Action potentials in action: Sufficient conditions for spikes in the FitzHugh-Nagumo system

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Outline

- Introduction
- The FitzHugh-Nagumo model
- Problem statement
- Definition of spikes
- Conditions for spikes
  - Dirac impulse
  - general impulse
- Periodic hybrid system
- Outlook

Introduction

- Excitable media: impulse $\to$ wave $\to$ cool-down
- Examples: wildfire, chemical oscillators and cardiac or neuronal tissue
- Hodkin-Huxley model: $4D$ non-linear dynamical system
- FitzHugh-Nagumo model: $2D$ non-linear dynamical system
- Reference: George Datseries and Ulrich Parlitz. *Nonlinear Dynamics*. Springer 2022

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The FitzHugh-Nagumo model

- FitzHugh-Nagumo equations:

$$\begin{aligned} \dot{u}(t) &= -bu(t)\,(u(t)-1)\,(u(t)-a) - w(t) + I(t), \\ \dot{w}(t) &= \varepsilon(u(t) - cw(t)), \end{aligned} \tag{0.1}$$

  with $a, b, c, \varepsilon \in \mathbb{R}$.

- Impulse: A measurable function $I : [t_0, t_0 + T] \to \mathbb{R}^+$

- Dirac Impulse: $I_0 \delta_{t_0}$, $(u(t_0), w(t_0)) \mapsto (u(t_0) + I_0, w(t_0))$

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- No impulse:

$$\begin{aligned} \dot{u} &= -bu\,(u-1)\,(u-a) - w, \\ \dot{w} &= \varepsilon(u - cw), \end{aligned} \tag{0.2}$$

- Nullclines determine global behaviour

$$\begin{aligned} \mathcal{N}_{\dot{u}}: &\quad w = -bu(u-a)(u-1) =: f(u), \\ \mathcal{N}_{\dot{w}}: &\quad w = \tfrac{1}{c}u \end{aligned} \tag{0.3}$$

- if $(1-a)^2 < \frac{4}{bc}$ unique globally stable fixpoint at origin
- if $(1-a)^2 > \frac{4}{bc}$ bistable; if $a, c > 0$, $b < 0$ periodic cycle

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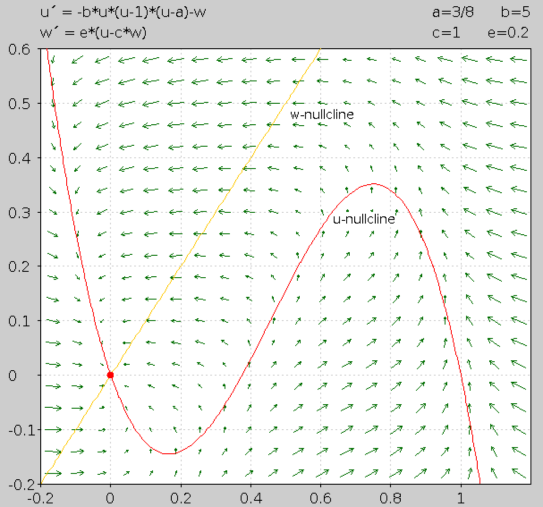

Figure: u-vs-w phase plot for $a = 3/8$, $b = 5$, $c = 1$, $\varepsilon = 0.2$

Problem statement

What kind of impulses to the fixpoint $(0, 0)$ will cause the system to spike, i.e. make an "excursion in phase space" with "large" u-value?

- How to define a spike rigorously?
- Calculate the exact value of a Dirac impulse leading to a spike for a fixed set of parameters
- What can be said about a general impulse?

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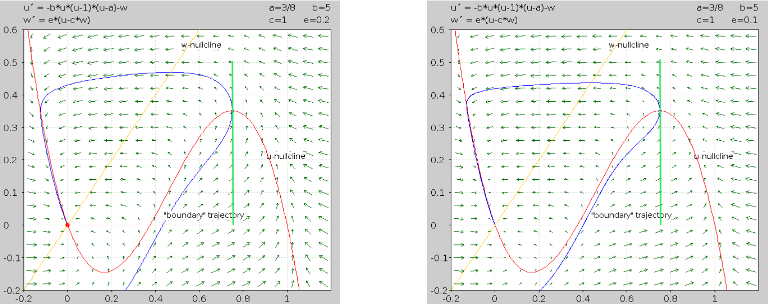
Definition of spikes

Definition

A trajectory $(u,w) : t \mapsto (u(t), w(t)) \ni \mathbb{R}^2$ *with* $(u(0), w(0)) = (0,0)$ *has a* **spike** *at time* t_s *if* $\dot{u}(t_s) > 0$ *and* $u(t_s) = u_s$ *with the spike threshold* $u_s := (\sqrt{a^2 - a + 1} + a + 1)/3$.



