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Master in Mathematics

Some Regularity Results for the Obstacle Problem

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Contents

1 Introduction

We will study the so called *obstacle problem*, namely

$$\text{minimize } J(v) = \int_\Omega \frac{1}{2}|\nabla v(x)|^2 dx \qquad \text{s.t. } v \geq \varphi \text{ and } v\big|_{\partial\Omega} = g. \tag{OP}$$

Visually, we can see the solution as an elastic membrane, fixed in some frame, spanning over an obstacle pushing from below. The membrane naturally minimizes its energy, at some parts it is touching the obstacle, at others it lays free in space. The obstacle problem and related subjects (Stefan problem, Hele-Shaw flow, ...) find wide-ranging applications in e.g. physics, control theory and finance. Here, we will focus on solutions to the *zero obstacle problem*,

$$\text{minimize } \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + fu\right) dx \qquad \text{s.t. } u \geq 0,\ u|_{\partial\Omega} = g - \varphi,\ f := -\Delta\varphi. \tag{ZOP}$$

We can think of the membrane now being submitted to a uniform force like gravity and a flat plate pushing against it from below. Solutions to the original problem can be obtained by adding back the obstacle, $v = u + \varphi$.

We notice that as $u \geq 0$, the domain can be split into two parts, $\{u > 0\}$ and the contact set $\{u = 0\}$. Especially, the a priori unknown boundary $\partial\{u > 0\}$ is of particular interest. First however, we study regularity of the solution itself and later investigate regularity of the free boundary using blow-ups. The regularity of the obstacle is of little interest for now, it is as smooth as necessary.

Theorem 1.1. (Existence and uniqueness) *Under the assumptions:*

1. *$\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain,*
2. *$f \in L^2(\Omega)$,*
3. *$C = \{w \in H^1(\Omega) : w \geq \phi$ in $\Omega, w\big|_{\partial\Omega} = g\} \neq \varnothing$,*

there exists a unique minimizer for (ZOP) *among all functions $v \in H^1(\Omega)$.*

Proof. Define $v := u + \varphi$, then

$$J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega |\nabla\varphi|^2 + \int_\Omega \nabla u \cdot \nabla\varphi = \frac{1}{2}\int_\Omega |\nabla u|^2 + \int_\Omega fu + C_\varphi,$$

with $f := -\Delta\varphi$. Hence u being a minimizer for (ZOP) is equivalent to v being a minimizer for (OP). Let us define

$$\theta_0 := \inf\left\{\int_\Omega |\nabla w|^2 dx : w \in H^1(\Omega), w\big|_{\partial\Omega} = g, w \geq \varphi \text{ in } \Omega\right\}.$$

Take now a minimizing sequence $v_k \subset C$ such that $J(v_k) \to \theta_0$ as $k \to \infty$. As $J(v_k)$ decreases as $k \to \infty$, $\|\nabla v_k\|_{L^2}$ is bounded and so by Poincaré inequality $\|v_k\|_{L^2}$ too, thus $\{v_k\}$ is uniformly bounded in $H^1(\Omega)$ and there exists a subsequence v_{k_j} converging to v weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ (Rellich-Kondrachov). This subsequence is again bounded in $H^1(\Omega)$, so by compactness of the trace operator there exists a subsubsequence (not relabeled) such that $v_{k_j}\big|_{\partial\Omega} \to v\big|_{\partial\Omega}$ in $L^2(\partial\Omega)$. Furthermore by the weak convergence in $H^1(\Omega)$ we have

$$\begin{aligned}
\|v\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)} = \|v\|_{H^1(\Omega)} &\leq \liminf_{j\to\infty} \|v_{k_j}\|_{H^1(\Omega)} = \liminf_{j\to\infty} \left(\|v_{k_j}\|_{L^2(\Omega)} + \|\nabla v_{k_j}\|_{L^2(\Omega)}\right) \\
\|\nabla v\|_{L^2(\Omega)} &\leq \liminf_{j\to\infty} \|\nabla v_{k_j}\|_{L^2(\Omega)} \qquad (\text{as } v_{k_j} \to v \text{ in } L^2(\Omega)) \\
J(v) &\leq \liminf_{j\to\infty} J(v_{k_j}),
\end{aligned}$$

and so v is a minimizer of the energy functional J.
Since $v_{k_j} \to v$ in $L^2(\Omega)$ and $v_{k_j} \geq \varphi$ in Ω we have $v \geq \varphi$ as well. For any two minimizers $v, v' \geq \varphi$ and $J(v) = J(v')$ we have

$$\begin{aligned}
J\left(\frac{v + v'}{2}\right) &= \int_\Omega \left|\frac{\nabla v + \nabla v'}{2}\right|^2 = \frac{1}{4}\int_\Omega |\nabla v|^2 + \frac{1}{4}\int_\Omega |\nabla v'|^2 + \frac{1}{2}\int_\Omega \nabla v \cdot \nabla v' \\
&\leq \frac{1}{4}J(v) + \frac{1}{4}J(v') + \frac{1}{2}\|\nabla v\|_{L^2}\|\nabla v'\|_{L^2} = \frac{1}{2}J(v) + \frac{1}{2}\|\nabla v\|_{L^2}^2 = J(v),
\end{aligned}$$

with equality if and only if $v = v'$. If $v \neq v'$ we get a contradiction with the minimality of v. □

Naturally we look for the Euler-Lagrange equation of the functional to determine properties of the minimizer.

Theorem 1.2. *Let Ω be a bounded Lipschitz domain, $f \in C^\infty(\Omega)$, $u \in H^1(\Omega)$ a minimizer of ZOP. Then*

$$\Delta u = f\chi_{\{u>0\}} \text{ in } \Omega. \tag{EL}$$

Proof. **Step 1:** We show that u minimizes (ZOP) if and only if u minimizes

$$J(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + fu^+ dx,$$

that is show $u = u^+ := \max\{u, 0\}$. Let $u^- := \min\{u, 0\}$. We know that $u \in H^1(\Omega) \implies u^+ \in H^1(\Omega)$, and it holds weakly

$$|\nabla u|^2 = |\nabla u^+ - \nabla u^-|^2 = |\nabla u^+|^2 + |\nabla u^-|^2 + 2|\nabla u^+||\nabla u^-| = |\nabla u^+|^2 + |\nabla u^-|^2,$$

since either $|\nabla u^+|$ or $|\nabla u^-|$ vanish; thus

$$\frac{1}{2}\int_\Omega |\nabla u^+|^2 + \int_\Omega fu^+ \le \frac{1}{2}\int_\Omega |\nabla u|^2 + \int_\Omega fu^+,$$

with equality if and only if $u = u^+$ a.e., hence u is non-negative.
Step 2: Take $\eta \in H^1_0(\Omega)$ and $\varepsilon > 0$, so $J(u + \varepsilon\eta) \ge J(u)$. Then

$$\begin{aligned} 0 &\le \lim_{\varepsilon\to 0} \frac{J(u+\varepsilon\eta) - J(u)}{\varepsilon} \\ &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(\frac{1}{2}\int_\Omega |\nabla u + \varepsilon\nabla\eta|^2 + \int_\Omega f(u+\varepsilon\eta)^+ - \int_\Omega \frac{1}{2}|\nabla u|^2 - \int_\Omega fu^+\right) \\ &= \int_\Omega \nabla u\cdot\nabla\eta + \lim_{\varepsilon\to 0}\int_\Omega f\frac{(u+\varepsilon\eta)^+ - u^+}{\varepsilon}. \end{aligned}$$

For analyzing the last term we have

$$\lim_{\varepsilon\to 0}\frac{(u+\varepsilon\eta)^+ - u^+}{\varepsilon} = \begin{cases} \eta & \text{in } \{u>0\} \\ \eta^+ & \text{in } \{u=0\}. \end{cases}$$

Taking the limit inside (dominated convergence theorem) gives

$$\int_\Omega \nabla u\cdot\nabla\eta + \int_\Omega f\eta\chi_{\{u>0\}} + \int_\Omega f\eta^+\chi_{\{u=0\}} \ge 0.$$

In the case $\eta \ge 0$,

$$\int_\Omega \nabla u\cdot\nabla\eta + \int_\Omega f\eta \ge 0 \quad \forall\eta\in H^1_0(\Omega), \eta\ge 0 \qquad \implies \Delta u \le f \text{ weakly},$$

and in the case $\eta \le 0$,

$$\int_\Omega \nabla u\cdot\nabla\eta + \int_\Omega f\chi_{\{u>0\}}\eta \ge 0 \quad \forall\eta\in H^1_0(\Omega), \eta\le 0 \qquad \implies \Delta u \ge f\chi_{\{u>0\}} \text{ weakly}.$$

To conclude,

$$f\chi_{\{u>0\}} \le \Delta u \le f \text{ in } \Omega,$$

and hence $\Delta u = f$ in $\{u > 0\}$.
Step 3: Since f is smooth, $\Delta u = f\chi_{\{u>0\}} \in L^2_{\text{loc}}(\Omega)$, so by the Calderón-Zygmund estimate [GT77, Theorem 9.9] we get $u \in W^{2,2}_{\text{loc}}(\Omega)$. By Stampaccia Lemma, $\Delta u = 0$ a.e. in the level set $\{u = 0\}$, and it follows that $\Delta u = f\chi_{\{u>0\}}$ weakly. □

2 Regularity of solution

From now on we consider the problem in a ball, i.e.

$$\begin{cases} u & \geq & 0 & \text{in } B_1, \\ \Delta u & = & f\chi_{u>0} & \text{in } B_1. \end{cases} \tag{2.1}$$

To study the regularity we use interior Schauder estimates (see [GT77]), which can be applied directly to (EL). Schauder estimates (for the Laplacian) are regularity results, telling that the solution is two derivatives "smoother" than the data. Another approach is the use of (ACF) monotonicity formulas as in [PSU12].

Theorem 2.1. *Let $u \in C^{2,\alpha}(B_1)$ satisfy $\Delta u = f$ with $f \in C^{0,\alpha}(B_1)$ and $0 < \alpha \leq 1$. Then*

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \leq C(n,\alpha)(\|u\|_{L^\infty(B_1)} + \|f\|_{C^{0,\alpha}(B_1)}).$$

Theorem 2.2. *Let $1 > \varepsilon > 0$ and $u \in C^2(B_1)$ satisfy $\Delta u = f$ in B_1 where $f \in L^\infty(B_1)$. Then for some universal constant $C(\varepsilon, n)$*

$$\|u\|_{C^{1,1-\varepsilon}(B_{1/2})} \leq C(\|u\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)}).$$

Here we do not gain two full derivatives and cannot wholly bound the $C^{1,1}$ norm. For a proof of Theorem 2.2, based on a modification of [Wan06], see Theorem A.2; it provides directly almost optimal regularity.

Remark 2.3. The assumptions $u \in C^{2,\alpha}(B_1)$ or $u \in C^2(B_1)$ are in fact redundant. This follows from a mollification argument and continuity of u, which could be shown by different means [FR22, Proposition 5.3].

Corollary 2.4. *The solution u to (2.1) is $C^{1,1-\varepsilon}$ inside Ω for every $\varepsilon > 0$, in particular u is continuous.*

Furthermore the following covering lemma is used extensively, telling us that it is enough to consider estimates in $B_{1/2}$, the proof can be found in the appendix (Lemma A.1) as well.

Lemma 2.5. (Covering Lemma) *Let u be a solution to $\Delta u = f$ in B_1. Suppose we have a uniform estimate of the following form ($\|\cdot\|_F$ is e.g. $\|\cdot\|_{L^\infty}$ or $\|\cdot\|_{C^{0,\alpha}}$):*

$$\|u\|_{C^{k,\alpha}(B_{r_1})} \leq C(r_1)\left(\|u\|_{L^\infty(B_1)} + \|f\|_{F(B_1)}\right).$$

Then a uniform estimate holds in a larger ball B_{r_2}, with $r_1 < r_2 < 1$, i.e.

$$\|u\|_{C^{k,\alpha}(B_{r_2})} \leq C(r_1)\left(\frac{1}{(1-r_2)^k}\|u\|_{L^\infty(B_1)} + \|f\|_{F(B_1)}\right)$$

Another standard result we need, before proving the optimal regularity, is Harnack's inequality, telling us that we can compare the supremum and infimum.

Theorem 2.6. (Harnack's inequality with RHS)
Let $f \in L^\infty(B_1)$ and $u \in H^1(B_1)$ such that $\Delta u = f$ and $u \geq 0$ in B^1. Then

$$\sup_{B_{1/2}} u \leq C(n)\left(\inf_{B_{1/2}} u + \|f\|_{L^\infty(B_1)}\right).$$

Furthermore a simple rescaling argument ($\Delta[u(rx)] = r^2 f(rx)$) and a translation give

$$\sup_{B_r(x_0)} u \leq C(n)(\inf_{B_r(x_0)} u + r^2\|f\|_{L^\infty(B_{2r}(x_0))}) \quad \text{for any } B_{2r}(x_0) \subset B_1.$$

Before showing the optimal regularity we show a growth condition on the solution, which is interesting on its own and will be essential later on.

Lemma 2.7. *Let u be a solution to (2.1) and $x_0 \in \bar{B}_{1/2}$ be any point in $\{u = 0\}$. Then*

$$0 \leq \sup_{B_r(x_0)} u \leq Cr^2 \qquad \forall r \in (0, 1/4),$$

for C depending only on n and $\|f\|_{L^\infty(B_1)}$.

