Lecture Notes on Real Analysis

Zhu Yun

Department of Mathematics
University of Nan Kai

October 12, 2025

Preface

This manuscript represents a systematic compilation of notes developed during my undergraduate studies in real analysis. The primary impetus for this undertaking was the consolidation of my understanding of measure-theoretic concepts and their applications in integration theory. This study employs standard real analysis methodologies as developed in Royden and Fitzpatrick (2010)[1], Rudin (1987/2019)[2], and Zhou and Sun (2014)[3].

The pedagogical approach adopted herein reflects my personal engagement with the material—particular attention is given to motivational examples, conceptual bridges between topics, and notational systems that facilitated my comprehension. While this perspective offers an alternative to standard treatments, it necessarily carries the limitations of individual interpretation.

It is important to note that this version of the manuscript remains a work in progress. The content is not yet comprehensive, and future revisions will incorporate additional material, corrections, and refinements.

Readers are cautioned that despite careful preparation, this work may contain inaccuracies or omissions. Critical feedback from the mathematical community would be greatly appreciated and may be directed to:

`catherinezhuyun@outlook.com`

It is my hope that this document, while personal in nature, might contribute to the mathematical discourse and aid students in their exploration of real analysis.

October 12, 2025

The Author

Contents

1 Prerequisites

1.1 Equinumerosity and cardinality

1.1.1 Basic definitions

Definition 1.1. *Let A, B be two sets, we say $A \sim B$ if $\exists f : A \to B$ bijective. $\sim$ is an equivalence relation.*

Theorem 1.2. *Let $\{A_\lambda\}_{\lambda\in\Lambda}, \{B_\lambda\}_{\lambda\in\Lambda}$ pairwise disjoint respectively, if $A_\lambda \sim B_\lambda, \forall\lambda \in \Lambda$, then $\bigcup_{\lambda\in\Lambda} A_\lambda \sim \bigcup_{\lambda\in\Lambda} B_\lambda$.*

Proof. For every $\lambda \in \Lambda$, $\exists f_\lambda : A_\lambda \to B_\lambda$ bijective, then let $f : \bigcup_{\lambda\in\Lambda} A_\lambda \to \bigcup_{\lambda\in\Lambda} B_\lambda$, $\forall x \in \bigcup_{\lambda\in\Lambda} A_\lambda, \exists! \lambda_0 \in \Lambda, x \in A_{\lambda_0}, f(x) := f_{\lambda_0}(x) \in B_{\lambda_0} \subseteq \bigcup_{\lambda\in\Lambda} B_\lambda$, then for every $y \in \bigcup_{\lambda\in\Lambda} B_\lambda, \exists! \lambda_1 \in \Lambda, y \in B_{\lambda_1}$, then $\exists! x \in A_{\lambda_1}, f(x) = f_{\lambda_1}(x) = y$, f is surjective, for every $a, b \in \bigcup_{\lambda\in\Lambda} A_\lambda, \exists! \lambda_a, \lambda_b \in \Lambda, a \in A_{\lambda_a}, b \in A_{\lambda_b}$, then $f(a) = f_{\lambda_a}(a), f(b) = f_{\lambda_b}(b)$, if $\lambda_a \neq \lambda_b$, then $B_{\lambda_a} \cap B_{\lambda_b} = \emptyset, f(a) \neq f(b)$, if $\lambda_a = \lambda_b$, then $f(a) = f_{\lambda_a}(a) \neq f_{\lambda_a}(b) = f(b)$, f is injective so that bijective. □

Definition 1.3. *If $A \sim B$, then $|A| = |B|$, where $|A|, |B|$ denote* ***cardinality*** *of A, B.*

1.1.2 Finite set, infinite set, countable set

Definition 1.4. *Given a set A, A is a finite set if $\exists n \in \mathbb{N}$ s.t. $A \sim \{1, ..., n\}$, else A is an infinite set; A is a countable set if $A \sim \mathbb{N}$.*

Proposition 1.5. *(i) Given any infinite set A, $\exists S \subseteq A$ s.t. $S \sim \mathbb{N}$.*
(ii) Given any countable set A, for every infinite subset $B \subseteq A$, B is a countable set.
(iii) The union of an at most countable collection of countable sets is itself countable.