(a) $a = \frac{3}{8}$, $b = 5$, $c = 1$, $\varepsilon = 0.2$

(b) $a = \frac{3}{8}$, $b = 5$, $c = 1$, $\varepsilon = 0.1$

Conditions for spikes

- Problem: boundary trajectory can only be found numerically
- Goal: find region A_ε analytically s.t. $(u, w) \in A_\varepsilon$ guarantees a spike

Figure: $a = 0.2$, $b = 5$, $c = 1$

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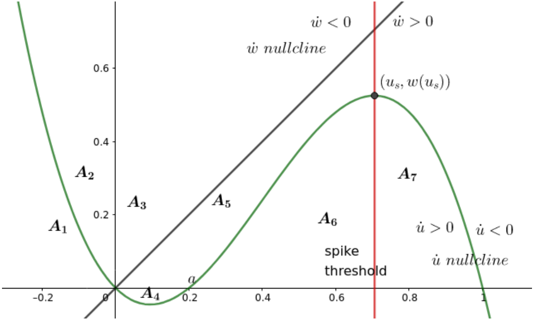

Figure: $a = 0.2$, $b = 5$, $c = 1$

Condition for spikes

Idea: estimate direction of flow

- if below $z_\lambda(u) = 1/(1 - c\epsilon\lambda)(f(u) - \varepsilon\lambda u)$ we have $\dot{u} \geq \lambda\dot{w}$
- if below $l_\lambda(u) := \frac{1}{\lambda}u + z(u_s) - \frac{1}{\lambda}u_s$, flow crosses spike threshold
- Let $A_\varepsilon = \bigcup_\lambda A_\lambda$

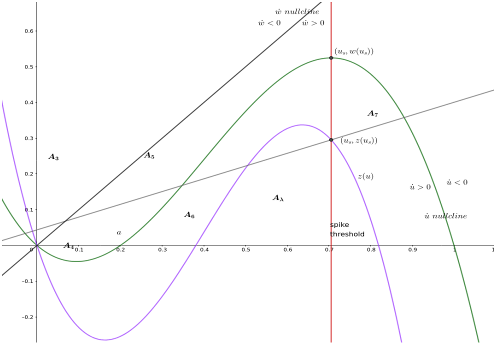

Figure: $a = 0.2$, $b = 5$, $c = 1$

Conditions for spikes - Dirac impulse

- $(u, w) \in A_\varepsilon$ is sufficient but not necessary condition!
- Why is A_ε easier to handle than the boundary trajectory?
- Q: For a fixed parameter set and fixed ε, determine $C_\varepsilon > a$
  s.t. $I_0\delta_0$ with $I_0 \geq C_\varepsilon$ causes a spike i.e. $(u(0+), w(0+)) \in A_\varepsilon$
  $\hookrightarrow$ find $\min_u\{A_\varepsilon \cap \{(u, w) : w = 0\}\}$

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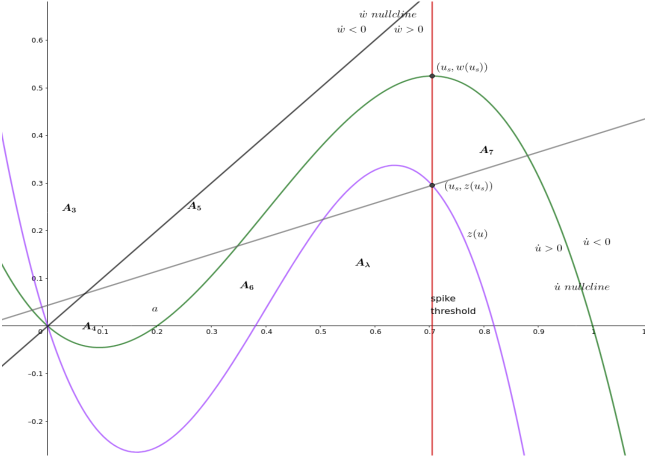

