Chapter2 Bernoulli Trials

Below S_n denotes the total number of successes, and has the binomial distribution with parameters n and p.

Bernoulli's law of Large Number

p does not depend on n,then for any $\epsilon > 0$

$$\lim_{n\to\infty} P\left\{\left|\frac{S_n}{n} - p\right| > \epsilon\right\} = 0$$

It follows at once from Chebyshev's inequality

Chebyshev's inequality

For all $n > 0$, $\epsilon > 0$

$$P\left\{|S_n - np| > n\epsilon\right\} \leq \frac{p(1-p)}{n\epsilon^2}$$

proof:

$$\begin{aligned} P\left\{|S_n - np| > n\epsilon\right\} &= \sum_{0\leq k\leq n, |k-np|>\epsilon} \binom{n}{k} p^k (1-p)^k \\ &\leq \sum_{0\leq k\leq n, |k-np|>\epsilon} \left(\frac{k-np}{n\epsilon}\right)^2 \binom{n}{k} p^k (1-p)^k \\ &\leq \frac{1}{(n\epsilon)^2} \sum_{k=0}^{n} (k-np)^2 \binom{n}{k} p^k (1-p)^k \\ &= \frac{varSn}{(n\epsilon)^2} = \frac{p(1-p)}{n\epsilon^2} \end{aligned}$$

The de Moivre-Laplace Central Limit Therom

p does not depend on n, then for all finite numbers $a < b$,

$$\lim_{n\to\infty} P\left\{a < \frac{Sn - np}{\sqrt{np(1-p)}} \leq b\right\} = P\{a < N(0,1) \leq b\}$$

to estimate $\binom{n}{k}$,we only need to estimate $k!$ for all k. Then we have result below

de Movire's Formula

There exists $\beta \in (0, +\infty)$ such that

$$n! \sim \beta n^{n+1/2} e^{-n} \qquad as\ n \to \infty$$

proof:
define: $f(n) = \frac{n!}{\beta n^{n+1/2} e^{-n}}$
$f(1) = e$, then we can write $f(n)$ as

$$f(n) = e \prod_{k=2}^{n} \frac{f(k)}{f(k-1)} = \exp\left\{1 + \sum_{k=2}^{n} \ln f(k) - \ln f(k-1)\right\}$$

as $k \to \infty$,

$$ln f(k) - \ln f(k-1) = 1 + \left(k - \frac{1}{2}\right) \ln\left(1 - \frac{1}{k}\right) \sim -\frac{1}{12k^2}$$

since

$$\sum_{k=2}^{\infty} \ln f(k) - \ln f(k-1) < \infty$$

we have the result.

proof of The de Moivre-Laplace CLT:
let $q = 1 - p$

$$\begin{aligned} P\left\{a < \frac{Sn - np}{\sqrt{np(1-p)}} \leq b\right\} &= \sum_{np + a\sqrt{npq} < k \leq np + b\sqrt{npq}} \binom{n}{k} p^k q^{n-k} \\ &= \sum_{a\sqrt{npq} < \lambda \leq b\sqrt{npq}} \binom{n}{\lambda + np} p^{\lambda + np} q^{nq-k} \end{aligned}$$

The index k is in $\{0, \ldots, n\}$, while λ runs over all real numbers of the form $k - np$, let us call any λ of this form an $n - admissible$ number.
According to the de Movire's Formula:
as $n \to \infty$:

$$\binom{n}{\lambda + np} \sim \frac{1}{\beta\sqrt{npq}} \left(\frac{\lambda}{n} + p\right)^{-\lambda - np} \left(q - \frac{\lambda}{n}\right)^{-nq + \lambda}$$

for all $n - admissible$ number $\lambda \in \left[a\sqrt{npq}, b\sqrt{npq}\right]$
Thus as $n \to \infty$:

$$P\left\{a < \frac{Sn - np}{\sqrt{np(1-p)}} \leq b\right\} \sim \frac{1}{\beta\sqrt{npq}} \sum_{a\sqrt{npq} < \lambda \leq b\sqrt{npq}} \left(\frac{\lambda}{n} + p\right)^{-\lambda - np} \left(q - \frac{\lambda}{n}\right)^{-nq + \lambda}$$

since as $n \to \infty$:

$$\left(\frac{\lambda}{n}+p\right)^{-\lambda-np}\left(q-\frac{\lambda}{n}\right)^{-nq+\lambda} \sim e^{-\lambda^2/2npq}$$

and by the definition of Riemann integrals:

$$P\left\{a<\frac{Sn-np}{\sqrt{np(1-p)}}\leq b\right\} \sim \frac{1}{\beta\sqrt{npq}}\int_{a\sqrt{npq}}^{b\sqrt{npq}} e^{-x^2/2npq}dx$$

that is

$$\lim_{n\to\infty} P\left\{a<\frac{Sn-np}{\sqrt{np(1-p)}}\leq b\right\} = \int_a^b \frac{e^{-\frac{x^2}{2}}}{\beta}dx = \frac{\sqrt{2\pi}}{\beta}\int_a^b \phi(x)dx$$

$\phi(x)$ demotes the standard-normal density function.

we then proof that $\beta=\sqrt{2\pi}$

By chebyshev's inequality:

$$1 \geq P\left\{-a<\frac{Sn-np}{\sqrt{np(1-p)}}\leq a\right\} \geq 1-\frac{1}{a^2}$$

for all $n>0$, $a>0$
let $n\to\infty$:

$$\frac{\beta}{\sqrt{2\pi}} \geq \int_{-a}^{a}\phi(x)dx \geq \frac{\beta}{\sqrt{2\pi}}(1-\frac{1}{a^2})$$

let $a\to\infty$:

$$\int_{-\infty}^{\infty}\phi(x)dx = \frac{\beta}{\sqrt{2\pi}} = 1$$

thus $\beta=\sqrt{2\pi}$

we verify that $\beta=\sqrt{2\pi}$, we have the Stiring formula:

Stiring Formula:

$$n! \sim n^{n+1/2}e^{-n}\sqrt{2\pi} \qquad as\ n\to\infty$$