Lund University, Numerical Analysis
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Numerical Analysis: Basic Course

Assignment Project

Problem 1

Consider the pendulum equation

$$\ddot{\alpha}(t) = -\frac{g}{l}\sin\alpha(t)$$

with $g = 9.81, l = 1$ and the initial conditions $\alpha(0) = \pi/2$ and $\dot{\alpha}(0) = 0$. You can transform this into an equivalent first order system in the form you are used to by considering $\dot{\alpha}(t)$ as an unknown function of its own. You then get the initial value problem

$$\dot{\mathbf{y}}(t) = \mathbf{f}(t, \mathbf{y}(t)), \quad \mathbf{y}(t_0) = (\pi/2, 0)^T$$

for $t \in [0, 5]$ with

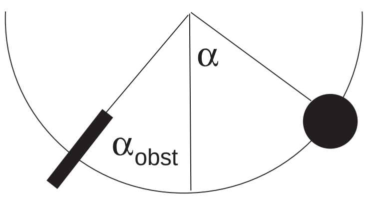
$$\mathbf{y}(t) = (\alpha(t), \dot{\alpha}(t))$$

and

$$\mathbf{f}(t, \mathbf{y}(t)) = \begin{pmatrix} \dot{\alpha}(t) \\ -\frac{g}{l}\sin\alpha(t) \end{pmatrix}.$$

The project is about solving this system using the methods you learned in the course.

As an additional difficulty, we place an obstacle (ett hinder, sv.) at an angle $\alpha_{\text{obst}} = -\frac{1}{6}\pi$, see figure. The obstacle will be hit at an unknown time t_{obst}. Let's say that it has then an angular velocity (also unknown) $\dot{\alpha}_{\text{obst}}$. When this obstacle is hit, we have to restart integration of the problem with new initial conditions $(\alpha_{\text{obst}}, -\dot{\alpha}_{\text{obst}})^T$.
Your task is to integrate the pendulum numerically over many periods and to show how the obstacle influences its trajectory. Furthermore we are interested in t_{obst}, α_{obst}, and $\dot{\alpha}_{\text{obst}}$.



a) Write down the nonlinear equation system you get in one time step when you use the trapezoidal rule

$$\mathbf{y}_i = \mathbf{y}_{i-1} + \frac{\Delta t}{2}(\mathbf{f}(t_{i-1}, y_{i-1}) + \mathbf{f}(t_i, y_i))$$

for the time integration of the above problem.

b) Write down Newton's method for such an equation system. What is a suitable initial guess at each time step? Discuss convergence of this iteration and of a fixed point iteration in one paragraph.

c) Choose a numerical method to solve these nonlinear equation systems in each time step.

d) Solve the problem without the obstacle and document your results.

e) Write a python program, which computes an interpolation polynomial for three consecutive solution points y_i, y_{i-1}, y_{i-2} .

f) Call this python function after every integration step after the first, so that you have in a typical step $t_{i-1} \to t_i$ the coefficients of a polynomial $p(t)$ available. Note that this polynomial is a vector valued function $p(t) = (p_\alpha(t), p_{\dot\alpha}(t))^T$ with two components one interpolating angles and another interpolating angular velocities.

g) Check if the integration passed the obstacle. To this end, you have to check if the function $p_\alpha(t) - \alpha_{\text{obst}} = 0$ has a solution in $[t_{i-1}, t_i]$. If you find a solution of this nonlinear equation, you call it t_{obst} . It is the time, when the obstacle is hit.

h) Compute α_{obst} and $\dot\alpha_{\text{obst}}$ by evaluating the polynomial.

i) Restart your integration by setting the initial time to t_{obst} and by setting new initial conditions (as given above).

j) Perform your integration over 5 sec and with an appropriate fixed step size h .

k) Make a plot of your result. A phase plot (α versus $\dot\alpha$) is quite instructive.

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