Lund University, Numerical Analysis **Spring Term 2, 2020**
Philipp Birken

Numerical Analysis: Basic Course

Assignment 3

Problem 1
A Newton-Cotes formula uses an interpolation polynomial on equidistant nodes in the integration interval to define a quadrature formula. In a closed Newton-Cotes formula, the end points are included. Thus, the trapezoidal rule is the closed Newton-Cotes formula of degree one.

Derive the closed Newton-Cotes formula of degree two, meaning that you use three interpolation nodes to define an interpolation polynomial of degree two, which is then integrated.

Problem 2
Consider Simpson's formula

$$\hat{I}_a^b(f) := \frac{(b-a)}{6}(f(a) + 4f(\frac{a+b}{2}) + f(b)).$$

Prove the following error estimate under the assumption that $f \in \mathcal{C}^4([a,b])$:

$$|\hat{I}_a^b(f) - I_a^b(f)| \leq \frac{(b-a)^5}{2880} \max_{\xi \in [a,b]} |f^{(4)}(\xi)|.$$

Problem 3
Write a program that computes an approximation of an integral $\int_{t_0}^{t_e} f(t)dt$ using an adaptive Simpson rule. To this end, write a function int_approx = adaptive_simpson(...) that computes $\hat{I}_a^b(f) \approx \int_a^b f(t)dt$ using Simpson's rule.

Then compute the approximation given by the composite Simpson rule with two intervals:

$$\begin{aligned} Q(f) :=& \hat{I}_a^{(a+b)/2}(f) + \hat{I}_{(a+b)/2}^b(f) \\ =& \frac{(b-a)}{2}\frac{1}{6}(f(a) + 4f(a+(b-a)/4) + 2f(\frac{a+b}{2}) + 4f(a+3(b-a)/4) + f(b)). \end{aligned}$$

It can then be shown that the difference between the two approximations divided by 15 is an error estimate for the quadrature error.

Thus, in case that $|\hat{I}_a^b(f) - Q(f)| > (b-a)\delta$ with $\delta = 15\epsilon/(t_e - t_0)$, the function recursively calls itself for the subintervals $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$. Initialise the method on the full integration interval.

As a test case, use $\int_{-1}^1 \sqrt{|x|}dx$ and $\epsilon = 0.000005$. Compute the exact integration error and the estimated error and plot the choice of intervals done by the method.

Return: **Tuesday, April 21st**