Proof. As $f\chi_{\{u>0\}} \in L^\infty(B_1)$ and $u \geq 0$ we can use (2.6) in $B_{2r}(x_0)$ to get

$$\sup_{B_r(x_0)} u \leq C\left(\inf_{B_r(x_0)} u + r^2\|f\chi_{\{u>0\}}\|_{L^\infty(B_{2r}(x_0))}\right).$$

Since $u(x_0) = 0$, the bound follows. $\square$

Finally we can proceed to show the optimal regularity.

Theorem 2.8. *Let $f \in C^\infty$ and u a solution to* (2.1)*. Then u is $C^{1,1}$ inside $B_{1/2}$ and*

$$\|u\|_{C^{1,1}(B_{1/2})} \leq C(n)(\|u\|_{L^\infty(B_1)} + \|f\|_{Lip(B_1)}).$$

Proof. We can assume that $K := \|u\|_{L^\infty(B_1)} + \|f\|_{Lip(B_1)} \leq 1$. (If that is not the case, divide (2.1) by K and then cancel K in the estimate after.) Let $x_1 \in \{u > 0\} \cap B_{1/2}$ and let x_0 be the closest point in $\partial\{u > 0\}$ to x_1, $\rho := |x_1 - x_0|$. Then $\Delta u = f$ in $B_\rho(x_1)$. We proceed by a rescaling argument. Define $\tilde{u}(y) = u(\rho y + x_1)$ and $\tilde{f} = f(\rho y + x_1)\rho^2$, so $\Delta\tilde{u} = \tilde{f}$ in $B_1(0)$. By Theorem 2.1, with $\alpha = 1$ ($\tilde{f}$ being smooth is clearly in $C^{0,1}$), we arrive at

$$\begin{aligned}
\|D^2\tilde{u}\|_{L^\infty(B_{1/2})} &\leq \|\tilde{u}\|_{C^{2,1}(B_{1/2})} \leq C(\|\tilde{u}\|_{L^\infty(B_1)} + \|\tilde{f}\|_{Lip(B_1)}) \\
\rho^2\|(D^2u)(\rho\cdot + x_1)\|_{L^\infty(B_{1/2})} &\leq C(\|u(\rho\cdot + x_1)\|_{L^\infty(B_1)} + \rho^2\|f(\rho\cdot + x_1)\|_{Lip(B_1)}) \\
\rho^2\|D^2u\|_{L^\infty(B_{\rho/2}(x_1))} &\leq C(\|u\|_{L^\infty(B_\rho(x_1))} + \rho^2\|f\|_{Lip(B_\rho(x_1))}) \\
\|D^2u\|_{L^\infty(B_{\rho/2}(x_1))} &\leq C\left(\frac{1}{\rho^2}\|u\|_{L^\infty(B_\rho(x_1))} + \|f\|_{Lip(B_\rho(x_1))}\right).
\end{aligned}$$

By Lemma 2.7 we have $\|u\|_{L^\infty(B_\rho(x_1))} \leq C\rho^2$, which then gives

$$\|D^2u\|_{L^\infty(B_{\rho/2}(x_1))} \leq C \implies |D^2u(x_1)| \leq C,$$

with C independent of x_1. Since $x_1 \in \{u > 0\} \cap B_{1/2}$ was arbitrary and $\nabla u \equiv 0$ on $\{u = 0\}$, $[u]_{C^{1,1}(B_{1/2})} \leq C$, and as we already have $\|u\|_{C^{1,1-\varepsilon}} \leq C$, $\|u\|_{C^{1,1}} \leq C$. $\square$

Proposition 2.9. (Nondegeneracy). *Assume that $f \geq c_0 > 0$ in B_1. Let u be a solution to* (2.1)*, then around a free boundary point $x_0 \in \partial\{u > 0\} \cap B_{1/2}$*

$$0 < cr^2 \leq \sup_{B_r(x_0)} u \leq Cr^2 \quad \forall r \in (0, 1/4).$$

Proof. It remains to show the left inequality; let $x_1 \in \{u > 0\}$ be a point close to x_0 and

$$\begin{aligned}
w(x) &:= u(x) - \frac{c_0}{2n}|x - x_1|^2, \\
\Delta w &= \Delta u - c_0 = f - c_0 \geq 0 \text{ in } \{u > 0\}.
\end{aligned}$$

Note that $w(x_1) = u(x_1) > 0$ and as $-\Delta w \leq 0$ in $\{u > 0\} \cap B_r(x_1)$, by the weak maximum principle, $\arg\max_x w \in \partial(\{u > 0\} \cap B_r(x_1))$ with the maximum being positive (as $w(x_1) > 0$). But on $\partial\{u > 0\}$, clearly $w(x) = 0 - \frac{c_0}{2n}|x - x_1|^2 < 0$, so the maximum is on $\partial B_r(x_1)$, i.e.

$$0 < \sup_{\partial B_r(x_1)} w = \sup_{\partial B_r(x_1)} u - \frac{c_0}{2n}r^2 \iff 0 < \frac{c_0}{2n}r^2 < \sup_{\partial B_r(x_1)} u = \sup_{B_r(x_1)} u.$$

Yet the RHS converges to $\sup_{B_r(x_0)} u$ as $x_1 \to x_0$ by continuity of u, thus $0 < cr^2 \leq \sup_{B_r(x_0)} u$. $\square$

We note that without the additional assumption $f \geq c_0$, we can construct examples with badly behaved free boundaries, i.e. non-smooth and infinite perimeter. The heuristic idea is here to use the fact that any closed set can be written as the zero set of a smooth function and perturb the obstacle accordingly.

3 The free boundary: Classification of blow-ups

After looking at the solution, in the next step we study the free boundary. As we already have $C^{1,1}$ regularity inside B_1, we rescale (from $B_{1/2} \Subset B_1$ to B_1) and consider now u solving

$$\begin{cases} u & \in & C^{1,1}(B_1), \\ u & \geq & 0, \\ \Delta u & = & \chi_{\{u>0\}}, \\ 0 & \text{is} & \text{a free boundary point.} \end{cases} \tag{P1}$$

Compared to the general case, we assume that $f \equiv 1$ and that the boundary point is the origin. By translation, it also hold for any other free boundary point (FBP) in B_1.

To study the geometry of the free boundary we consider a rescaling ("zooming in") at a FBP x_0,

$$u_r(x) := \frac{u(x_0 + rx)}{r^2}.$$

The so called blow-up, u_0, we get when taking the limit as $r \to 0$. The existence of the limit will be the first step, then using homogeneity and convexity of the blow-up, it can be classified into two categories. Usually it is easier to show certain properties for the free boundary of the blow-ups, and then transfer those back to the free boundary of the actual solution u.

Proposition 3.1. *Let u be a solution to* (P1) *and define $u_r(x) := \frac{u(rx)}{r^2}$. Then, for any sequence $r_k \to 0$, there exists a subsequence r_{kj} such that*

$$u_{r_{kj}} \to u_0 \qquad \text{in the } C^1_{\text{loc}}(\mathbb{R}^n) \text{ norm},$$

where u_0 (the blow-up at 0*) satisfies the* global problem

$$\begin{cases} u_0 & \in & C^{1,1}_{\text{loc}}(\mathbb{R}^n), \\ u_0 & \geq & 0, \\ \Delta u_0 & = & \chi_{\{u_0>0\}}, \\ 0 & \text{is} & \text{a free boundary point.} \end{cases} \tag{GP}$$

Proof. Start by noticing that $0 < cr^2 \leq \sup_{B_r(0)} u(x) \leq Cr^2$, so

$$\frac{1}{C} \leq \sup_{B_1} u_r \leq C, \qquad \text{for some } C.$$

Furthermore as $u \in C^{1,1}(B_1)$, we have

$$\|D^2 u_r\|_{L^\infty(B_{1/(2r)})} = \|D^2 u\|_{L^\infty(B_{1/2})} \leq C \qquad \forall r > 0,$$

i.e. $\{u_{r_k}\}$ is uniformly bounded in $C^{1,1}(K)$, with K compact. By Arzelà-Ascoli (see e.g. [FR22, Theorem 1.7]) there exists a subsequence r_{k_j} such that

$$u_{r_{k_j}} \to u_0 \in C^{1,1}(K) \qquad \text{in the } C^1_{\text{loc}}(\mathbb{R}^n) \text{ norm}.$$

By [FR22, (H8)] we also have that $\|D^2 u_0\|_{L^\infty(K)} \leq \|u_0\|_{C^{1,1}(K)} \leq C$ and as each $u_{r_{k_j}}(x) \geq 0$ in K, $\lim_j \sup_K |u_{r_{k_j}} - u_0| = 0$ it holds that $u_0 \geq 0$ in K. Now let us check that $\Delta u_0 = 1$ in $\{u_0 > 0\}$. If we can show for any compact set K that $\Delta u_0 = 1$ in $\{u_0 > 0\} \cap K$, it also holds in $\{u_0 > 0\}$. Take $\eta \in C_c^\infty(\{u_0 > 0\} \cap K)$. As $\sup_K |u_{r_{k_j}} - u_0| \to 0$, for large j, $u_{k_j} > 0$ in $\operatorname{spt} \eta$ and as $u_{r_{k_j}} \to u_0$ in $C^1(K)$,

$$-\int_{\mathbb{R}^n} \eta = -\int_{\mathbb{R}^n} \Delta u_{r_{k_j}} \eta = \int_{\mathbb{R}^n} \nabla u_{r_{k_j}} \cdot \nabla \eta \to \int_{\mathbb{R}^n} \nabla u_0 \cdot \nabla \eta \qquad \forall \eta \in C_c^\infty(\{u_0 > 0\} \cap K).$$

Lastly, to show that 0 is a FBP for u_0, we know that $\lim u_{r_{k_j}}(0) = 0$ (by C^1 convergence), but also for any $\rho \in (0,1)$

$$\begin{aligned} &\frac{1}{C} \leq \sup_{B_1} u_r \leq C, \qquad \text{for some } C \\ \iff \quad &\frac{1}{C} \leq \sup_{B_1} \frac{u(\rho r_{k_j} x)}{\rho^2 r_{k_j}^2} \leq C \quad \iff \quad \frac{\rho^2}{C} \leq \sup_{B_\rho} \frac{u(r_{k_j} x)}{r_{k_j}^2} = \sup_{B_\rho} u_{r_{k_j}} \leq C\rho^2. \end{aligned}$$

Thus $u_{r_{k_j}}$ is not identically 0 in any ball B_ρ and so passing to the limit yields that 0 is a FBP for u_0. $\square$

Instead of looking directly at the solution u, which might be quite complicated, we study u_0 first. First we note that a blow-up is necessarily a solution to (GP), but not vice versa. A global solution coming from a blow-up can actually be classified; the advantage to the original solution u are homogeneity and convexity, which enable us to classify the blow-ups completely. There are several approaches to show convexity of u_0. The original Caffarelli approach [Caf98] going directly from global solutions to convexity, using an interior estimate ($D_{ee}u \geq -\alpha$ in B_r implies $D_{ee}u \geq -\alpha + C\alpha^M$ in $B_{r/2}$) and an induction argument to arrive at $D_{ee}u(x) \geq -\frac{C}{|\log(|x|)|^\varepsilon}$ and then passing to the limit. Another way is [PSU12, Theorem 5.1], arguing by contradiction; assume $\partial_{ee}u_0 < 0$, employ the already existing lower bound $\partial_{ee}u_0 = -m$, integrate twice and arrive at u_0 being a non-positive second degree polynomial. Our approach makes use of the homogeneity of the blow-up; once we have homogeneity, convexity follows straightforwardly. Classically, to show homogeneity, Weiss' monotonicity formula [Wei99] is used. Very briefly, a straight computation shows that the so called Weiss energy

$$W_u(r) = \frac{1}{r^{n+2}} \int_{B_r} \frac{1}{2}|\nabla u|^2 + u dx - \frac{1}{r^{n+3}} \int_{\partial B_r} u^2 dx$$

is increasing in r, bounded below and $W_{u_r}(\rho) = W_u(\rho r)$, thus for $u_{r_j} \to u_0$

$$W_{u_0}(\rho) = \lim_{r_j \to 0} W_{u_{r_j}}(\rho) = \lim_{r_j \to 0} W_u(\rho r_j) = W_u(0).$$

Hence $W_{u_0}(\rho)$ is constant which implies 2-homogeneity. However here we show homogeneity differently, not based on a monotonicity formula, but by using the approach in [AP86, Spr83].

Theorem 3.2. *Let u_0 be a blow-up at the origin of a solution u to* (P1)*, i.e. $u_0(x) = \lim_{\rho_k \to 0} u_{\rho_k}(x) = \lim_{\rho_k \to 0} \frac{u(\rho_k x)}{\rho_k^2}$, fulfilling necessarily* (GP)*. Then $u_0(rx) = r^2 u_0(x)$, so u_0 is 2-homogeneous.*

Proof. Start by changing to polar coordinates, $u_0(x) = u_0(r\theta) = u_0(r,\theta)$, we want to show $u_0(r,\theta) = r^2 u_0(1,\theta)$ with $r \in \mathbb{R}^+$ and $\theta \in \mathbb{S}^{n-1}$. We define for $\tau > \tau_0 = -\log(1/2)$ the following function:

$$\psi(\tau,\theta) := \frac{u(e^{-\tau},\theta)}{e^{-2\tau}} = \frac{u(r(\tau),\theta)}{r(\tau)^2} = \frac{u(r(\tau)\theta)}{r(\tau)^2} \quad \text{with } r(\tau) = e^{-\tau} \Leftrightarrow \tau(r) = -\log r.$$

Note also that $\tau \in (\tau_0, \infty) \Leftrightarrow r \in (0, 1/2)$, we need to stay inside to use the regularity of u.