Proposition 1.6. *Let A be an infinite set, B is at most countable, then $A \sim A \bigcup B$.*

Proof. By 1.5, $\exists S \subseteq A$, $S \sim \mathbb{N} \sim S \bigcup B$, then $A = (A \setminus S) \sqcup S, A \bigcup B = (A \setminus S) \sqcup (S \bigcup B)$, by 1.2, $A \sim A \bigcup B$. □

1.1.3 Continuum cardinality

Example 1.7. $[0, 1]$ *is uncountable.*

Proof. Assume $[0, 1] = \{a_1, a_2, ..., a_n, ...\}$, since $\forall x, y \in [0, 1], x < y$, $\exists x_1, y_1 \in [0, 1]$ s.t. $x < x_1 < y_1 < y$, then $\exists I_1 \subseteq [0, 1]$ a nondegenerate closed interval s.t. $a_1 \notin I_1$, similarly, $\exists I_2 \subseteq I_1, a_2 \notin I_2$,...,$I_n \subseteq I_{n-1}, a_n \notin I_n$, then we have $\{I_n\}_{n=1}^{\infty}$ closed and monotonically decrease, $[0, 1]$ is bounded, so that $\exists b \in \bigcap_{n=1}^{\infty} I_n, b \notin \{a_1, a_2, ..., a_n, ...\}$, a contradiction, $[0, 1]$ is uncountable. □

Definition 1.8. *Given a set A, we say A has continuum cardinality if $A \sim [0, 1]$.*

Example 1.9.

$$[0, 1] \sim [a, b] \sim [a, b) \sim (a, b] \sim (a, b) \sim (-\frac{\pi}{2}, \frac{\pi}{2}) \overset{\tan x}{\sim} \mathbb{R}.$$

Definition 1.10. *Given $n \geq 2$, if $\{a_k\}_{k=1}^{\infty}$ satisfies $a_k \in \{0, 1, ..., n-1\}, \forall k \geq 1$, then we say $\{a_k\}_{k=1}^{\infty}$ is a n-variable sequence. Let $I = \{k : a_k \neq 0, k \geq 1\}$, then we say $\{a_k\}_{k=1}^{\infty}$ is a finite n-variable sequence if I is finite, we say $\{a_k\}_{k=1}^{\infty}$ is an infinite n-variable sequence if $I \sim \mathbb{N}$.*

Example 1.11. *Let $n \geq 2$, then the set E_n of all n-variable sequence has continuum cardinality.*

Proof. Since $E_n = I_n \bigsqcup F_n$, where I_n denotes the set of all infinite n-variable sequences, F_n denotes the set of all finite n-variable sequences. F_n countable, I_n infinite, then by 1.6 we have $E_n \sim I_n$. For every $\{a_k\}_{k=1}^{\infty} \in I_n$, define

$$f(\{a_k\}_{k=1}^{\infty}) = \sum_{k=1}^{\infty} \frac{a_k}{n^k}$$

then $f : I_n \to (0, 1]$ bijection, $E_n \sim I_n \sim (0, 1]$. □

Corollary 1.12. *Let A be a countable set, then $\mathcal{P}(A)$ has continuum cardinality. (By 1.11)*

Theorem 1.13. *The Cartesian product of at most countably many sets, each of cardinality of the continuum, also has cardinality of the continuum.*

Proof. We only need to prove that $X \sim E_2$, where $X = \prod_{n=1}^{\infty} X_n$, $X_n = E_2, \forall n \geq 1$ and E_2 is the set of all 2-variable sequences. Then by Cantor diagonal argument, we define

$$\begin{aligned} f :& X \to E_2 \\ & x \mapsto \{x_1^1, x_2^1, x_1^2, ...\} \end{aligned}$$

then f is a bijection, $X \sim E_2$. □

Remark 1.14. *Let $A = \{0, 1\}$, we have $\prod_{i=1}^{\infty} A \sim E_2 \sim [0, 1]$.*

Example 1.15.

$$\mathbb{R}^{\infty} \sim \mathbb{R}^2 \sim \mathbb{R}^3 \sim [0, 1]$$

1.1.4 Comparability of cardinal numbers

Lemma 1.16. *Let $\{A_n\}_{n=1}^{\infty}$ be a monotone decreasing sets sequence, then*

$$A_1 = (\bigcap_{i=1}^{\infty} A_i) \cup (\bigcup_{j=1}^{\infty} (A_j - A_{j+1})).$$

Theorem 1.17. *Let A_0, A_1, A_2 be three sets satisfying $A_2 \subseteq A_1 \subseteq A_0$, if $A_0 \sim A_2$, then $A_0 \sim A_1$.*

