On the continuous and discrete Gradient Conjecture

Semester project in CSE

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Outline

1. Introduction and known results
2. Gradient conjectures in comparison
3. The proof of Kurdyka [KMP99]
4. Problems with a direct discretization
5. Dynamical Systems
   5.1 Continuous Case
   5.2 Discrete Case

Introduction and known results

- continuous gradient descent flow:

$$x'(t) = -\nabla f(x(t)), \quad x(0) = x_0$$

- discrete gradient descent:

$$x_{k+1} = x_k - \alpha_k \nabla f(x_k), \quad x_0 \text{ given}$$

Introduction and known results

- Convergence? Yes, thanks to Lojasiewicz [Loj84] (1984, *Trajectoires du gradient d'une fonction analytique*)
  Thm: **if** f real-analytic, convergence to critical point x^*
  or $|x(t)| \to \infty$.
  $\hookrightarrow$ Lojasiewicz inequality: $|\nabla f| \geq c|f|^{\rho}, \quad \rho < 1$

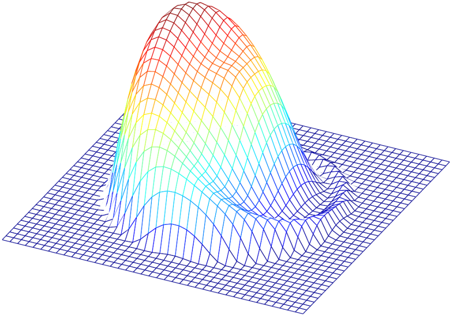

Figure: Mexican hat from [AMA05]

Introduction and known results

- Convergence also for discrete case
  (2005, P.-A. Absil, R. Mahony and B. Andrews: *Convergence of the iterates of descent methods for analytic cost functions*) [AMA05]

- infinite dimensional Lojasiewicz inequalities for PDEs

Gradient conjectures in comparison

- (1) Gradient Conjecture (GC): $x'(t) = -\nabla f(x(t))$, $x(t) \to x^*$

$$\text{Then} \qquad \lim_{t\to\infty} \frac{x(t)-x^*}{|x(t)-x^*|} \qquad \text{exists.}$$

$\hookrightarrow$ trajectory "does not wiggle"

- proven by K. Kurdyka, T. Mostowski, and A. Parusinski in *Proof of the gradient conjecture of R. Thom* [KMP99]

- (2) strong GC: $\lim_{t\to\infty} \frac{x'(t)}{|x'(t)|}$ exists ? (implies GC)

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Gradient conjectures in comparison

- (3) length-distance convergence:

$$\lim_{t\to\infty} \frac{\sigma(t)}{|x(t)|} = \lim_{t\to\infty} \frac{\int_t^\infty |x'(t)|}{|x(t)|} \to 1$$

- holds for f analytic (Corollary in [KMP99] proof)
- strong GC $\implies$ length-distance convergence
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Gradient conjectures in comparison

- Counterexample: $f \in C^\infty$ and $x(t)$ solution to $x' = -\nabla f(x)$
  length-distance convergence
  but converging in spiral

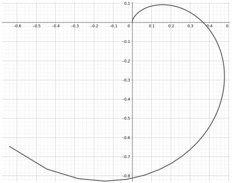

Figure: The curve γ

- $\gamma(t) = (r(t), \theta(t)) = \left(\frac{1}{t}, \log(\log(t))\right)$
- $f(r, \theta) = e^{\frac{-1}{r}} \left(1 - \frac{\log(1/r)}{r + r^2 \log(1/r)^2} \sin(\theta - \log(\log(1/r))\right)$
- $\implies -\nabla f\big|_{\gamma(t)} \parallel \gamma'(t)$, thus gradient flow follows γ.

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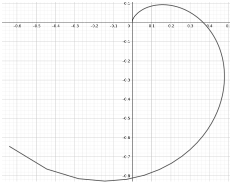

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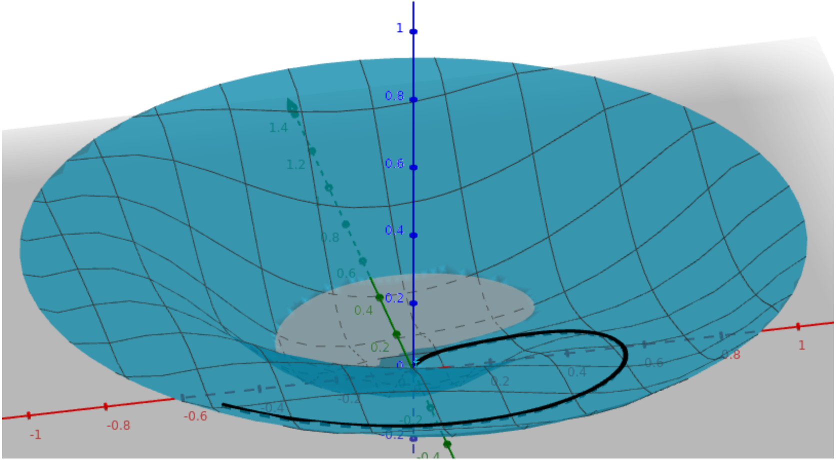

Figure: f and a gradient descent trajectory

The proof of [KMP99]

- split up into ∇f into $\partial_r f$ and $\nabla' f$ and stratify
- $F(x(s)) = \frac{f}{r^\ell} = \frac{f(x(s))}{|x(s)|^\ell}$ has limit a as $s \to s_0$ $(x(s) \to 0)$
  $\hookrightarrow$ ℓ characteristic to f
- strong Lojasiewicz inequality: $r|\nabla F| \geq |F - a|^\rho$ on some strata

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- $g(x(s)) := F - a - r^\alpha$ is *control function*

  $\hookrightarrow$ bounded and $\frac{dg}{d\tilde{s}} \geq |g|^\xi, \xi < 1$ $\quad$ ($\tilde{s}(s)$ is arclength of $\frac{x(s)}{|x(s)|}$)

- Lojasiewicz argument: $\frac{dg(x(\tilde{s}))}{d\tilde{s}} \geq c|g(x(\tilde{s}))|^\xi \implies \tilde{x}(s)$ finite

  $\hookrightarrow$ in original [Loj84] theorem: $\frac{df}{ds} \geq |\nabla f| \geq c|f|^\rho \implies x(s)$ finite

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* for F, strong Lojasiewicz, but only on strata $(|\partial_r F| \ll |\nabla' F|)$

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Dynamical Systems: Continuous case

- Hartman-Grobman: $x'(t) = u(x(t))$. Around hyperbolic fixpoint, flows of linear and non-linear system are homeomorphic.
- Hartman: If $Du(0) < 0$, then trajectories of linear and non-linear system are diffeomorphic.
- for gradient descent: $H_f(0) > 0$
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- Sebastian van Strien, 1990 in *Smooth linearization of hyperbolic fixed points without resonance conditions* [vS90]: If F is C^2 diffeo,

$$F(x) = \psi^{-1} \circ DF(0) \circ \psi(x), \quad \text{with } \psi, \psi^{-1} \text{ differentiable at } 0.$$

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Thank you for listening!

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