Step 1: We have the following properties from u also on ψ:

- $\{\psi > 0\} = \{u > 0\}$ and ψ is smooth in $\{\psi > 0\}$, because u is smooth there.
- $\|\psi\|_{L^\infty(\tau_0,\infty\times\mathbb{S}^{n-1}))} = \sup_{\tau\in(\tau_0,\infty),\theta\in\mathbb{S}^{n-1}} \frac{u(r(\tau),\theta)}{r(\tau)^2} = \sup_{r\in(\tau_0,\rho_0),\theta\in\mathbb{S}^{n-1}} \frac{u(r(\tau),\theta)}{r(\tau)^2} \leq \frac{Cr(\tau)^2}{r(\tau)^2} = C$ as Proposition 2.9 holds inside B_1, so ψ is bounded.
- $\psi \in C^{1,1}((\tau_0,\infty) \times \mathbb{S}^{n-1})$, follows from optimal $C^{1,1}$ regularity of u inside B_1.

Step 2: $\psi_\tau = \frac{\partial}{\partial\tau}\psi$ and $\psi_\theta = \frac{\partial}{\partial\theta}\psi$ are continuous and uniformly bounded in $(\tau_0,\infty) \times \mathbb{S}^{n-1}$:

$$|\psi_\theta(\tau,\theta)| = \frac{|\nabla_\theta u(r(\tau)\theta)|}{r(\tau)} = e^\tau |\nabla_\theta u(e^{-\tau}\theta)| \leq e^\tau C e^{-\tau} = C \quad \text{as } \nabla u \in Lip, \text{ thus } |\nabla u(x)| \leq C|x|,$$
$$|\psi_\tau(\tau,\theta)| = |-2r(\tau)^{-3}r'(\tau)u(r(\tau),\theta) + r(\tau)^{-2}\nabla u(r(\tau),\theta)r'(\tau)| \leq 2|\psi(\tau,\theta)| + |e^\tau \nabla u(\tau,\theta)| \leq C.$$

Step 3: Let $u(x) = u(r,\theta)$, so $\Delta u = \chi_{\{u>0\}}$ becomes

$$\Delta u = u_{rr} + \frac{n-1}{r}u_r + \frac{1}{r^2}\Delta_\theta u = \chi_{\{u>0\}} = \chi_{\{\psi>0\}}.$$

We calculate

$$\begin{aligned}
\psi_\tau &= u_r(r,\theta)r'(\tau)r(\tau)^{-2} - 2u(r,\theta)r(\tau)^{-3}r'(\tau) = 2u(r,\theta)r(\tau)^{-2} - u_r(r,\theta)r(\tau)^{-1},\\
\psi_{\tau\tau} &= u_{rr} + 4ur^{-2} - 3u_r r^{-1},\\
\Delta_\theta\psi &= \Delta_\theta u r^{-2}.
\end{aligned}$$

So that, equation $\Delta u = \chi_{u>0}$ gives:

$$\psi_{\tau\tau} - (n+2)\psi_\tau + \Delta_\theta \psi + 2n\psi = u_{rr} + \frac{n-1}{r} u_r + \frac{1}{r^2}\Delta_\theta u = \chi_{\{\psi>0\}}.$$

Step 4: Multiply the equation by ψ_τ and integrate over $(\tau_0, \infty) \times \mathbb{S}^{n-1}$.

$$\int_{\tau_0}^{\infty} \int_{\mathbb{S}^{n-1}} \psi_{\tau\tau}\psi_\tau - (n+2)\psi_\tau^2 + \Delta_\theta \psi \psi_\tau + 2n\psi\psi_\tau = \int_{\tau_0}^{\infty} \int_{\mathbb{S}^{n-1}} \chi_{\{\psi>0\}}\psi_\tau$$

$$\Leftrightarrow C \int_{\tau_0}^{\infty} \int_{\mathbb{S}^{n-1}} \psi_\tau^2 = \int_{\tau_0}^{\infty} \int_{\mathbb{S}^{n-1}} \psi_{\tau\tau}\psi_\tau + \int_{\tau_0}^{\infty} \int_{\mathbb{S}^{n-1}} \Delta_\theta \psi + \int_{\tau_0}^{\infty} \int_{\mathbb{S}^{n-1}} 2n\psi\psi_\tau - \int_{\tau_0}^{\infty} \int_{\mathbb{S}^{n-1}} \chi_{\{\psi>0\}}\psi_\tau$$

We want to upperbound the terms on the RHS, to show that $\psi_\tau \in L^2((\tau_0, \infty) \times \mathbb{S}^{n-1})$.

$$\int_{\tau_0}^{\infty} \int_{\mathbb{S}^{n-1}} \psi_{\tau\tau}\psi_\tau = \int_{\mathbb{S}^{n-1}} \frac{1}{2}\psi_\tau^2 \Big|_{\tau_0}^{\infty} \leq \int_{\mathbb{S}^{n-1}} (\sup \psi_\tau)^2 \leq C, \quad \text{as } \psi_\tau \text{ has an uniform bound,}$$

$$\int_{\tau_0}^{\infty} \int_{\mathbb{S}^{n-1}} \Delta_\theta \psi = \int_{\tau_0}^{\infty} \psi_\theta \psi_\tau \Big|_{\mathbb{S}^{n-1}} - \int_{\tau_0}^{\infty} \int_{\mathbb{S}^{n-1}} \psi_\theta \psi_{\tau\theta} \leq \int_{\mathbb{S}^{n-1}} \frac{1}{2}\psi_\theta^2 \Big|_{\tau_0}^{\infty} \leq \int_{\mathbb{S}^{n-1}} (\sup \psi_\theta)^2 \leq C,$$

$$\int_{\tau_0}^{\infty} \int_{\mathbb{S}^{n-1}} \psi\psi_\tau = \int_{\mathbb{S}^{n-1}} \frac{1}{2}\psi^2 \Big|_{\tau_0}^{\infty} \leq C \int_{\mathbb{S}^{n-1}} \leq C, \quad \text{as } \psi \text{ is bounded,}$$

$$\int_{\tau_0}^{\infty} \int_{\mathbb{S}^{n-1}} \chi_{\{\psi>0\}}\psi_\tau = \int_{\mathbb{S}^{n-1}} \psi \Big|_{\tau_0}^{\infty} \leq C.$$

Despite e.g. $\psi_{\tau\tau}$ not being continuous, the application of the fundamental theorem of calculus is justified as ψ_τ is Lipschitz in $(\tau_0, \infty) \times \mathbb{S}^{n-1}$ and there is no need for an approximation argument.

Step 5: Let ρ_k be a blow-up sequence, from Proposition 3.1, i.e. $u_{\rho_k} \to u_0$ locally uniformly in $\mathbb{R}^n$, and define $\tau_k := -\log \rho_k$ where $\tau_k \to \infty$. Now let $s \in [-N, N]$, then

$$\|\psi(\tau_k + s, \theta) - u_0(1,\theta)\|_{L^2((-N,N)\times\mathbb{S}^{n-1})} \leq \|\psi(\tau_k + s, \theta) - \psi(\tau_k, \theta)\|_{L^2} + \|\psi(\tau_k, \theta) - u_0(1,\theta)\|_{L^2} = I + II.$$

As $\psi(\tau_k, \theta) = \frac{u(\rho_k \cdot 1, \theta)}{\rho_k^2} \to u_0(1,\theta)$, we have $\psi(\tau_k, \theta)$ converging locally uniformly to $u_0(1,\theta)$, so II converges to zero. For the first term, we use the estimate from Step 4, $\psi_\tau \in L^2((\tau_0, \infty) \times \mathbb{S}^{n-1})$, in particular $\int_M^\infty \psi_t^2 \to 0$ as $M \to \infty$, thus

$$\begin{aligned}
\|\psi(\tau_k + s, \theta) - \psi(\tau_k, \theta)\|_{L^2} &= \int_{-N}^{N} \int_{\mathbb{S}^{n-1}} |\psi(\tau_k + s, \theta) - \psi(\tau_k, \theta)|^2 = \int_{-N}^{N} \int_{\mathbb{S}^{n-1}} \left(\int_{\tau_k}^{\tau_k+s} \psi_t \right)^2 \\
&\leq \int_{-N}^{N} \int_{\mathbb{S}^{n-1}} s \int_{\tau_k}^{\tau_k+s} \psi_t^2 \leq \int_{-N}^{N} \int_{\mathbb{S}^{n-1}} N \int_{\tau_k}^{\infty} \psi_t^2 \\
&\leq CN^2 \int_{\tau_k}^{\infty} \psi_t^2 \longrightarrow 0 \quad \text{as } k \to \infty.
\end{aligned}$$

Hence for any $N \in \mathbb{R}$, $\psi(\tau_k + s, \theta) \to u_0(1,\theta)$ in $L^2((-N,N) \times \mathbb{S}^{n-1})$. Converting back to u we have

$$\frac{u(\rho_k e^{-s}, \theta)}{\rho_k^2 (e^{-s})^2} = \frac{u(e^{-\tau_k - s}, \theta)}{e^{-2\tau_k - 2s}} = \psi(\tau_k + s, \theta) \to u_0(1,\theta) \quad \text{in } L^2((-N,N) \times \mathbb{S}^{n-1})$$

$$\Leftrightarrow u_{\rho_k}(r,\theta) \to r^2 u_0(1,\theta) \quad \text{in } L^2((e^{-N}, e^N) \times \mathbb{S}^{n-1}).$$

But also $u_{\rho_k}(r,\theta) \to u_0(r,\theta)$ and so by continuity of u_0 we have $u_0(r,\theta) = r^2 u_0(1,\theta)$ for $r \in (e^{-N}, e^N)$. Since N was arbitrary, it holds for any $r \in (0, \infty)$. □

Finally we can prove convexity, yet we need a short lemma, stating that a non-positive superharmonic function w in $\{w < 0\}$ is superharmonic everywhere. For a proof see [FR22, Lemma 5.21].

Lemma 3.3. *Let $\Lambda \subset B_1$ be closed and $w \in H^1(B_1) \cap C(B^1)$ such that $w \geq 0$ in Λ and w weakly superharmonic in $B_1 \backslash \Lambda$. Then $\min\{w, 0\}$ is weakly superharmonic in B_1.*

Theorem 3.4. *Let $u_0 \in C^{1,1}$ be a solution to* (GP) *that is 2-homogeneous. Then u_0 is convex, i.e. $D^2 u_0 \geq 0$.*

Proof. For $e \in \mathbb{S}^{n-1}$ fixed, define $w_0 := \min\{\partial_{ee} u_0, 0\}$ We want to show that w_0 is superharmonic in $\mathbb{R}^n$, as defined for L^1_{loc} functions, i.e. $r \mapsto \fint_{B_r(x)} u(y)dy$ is non-increasing for $r \in (0, \text{dist}(x, \partial B_1))$. Define now for $t > 0$

$$\delta_t^2 u_0(x) := \frac{u_0(x + te) + u_0(x - te) - 2u_0(x)}{t^2}.$$

Using $\Delta u_0 = \chi_{\{u_0 > 0\}}$,

$$\Delta \delta_t^2 u_0 = \frac{1}{t^2} \left(\chi_{\{u_0(\cdot + te)\}} + \chi_{\{u_0(\cdot - te)\}} - 2 \right) \leq 0 \qquad \text{in } \{u_0 > 0\},$$

so $\delta_t^2 u_0$ is weakly superharmonic in $\{u_0 > 0\}$. Moreover $\delta_t^2 u_0 \geq 0$ in $\{u_0 = 0\}$ and $\delta_t^2 u_0 \in C^{1,1}(B_1)$, so by Lemma 3.3, $w_t := \min\{\delta_t^2 u_0, 0\}$ is weakly superharmonic. Thus, also the weaker definition (non-increasing mean value) applies. As $u_0 \in C^{1,1}$, $\delta_t^2 u_0$ and thereby w_t is uniformly bounded and converges pointwise to w_0. By [FR22, Lemma 1.16] it follows that w_0 is superharmonic in the sense of the non-increasing mean value property. By [FR22, Lemma 1.17] we have lower semicontinuity of w_0 almost everywhere. Now as u_0 is 2-homogeneous, $\partial_{ee} u_0$ and so w_0 are 0-homogeneous (i.e. $f(\lambda x) = f(x)$). It has thus a minimum at $y_0 \in B_1$, however $\fint_{B_r(y_0)} w_0$ is non-increasing in r, so w_0 has to be constant and equal to its value on the free boundary, i.e. $w_0 \equiv 0$. This in turn implies that $\partial_{ee} u_0 \geq 0$. □

Now that we have convexity we proceed with the classification of blow-ups. Before we had almost no information on u_0, now we are able to show that there are only two possible forms.

Theorem 3.5. (Classification Theorem) *Let u be a solution to* (P1) *and u_0 be any blow-up of u at 0, i.e. a global solution.*

1. *If $\{u_0 = 0\}$ has non-empty interior, then*

$$u_0(x) = \frac{1}{2}(x \cdot e)_+^2 \qquad \text{for some } e \in \mathbb{S}^{n-1}.$$

2. *If $\{u_0 = 0\}$ has empty interior, then*

$$u_0(x) = \frac{1}{2} x^T A x \qquad \text{for some } A \geq 0, \ \text{tr}A = 1.$$

Before proving the main theorem we will need several lemmata (three for the first case, one for the second case) concerning convexity and harmonic functions.

