# Chapter3 Measure Theory

## *Measure Space*

why $\sigma$-algebra?
We construct a $\sigma$-algebra from $\Omega$ to make sure this set is closed under sets operations (intersection, complementation,.. ). Thus the measure defined on it can be well-defined. The reason we don't just define the measure on the $\Omega$ is that $\Omega$ is too large there are some redundant elements we are not care about.

Th. $\sigma$-algebras(respectively, algebras) are closed under countable( respectively, finite) intersections.

just check the definition of $\sigma$-algebra.

*this theorem suggests that for any set $\mathcal{A} \subset \Omega$ , we can generate a smallest $\sigma$-algebra that contains $\mathcal{A}$ by taking intersection of all the $\sigma$-algebras containing $\mathcal{A}$*

Lemma Let$(\Omega, \mathcal{F}, \mu)$ denote a measure space.
(1) (Continuity from below) if $A_1 \subseteq A_2 \subseteq \ldots$ are all measurable then $\mu(A_n) \uparrow \mu(\bigcup_{m=1}^{\infty} A_m)$ as $n \to \infty$.
(1) (Continuity from above) if $A_1 \supseteq A_2 \supseteq \ldots$ are all measurable and $\mu(A_n) < \infty$ for some $n$ , then $\mu(A_n) \downarrow \mu(\bigcup_{m=1}^{\infty} A_m)$ as $n \to \infty$.

these are results follow from Lebesgue Measure theory.

## *Lebesgue Measure*

### *The Carathéodory Extension Theorem*

Suppose $\Omega$ is a set, and $\mathcal{A}$ denotes an *algebra* of subsets of $\Omega$. Given a *countably additive* set function $\mu$ on $\mathcal{A}$, there exists a measure $\overline{\mu}$ on $(\Omega, \sigma(\mathcal{A}))$, such that $\mu(E) = \overline{\mu}(E)$ for all $E \in \mathcal{A}$.
Suppose, in addition, that there exist $\Omega_1 \subseteq \Omega_2 \subseteq \ldots$ in $\mathcal{A}$ such that $\bigcup_{m=1}^{\infty} \Omega_m = \Omega$ and $\mu(\Omega_i) < \infty$ for all $i \geq 1$. Then the extension $\overline{\mu}$ of $\mu$ is *unique* and $\overline{\mu}$ is a $\sigma - finite$ measure with $\mu(\Omega) = \overline{\mu}(\Omega)$.

This inspires us that, once we can define a countably additive set function $\mu$ on an algebra $\mathcal{A}$, we can extend it to become a measure on $(\Omega, \sigma(\mathcal{A}))$

the way we construct Lebesgue's measure on $(0,1]^d$ for $d \geq 2$ is just follow the theorem. we define measure of hypercube as

$$m\left\{x \in (0,1]^d : a_j < x_j < b_j\right\} = \prod_{j=1}^{d}(b_j - a_j)$$

since the collection of all finite unions of hypercubes is an algebra that generates $\mathcal{B}((0,1]^d)$
thus the definition can be extended to $\mathcal{B}((0,1]^d)$.

Th.(the definition of density function of $\mu$)
Suppose $f : R^d \to R_+$ is a *continuous function such that the Riemann integral $\int_{R^d} f(x)dx = 1$*. Given $a, b \in R^d$ with $a_j \leq b_j$ for all $j \leq d$, consider the hypercube $C := (a_1, b_1] \times \cdots \times (a_d, b_d]$, and define $\mu(C) = \int_C f(x)dx$. Then $\mu$ extends uniquely to a probability measure on $(R^d, \mathcal{B}(R^d))$, and $f$ is called the density function of $\mu$.

mark: the collection of all finite unions of disjoint hypercubes of the type mentioned is an algebra of subsets of $R^d$. We can do as we did when we constructed the Lebesgue measure.

# *Completion*

Thinking:
Consider the $([0,1], \mathcal{B}[0,1], m)$, where $m$ is the Lebesgue's measure. We can prove that the Cantor Set $\mathcal{C}$ satisfies:

1. $\mathcal{C} \in \mathcal{B}[0,1]$
2. $m(\mathcal{C}) = 0$
3. the cardinality of $\mathcal{C} >$ the cardinality of $[0,1]$
   these suggests that there exist many subsets of $\mathcal{C}$ which are not borel sets! For these subsets, they are not measurable. But logically their measure are 0.
   There inspires us to attach a strong limitation to prevent these happening.

*Completion Definition*
Given a measure space $(\Omega, \mathcal{F}, \mu)$, a measurable set $E$ is $null$ if $\mu(E) = 0$. The $\sigma-$algebra $\mathcal{F}$ is said to be $complete$ if *all subsets of null sets are themselves measurable and null*. When $\mathcal{F}$ is complete, we say also that $(\Omega, \mathcal{F}, \mu)$ is complete.

