$

Remark 3.10. Again as $T \to 0$ we recover the conditions for a δ-impulse (i.e. $\beta = 0$), so usually "good" impulses have a high amplitude with small support ($0 < T \ll 1$) to achieve the above bounds. Since $C_\beta(\varepsilon) < u_s$ by construction, we can assure existence of impulses satisfying the conditions; they do not exclude each other if T sufficiently small and $\int_0^T I(t)$ large.

Example 3.11. Let $a = 3/8, b = 5, c = 1$ and $\varepsilon = 0.1$. We provide an example of a block impulse, $I(t) = I_0 \cdot \chi_{[0,T]}(t)$, that is calculating I_0 and T that cause a u-spike. Then three conditions are

(i) $\int_0^T I(t) \le (1 - T^2\varepsilon)u_s - Tf(u_s) \iff TI_0 \le (1 - \frac{T^2}{10})\frac{3}{4} - T\frac{45}{64}$,

(ii) $\beta := T\varepsilon u_s \le f(u_s) \iff \beta := \frac{3T}{40} \le \frac{45}{64}$,

(iii) $\int_0^T I(t) \ge \left(1 - \frac{T^2\varepsilon}{1-T^2\varepsilon}\right)^{-1} \left(C_\beta(\varepsilon) + \frac{T^3\varepsilon f(u_s)}{1-T^2\varepsilon} - T\min_{u\in[0,u_s]} f(u)\right)$
$\iff TI_0 \ge \left(1 - \frac{T^2}{10-T^2}\right)^{-1} \left(C_\beta(0.1) + \frac{T^3\frac{45}{64}}{10-T^2} + \frac{125}{864}T\right)$.

Set now $T = 0.1$, i.e. (ii) is clearly satisfied, and we are left with

(i) $I_0 \le 10 \cdot \left(\frac{999}{1000}\frac{3}{4} - \frac{45}{640}\right) = \frac{10863}{1600}$,

(iii) $I_0 \ge 10 \cdot \frac{999}{998}\left(C_{\frac{3}{400}}(0.1) + \frac{1}{1408} + \frac{125}{8640}\right)$.

Note that it is not necessary to calculate $C_{\frac{3}{400}}(0.1)$ exactly, an upper bound of $C_{\frac{3}{400}}(0.1) \le \frac{1}{2}$ (see Figure 8b) suffices. Hence $I_0 \ge \frac{9990}{998} \cdot \frac{3}{5}$ directly implies (ii). The two inequalities imply in turn

$$\frac{10863}{1600} \ge 6.7 \ge I_0 \ge 6.1 \ge \frac{9990}{998}\frac{3}{5}, \tag{3.12}$$

thus e.g. $I_0 = 6.5$ is valid. To conclude a block impulse of form $I(t) = 6.5\chi_{[0,0.1]}(t)$ to the state $(0, 0)$ causes a u-spike.

4 Repeated impulses, hybrid system and periodicity

After studying the trajectory of u-spikes close to initial time, we will study what happens when trajectories in (FHN) return to the global fix point at the origin. After crossing the u-spike threshold, it will cross $\mathcal{N}_u$ on the right, then loop back go leftwards, cross $\mathcal{N}_w$ and $\mathcal{N}_u$ to the left.

Proposition 4.1. *Any u-spike will approach the origin from the first quadrant below $\mathcal{N}_u$ and never crosses the u-axis for any finite t.*

Proof. Note first that the flow on $\mathcal{N}_u$ in the second quadrant is only vertical downwards, any trajectory crosses $\mathcal{N}_u$ only once. At the equilibrium at the origin are two invariant manifolds tangent to the eigenspaces spanned by the Jacobian matrix J of the system. The trajectory approaches the origin along the slow manifold $\mathcal{M}_{\text{slow}}$ (the one tangent to the eigenspace of the larger eigenvalue, i.e. $(\sqrt{3209} - 67)/16, 1))$ from above. In other words, the trajectory is trapped between $\mathcal{M}_{\text{slow}}$ and $\mathcal{N}_u$. Moreover, as the flow on the negative u-axis is downward, $\mathcal{M}_{\text{slow}}$ lies in one direction entirely between $\mathcal{N}_u$ and the u-axis. Hence the incoming trajectory will be squeezed between and $\mathcal{N}_u$. □

Remark 4.2. As $\varepsilon \to 0$, $\mathcal{M}_{\text{slow}}$ will converge to $\mathcal{N}_u$ pointwise, by Fenichel theorems.

This enables us to construct a periodic orbit of the hybrid system in the following steps:

1. Give a δ-impulse of $I_0 \ge C_0(\varepsilon)$.
2. Wait until the trajectory comes back and intersects the line $w = \beta$ at the point $(u(t_1), \beta)$.

3. Send a δ-impulse of size $u(t_1) + I_1$ with $I_1 \geq C_\beta(\varepsilon)$.

4. Wait again until this trajectory comes back and crosses $y = \beta$ at the point $(u(t_2), \beta)$ send another δ-impulse of size $u(t_2) + I_1$.

5. Repeat step 4 each time the trajectory comes back and crosses $y = \beta$ at the point $(u(t_2), \beta)$.

We have now a loop at height β, where at each circulation we have to inject a δ-pulse of size $u(t_2)+I_1$. The period of the cycle $\tau(I_1)$ is determined implicitly. Note that intuitively for a fixed I_1, when $\beta \to 0$ this period gets larger, $\tau_{I_1}(\beta) \to \infty$. This is due to the fact that the trajectory approaches the origin "monotonically" by Proposition 4.1.

Proposition 4.3. *Let $\beta > 0$ be fixed. Then there exists a (not necessarily unique) δ-impulse of size I_* and a corresponding cycle that minimizes the period. That is $\tau(I_*) \leq \tau(I)$ for any other $I \leq u_s$.*

Proof. This follows from the continuity of τ and the fact that I lies in a compact interval, the interval from the true threshold value $\tilde{C}_\beta(\varepsilon)$ (see Remark 3.4) to u_s. $\square$

4.1 Further perspectives and open questions

A lot remains to be said about the analytical properties of (FHN). Some possible directions are:

- Is there a region $B(\varepsilon) \subset A_5$, such that if $(u(T), w(T)) \in B(\varepsilon)$ after an impulse, it is guaranteed that no u-spike will occur? So far it is only known that $\partial B(\varepsilon)$ has to lie left of $\tilde{C}(\varepsilon)$.

- Can one calculate e.g. $\tilde{C}_0(\varepsilon)$ through other means then to backtrack the trajectory going through $(u_s, f(u_s))$?

- Can the conditions for arbitrary impulses be improved using more sophisticated estimates?

- Is the minimizing impulse and periodic orbit in Proposition 4.3 unique ?

- Let the size of a δ-impulse be fixed and sufficiently large (not including the first impulse to start the cycle). Calculate the period τ depending on β. Will it decrease as β increases? What is its minimum?

- Is there a notion of maximum response for Dirac impulses? In other words, the maximal u-elongation, u_M of a trajectory (which actually is attained on $\mathcal{N}_u$) depends on the size of the impulse I_0. For what impulse would the absolute response $u_M - I_0$ or the relative response $\frac{u_M}{I_0}$ be maximized?

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