Lemma 3.6. (Hopf Lemma) *Let Ω satisfy the interior ball condition (i.e. every point on the boundary can be touched from the inside with a ball of radius ρ) and $u \in C(\bar{\Omega})$ a positive harmonic function in* $\text{cl}(\Omega \cap B_1)$ *with $u \geq 0$ on $\partial\Omega \cap B_1$. Then for some $c_0 > 0$*

$$u(x) \geq c_0 d_\Omega(x) \quad \forall x \in \Omega \cap B_1,$$

where $d_\Omega(x) := \text{dist}(x, \Omega^c)$.

Proof. Firstly, as $u > 0$ and it is continuous in $\Omega \cap B_1$, a bounded domain, we have

$$u \geq c_1 > 0 \text{ in } \left\{ d_{\Omega \cap B_1} \geq \frac{\rho_0}{2} \right\} =: \Omega', \quad \text{ for some small } c_1, \rho_0 > 0.$$

(As the minimum of u in the compact set Ω' is strictly positive.) Let w be the solution to

$$\begin{array}{rcll} \Delta u &=& 0 & \text{in } B_{\rho_0} \backslash B_{\rho_0/2} =: A_{\rho_0}(0) \\ w &=& 0 & \text{on } \partial B_{\rho_0} \\ w &=& 1 & \text{on } \bar{B}_{\rho_0/2}, \end{array}$$

which can be given explicitly using the fundamental solution, i.e.

$$w(x) = \begin{cases} \frac{\rho_0^{n-2}}{(2^{n-2}-1)} \frac{1}{|x|^{n-2}} - \frac{1}{2^{n-2}-1}, & n \geq 3 \\ \frac{1}{\log(2)} \log\left(\frac{1}{|x|}\right) - \frac{-\log(\rho_0)}{\log(2)}, & n = 2. \end{cases}$$

Next we want to show that in both cases $w(x) \geq c_2(\rho_0 - |x|)$ in B_{ρ_0} with c_2 depending only on ρ_0 and the dimension. The inequality clearly holds in $B_{\rho_0/2}$ whenever $c_2 \leq \frac{2}{\rho_0}$, otherwise

$$\left\{ \begin{array}{ll} \frac{\rho_0^{n-2}}{(2^{n-2}-1)} \frac{1}{r^{n-2}} - \frac{1}{2^{n-2}-1} & \geq c_2(\rho_0 - r) \\ \frac{1}{\log(2)} \log\left(\frac{1}{r}\right) - \frac{-\log(\rho_0)}{\log(2)} & \geq c_2(\rho_0 - r) \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{ll} \frac{\rho_0^{n-2}}{r^{n-2}} & \geq (2^{n-2}-1)c_2(\rho_0 - r) + 1 \\ \log\left(\frac{\rho_0}{r}\right) & \geq \log(2)c_2(\rho_0 - r). \end{array} \right\}$$

As we have equality for $r = \rho_0$, and all quantities involved are positive and monotonically decreasing for $r \in (0, \rho_0)$, it suffices to show that the derivative on the LHS is smaller than on the RHS (steeper decrease),

$$\left\{ \begin{array}{lll} (2-n)\frac{\rho_0^{n-2}}{r^{n-3}} & \leq & (1-2^{n-2})c_2 \\ \frac{-1}{r} & \leq & -\log(2)c_2 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{llll} c_2 & \leq & \frac{n-2}{2^{n-2}-1}\frac{1}{\rho_0}, & n \geq 3 \\ c_2 & \leq & \frac{1}{\log(2)\rho_0}, & n = 2. \end{array} \right\}$$

We now want to show that the growth condition holds everywhere in $\Omega \cap B_1$. The only issue are values close to $\partial(\Omega \cap B_1)$, because in Ω' we already have

$$u(x) \geq c_1 \geq \frac{c_1}{M} \operatorname{dist}(x, \partial\Omega) \geq c_0 d_\Omega(x),$$

where $M = \max_{x \in \bar{B}_1} \operatorname{dist}(x, \partial\Omega)$.

Take now $x_1 \in (\Omega \cap B_1) \backslash \Omega'$ and let $\bar{x}_1$ be its closest boundary point, i.e. $|x_1 - \bar{x}_1| = \operatorname{dist}(x_1, \partial\Omega)$. By the interior sphere condition, we can find $B_{\rho_0}(x_0)$ for some x_0, touching at $\bar{x}_1$, the points $x_1, \bar{x}_1, x_0$ are co-linear and so $\rho_0 - |x_1 - x_0| = \operatorname{dist}(x_1, \partial\Omega)$.

Consider now $c_1 w(x+x_0)$ in $A_{\rho_0}(x_0) \subset \Omega \cap B_1$. As $w(x+x_0) \leq 1$ on $\partial A_{\rho_0}(x_0)$ (i.e. $w(x+x_0) = 0$ on $\partial B_{\rho_0}(x_0)$ and $w(x+x_0) = 1$ on $\partial B_{\rho_0/2}(x_0)$) and $u \geq c_1$ on $\partial B_{\rho_0/2}(x_0)$, because $\operatorname{dist}(x_0, \partial(\Omega \cap B_1)) \geq \rho/2$), it follows that $c_1 w(x+x_0) \leq u$ on $\partial A_{\rho_0}(x_0)$. Clearly also $c_1 w(x+x_0) = c_1 \leq u$ in $B_{\rho_0/2}(x_0)$. Thus by the comparison principle (here we need harmonicity up to the boundary) and the previous estimate,

$$u(x) \geq c_1 w(x+x_0) \geq c_1 c_2(\rho_0 - |x - x_0|) \geq c_1 c_2 \operatorname{dist}(x, \partial\Omega) \geq c_0 d_\Omega(x) \quad \forall x \in B_{\rho_0}(x_0),$$

in particular the estimate holds for x_1. Since x_1 was arbitrary, $u(x) \geq c_0 d_\Omega(x)$ in $\Omega \cap B_1$. □

The next Lemma provides a relation between convexity and harmonicity.

Lemma 3.7. *Let Σ be a closed convex cone with non-empty interior with its vertex at the origin. Let $w \in C(\mathbb{R}^n)$ be a 1-homogeneous function satisfying $\Delta w = 0$, $w > 0$ in Σ^c and $w = 0$ in Σ. Then Σ is a half space.*

Proof. As Σ is convex, there exists $H = \{x \cdot e\}$ such that $H \subset \Sigma^c$. Define now $v := (x \cdot e)_+$, which is positive in H and vanishes in H^c and on ∂H. By assumption $w > 0$ in Σ^c, which satisfies the interior ball condition, so by the Hopf Lemma 3.6, $w \geq c_0 \operatorname{dist}(x, \Sigma)$ in $\Sigma^c \cap B_1$. It is easy to see that $\operatorname{dist}(x, \Sigma)$ is also 1−homogeneous, as Σ is a cone. It follows that $w \geq c_0 \operatorname{dist}(x, \Sigma)$ everywhere in Σ^c.

Note also that $\operatorname{dist}(x, \Sigma) \geq \operatorname{dist}(x, H^c) = v$, due to $\Sigma \subset H^c$. Hence $w \geq c_0 v$. Define now

$$c_* := \sup\{c > 0 : w \geq cv \text{ in } \Sigma^c\},$$

which is well defined as $c_* \geq c_0 > 0$. We want to show that the nonnegative function $w - c_* v$ is identically zero. Assume it is not. Because it is harmonic in Σ^c, by the strong maximum principle $w - c_* v > 0$ in Σ^c and in particular also in H. Again by the Hopf Lemma 3.6,

$$\begin{aligned} w(x) - c_* v(x) \geq c_1 \operatorname{dist}(x, H^c) = c_1 v(x) \quad &\text{in } H \\ \Leftrightarrow w - (c_* + c_1)v \geq 0 \quad &\text{in } H \\ \Leftrightarrow w - (c_* + c_1)v \geq 0 \quad &\text{in } \Sigma^c, \end{aligned}$$

but as $c_1 > 0$ that contradicts the maximality of c_*. Therefore $w - c_* v \equiv 0$ and so $\Sigma = H$, a half space. □

The following lemma tells us that the Laplacian Δ does not notice sets of co-dimension 1.

Lemma 3.8. *Assume that $\Delta u = 1$ in $\mathbb{R}^n \backslash \partial H$ with $\partial H = \{x_1 = 0\}$. If $u \in C^1(\mathbb{R}^n)$, then $\Delta u = 1$ in $\mathbb{R}^n$.*

Proof. First let w be the solution to $\Delta w = 1$ in B_R, $w = u$ on ∂B_R and define $v := u - w$. Then v is harmonic in $B_R \backslash \partial H$ and $v = 0$ on ∂B_R. Furthermore as u (by continuity) and w (by maximum principle) are bounded on $\partial H \cap B_R$, v is bounded as well. Thus

$$v(x) \leq \kappa(2R - |x_1|) =: \kappa \quad \text{ in } B_R, \text{ if } \kappa \text{ is large.}$$

Define $\kappa^* := \inf\{\kappa \geq 0 : v(x) \leq \kappa(2R - |x_1|) \quad \forall x \in B_R\}$.

Assume that $\kappa^* > 0$, then by continuity of v and $(2R - |x_1|)$ there exists a point $p \in \bar{B}_R$, such that $v(p) = \kappa^*(2R - |p_1|)$. As $v = 0$ on ∂B_R, the point p lies inside B_R. Furthermore as $v \in C^1$(because u is C^1 by assumption and w is C^1 being the solution of the Poisson problem) and $\frac{\partial}{\partial x^i}(2R - |x_1|)\Big|_{x \in \partial H}$ does not exist for any $i \geq 2$ (the function has a wedge) it follows that $p \in B_R \backslash \partial H$.
However the harmonic function $\phi := \kappa^*(2R - |x_1|) - v \geq 0$ attaining its minimum 0 at the interior point p leads to the contradiction $\phi \equiv 0$ by the strong maximum principle. Therefore $\kappa^* = 0$ and so $v \leq 0$ in B_R, equally taking $-v$ gives $v \geq 0$. So $v \equiv 0$ and $u \equiv w$. □

The last lemma states that if a convex function stays constant on a line, then it stays always constant in direction of the direction vector of that line.

Lemma 3.9. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be convex such that for some $e \in \mathbb{S}^{n-1}$ we have $\{te : t \in \mathbb{R}\} \subset \{u = 0\}$, i.e. the contact set contains a straight line.*
Then $u(x + te) = u(x), \quad \forall x \in \mathbb{R}^n \quad \forall t \in \mathbb{R},$ that is, u is constant in the direction of e.

Proof. Without loss of generality assume $e = e_n$ and write $x = (x', x_n)$, so $u(0, x_n) = 0$ for any x_n. Thus we want to prove that for any x_n, $u(x', x_n) = u(x', 0)$ $\forall x' \in \mathbb{R}^{n-1}$. As u is convex

$$\begin{aligned}(1-\mu)u(x', x_n) + \mu u(0, x_n + M) &\geq u[(1-\mu)x' + \mu 0, (1-\mu)x_n + \mu x_n + \mu M] \\ &= u[(1-\mu)x', x_n + \mu M] \\ \implies (1-\mu)u(x', x_n) &\geq u[(1-\mu)x', x_n + \mu M].\end{aligned}$$

Choose $M = \lambda/\mu$ and passing to the limit as $\mu \to 0$ (u is continuous)

$$u(x', x_n) \leftarrow (1-\mu)u(x', x_n) \geq u((1-\mu)x', x_n + \lambda) \to u(x', x_n \mu M).$$

As this holds for any x_n, let $\tilde{x}_n := x_n + \lambda$, then

$$u(x', x_n + \lambda) = u(x', \tilde{x}_n) \geq u(x', \tilde{x}_n + \lambda) = u(x', x_n) \implies u(x', x_n) = u(x', x_n + \lambda),$$

and the fact that λ was arbitrary finishes the proof. □

Eventually we can prove the classification result.

Proof. (of Theorem 3.5, the classification theorem)
Case 1: The set $\{u_0 = 0\}$ has non-empty interior, it follows by convexity and 2-homogeneity that $\{u_0 = 0\} = \Sigma$ is a closed convex cone. We claim now that $\partial_\tau u_0 \geq 0$ for any $\tau \in \mathbb{S}^{n-1}$ with $-\tau \in \mathring{\Sigma}$. As Σ was a convex cone with non-empty interior, for any fixed $x \in \mathbb{R}^n$, there exists $t_0 \ll -1$ such that $u_0(x + \tau t) = 0$ $\forall t \leq t_0$. By convexity, $\partial_t u_0(x + \tau t) = \nabla u_0(x + \tau t) \cdot \tau$ is nondecreasing in t and zero for $t \leq t_0$, so $\partial_\tau u_0 = \partial_t u_0 \geq 0$.

