Lund University, Numerical Analysis **Spring Term 2, 2020**
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Numerical Analysis: Basic Course

Assignment 4

Problem 1
Prove the following theorem:

Let the quadrature formula $\hat{I}_{t_{i-1}}^{t_i}$ be given by the coefficients $(b_i, c_i)_{i=1}^s$. Then it holds that if $\sum_{i=1}^s b_i c_i^{q-1} = \frac{1}{q}$, $q = 1, ..., r$ for $r \leq k$, then the method $\hat{I}_{t_{i-1}}^{t_i}$ is of order k.

Hint: Prove the theorem first for an integral over the interval $[0, 1]$, then make an argument for a general interval.

Problem 2
The polynomials

$$P_s(x) := \frac{d^s}{dx^s}[(x^2 - 1)^s], \quad s = 0, 1, ...$$

are called Legendre polynomials. In this form, they can be used to construct quadrature formulas of optimal order (Gauß quadratures) on the interval [-1,1]. On other intervals, they need to be scaled, making the derivation more difficult.

a) Write down the polynomials for $s = 0, 1, 2$ in monomial form and prove for these, that they have exactly s roots in (0,1). (Note: This is actually true for all s).

b) Prove: For $k = 0, ..., s$, $x = \pm 1$ is a root of multiplicity $s - k$ of $Q_k(x) := \frac{d^k}{dx^k}[(x^2 - 1)^s]$.

c) Show using partial integration: For all $q \in \mathcal{P}_{s-1}$ we have

$$\int_{-1}^{1} q(x) P_s(x) dx = 0$$

d) Let $x_1, ..., x_s$ be the zeros of P_s. Let the weights b_i be chosen such that for all $p \in \mathcal{P}_{s-1}$ we have:

$$\hat{I}_{-1}^{1}(p) = \sum_{i=1}^{s} b_i p(x_i) = \int_{-1}^{1} p(x) dx.$$

Prove: The Gauß quadrature method $\hat{I}_{-1}^{1}(f) = \sum_{i=1}^s b_i f(x_i)$ has order $2s$.

Hint: Use again the relation $f = qP_s + p$ for $f \in \mathcal{P}_{2s-1}$, $q, p \in \mathcal{P}_{s-1}$.

Problem 3

a) Design a numerical experiment to get information about the order of a given quadrature formula.

b) Apply this procedure to the mid point rule, the two stage Gauss quadrature formula and the formula $\hat{I}_a^b(f) = (b-a)f(a)$.

c) Consider the composite form of these formulas with an equidistant spacing of intervals with $h = (b-a)/n$. Apply it to the test integrals $\int_{-1}^{1} \sqrt{|x|}dx$, $\int_0^1 e^{x^2} dx$ and $\int_{-1}^{1} \operatorname{sign} x dx$ with varying h. Plot the integration error over h.

d) State what you expect to come out of these computations and why and compare it with the actual result.

Return: **Tuesday, April 28th**