Proof. Since $A_0 \sim A_2$, then $\exists h : A_0 \to A_2$ bijection, $A_2 = h(A_0)$, then define $A_{n+2} = h(A_n), \forall n \geq 0$, we have $\{A_n\}_{n=0}^{\infty}$, note that $A_0 \supseteq A_1, h(A_0) \supseteq h(A_1), A_2 \supseteq A_3$, then $A_{2k} \supseteq A_{2k+1}, \forall k \geq 0$, similarly $A_{2k+1} \supseteq A_{2k+2}, \forall k \geq 1$, then $\{A_n\}_{n=1}^{\infty}$ is monotone decreasing. Let $A_{-1} = \bigcap_{n=0}^{\infty} A_n = \bigcap_{k=0}^{\infty} A_{2k} = \bigcap_{k=1}^{\infty} A_{2k-1}$, by 1.16, $A_0 = A_{-1} \cup [\bigcup_{k=0}^{\infty} (A_{2k} - A_{2k+2})], A_1 = A_{-1} \cup [\bigcup_{k=0}^{\infty} (A_{2k+1} - A_{2k+3})]$, also, since $\forall k \geq 0$, $h(A_{2k} - A_{2k+1}) = A_{2k+2} - A_{2k+3}$ so that $(A_{2k} - A_{2k+1}) \sim (A_{2k+2} - A_{2k+3})$, then by 1.2, $(A_{2k+1} - A_{2k+2}) \cup (A_{2k} - A_{2k+1}) \sim (A_{2k+1} - A_{2k+2}) \cup (A_{2k+2} - A_{2k+3})$ i.e. $(A_{2k} - A_{2k+2}) \sim (A_{2k+1} - A_{2k+3})$, by 1.2, $A_0 \sim A_1$. □

Definition 1.18. *Given sets A, B, we say $|A| \leq |B|$ if $\exists S \subseteq B$ s.t. $A \sim S$.*

Theorem 1.19. *(i) $\forall A$ set, $|A| \leq |A|$.*
(ii) If $|A| \leq |B|, |B| \leq |C|$, then $|A| \leq |C|$.
(iii) If $|A| \leq |B|, |B| \leq |A|$, then $|A| = |B|$.

Proof. (iii) $\exists A_1 \subseteq A, B_1 \subseteq B$ s.t. $A \sim B_1, B \sim A_1$, then $|A| = |B_1| \leq |B| = |A_1|, \exists A_2 \subseteq A_1$ s.t. $A \sim A_2$, then since $A \supseteq A_1 \supseteq A_2$, by 1.17, $A \sim A_1$ so that $A \sim A_1 \sim B$, $|A| = |B|$. □

Example 1.20. *The set $F_{[0,1]}$ of all continuous functions on $[0, 1]$ has continuum cardinality c.*

Proof. Consider $f_c : [0, 1] \to \{c\}, f_c \equiv c, \forall c \in \mathbb{R}$, then $|F_{[0,1]}| \geq c$. Also, let $[0, 1] \cap \mathbb{Q} = \{q_1, q_2, ..., q_n, ...\}$, then let $g : F_{[0,1]} \to \mathbb{R}^{\infty}, g(f) = (f(q_1), f(q_2), ..., f(q_n), ...)$ injective so that $|F_{[0,1]}| \leq c$, by 1.19, $|F_{[0,1]}| = c$. □

Theorem 1.21. *Let μ be a cardinality, then $\mu < 2^{\mu}$, i.e. there is no set of greatest cardinality.*

Proof. Let A be a set with $|A| = \mu$, then $|A| = |\{\{x\} : x \in A\}| \leq |\mathcal{P}(A)|$. Assume $\exists f : A \to \mathcal{P}(A)$ bijection, consider $A^* = \{x : x \notin f(x), \forall x \in A\}$, then $\exists! z \in A, f(z) = A^*$, if $z \in A^*, z \in A \setminus A^*$, else $z \notin A^*, z \in A^*$, both contradiction, then $\nexists f$, $|A| < |\mathcal{P}(A)|$. □

1.2 Topology in $\mathbb{R}^n$

Definition 1.22. *Let $G \subseteq \mathbb{R}$ be an open subset, (a, b) an open interval, we say (a, b) is a constructing interval of G, if $(a, b) \subseteq G, a \notin G, b \notin G$.*