Let $w := \partial_\tau u_0 \geq 0$, which is not identically zero for at least some $\tau \in \mathbb{S}^{n-1}$ with $-\tau \in \mathring{\Sigma}$. As $\Delta u_0 = 1$ in Σ^c, w is harmonic there, it follows that $w > 0$ in Σ^c. Since the derivative of a 2-homogeneous function is 1-homogeneous, we can apply Lemma 3.7 on w, which implies that Σ is a closed half space. Clearly Σ, in particular $\partial\Sigma$, contains now straight lines of the form $\{t\bar{e} : t \in \mathbb{R}\}$, so by Lemma 3.9 u_0 remains constant parallel to those lines, i.e. it is a 1D function. Write thus $u_0(x) = U(x \cdot e)$ with $U : \mathbb{R} \to \mathbb{R}$, one point on each line suffices to give its value, also $e \perp \bar{e}$. By properties of u_0 (see (GP)), $U''(t) = 1$ for $t < 0$ and $U(t) = 0$ for $t \leq 0$. Hence $U(t) = \frac{1}{2}t_+^2$ and $u_0(x) = \frac{1}{2}(x \cdot e)_+^2$.

Case 2: Now let us suppose that $\{u_0 = 0\}$ has empty interior, and by convexity $\{u_0 = 0\}$ is thus contained in an $n-1$ dimensional hyperplane ∂H. This follows from the contact set $\{u_0 = 0\}$ of a convex function being convex itself. If it contained any other point not on the straight line given by Lemma 3.9, then it would also contain the convex hull of this point and the line, contradicting the fact that it has empty interior. Moreover $\Delta u_0 = 1$ in $\mathbb{R}^n \backslash \partial H$, so by Lemma 3.8, $\Delta u_0 = 1$ in $\mathbb{R}^n$. All second derivatives of u_0 are harmonic and bounded ($u_0 \in C^{1,1}$), so they are constant. Hence u_0 is a quadratic polynomial and since $u_0(0) = 0$, $\nabla u_0(0) = 0$ and $u_0 \geq 0$, only the quadratic terms remain, that is $u_0(x) = \frac{1}{2}x^T Ax$ with $A \geq 0$. Also since $\Delta u_0 = 1$ we have $\text{tr}A = 1$. □

4 Regularity of the free boundary

The main goal is now to use the information from the blow-up (convexity and classification) to say something about the behaviour on the free boundary of a scaled version u_r, with r small enough, and so the original solution u. The rough idea is as follows:

1. Split up the free boundary of u, $\partial\{u > 0\}$, into points with positive density (regular points) and zero density (singular points).
2. Show that u has a blow-up $u_0 = \frac{1}{2}(x \cdot e)_+^2$ for some $e \in \mathbb{S}^{n-1}$ at regular points.
3. Show that u_0 and its derivative are sufficiently "close" to a scaled version of u.
4. Show that the free boundary of the scaled version u_{r_0} around 0 is contained in a strip.
5. Show that there exists a cone of directions τ, in where u_r is non-decreasing, that is partially transferring monotonicity from u_0 to u_r.
6. In particular the free boundary of u_{r_0} is Lipschitz, which implies that the free boundary of u is also Lipschitz.
7. Use Boundary Harnack to show $C^{1,\alpha}$ regularity around the origin.
8. Use Higher order Boundary Harnack repeatedly to show C^∞ regularity around the origin.

Definition 4.1. We say x is a *regular (free boundary) point* if

$$\limsup_{r \to 0} \frac{|\{u = 0\} \cap B_r(x)|}{|B_r|} > 0,$$

that is the contact set $\{u = 0\}$ has positive density at x.

First we show that for regular points we have a blow-up of the form $u_0 = \frac{1}{2}(x \cdot e)_+^2$.

Lemma 4.2. *Let u be a solution to the classical obstacle problem in the ball* (P1) *where* 0 *is a regular point as in Definition 4.1. Then there exists a blow-up u_0 at* 0 *of the form*

$$u_0(x) = \frac{1}{2}(x \cdot e)_+^2, \quad e \in \mathbb{S}^{n-1}.$$

Proof. Start by taking a sequence $r_k \to 0$ where

$$\lim_{r_k \to 0} \frac{|\{u = 0\} \cap B_{r_k}|}{|B_{r_k}|} \geq \theta > 0.$$

By Proposition 3.1 there exists a subsequence r_{k_j} such that $u_{r_{k_j}} \to u_0$ in the $C^1_{\text{loc}}(\mathbb{R}^n)$ norm.

Now assume that $\{u_0 = 0\}$ has empty interior. Then it is contained in a hyperplane, and without loss of generality let that hyperplane be $\{x_1 = 0\}$. By continuity of u_0, $\forall \delta > 0$ there exists $\varepsilon > 0$ such that

$$u_0 \geq \varepsilon > 0 \text{ in } \{|x_1| > \delta\} \cap B_1.$$

By the $C^1_{\text{loc}}(\mathbb{R}^n)$ convergence, we have uniform convergence in B_1, i.e. there exists $r_{k_{j_0}} > 0$ such that

$$u_{r_{k_j}} \geq \frac{\varepsilon}{2} > 0 \text{ in } \{|x_1| > \delta\} \cap B_1 \quad \forall k_j \geq k_{j_0}.$$

Hence

$$\begin{aligned}
\{u_{r_{k_j}} = 0\} \subset \{|x_1| \leq \delta\} \cap B_1 \implies & \frac{|\{u_{r_{k_j}} = 0\} \cap B_1|}{|B_1|} \leq \frac{|\{|x_1| \leq \delta\} \cap B_1|}{|B_1|} \leq C\delta \\
\implies & \frac{|\{u = 0\} \cap B_{r_{k_j}}|}{|B_{r_{k_j}}|} = \frac{|\{u_{r_{k_j}} = 0\} \cap B_1|}{|B_1|} < C\delta,
\end{aligned}$$

with C depending only on the dimension n. However $\delta > 0$ was arbitrary so

$$\lim_{r_{k_j} \to 0} \frac{\left|\{u = 0\} \cap B_{r_{k_j}}\right|}{|B_{r_{k_j}}|} = 0,$$

which contradicts the fact that 0 was a regular point. Thus $\{u_0 = 0\}$ must have nonempty interior and can be written as $u_0(x) = \frac{1}{2}(x \cdot e)_+^2$ for some $e \in \mathbb{S}^{n-1}$ by Theorem 3.5. $\square$

From the convergence of the rescalings to the blow-ups, the next proposition follows.

Proposition 4.3. *Assume u is a solution to* (P1) *with* 0 *a regular FBP. Then there exists $r_0 > 0$ and $e \in \mathbb{S}^{n-1}$ such that*

$$|u_{r_0}(x) - \frac{1}{2}(x \cdot e)_+^2| \leq \varepsilon, \quad |\partial_\tau u_{r_0}(x) - (x \cdot e)_+(\tau \cdot e)| \leq \varepsilon \quad \forall \tau \in \mathbb{S}^{n-1} \quad \forall x \text{ in } B_1.$$

Proof. This follows essentially from the $C^1_{\rm loc}(\mathbb{R}^n)$ convergence along a subsequence r_j of $u_{r_j} \to \frac{1}{2}(x\cdot e)_+^2$ and the fact that $\partial_\tau \left(\frac{1}{2}(x \cdot e)_+^2\right) = (x \cdot e)_+(\tau \cdot e)$. □

The next lemma gives a way to extend nonnegativity of a harmonic function on a compact subset of the domain to the whole domain. We use that fact later on the partial derivatives of u_0 and u_r, which are close by Proposition 4.3.

Lemma 4.4. *Let u be a solution to* (P1) *and its rescaled version u_{r_0}, with $\Omega = \{u_{r_0} > 0\}$. Assume that $w \in C(B_1)$ satisfies:*

- *w is bounded and harmonic in $\Omega \cap B_1$,*
- *$w = 0$ on $\partial\Omega \cap B_1$,*
- *$w \geq -c_1$ in N_δ and $w \geq C_2 > 0$ in $\Omega \backslash N_\delta$ where $N_\delta := \{x \in B_1 : \mathrm{dist}(x, \partial\Omega) < \delta\}$.*

Then, provided that c_1/C_2 and $\delta > 0$ are sufficiently small, $w \geq 0$ in $B_{1/2} \cap \Omega$.

Proof. We only need to show that $w \geq 0$ in $N_\delta \cap \Omega \cap B_{1/2}$. Assume by contradiction that for a point inside, $w(y_0) < 0$. Define in $B_{1/4}(y_0)$,

$$v(x) := w(x) - \gamma \left(u_{r_0}(x) - \frac{1}{2n}|x - y_0|^2 \right).$$

In $B_{1/4}(y_0) \cap \Omega$, $\Delta v = 0$ and $v(y_0) < 0$, so by the maximum principle the negative minimum of v is achieved on $\partial(B_{1/4}(y_0) \cap \Omega)$. Split up this boundary into three parts:

$$S_1 := \partial\Omega \cap B_{1/4}(y_0), \quad S_2 := \partial B_{1/4}(y_0) \cap N_\delta, \quad S_3 := \partial B_{1/4}(y_0) \cap (\Omega \backslash N_\delta).$$

Denote $C_0 := \|u_{r_0}\|_{C^{1,1}(B_1)}$.

1. Clearly $S_1 \subset \partial\Omega \cap B_1$, so $v(x) = \frac{\gamma}{2n}|x - y_0|^2 > 0$.
2. On S_3 we have

$$v(x) = w(x) - \gamma u_{r_0}(x) + \frac{\gamma}{32n} \geq C_2 - C_0\gamma \geq 0, \quad \text{if } \gamma \leq C_2/C_0.$$

3. On $S_2 = \partial B_{1/4}(y_0) \cap N_\delta$ we first rewrite $u_{r_0}(x)$ using its Taylor expansion centered at $y \in \partial\Omega$.

$$\begin{aligned} u_{r_0}(x) &= u_{r_0}(y) + Du_{r_0}(y)(y - x) + D^2 u_{r_0}(y)\xi^2 \quad \text{ with } \xi \text{ in the segment } (y, x) \\ &\implies |u_{r_0}(x)| = |D^2 u_{r_0}(y)\xi^2| \leq \|D^2 u_{r_0}(y)\|\xi^2 \leq C_0\delta^2. \end{aligned}$$

Thus we want to show that for certain δ and γ

$$v \geq -c_1 - C_0\gamma\delta^2 + \frac{\gamma}{32n} \geq 0.$$

Take first $\delta \leq \frac{1}{\sqrt{64nC_0}}$, then

$$v \geq -c_1 + \gamma \left(\frac{\gamma}{32n} - C_0\delta^2 \right) \geq -c_1 + \gamma\frac{1}{64n} \geq 0, \quad \text{ if } \gamma \geq c_1 64n.$$

Now we have the following condition

$$\gamma : c_1 64n \leq \gamma \leq C_2/C_0,$$

which makes sense only if $c_1/C_2 \leq (64nC_0)^{-1}$.

However this implies that the negative minimum of v is actually nonnegative and so $w \geq 0$ in $B_{1/2} \cap \Omega$. $\square$

Now we can show that the solution u_r is actually nondecreasing in a conce near the origin.

Proposition 4.5. *Let u be a solution to* (P1) *and* 0 *a regular FBP. Then there exists $r_0 > 0$ and $e \in \mathbb{S}^{n-1}$ such that*

$$\partial_\tau u_{r_0} \geq 0 \text{ in } B_{1/2}, \qquad \forall \tau \in \mathbb{S}^{n-1} \text{ s.t. } \tau \cdot e \geq \frac{1}{2}.$$

Proof. Denote $\Omega = \{u_{r_0} > 0\}$.
Step 1: We start by showing that the free boundary $\partial\Omega$ is contained in a strip $\{x : |x \cdot e| \leq C(n)\sqrt{\varepsilon}\}$ of width $2C(n)\sqrt{\varepsilon}$. That is we want to show that

$$u_{r_0} > 0 \text{ in } \{x \cdot e > C\sqrt{\varepsilon}\},$$
$$u_{r_0} = 0 \text{ in } \{x \cdot e < -C\sqrt{\varepsilon}\}.$$

We have from Proposition 4.3, $|u_{r_0}(x) - \frac{1}{2}(x \cdot e)_+^2| \leq \varepsilon$, in other words for $x \cdot e > C\sqrt{\varepsilon}$,

$$u_{r_0} \geq \frac{1}{2}(x \cdot e)_+^2 - \varepsilon > \frac{1}{2}(C\sqrt{\varepsilon})^2 - \varepsilon = \frac{C^2}{4} > 0 \quad \text{if } C \text{ sufficiently large.}$$

For the second statement, assume there exists a FBP x_0 in $\{x \cdot e < -C\sqrt{\varepsilon}\}$. By the nondegeneracy condition for a FBP

$$\sup_{B_{C\sqrt{\varepsilon}}(x_0)} u_{r_0} = \sup_{B_{C\sqrt{\varepsilon}}(x_0)} \frac{u(r_0 \cdot)}{r_0^2} = \frac{1}{r_0^2} \sup_{B_{C\sqrt{\varepsilon}r_0}(x_0)} u \geq \frac{1}{r_0^2} c(C\sqrt{\varepsilon}r_0)^2 = c(C\sqrt{\varepsilon})^2 > 2\varepsilon,$$

with $c = \frac{1}{2n}$ (c depends on the assumption $f \geq c_0 > 0$ (here $f \equiv 1$), and n). Choosing then $C > 2\sqrt{n}$ suffices, take ε small enough so that $\bar{B}_{C\sqrt{\varepsilon}}(x_0) \subset B_1$. However now for x close to x_0 in $\{x \cdot e < -C\sqrt{\varepsilon}\}$,

$$|u_{r_0}(x) - \frac{1}{2}(x \cdot e)_+^2| = |u_{r_0}(x)| = |2\varepsilon| = 2\varepsilon \leq \varepsilon,$$

which contradicts the statement from Proposition 4.3. Hence $\partial\Omega \subset \{|x \cdot e \leq C\sqrt{\varepsilon}\}$.

