# Chapter4 Integration

Throughout $(\Omega, \mathscr{F}, \mu)$ denotes a measure space.

## *Measurable Functions*

Def(Random variables)

A function $f : \Omega \to R^n$ is (Borel) $measurable$ if $f^{-1}(E) \in \mathscr{F}$ for all $E \in \mathscr{B}(R^n)$. Measurable functions on probability space are often referred to as $random\ variables$.

Lemma( an easier way to check the measurability)

If $\mathscr{A}$ is an *algebra that generates $\mathscr{B}(R^n)$* and $f^{-1}(A) \in \mathscr{F}$ for all $A \in \mathscr{A}$, then $f : \Omega \to R^n$ is measurable.

Proof: check that $\left\{\mathscr{A} \in \mathscr{B}(R^n) : f^{-1}(A) \in \mathscr{F}\right\}$ is a monotone class containing $\mathscr{A}$. Using the monotone class theorem.

## *Modes of Convergence*

Throughout, $f_1, f_2, \cdots : \Omega \to R$ are measurable.

Def(convergence almost everywhere)

We say that $f_n$ convergences to $f$ $\mu - almost\ everywhere$ $(\mu - a.e.)$ if

$$\mu\left(\left\{\omega \in \Omega : \lim_{n\to\infty} \sup |f_n(\omega) - f(\omega)| > 0\right\}\right) = 0$$

equivalently

$$\mu\left(\left\{\bigcup_{l=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} |f_n(\omega) - f(\omega)| > \frac{1}{l}\right\}\right) = 0$$

when $(\Omega, \mathscr{F}, \mu)$ is probability space and $X, X_1, X_2, \ldots$ are random variables on this space, we say instead that $X_n$ convergences to $X$ $almost\ surely$ (a.s)

Def(convergence $L^p(\mu)$ and in measure)

We say that $f_n \to f$ in $L^p(\mu)$ if $\lim_{n\to\infty} \|f_n - f\|_p = 0$

also, $f_n \to f$ in measure if $\lim_{n\to\infty} \mu\{|f_n - f| \geq \epsilon\} = 0$

Th.

Either a.e.-convergence or $L^p$-convergence implies convergence in measure.

Conversely, if $sup_{j\geq n} |f_j| \to 0$ in measure, then $f \to 0$ almost everywhere.

*Markov's Inequality*

If $f \in L^1(\mu)$, then for all $\lambda > 0$,

$$\mu\{|f| > \lambda\} \leq \frac{1}{\lambda}\int_{\{|f|>\lambda\}} |f| d\mu \leq \frac{\|f\|_1}{\lambda}$$

proof

$$\int_{\{|f|>\lambda\}} |f| d\mu \geq \int_{\{|f|>\lambda\}} \lambda d\mu = \lambda\mu\{|f| > \lambda\}$$

this yields the first inequality.
the second one trivial to check.

applying the result to $L^p$ space we have
*Chebyshev's Inequality*
for all $\lambda > 0$, $p > 0$ and $f \in L^p(\mu)$,

$$\mu\{|f| > \lambda\} \leq \frac{1}{\lambda^p}\int_{\{|f|>\lambda\}} |f|^p d\mu \leq \frac{\|f\|_p}{\lambda^p}$$

Proof of Th.
The chebyshev's inequality suggests that $L^p$-convergence implies convergence in measure.

$f_n \to f$ a.e. iff $\mu\left(\left\{\bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty}|f_n(\omega) - f(\omega)| \geq \epsilon\right\}\right) = 0$, for all $\epsilon > 0$
since $\mu$ is continuous from above,

$$f_n \to f \ a.e. \quad \Longleftrightarrow \quad \lim_{N\to\infty}\mu(\bigcup_{n=N}^{\infty}|f_n(\omega) - f(\omega)| \geq \epsilon) = 0 \quad \forall\epsilon > 0$$

because $\mu\{|f_n - f| \geq \epsilon\} = 0 \leq \mu(\bigcup_{n=N}^{\infty}|f_n(\omega) - f(\omega)| \geq \epsilon)$
that is if $f_n \to f$ a.e. then $f_n \to f$ in measure

finally, if $sup_{j\geq n}|f_j| \to 0$ in measure them

$$\mu\left(\left\{\bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty}\{sup_{j\geq n}|f_j|\} \geq \epsilon\right\}\right) = \lim_{n\to\infty}\mu(sup_{j\geq n}|f_j| \geq \epsilon) = 0$$

for all $\epsilon > 0$
thus $\limsup_{m\to\infty}|f_m| \leq \limsup_n sup_{j\geq n}|f_j| < \epsilon$ a.e.
Let $N(\epsilon)$ denotes *the set of $\omega$'s for which the inequality fails*,
since $\lim_m |f_m| < \epsilon$ a.s., for all $\epsilon \in Q_+$

$$\mu\left\{\bigcup_{\epsilon\in Q_+} N(\epsilon)\right\} = 0$$

as $Q$ is dense in $R$
thus $f_n \to 0$ a.e.

## *The Radon-Nikodým Theorem*

Q: Given two measures $\mu$ and $\nu$ , one can ask, "when can we find a function $\pi_\star$ such that for *all measurable sets $A$*, $\nu(A) = \int_A \pi_\star d\mu$ ?"
Suppose $\mu$ denotes the Lebesgue's measure and the function $\pi_\star$ is a probability density function, then the prescription $\nu(A) := \int_A \pi_\star d\mu$ defines a probability measure $\nu$.