Figure: $a = 0.2$, $b = 5$, $c = 1$

Conditions for spikes - Dirac impulse

- find $\min_u\{A_\varepsilon \cap \{(u,w) : w = 0\}\}$
- solve for roots $r_z(\lambda)$ of $z_\lambda(u)$ (quadratic) and $r_l(\lambda)$ of $l_\lambda(u)$ (linear)
  $\hookrightarrow$ algebraic expressions involving all parameters, ε and λ
  $\hookrightarrow r_z(\lambda) = A - \sqrt{B - C\lambda\varepsilon}, \quad r_l(\lambda) = D - \frac{\lambda}{1-\lambda\varepsilon}(E - F\lambda\varepsilon)$
- $C_\varepsilon = \min_\lambda \max\{r_z(\lambda), r_l(\lambda)\}$, $\lambda_* = \operatorname{argmin}_\lambda \max\{r_z(\lambda), r_l(\lambda)\}$
  $\hookrightarrow$ easier to solve than ODE
- either $r_l(\lambda_*) = r_z(\lambda_*)$ or $\lambda_* = \operatorname{argmin}_\lambda r_l(\lambda)$
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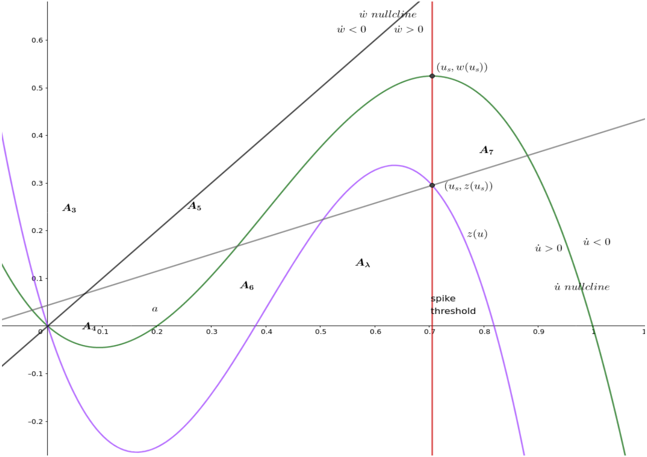
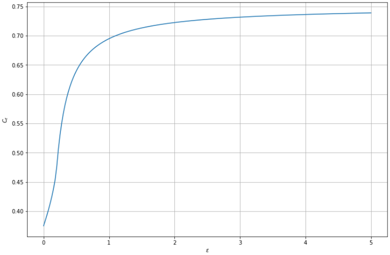


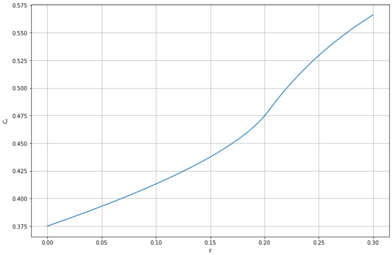
Figure: $a = 0.2$, $b = 5$, $c = 1$

Conditions for spikes - Dirac impulse

- when $\epsilon \to 0$ then $C_\epsilon \to a = \frac{3}{8}$ linearly
- when $\epsilon \to \infty$ then $C_\epsilon \to u_s = \frac{3}{4}$ asymptotically



(a) ε vs C_ε



(b) ε vs C_ε

Figure: $a = \frac{3}{8}$, $b = 5$, $c = 1$

Conditions for spikes - Dirac impulse

- similar constant $C_{\varepsilon,\beta}$ for Dirac impulse to $(u, w) = (0, \beta)$
  $\hookrightarrow$ boundary of A_ε

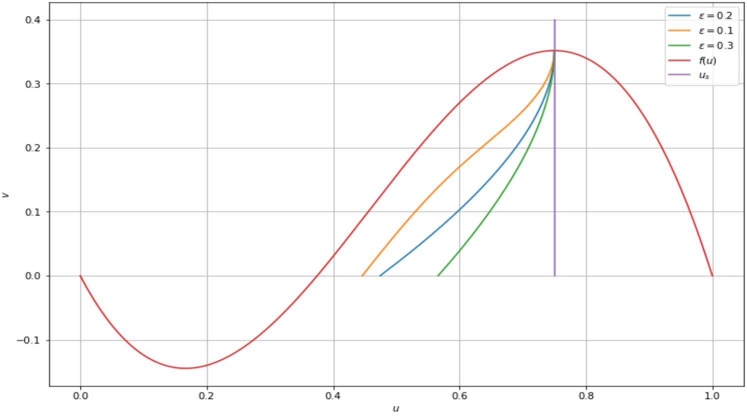
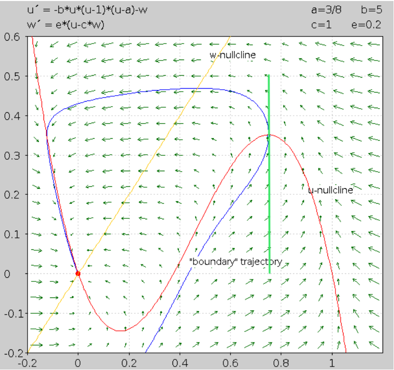