Step 2: Define now $w := \partial_\tau u_{r_0}$, the first two conditions from Lemma 4.4 are clearly satisfied, as $u_{r_0} \in C^{1,1}(B_1)$ and harmonic in Ω. The last condition requires a more delicate argument. We note that from Proposition 4.3 we also have

$$|\partial_\tau u_{r_0}(x) - (x \cdot e)_+(\tau \cdot e)| \leq \varepsilon.$$

Moreover, let $\delta > 0$ (specified later) and $N_\delta := \{x \in B_1 : \operatorname{dist}(x, \partial\Omega) < \delta\}$, then

$$\varepsilon \geq (x \cdot e)_+(\tau \cdot e) - w \geq \frac{1}{2}(x \cdot e)_+ - w \implies w \geq w - \frac{1}{2}(x \cdot e)_+ \geq -\varepsilon.$$

Now note that $\{|x \cdot e| < \delta - C\sqrt{\varepsilon}\} \subset N_\delta$ (for e.g. $\delta \geq 2C\sqrt{\varepsilon}$) and inside $(\Omega \backslash N_\delta) \cap B_1 = (B_1 \backslash N_\delta) \cap \Omega = (B_1 \cap \Omega) \backslash N_\delta$ we have

$$w \geq \frac{1}{2}(x \cdot e)_+ - \varepsilon \geq \frac{1}{2}\delta - \frac{1}{2}C\sqrt{\varepsilon} - \varepsilon \geq \frac{1}{4}\delta,$$

provided that $\delta \geq 2C\sqrt{\varepsilon} + 4\varepsilon$. What remains, before using Lemma 4.4, is to check the ratio between $c_1 = \varepsilon$ and $C_2 = \delta/4$ and ensure that δ is sufficiently small (here the condition from the Lemma was $\delta \leq (64n\|u_{r_0}\|_{C^{1,1}(B_1)})^{-1/2}$). We have

$$\frac{1}{64n\|u_{r_0}\|_{C^{1,1}(B_1)}} \geq \frac{c_1}{C_2} = \frac{\varepsilon}{\delta/4} \implies 256n\|u_{r_0}\|_{C^{1,1}(B_1)}\varepsilon \leq \delta,$$

thus we can conclude the condition on δ as follows:

$$\max\{2C\sqrt{\varepsilon} + 4\varepsilon, 256n\|u_{r_0}\|_{C^{1,1}(B_1)}\varepsilon\} \leq \delta \leq (64n\|u_{r_0}\|_{C^{1,1}(B_1)})^{-1/2}$$

Note it is important to first pick ε small enough, then we get r_0 and lastly δ. Now apply Lemma 4.4 and deduce that $\partial_\tau u_{r_0} = w \geq 0$ in $B_{1/2} \cap \Omega$. Since τ with $\tau \cdot e \geq 1/2$ was arbitrary and $\partial_\tau u_{r_0} = 0$ in Ω^c the proof is complete. $\square$

What this proposition tells us is that as $\partial_\tau u_0 = (x \cdot e)_+(\tau \cdot e)$ is nonnegative in the halfspace $\tau \cdot e$ (i.e. u_0 is non-decreasing), we can get a partial result for u_{r_0}. In fact u_{r_0} is non-decreasing, not in the whole half space but in a cone centered at the origin. This has the far reaching consequence:

Theorem 4.6. *Let u be a solution to* (P1) *and assume that the FBP at the origin regular. Then there exists r_0 such that $\partial\{u_{r_0} > 0\}$ is Lipschitz in $B_{1/2}$ (equivalently, $\partial\{u > 0\}$ is Lipschitz in $B_{r_0/2}$).*

Proof. Take $x_0 \in B_{1/2} \cap \partial\{u_{r_0} > 0\}$ and define

$$\begin{aligned} \Theta &:= \{\tau \in \mathbb{S}^{n-1} : \tau \cdot e > 1/2\}, \\ \Sigma_1 &:= \{x \in B_{1/2} : x = x_0 - t\tau, \text{ with } \tau \in \Theta, t > 0\}, \\ \Sigma_2 &:= \{x \in B_{1/2} : x = x_0 + t\tau, \text{ with } \tau \in \Theta, t > 0\}, \end{aligned}$$

where r_0 and e are given by Proposition 4.5.

First, since $u_{r_0}(x_0) = 0$, $u_{r_0} \geq 0$ and $\partial_\tau u_{r_0} \geq 0$,

$$\frac{d}{dt} u_{r_0}(x_0 - t\tau) = -\partial_\tau u_{r_0}(x_0 - t\tau) \leq 0 \implies u_{r_0}(x_0 - t\tau) = 0 \quad \forall t > 0, \forall \tau \in \Theta,$$

so $u_{r_0} \equiv 0$ in Σ_1 and there cannot be another FBP in Σ_1.

Secondly assume $u_{r_0}(x_2) = 0$ for some $x_2 \in \Sigma_2$, then $u_{r_0} \equiv 0$ in $\Sigma_2' := \{x \in B_{1/2} : x = x_2 - t\tau\}$, which contains also x_0. But by the first argument, x_0 would then not be a FBP, so $u_{r_0} > 0$ in Σ_2. Thus $\partial\{u_{r_0} > 0\} \cap B_{1/2}$ satisfies an exterior and an interior cone condition and can thereby be written as a Lipschitz function. Notice that we have actually control over the Lipschitz constant as the cone is of the form $\{x \cdot e = \frac{1}{2}\}$, the Lipschitz constant is $L = \sqrt{3}/2$. □

To proceed from Lipschitz regularity to C^∞ regularity two deep results about the comparability of harmonic functions are needed. For proofs we refer to [DS20, DSS14].

Theorem 4.7. (Boundary Harnack) *Let Ω be a Lipschitz domain. Let w_1, w_2 be two positive harmonic functions in $B_1 \cap \Omega$ i.e. for $i = 1, 2$*

$$\begin{cases} \Delta w_i &= 0 & \text{in } \Omega \cap B_1 \\ w_i &= 0 & \text{on } \partial\Omega \cap B_1 \\ C_0^{-1} &\leq \|w_i\|_{L^\infty(B_{1/2})} \leq C_0, & \text{for some } C_0 > 0. \end{cases}$$

Then for some $\alpha > 0$ small and $C = C(C_0, n, \Omega)$:

(i) $\frac{1}{C} w_2 \leq w_1 \leq C w_2$ *in $\bar{\Omega} \cap B_{1/2}$, that is w_1 and w_2 are comparable in $\bar{\Omega}$.*

(ii) $\left\| \frac{w_1}{w_2} \right\|_{C^{0,\alpha}(\bar{\Omega} \cap B_{1/2})} \leq C$, *their quotient is α-Hölder continuous.*

Theorem 4.8. (Higher order boundary Harnack) *Let Ω be a $C^{k,\alpha}$ domain ($k \geq 1$, $0 < \alpha < 1$), let w_1, w_2 with $w_2 > 0$ in Ω such that*

$$\begin{cases} \Delta w_i &= 0 & \text{in } \Omega \cap B_1 \\ w_i &= 0 & \text{on } \partial\Omega \cap B_1 \\ C_0^{-1} &\leq \|w_i\|_{L^\infty(B_{1/2})} \leq C_0, & \text{for some } C_0 > 0. \end{cases}$$

Then for some constant $C = C(n, k, \alpha, C_0, \Omega)$

$$\left\| \frac{w_1}{w_2} \right\|_{C^{k,\alpha}(\bar{\Omega} \cap B_{1/2})} \leq C.$$

We can now lift the Lipschitz regularity to $C^{1,\alpha}$ regularity.

Theorem 4.9. *Let u be a solution to* (P1) *such that the origin is a regular free boundary point. Then there exists $r_0 > 0$ such that $\partial\{u_{r_0} > 0\}$ is $C^{1,\alpha}$ in $B_{1/4}$ and thereby $\partial\{u > 0\}$ is $C^{1,\alpha}$ in $B_{r_0/4}$ with $\alpha > 0$ small.*

Proof. Denote $\Omega = \{u_{r_0} > 0\}$, we can rotate Ω (does not change anything about smoothness) such that $\partial\Omega = \{x : x_n = g(x_1, ..., x_{n-1})\}$ for some Lipschitz function g. Define now

$$\begin{aligned} w_1 &:= \partial_{e_i} u_{r_0} + \partial_{e_n} u_{r_0}, \quad i = 1, ..., n-1, \\ w_2 &:= \partial_{e_n} u_{r_0}. \end{aligned}$$

As by Proposition 4.5 $\partial_{e_n} u_{r_0} \geq 0$, $w_2 \geq 0$. For w_1, note that $\partial_{e_i} + \partial_{e_n} = \partial_{e_i+e_n} = \sqrt{2}\partial_\tau$ where $\tau \cdot e_n = 1/\sqrt{2} > 1/2$, so $w_1 \geq 0$ as well. Since $\Delta u_{r_0} = 1$ in Ω, w_1 and w_2 are harmonic. Now $w_i = 0$ on $\partial\Omega \cap B_1$ (as both e_n and $e_i + e_n$ point inwards the cone) and by harmonicity (maximum principle) $w_i > 0$ in $\Omega \cap B_{1/2}$. Thus $C_0^{-1} \leq \|w_i\|_{L^\infty(B_{1/4})} \leq C_0$ for some C_0, and we apply the boundary Harnack inequality (Theorem 4.7) to get

$$\begin{aligned} C \geq \left\| \frac{w_1}{w_2} \right\|_{C^{0,\alpha}(\bar{\Omega} \cap B_{1/4})} &= \left\| 1 + \frac{\partial_{e_i} u_{r_0}}{\partial_{e_n} u_{r_0}} \right\|_{C^{0,\alpha}(\bar{\Omega} \cap B_{1/4})} \\ \implies \left\| \frac{\partial_{e_i} u_{r_0}}{\partial_{e_n} u_{r_0}} \right\|_{C^{0,\alpha}(\bar{\Omega} \cap B_{1/4})} &\leq C. \end{aligned}$$

Remember that inside $\{u_{r_0} > 0\}$, u_{r_0} is already smooth, so the normal vector to the level set $\{u_{r_0} = t > 0\}$ is well defined and given by

$$\nu^i(x) = \frac{\partial_{e_i u_{r_0}}}{|\nabla u_{r_0}|} = \frac{\partial_{e_i} u_{r_0} / \partial_{e_n} u_{r_0}}{\sqrt{1 + \sum_{j=1}^{n-1} (\partial_{e_i} u_{r_0} / \partial_{e_n} u_{r_0})^2}} \qquad i = 1, ..., n.$$

This is a $C^{0,\alpha}$ function thanks to the bound from the boundary Harnack. Passing to the limit in t gives that the free boundary is then $C^{1,\alpha}$. □

Using now the higher order boundary Harnack we can establish full regularity of the free boundary for regular points.

Theorem 4.10. *Let u be a solution to* (P1) *such that the origin is a regular free boundary point. Then $\partial\{u > 0\}$ is C^∞ in a neighborhood around the origin.*

Proof. From the previous theorem we already have a certain r_0', so $\partial\{u_{r_0} > 0\}$ can be expressed as the graph of a $C^{1,\alpha}$ function $x_n = g(x_1, ..., x_{n-1})$ in B_1, i.e. $\partial_{e_n} u_{r_0} > 0$ in $\{u_{r_0} > 0\} \cap B_{1/4}$. Now use the higher order boundary Harnack (Theorem 4.8) with $w_1 = \partial_{e_i} u_{r_0}$ and $w_2 = \partial_{e_n} u_{r_0}$ to get

$$\left\| \frac{\partial_{e_i} u_{r_0}}{\partial_{e_n} u_{r_0}} \right\|_{C^{1,\alpha}(\bar{\Omega} \cap B_{1/2})} \leq C.$$

Using an argument similar to Lemma 2.5, with $k = 1$, we get

$$\left\| \frac{\partial_{e_i} u_{r_0}}{\partial_{e_n} u_{r_0}} \right\|_{C^{1,\alpha}(\bar{\Omega} \cap B_{1-\delta})} \leq \frac{1}{1-\delta} C \qquad \forall \delta < 1.$$

Again the normal vector of the level set $\{u_{r_0} = t\}$,

$$\nu^i(x) = \frac{\partial_{e_i u_{r_0}}}{|\nabla u_{r_0}|} = \frac{\partial_{e_i} u_{r_0} / \partial_{e_n} u_{r_0}}{\sqrt{1 + \sum_{j=1}^{n-1} (\partial_{e_i} u_{r_0} / \partial_{e_n} u_{r_0})^2}} \quad i = 1, ..., n,$$

inherits the smoothness and is so a $C^{1,\alpha}$ function in $B_{1-\delta}$.
Passing to the limit as $t \to 0$ and as $\delta > 0$ was arbitrary gives that the free boundary is $C^{2,\alpha}$ inside B_1. Iterating this argument and using the higher order boundary Harnack in each step gives that the free boundary is actually $C^{k,\alpha}$ inside B_1, $\forall k \in \mathbb{N}$, thus C^∞ around the origin. □

5 The singular points

Definition 5.1. We say $x = 0$ is a *singular (free boundary) point* if

$$\limsup_{r_k \to 0} \frac{|\{u = 0\} \cap B_{r_k}|}{|B_{r_k}|} = 0,$$

for any sequence $r_k \to 0$.

