Assignment 3 –

NUMB11/FMNN01: Numerical Linear Algebra

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The purpose of these exercises is practice the least squares method and to make some studies related to the concept of a condition number.

Task 1

A model of a signal is given by the formula

$$y(x) = a\sin(x) + b\cos(x) + c\sin(2x) + d\cos(2x)$$

with unknown coefficients a, b, c, d. The signal was measured at 200 equidistant sampling points in $[0, 4\pi]$. These measurements are available in the data file `signal.dat`. To save memory they were recorded in low precision. Write down a least squares approach to determine these coefficients from the measurement.

Now write a code that solves this problem. Use for this an approach based on QR-factorization and another approach based on a singular value decomposition. Compare the results and the computational time. Plot the signal.
Note: In Python a least squares solution is computed by the function `scipy.linalg.lstsq`. Do not use this function for this task. But feel free to use the functions `scipy.linalg.qr` and `scipy.linalg.svd`. For MATLAB users the corresponding rules apply.

Task 2

The condition number of a matrix gives a sharp estimate of the sensitivity of x with respect to perturbations of b when solving $Ax = b$, this means there exists a right hand side b and a perturbation δb such that

$$\frac{\|\delta x\|_2}{\|x\|_2} = \kappa_2(A)\frac{\|\delta b\|_2}{\|b\|_2}$$

(Note the equal sign!). Give a vector pair $(b, \delta b)$ for which this equality holds. Hint, express these vectors in terms of left singular vectors.

Task 3

Hilbert matrices are notoriously ill conditioned. Verify your result from Task 3 by solving a linear system with a 50×50 Hilbert matrix and a worst case b and δb. Hilbert matrices and their exact inverses can be constructed in MATLAB by `hilb` and `invhilb` and in Python by the commands `scipy.linalg.hilbert` and `scipy.linalg.invhilbert`.

Task 4

Let

$$A = \begin{pmatrix} a_{11} & w^{\mathrm{T}} \\ w & A_1 \end{pmatrix}$$

be a $n \times n$ positive definite matrix. Show that a_{11} is strictly positive and that the $(n-1) \times (n-1)$ submatrix is positive definite.

Task 5

Show that a strictly diagonally dominant matrix A is invertible. Hint: Show that there exists no vector $u \neq 0$ which solves $Au = 0$. For this assume $u \neq 0$ and show that u cannot have all elements of the same size (in absolute value) and that furthermore there is no element u_i with $|u_i| \geq |u_j|$ for all j.