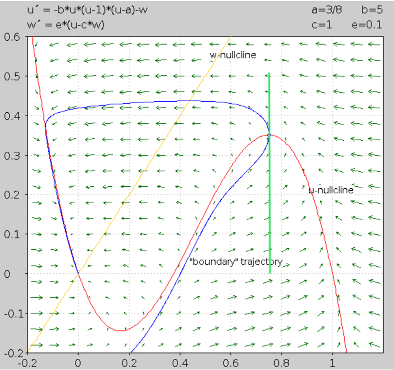
Figure: Boundaries of A_ε for different ε

Conditions for spikes - Dirac impulse

- significantly worse than "exact" boundary trajectory



(a) $\varepsilon = 0.2$



(b) $\varepsilon = 0.1$

Figure: $a = \frac{3}{8}$, $b = 5$, $c = 1$

Conditions for spikes - general impulse

Theorem

Let $I(t)$ be an impulse of finite length T to the steady state $(0,0)$ and $\beta := T\varepsilon u_s$. If the following conditions

1. $\int_0^T I(t) \leq (1 - T^2\varepsilon)u_s - Tf(u_s)$ (upper bound 1),
2. $T\varepsilon u_s \leq f(u_s)$ (upper bound 2),
3. $\int_0^T I(t) \geq \left(1 - \frac{T^2\varepsilon}{1-T^2\varepsilon}\right)^{-1} \left(C_{\varepsilon,\beta} + \frac{T^3\varepsilon f(u_s)}{1-T^2\varepsilon} - T\min_{u\in[0,u_s]} f(u)\right)$ (lower bound),

are satisfied, then the trajectory has a spike for some $t_s > T$.

- upper bounds give $u(T) \leq u_s$, $w(T) \leq \beta \leq f(u_s)$, lower bound $u(T) \geq C_{\varepsilon,\beta}$
- if $T \to 0$, recover $C_{\varepsilon,0}$

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Periodic hybrid system

- What if there is a second impulse? When?
- send second Dirac impulse when returning to height β; $I(\beta, u)$.

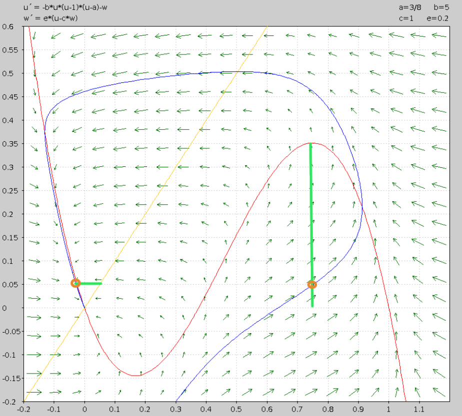

Figure: periodic hybrid system

Outlook

- Is there a region where it is guaranteed that NO spike occurs?
- How does the difference between the "exact" boundary and ∂A_ε change with ε? Does the flow estimations become "better" or "worse"?
- definition of spikes based on convexity of trajectory
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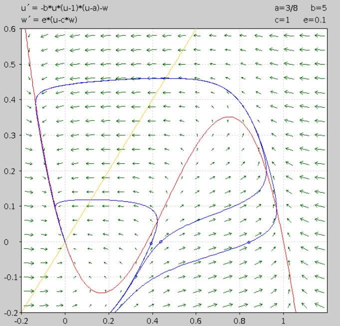
Outlook

- for "large" ε only convex trajectories (threshold ε_0 ?)
  $\hookrightarrow$ Mathieu Desroches and Mike Jeffrey. *Canards and curvature: the smallness of ε in slow-fast dynamics*



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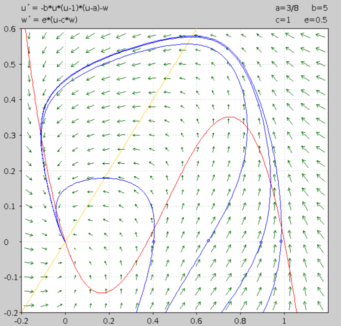

(b) $\varepsilon = 0.5$

Figure: $a = \frac{3}{8}$, $b = 5$, $c = 1$