We can always ensure completeness.
Th. Given a measure space $(\Omega, \mathcal{F}, \mu)$, there exists a complete $\sigma-$algebra $\mathcal{F}' \supseteq \mathcal{F}$ and a

measure $\mu'$ on $(\Omega, \mathcal{F}')$ such that $\mu$ and $\mu'$ agree on $\mathcal{F}$.
The measure space $(\Omega, \mathcal{F}', \mu')$ is called the completion of $(\Omega, \mathcal{F}, \mu)$.

How to construct?
For any two sets $A$ and $B$ define:

$$\mathcal{F}' = \{A \subseteq \Omega : \exists B, N \subseteq \mathcal{F} \ such \ that \ \mu(N) = 0 \ and \ A \Delta B \subseteq N\}$$

such $\mathcal{F}'$ is a $\sigma-$algebra.
Define measure $\mu'$:
For any $A \in \mathcal{F}'$ define $\mu'(A) := \mu(B)$, where $B \in \mathcal{F}$ is a set such that for a null set $N \in \mathcal{F}$, $A \Delta B \subseteq N$.
We can prove that $\mu'$ is a measure on $(\Omega, \mathcal{F}')$.

# *Proof of Carathéodory's Theorem*

Throughout, $\Omega$ is a set and $\mathcal{A}$ is an algebra of subsets of $\Omega$.

Definition (monotone class)
A collection of subsets of $\Omega$ is a $monotone \ class$
if it is closed under increasing countable unions and decreasing countable intersections.

Prop. An arbitrary intersection of monotone classes is a monotone class. In particular, there exists a smallest monotone class containing $\mathcal{A}$.

the smallest monotone class containing $\mathcal{A}$ is written as mc($\mathcal{A}$), and is called the monotone class generated by $\mathcal{A}$.

## *The Monotone Class Theorem*

Any monotone class that contains $\mathcal{A}$ also contains $\sigma(\mathcal{A})$. In other words, mc($\mathcal{A}$)=$\sigma(\mathcal{A})$.

proof:
since $\sigma(\mathcal{A})$ is a monotone class, then $\sigma(\mathcal{A}) \supseteq mc(\mathcal{A})$.
It suffices to prove that $\sigma(\mathcal{A}) \subseteq mc(\mathcal{A})$.
Consider the following monotone classes:

$$\mathscr{C}_1 := \{E \in \sigma(\mathcal{A}) : E^c \in mc(\mathcal{A})\}$$

$$\mathscr{C}_2 := \{E \in \sigma(\mathcal{A}) : \forall F \in mc(\mathcal{A}), E \cup F \in mc(\mathcal{A})\}$$

$$\mathscr{C}_3 := \{E \in \sigma(\mathcal{A}) : \forall F \in \mathcal{A}, E \cup F \in mc(\mathcal{A})\}$$

$\mathscr{C}_1$ is a monotone class contains $\mathcal{A}$, thus $\mathscr{C}_1 \supseteq mc(\mathcal{A})$
This means that $mc(\mathcal{A})$ is closed under complementation.

$\mathscr{C}_3$ is a monotone class contains $\mathcal{A}$, thus $\mathscr{C}_3 \supseteq mc(\mathcal{A})$
By reversing the roles of $E$ and $F$ in the definition of $\mathscr{C}_3$ we can see that $\mathscr{C}_2 \supseteq mc(\mathcal{A})$.
Thus $\mathcal{A}$ is closed under finite unions and is therefore a $\sigma-$algebra.
Consequently, we have $\sigma(\mathcal{A}) \subseteq mc(\mathcal{A})$.

Proof of Carathéodory's Theorem(Existence):

We first define $\overline{\mu}(E)$ for all $E \subseteq \Omega$. This defines a set function $\overline{\mu}$ on the Power set $\mathscr{P}(\Omega)$ of $\Omega$ which may be too big a $\sigma-$algebra in the sense that $\overline{\mu}$ may fail to be countably additive on $\mathscr{P}(\Omega)$. However it will be additive on $\sigma(\mathcal{A})$.