Note that here the definition has to hold for all sequences r_k, if there were only one sequence such that Definition 4.1 applies then the free boundary is smooth around the origin and so Definition 4.1 has to hold for each sequence.

Lemma 5.2. *Let u be a solution to* (P1) *such that Definition 5.1 holds (the origin is a singular point). Then for every blow-up u_0, $|\{u_0 = 0\}| = 0$.*

Proof. Let u_{r_k} be rescalings such that $u_{r_k} \to u_0$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$. As $\Delta u = \chi_{\{u>0\}}$ we have $\Delta u_{r_k} = \chi_{\{u_{r_k}>0\}}$ weakly in B_1. Clearly $|\{u_{r_k} = 0\} \cap B_1| = |\{u = 0\} \cap B_{r_k}| \to 0$, so

$$\int_{B_1} \nabla u_0 \eta = \lim_k \int_{B_1} \nabla u_{r_k} \nabla \eta = \lim_k \int_{B_1} \chi_{\{u_{r_k}>0\}} \eta = \lim_k \int_{\{u_{r_k}>0\} \cap B_1} \eta + \lim_k \int_{\{u_{r_k}>0\} \cap B_1} \eta = \int_{B_1} \eta.$$

Because $\{u_0 = 0\}$ is a convex set, it has to be contained in a hyperplane. Assume for the sake of contradiction, that it contained n points $x_1, \dots, x_n$ not all in a hyperplane, then by convexity of u_0, for each point x in the convex hull of $\mathrm{co}(\{x_i\}_{i=1}^n)$, $u_0(x) = 0$. Pick some $\bar{x}$ in the convex hull, then for r small enough, $B_r(\bar{x})$ small such that $B_r(\bar{x}) \subseteq \mathrm{co}(\{x_i\}_{i=1}^n)$. Yet $u_0 \equiv 0$ in $B_r(\bar{x})$ contradicts $\Delta u_0(\bar{x}) = 1$ and so $\{u_0 = 0\}$ is contained in a hyperplane and thus $|\{u_0 = 0\}| = 0$. □

Proposition 5.3. *For a solution u to* (P1) *we have the following dichotomy:*

(i) The origin is a regular point and all blow-ups are of the form $u_0(x) = \frac{1}{2}(x \cdot e)_+^2$, with $e \in \mathbb{S}^{n-1}$.

(ii) The origin is a singular point and all blow-ups are of the form $u_0(x) = \frac{1}{2}x^T Ax$, where A has trace 1 *and is positive semidefinite.*

Proof. If the origin is a regular point, by smoothness the blow-up is unique, so apply Lemma 4.2. If the origin is a regular point then by 5.2 we are in the second case of Theorem 3.5. □

Now that we have characterized all possible blow-ups, it remains to show that the blow-ups are unique, i.e. there are not two different sequence $r_k, r_l \to 0$ converging to different blow-ups. We will make use of another monotonicity formula, introduced in [Mon03]. For a proof see [PSU12, Ser19].

Theorem 5.4. (Monneau's monotonicity formula) *Let u be a solution to* (P1) *with* 0 *as a singular free boundary point. Let q be a homogenous quadratic polynomial such that $q \geq 0$, $q(0) = 0$ and $\Delta q = 1$. Then*

$$M_{u,q}(r) := \frac{1}{r^{n+3}} \int_{\partial B_r} (u - q)^2$$

is non-decreasing in r, i.e. $\frac{d}{dr} M_{u,q}(r) \geq 0$.

Theorem 5.5. *Let u be a solution to* (P1) *and the origin a singular point. Then the blow-up is unique and is given by $p_2(x) = \frac{1}{2}x^T Ax$ with $A \geq 0$ and $\Delta p_2 = 1$. Furthermore $u(x) = p_2(x) + o(|x|^2)$ and $\nabla u(x) = \nabla p_2(x) + o(|x|)$.*

Proof. By Proposition 5.3 we have a subsequence r_k such that u_{r_k} converges (in C^1_{loc}) to some polynomial p fulfilling the conditions for Theorem 5.4. As $M_{u,p}(r)$ is nonnegative, the limit $\lim_{r\to 0} M_{u,p}(r) := M_{u,p}(0^+)$ exists. Also by homogeneity of p,

$$\frac{u(r_k x)}{r_k^2} - p(x) = \frac{u(r_k x) - p(r_k x)}{r_k^2} \to 0 \quad \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^n) \implies \frac{1}{r_k^2} \|u - p\|_{L^\infty(B_{r_k})} \to 0.$$

Which in turn implies

$$M_{u,p}(r_k) \leq \frac{1}{r_k^{n+3}} \int_{\partial B_{r_k}} \|u - p\|^2_{L^\infty(B_{r_k})} = \frac{1}{r_k^2} \|u - p\|_{L^\infty(B_{r_k})} \frac{1}{r^{n-1}} \int_{\partial B_{r_k}} dx \to 0,$$

that is $M_{u,p}(0^+) = 0$. Assume now that there were another subsequence $r_j \to 0$ such that $u_{r_j} \to q \neq p$ in C^1_{loc}. It holds then equally $M_{u,q}(0^+) = 0$ and so

$$\begin{aligned} \int_{\partial B_1} (p - q)^2 &= \frac{1}{r^{n+3}} \int_{\partial B_r} (p - q)^2 = \frac{1}{r^{n+3}} \int_{\partial B_r} (p - u + u - q)^2 \\ &\leq \int_{\partial B_r} 2(p - u)^2 + 2(u - q)^2 = 2M_{u,p}(r) + 2M_{u,q}(r) \longrightarrow 0. \end{aligned}$$

Thus $p \equiv q$. Lastly we show $\frac{\|u-p\|_{L^\infty(B_r)}}{r^2} \to 0$. Suppose there were a sequence $r_k \to 0$ such that

$$\frac{\|u-p\|_{L^\infty(B_{r_k})}}{r_k^2} \geq c > 0.$$

Then by Proposition 3.1 there exists a subsequence r_{k_l} such that $u_{r_{k_l}} \to u_0$, so $\|u_0 - p\|_{L^\infty(B_1)} \geq c > 0$. This contradicts $u_0 = p$, the uniqueness of the blow-up and so $u(x) = p_2(x) + o(|x|^2)$ and also $\nabla u(x) = \nabla p_2(x) + o(|x|)$. □

Corollary 5.6. *At singular points, blow-ups depend continuously on the free boundary points, i.e. the map $x_0 \mapsto p_{x_0}$ is continuous and so for two singular free boundary points x_0, x_1 and $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$|x_1 - x_0| < \delta \implies \sup_{B_1} |p_{x_0} - p_{x_1}| < \varepsilon.$$

Proof. Take $\varepsilon' > 0$ fixed and let r_0 such that

$$\int_{\partial B_1} \left(\frac{u(x_0 + r_0 x)}{r_0^2} - p_{x_0}(x) \right)^2 dx < \varepsilon'/4.$$

Now by continuity of u in B^1, take $\delta > 0$ sufficiently small

$$|x_0 - x_1| < \delta \implies |u(x_0 + r_0 x) - u(x_1 + r_0 x)| < \sqrt{\varepsilon'}/2,$$

then

$$\begin{aligned} \int_{\partial B_1} \left(\frac{u(x_1 + r_0 x)}{r_0^2} - p_{x_0}(x) \right)^2 &\leq 2 \int \left(\frac{u(x_1 + r_0 x) - u(x_0 + r_0 x)}{r_0^2} \right)^2 + 2 \int \left(\frac{u(x_0 + r_0 x)}{r_0^2} - p_{x_0}(x) \right)^2 \\ &< \varepsilon'/2 + \varepsilon'/2 = \varepsilon'. \end{aligned}$$

However by Monneau's monotonicity formula (Theorem 5.4)

$$\int_{\partial B_1} \left(\frac{u(x_1 + r x)}{r^2} - p_{x_0}(x) \right)^2 \leq \int_{\partial B_1} \left(\frac{u(x_1 + r_0 x)}{r_0^2} - p_{x_0}(x) \right)^2 < \varepsilon' \quad \forall r \leq r_0,$$

and thus passing to the limit ($r \to 0$) gives

$$\int_{\partial B_1} (p_{x_1}(x) - p_{x_0}(x))^2 \leq \varepsilon'.$$

As we are in the finite dimensional space of quadratic polynomials, all norms are equivalent and so

$$\|p_{x_1} - p_{x_0}\|_{L^\infty(B_1)} \leq C \|p_{x_1} - p_{x_0}\|_{L^2(\partial B_1)} < C\varepsilon' =: \varepsilon.$$

□

We are now able to show that the set of singular points still behaves "nicely", an important result that goes back to [Caf98]. Studying the size and regularity of the singular set is an area of active research with yearly new results.

Theorem 5.7. *Let u be any solution to* (P1)*. Denote by Σ the set of singular points in B_1. Then Σ is locally contained in a C^1 manifold of dimension $n-1$.*

Proof. Let $K \Subset B_1$ and $E = K \cap \Sigma$, which is compact as Σ is closed. Its complement, the set of regular points is open as $\partial\{u > 0\}$ is locally smooth at regular points. Let $f : \Sigma \to \mathbb{R}$ be given by $f \equiv 0$. We use Whitney's extension theorem (see [Fed96, Theorem 3.1.14] or [PSU12, Lemma 7.10]) to find a C^2 extension that matches with the derivatives of the polynomial blow-ups at singular points. Thus define the family of quadratic polynomials $\{q_{x_0}(x)\}_{x_0 \in E}$ by $q_{x_0}(x) = p_{x_0}(x - x_0)$, where p_{x_0} is the blow-up at x_0 given by Theorem 5.5. Clearly the first condition of Whitney's theorem is fulfilled, we have

$$q_{x_0}(x_0) = p_{x_0}(0) = 0 = f(x_0).$$

Let us show

$$|D^k q_{x_0}(x_1) - D^k q_{x_1}(x_1)| = o(|x_0 - x_1|^{2-k}) \quad \text{for } k = 0, 1, 2.$$

From Theorem 5.5 we have

$$\begin{aligned}
k=0: \quad & |q_{x_0}(x_1)-q_{x_1}(x_0)| = |q_{x_0}(x_1-x_0)-q_{x_1}(x_1-x_1)| = |q_{x_0}(x_1-x_0)| = |u(x_1)-q_{x_0}(x_1-x_0)| \\
& = |u(x_1)-p_{x_0}(x_1)| = o(|x_1-x_0|^2), \\
k=1: \quad & |\nabla q_{x_0}(x_1)-\nabla q_{x_1}(x_1)| = |\nabla p_{x_0}(x_1-x_0)| = |\nabla u(x_1)-\nabla p_{x_0}(x_1-x_0)| \\
& = |\nabla u(x_1)-\nabla p_{x_0}(x_1)| = o(|x_0-x_1|), \\
k=2: \quad & |D^2 q_{x_0}(x_1)-D^2 q_{x_1}(x_1)| = |A_{x_0}-A_{x_1}| = o(1) \quad \text{by Corollary 5.6 .}
\end{aligned}$$

Hence by Whitney's extension theorem we have a C^2 extension, f, on $\mathbb{R}^n$. Since f was matching the quadratic polynomial up to the second derivative on the singular set we have

$$\Sigma \cap K \subset \{\nabla f = 0\} = \bigcap_{i=1}^{n} \{\partial_{x_i} f = 0\}.$$

Take now $\bar{x} \in \Sigma$, we have $D^2 f(\bar{x}) = A_{\bar{x}} \neq 0$, which has at least one eigenvalue λ not equal to zero, i.e. there is at least one direction such that the blow-up behaves quadratically. Without loss of generality (arrange coordinate axes accordingly), assume that e_n is the corresponding eigenvector. Consider now $\partial_{x_n} f : \mathbb{R}^n \to \mathbb{R}$, the restriction of ∇f to its last coordinate. Since

$$\frac{\partial}{\partial x_n} \partial_{x_n} f(\bar{x}) = \det D^2_{(x_n)} f(\bar{x}) = \lambda \neq 0,$$

the implicit function theorem applies. Thus we have a unique C^1 function $g : U \subset \mathbb{R}^{n-1} \to \mathbb{R}$ such that $g(\bar{x}_1, ..., \bar{x}_{n-1}) = \bar{x}_n$ and $\partial_{x_n} f(x_1, ..., x_{n-1}, g(x_1, ..., x_{n-1})) = 0$ in U. Since $\partial_{x_n} f$ and g are C^1 in U, Σ is contained in a C^1 manifold of dimension $(n-1)$ defined implicitly by

$$h(x_1, ..., x_{n-1}) := \partial_{x_n} f(x_1, ..., x_{n-1}, g(x_1, ..., x_{n-1})) = 0$$

□

As a fairly recent result, [Mon03] states that generically (in $\mathbb{R}^2$) we do not expect any singularities in the free boundary. In other words for most boundary data, the free boundary is smooth.

Theorem 5.8. *Let g be an arbitrary fixed function on Ω, where $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain. Let u_λ be the solution to*

$$\begin{cases} \Delta u_\lambda = \chi_{\{u_\lambda > 0\}} & \text{in } \Omega \\ u_\lambda = g + \lambda & \text{on } \partial\Omega \end{cases}$$

Then for almost every $\lambda > 0$, the free boundary $\partial\{u_\lambda > 0\}$ is a smooth manifold.