Def(absolutely continuous)
Given two measures $\mu$ and $\nu$ on $(\Omega, \mathscr{F})$, we say that $\nu$ is $\textit{absolutely continuous}$ with respect to $\mu$ (written $\nu \ll \mu$) if $\nu(A) = 0$ for all $A \in \mathscr{F}$ such that $\mu(A) = 0$.

*The Radon-Nikodým Theorem*
If $\nu \ll \mu$ are two *finite measures* on $(\Omega, \mathscr{F})$, then there exists a non-negative $\pi_\star \in L^1(\mu)$ such that $\int f d\nu = \int f\pi_\star d\mu$ for all *bounded measurable functions* $f : \Omega \to R$. Furthermore, $\pi_\star$ is unique up to a $\mu$-null set.

Remark:
Frequently we write $\pi_\star := \frac{d\nu}{d\mu}$. The function $\pi_\star$ is called the Radon-Nikodým derivative of $\nu$ with respect to $\mu$.

Proof:
First, we proof the theorem under the stronger domination condition that $\nu(A) \le \mu(A)$ for all $A \in \mathscr{F}$

Step1. The Case $\nu \le \mu$
Consider the linear functional $\mathscr{L}(f) = \int f d\nu$ that acts on all $f \in L^1(\nu)$. By the Cauchy-Schwarz inequality,

$$|\mathscr{L}(f)|^2 \le \nu(\Omega) \int |f|^2 d\mu$$

Hence $\mathscr{L}$ is bounded linear functional on $L^2(\mu)$
Since the *$L^2(\mu)$ is a Hilbert Space, using the Reisz represent Theorem*, there exists a $\mu$-almost everywhere unique $\pi \in L^2(\mu)$ such that $\int f d\nu = \int f\pi d\mu$ for all $f \in L^2(\mu)$
choose an $\alpha > 0$ and replace $f$ by $\mathit{1}_{\{\pi \le -\alpha\}}$, then,

$$\nu\{\pi \le -\alpha\} = \int 1_{\{\pi \le -\alpha\}} d\nu = \int 1_{\{\pi \le -\alpha\}} \pi d\mu \le -\alpha\mu\{\pi \le -\alpha\} \le 0$$

this happens only when $\mu\{\pi \le -\alpha\} = 0$

It follows from right continuity of $\mu$ that $\pi \ge 0$ a.e.$[\mu]$

That is we have established the theorem for all $f \in L^2(\mu)$.

*Especially, the theorem holds for all simple functions*.

By the monotone convergence theorem this fact holds for all measurable $f \ge 0$, using the same construction from establishing the integral, we see that the theorem follows with $\pi_\star = \pi$.

Step2. General $v, \mu$.

Since $\nu \le (\mu+\nu)$, Step1 exacts a $(\mu+\nu)$-a.e. unique and non-negative $\pi \in L^2(\mu+\nu)$ such that $\int f d\nu = \int f\pi d(\mu+\nu)$ for all $f \in L^2(\mu+\nu)$.

that is $\int f(1-\pi)d\nu = \int f\pi d\mu$

Replace $f$ by $1_{\{\pi \ge 1\}}$, we have $\mu(\pi \ge 1) = 0$. Consequently,

$$\int_{\pi \le 1} f(1-\pi)d\nu = \int_{\pi \le 1} f\pi d\mu \quad \forall f \in L^2(\mu+\nu)$$

By the monotone convergence theorem this fact holds for all measurable $f \ge 0$.

Replace $f$ by $f(1-\pi)^{-1}1_{\{\pi<1\}}$, and consider $\Pi := \pi(1-\pi)^{-1}1_{\{\pi<1\}}$

Then $\int_{\pi \le 1} f d\nu = \int_{\pi \le 1} f\Pi d\mu$ for all measurable $f \ge 0$.

*Since $\nu \ll \mu$ ,then $\mu(\pi \ge 1) = 0$ implies that $\nu(\pi \ge 1) = 0$*, hence for all measurable $f \ge 0$,

$$\int f d\nu = \int f\Pi d\mu$$

Plug in $f \equiv 1$, we have $\int \Pi d\mu = \nu(\Omega) < \infty$

It follows that $\Pi \in L^1(\mu)$, and the theorem concludes with $\pi_\star = \Pi$.

Step3. Uniqueness.

Suppose that there exists $\pi_\star, \pi^\star \in L^1(\mu)$ such that $\int f\pi_\star d\mu = \int f\pi^\star d\mu$ for all bounded measurable functions $f : \Omega \to R$.

we shall prove that $\pi_\star = \pi^\star$ a.e.$[\mu]$.

fix $\epsilon > 0$ and define $f = 1_{A(\epsilon)}$, where $A(\epsilon) := \{\omega \in \Omega : \pi^\star(\omega) \ge \pi_\star(\omega) + \epsilon\}$,then,

$$\int f\pi_\star d\mu = \int f\pi^\star d\mu \ge \int f(\pi_\star + \epsilon)d\mu = \int f\pi_\star d\mu + \epsilon\mu(A(\epsilon))$$

since $\int f\pi_\star d\mu \le \|\pi_\star\|_{L^1(\mu)} < \infty$, this suggests that $\mu(A(\epsilon)) = 0$ for all $\epsilon > 0$. By the continuity properties of measures

$$0 = \lim_{\epsilon \to 0, \epsilon \in Q} \mu(A(\epsilon)) = \mu\left(\bigcup_{\epsilon > 0, \epsilon \in Q} A(\epsilon)\right).$$

this shows that $\mu(A(\epsilon)) = 0$ a.e. That is $\pi^\star \leq \pi_\star$ a.e.$[\mu]$
Reverse the roles of $\pi^\star$ and $\pi_\star$ shows that $\pi^\star = \pi_\star$ a.e.$[\mu]$.