For all $E \subseteq \Omega$, define:

$$\overline{\mu}(E) := inf \left\{ \sum_{n=1}^{\infty} \mu(E_n) : \forall j \geq 1, E_j \in \mathcal{A} \; and \; E \subseteq \bigcup_{n=1}^{\infty} E_n \right\}$$

*Step 1. Countable Subadditivity of $\overline{\mu}$*

It is suffices to prove that $\overline{\mu}(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \overline{\mu}(A_n)$ for all $A_1, A_2, \cdots \subseteq \Omega$.
Consider any collection $\{A_{j,n}\}$ of elements of $\mathcal{A}$ such that $A_n \subseteq \bigcup_{n=1}^{\infty} A_{j,n}$, by the definition of $\overline{\mu}$,

$$\overline{\mu}\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} \mu(A_{j,n})$$

also by the definition of inf, for any $\epsilon > 0$, we can choose $A_{j,n}$ such that,

$$\sum_{j=1}^{\infty} \mu(A_{j,n}) \leq \frac{\epsilon}{2^n} + \overline{\mu}(A_n)$$

whence,

$$\overline{\mu}\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \epsilon + \sum_{n=1}^{\infty} \overline{\mu}(A_n)$$

since $\epsilon > 0$ is arbitrary, this yields the countable subadditivity of $\overline{\mu}$.

*Step 2. $\overline{\mu}$ extends $\mu$*

It is suffices to prove that $\overline{\mu}$ and $\mu$ agree on $\mathcal{A}$.
$\overline{\mu}(E) \leq \mu(E)$ for all $E \subseteq \mathcal{A}$.
we seek to prove the converse inequality.
Consider a collection of $E_1, E_2, \ldots$ of elements of $\mathcal{A}$ that cover $E$.

For any $\epsilon > 0$, we can choose $E_n$ such that $\sum_{n=1}^{\infty} \mu(E_n) \leq \overline{\mu}(E) + \epsilon$.
Since $\mu$ is countably additive on $\mathcal{A}$,

$$\mu(E) \leq \mu(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \mu(E_n) \leq \overline{\mu}(E) + \epsilon.$$

since $\epsilon > 0$ is arbitrary, Step2 is completed.

*Step 3. Countable Additivity*

we now complete our proof by showing that the restriction of $\overline{\mu}$ to $\sigma(\mathcal{A})$ is countable additivity.
thanks to step1, it is suffices to prove that $\overline{\mu}(\bigcup_{n=1}^{\infty} A_n) \geq \sum_{n=1}^{\infty} \overline{\mu}(A_n)$, for all disjoint $A_1, A_2, \cdots \in \sigma(\mathcal{A})$.
Consider,

$$\mathscr{M} = \{E \in \Omega : \forall F \in \mathcal{A}, \overline{\mu}(E) = \overline{\mu}(E \cap F) + \overline{\mu}(E \cap F^c)\}$$

According to step2, $\mathscr{M}$ contains $\mathcal{A}$.
By the Countable Subadditivity of $\overline{\mu}$,

$$\mathscr{N} := \{E \in \Omega : \forall F \in \mathcal{A}, \overline{\mu}(E) \geq \overline{\mu}(E \cap F) + \overline{\mu}(E \cap F^c)\} = \mathscr{M}$$

We can prove that *$\mathscr{N}$ is a monotone class containing $\mathcal{A}$*.
thanks to the Monotone Class Theorem, $mc(\mathcal{A}) = \sigma(\mathcal{A}) \subseteq \mathscr{N}$
this suggests that $\overline{\mu}$ is finitely additive on $\sigma(\mathcal{A})$, since $A_n$ is disjoint, it follows that,

$$\overline{\mu}\left(\bigcup_{n=1}^{\infty} A_n\right) \geq \overline{\mu}\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N} \overline{\mu}(A_n)$$

for every $N \geq 1$.
whence the Carathéodory's Theorem(Existence) follows from letting $N \to \infty$.

Proof of Carathéodory's Theorem(Uniqueness):
If $\mu(\Omega) < \infty$,
Suppose there were two extensions $\overline{\mu}$ and $\nu$, and define

$$\mathscr{C} := \{E \in \sigma(\mathcal{A}) : \overline{\mu}(E) = \nu(E)\}$$

we can check that $\mathscr{C}$ is a monotone class that contains $\mathcal{A}$
thus $mc(\mathcal{A}) = \sigma(\mathcal{A}) \subseteq \mathscr{C}$
that is $\overline{\mu}(E) = \nu(E)$ for all $E \subseteq \sigma(\mathcal{A})$
If $\mu(\Omega) = \infty$ and $\bigcup_{m=1}^{\infty} \Omega_m = \Omega$, $\mu(\Omega_i) < \infty$ for all $i \geq 1$.
for every $n > 0$, $E \in \sigma(\mathcal{A})$, as we proved before

$$\overline{\mu}(E \cap \Omega_n) = \nu(E \cap \Omega_n)$$

let $n \to \infty$, since $E \cap \Omega_n \to E$, using the Continuity from below of measure,

$$\overline{\mu}(E) = \lim_{n\to\infty} \overline{\mu}(E \cap \Omega_n) = \lim_{n\to\infty} \nu(E \cap \Omega_n) = \nu(E)$$