A Appendix

Here we will provide some proofs to basic tools from PDE theory.

Lemma A.1. (Covering Lemma) *Let u be a solution to $\Delta u = f$ in B_1. Suppose we have a uniform estimate of the following form ($\|\cdot\|_F$ is e.g. $\|\cdot\|_{L^\infty}$ $\|\cdot\|_{C^{0,\alpha}}$):*

$$\|u\|_{C^{k,\alpha}(B_{r_1})} \le C(r_1)\left(\|u\|_{L^\infty(B_1)} + \|f\|_{F(B_1)}\right).$$

Then a uniform estimate holds in a larger ball B_{r_2}, with $r_1 < r_2 < 1$, i.e.

$$\|u\|_{C^{k,\alpha}(B_{r_2})} \le C(r_1)\left(\frac{1}{(1-r_2)^k}\|u\|_{L^\infty(B_1)} + \|f\|_{F(B_1)}\right).$$

Proof. Let $r := (1-r_2)r_1$, so B_{r_2} can be covered by a finite collection of balls $\{B_r(x_i)\}_{i=1}^N$ with $x_i \in B_{r_2}$, each contained in B_1. Consider now for each i, $B_{r/r_1}(x_i) \subset B_1$. Since $r = (1-r_2)r_1$,

$$\tilde{u}(x) = u((1-r_2)x), \quad \Delta\tilde{u} = (1-r_2)^2 f((1-r_2)\cdot) = (1-r_2)^2\tilde{f}.$$

We get by interpolation inequalities [GT77, Lemma 6.35] and scaling

$$\begin{aligned}
(1-r_2)^k\|u\|_{C^{k,\alpha}(B_r(x_i))} &= (1-r_2)^k\left(\|u\|_{C^k(B_r(x_i))} + [D^k u]_{C^{0,\alpha}(B_r(x_i))}\right)\\
&\le (1-r_2)^k\left(\|u\|_{L^\infty(B_r(x_i))} + 2[D^k u]_{C^{0,\alpha}(B_r(x_i))}\right)\\
&\le (1-r_2)^k\left(\|u\|_{L^\infty(B_1)} + 2[D^k u((1-r_2)\cdot)]_{C^{0,\alpha}(B_{r_1}(x_i))}\right)\\
&\le (1-r_2)^k\left(\|u\|_{L^\infty(B_1)} + 2\frac{1}{(1-r_2)^k}[D^k(u((1-r_2)\cdot))]_{C^{0,\alpha}(B_{r_1}(x_i))}\right)\\
&\le \|u\|_{L^\infty(B_1)} + 2[D^k\tilde{u}]_{C^{0,\alpha}(B_{r_1}(x_i))}\\
&\le \|u\|_{L^\infty(B_1)} + 2\|\tilde{u}\|_{C^{k,\alpha}(B_{r_1}(x_i))}\\
&\le \|u\|_{L^\infty(B_1)} + 2C(r_1)\left(\|u\|_{L^\infty(B_1)} + \|(1-r_2)^k\tilde{f}\|_{F(B_1)}\right)\\
&\le C(r_1)\left(\|u\|_{L^\infty(B_1)} + (1-r_2)^k\|f\|_{F(B_1)}\right).
\end{aligned}$$

Dividing by $(1-r_2)^k$ yields the estimate

$$\|u\|_{C^{k,\alpha}(B_r(x_i))} \le C(r_1)\left(\frac{1}{(1-r_2)^k}\|u\|_{L^\infty(B_1)} + \|f\|_{F(B_1)}\right).$$

Since B_{r_2} can be covered by a finite collection of balls,

$$\|u\|_{C^{k,\alpha}(B_{r_2})} \le \sum_{i=1}^N \|u\|_{B_r(x_i)} \le NC(r_1)\left(\frac{1}{(1-r_2)^k}\|u\|_{L^\infty(B_1)} + \|f\|_{F(B_1)}\right) \le C.$$

□

Theorem A.2. (Schauder estimate) *Let $0 < \varepsilon < 1$ and $u \in C^2(B_1)$ satisfy $\Delta u = f$ in B_1 where $f \in L^\infty(B_1)$. Then for some universal constant $C(\varepsilon, n)$*

$$\|u\|_{C^{1,1-\varepsilon}(B_{1/2})} \le C(\|u\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)}).$$

Proof. We want to prove

$$|Du(z) - Du(y)| \le C|z-y|^{1-\varepsilon}(\|u\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)}) \qquad \forall y, z \in B_{1/2}.$$

To simplify the proof we show the estimate in a smaller ball $B_{1/16}$ and then use the covering lemma (Lemma A.1) to get the estimate in $B_{1/2}$. By a translation(radius doubles) we can assume $y = 0$ and by dividing the solution by $\|u\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)}$ we can assume $\|u\|_{L^\infty(B_1)} \le 1$ and $\|f\|_{L^\infty(B_1)} \le 1$. Thus we set out to prove

$$|Du(z) - Du(0)| \le C|z|^{1-\varepsilon} \qquad \forall z \in B_{1/8}.$$

We will use the following well known derivative estimate for harmonic functions, i.e.

$$\Delta w = 0 \text{ in } B_r \implies \|D^\kappa w\|_{L^\infty(B_{r/2})} \leq C r^{-\kappa} \|w\|_{L^\infty(B_r)}. \tag{A.1}$$

Step 1: For $k = 0, 1, \ldots$ let u_k be the solution to

$$\begin{cases} \Delta u_k &= 0 \quad \text{in } B_{2^{-k}}, \\ u_k &= u \quad \text{on } \partial B_{2^{-k}}. \end{cases}$$

As $\Delta(u_k - u) = -f$ by the maximum principle ([FR22, Lemma 1.14]) and $f \in L^\infty(B_1)$

$$\|u_k - u\|_{L^\infty(B_{2^{-k}})} \leq C(2^{-k})^2 \|f\|_{L^\infty(B_{2^{-k}})} \leq C 2^{-2k}, \tag{A.2}$$

thus by triangle inequality we have $\|u_{k+1} - u_k\|_{L^\infty(B_{2^{-k-1}})} \leq C 2^{-2k}$. As $u_{k+1} - u_k$ is harmonic, by (A.1),

$$\|D(u_{k+1} - u_k)\|_{L^\infty(B_{2^{-k-1}})} \leq C 2^{k+1} \|u_{k+1} - u_k\|_{L^\infty(B_{2^{-k-1}})} \leq C 2^{-k}.$$

Now given $z \in B_{1/8}$ choose $k \in \mathbb{N}$ such that $2^{-k-3} \leq |z| \leq 2^{-k-2}$. We will estimate each term separately in

$$|Du(z) - Du(0)| \leq |Du_k(0) - Du(0)| + |Du_k(0) - Du_k(z)| + |Du_k(z) - Du(z)|.$$

Step 2: ($|Du_k(0) - Du(0)| \leq C 2^{-k}$)
Start by taking $\tilde{u} := u(0) + x \cdot Du(0)$, since $u \in C^2$, $\|\tilde{u} - u\|_{L^\infty(B_r)} = o(r)$. Now noticing that $D\tilde{u}(x) = Du(0)$ and that $\tilde{u} - u_k$ is harmonic, we can use (A.1) and the to get maximum principle

$$\begin{aligned} |Du_k(0) - Du(0)| &\leq \|D(u_k - \tilde{u})\|_{L^\infty(B_{2^{-k-1}})} \leq C 2^k \|u_k - \tilde{u}\|_{L^\infty(B_{2^{-k}})} \leq C 2^k \|u - \tilde{u}\|_{L^\infty(\partial B_{2^{-k}})} \\ &= C 2^k o(2^{-k}) \xrightarrow{k \to \infty} 0 \implies Du(0) = \lim_{k \to \infty} Du_k(0). \end{aligned}$$

This enables us to estimate

$$|Du_k(0) - Du(0)| \leq \sum_{j=k}^{\infty} |Du_j(0) - Du_{j+1}(0)| \leq C \sum_{j=k}^{\infty} 2^{-j} = C 2^{-k}.$$

Step 3: ($|Du_k(z) - Du(z)| \leq C 2^{-k}$)
Take solutions v_j of

$$\begin{cases} \Delta v_j &= 0 \quad \text{in } B_{2^{-j}}(z), \\ v_j &= u \quad \text{on } \partial B_{2^{-j}}(z). \end{cases}$$

Rewrite, by using triangle inequality,

$$|Du_k(z) - Du(z)| \leq |Du_k(z) - Dv_k(z)| + |Dv_k(z) - Du(z)| =: I + II.$$

Term II can be bounded by $C 2^{-k}$ like before (consider Taylor expansion $\tilde{u}(x) = u(z) + xDu(z)$), but term I is more delicate. First note that $B_{2^{-k-1}}(z) \subset B_{2^{-k}} \cap B_{2^{-k}}(z)$ because $|z| \leq 2^{-k-2}$, so in $B_{2^{-k-2}}(z)$ using the harmonicity of $u_k - v_k$ in $B_{2^{-k-1}}(z)$,

$$\begin{aligned} I = |Du_k(z) - Dv_k(z)| &\leq \|D(u_k - v_k)\|_{L^\infty(B_{2^{-k-2}}(z))} \leq C 2^{k+1} \|u_k - v_k\|_{L^\infty(B_{2^{-k-1}}(z))} \\ &\leq C 2^{k+1} (\|u_k - u\|_{L^\infty(B_{2^{-k-1}}(z))} + \|v_k - u\|_{L^\infty(B_{2^{-k-1}}(z))}) \leq C \|f\|_{L^\infty(B_{2^{-k}})} 2^{-k} = C 2^{-k}, \end{aligned}$$

where in the last inequality we use $\Delta(u_k - u) = \Delta(v_k - u) = -f$ in $B_{2^{-k-1}}(z)$ and the maximum principle as in (A.2).

Step 4: ($|Du_k(z) - Du_k(0)| \leq 2^{-k(1-\varepsilon)}$)
Define for $j = 1, \ldots, k$, $h_j := u_j - u_{j-1}$ which are harmonic in $B_{2^{-j-1}}$. By the mean value theorem and (A.1) we get

$$\begin{aligned} \frac{|Dh_j(z) - Dh_j(0)|}{|z|} &\leq \|D^2 h_j\|_{L^\infty(B_{2^{-k-2}})} \leq C 2^{2j} \|h_j\|_{L^\infty(B_{2^{-j}})} \\ &\leq C 2^{2j} \|u_j - u_{j-1}\| \leq C 2^{2j} 2^{-2j} \leq C. \end{aligned}$$

Note that we also have

$$|z|^{-1}|Du_0(z) - Du_0(0)| \leq \|D^2 u_0\|_{L^\infty(B_{1/2})} \leq C\|u_0\|_{L^\infty(B_1)}$$
$$\|u_0\|_{L^\infty(B_1)} - \|u\|_{L^\infty(B_1)} \leq \|u_0 - u\|_{L^\infty(B_1)} \leq C \quad \text{(use (A.2) with } k=0) \implies \|u_0\|_{L^\infty(B_1)} \leq C.$$

Which in turn gives

$$\begin{aligned}|Du_k(0) - Du_k(z)| &\leq |Du_0(z) - Du_0(0)| + \sum_{j=1}^{k} |Dh_j(z) - Dh_{j-1}(z)| \\ &\leq C|z|\|u_0\|_{L^\infty(B_1)} + C|z|\sum_{j=1}^{k} C' \leq C|z| + C|z|k \leq C2^{-k}(1+k) \leq C2^{-k(1-\varepsilon)}.\end{aligned}$$

Note that in the last inequality we can not bound by $C2^{-k}$, but by $C2^{-k}f(k)$ with $f(k)$ positive and superlinear, here $f(k) = 2^{\varepsilon k}$. That is the reason why we can not have a bound in $\|\cdot\|_{C^{1,1}}$ ($\varepsilon = 0$).

Step 5: We collect the estimates from Step 1, Step 2 ($2^{-k} \leq C2^{-k(1-\varepsilon)}$) and Step 3 and use the lower bound $2^{-k-4} \leq |z|$ to get

$$\begin{aligned}|Du(z) - Du(0)| &\leq |Du_k(0) - Du(0)| + |Du_k(0) - Du_k(z)| + |Du_k(z) - Du(z)| \\ &\leq C2^{-k(1-\varepsilon)} = C(2^{-k})^{1-\varepsilon} = C^\varepsilon C^{1-\varepsilon}(2^{-k})^{1-\varepsilon} \\ &= C^\varepsilon(C2^{-k})^{1-\varepsilon} = C^\varepsilon(2^{-k-4})^{1-\varepsilon} \leq C|z|^{1-\varepsilon}.\end{aligned}$$

Therefore in $B_{1/8}$ it holds

$$[Du]_{C^{0,1-\varepsilon}(B_{1/8})} \leq C.$$

So by interpolation inequalities [GT77, Lemma 6.35] ,

$$\begin{aligned}\|u\|_{C^{1,1-\varepsilon}(B_{1/8})} &= \|u\|_{C^1(B_{1/8})} + [Du]_{C^{0,1-\varepsilon}(B_{1/8})} \\ &\leq C\|u\|_{L^\infty(B_{1/8})} + 2\,[Du]_{C^{1,1-\varepsilon}(B_{1/8})} \leq C,\end{aligned}$$

and by the Covering Lemma A.1) and rescaling by $\|u\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)}$ the proof is finished. □

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