Lemma 1.23. *Let $G \subseteq \mathbb{R}$ be an open subset, then every $x \in G$, $\exists(a, b)$ a constructing interval of G s.t. $x \in (a, b)$.*

Proof. Given any $x \in G$, let $a = \inf\{a' : (a', x) \subseteq G\}, b = \sup\{b' : (x, b') \subseteq G\}$, since G open, $\exists\epsilon > 0, (x - \epsilon, x + \epsilon) \subseteq G, x - a, b - x > 0$, then $\forall y \in (a, x), \exists a < y' < y, y \in (y', x) \subseteq G$ so that $(a, x) \subseteq G$, similarly $(x, b) \subseteq G$, $x \in (a, b) \subseteq G, a \notin G, b \notin G$. □

Corollary 1.24. *Let $G \subseteq \mathbb{R}$ be an open subset, then G is the union of at most countable many pairwise disjoint open intervals.*

Theorem 1.25. *Let $F \subseteq \mathbb{R}$, then F is complete if and only if $\mathbb{R} \setminus F$ is the union of at most countable many pairwise disjoint open intervals with no common endpoints.*

Theorem 1.26. *Cantor set is complete, nowhere dense and has continuum cardinality.*

Proof. (1) C has no inner point, so that C is nowhere dense.

Assume $x \in C$ is a inner point, then $\exists\epsilon > 0, (x - \epsilon, x + \epsilon) \subseteq C$, $\exists n \geq 1, 0 \leq k \leq (3^n - 1)/2$, s.t. $[(2k)/3^n, (2k + 1)/3^n] \subseteq (x - \epsilon, x + \epsilon)$, a contradiction.

(2) C has continuum cardinality.

For every $x \in C$, $\exists!\{a_1, a_2, ..., a_n, ...\} \in I_3, a_k \in \{0, 2\}, \forall k \geq 1$ s.t. $x = \sum_{n=1}^{\infty} a_n/3^n$, then let $f : C \to I_2, f(x) = \{a_1/2, a_2/2, ..., a_n/2, ...\}$ a bijection, then $|C| = c$. □

Theorem 1.27. *Let $I \subseteq \mathbb{R}^n$ be a cube, length λ, then I can be expressed as the union of finite many cubes length $\lambda/2$.*

Theorem 1.28. *Every open set in $\mathbb{R}^n$ can be expressed as the union of countable many pairwise disjoint half-open cube.*

1.3 Example and exercise

Example 1.29. *Let $f(x), f_n(x)(n \geq 1)$ be real-valued functions on $\mathbb{R}$, prove that:*

$$\{x : \lim_{n\to\infty} f_n(x) = f(x)\} = \bigcap_{r=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty}\{x : |f_k(x) - f(x)| < \frac{1}{r}\}.$$

Proof.

$$\begin{aligned}\bigcap_{r=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty}\{x : |f_k(x) - f(x)| < \frac{1}{r}\} &= \bigcap_{r=1}^{\infty}\liminf_{k\to\infty}\{|f_k(x) - f(x)| < \frac{1}{r}\}\\ &= \bigcap_{r=1}^{\infty}\{x : \exists N_x > 0, \forall n > N_x, |f_n(x) - f(x)| < \frac{1}{r}\}\\ &= \{x : \lim_{n\to\infty} f_n(x) = f(x)\}.\end{aligned}$$

□

Example 1.30. *Prove: (i) For any sets A, B, $\exists C$ a set s.t. $A\Delta C = B$.*
(ii) $A\Delta(B\Delta C) = (A\Delta B)\Delta C$.
(iii) $A_1\Delta A_2\Delta ...\Delta A_n = \{x : 2 \nmid |\{A_k : x \in A_k, 1 \leq k \leq n\}|\}$.

Proof. (i) $C = A\Delta B$.
(ii) $A\Delta(B\Delta C) = (A \cap B \cap C) \cup (A \setminus (B \cup C)) \cup (B \setminus (A \cup C)) \cup (C \setminus (A \cup B)) = (A\Delta B)\Delta C$.
(iii) $n = 1$, trivial. Assume $\leq n - 1$ satisfy the condition, then

$$\begin{aligned}A_1\Delta ...\Delta A_{n-1}\Delta A_n &= \{x : x \in A_1\Delta ...\Delta A_{n-1}, x \notin A_n\} \cup \{x : x \in A_n, x \notin A_1\Delta ...\Delta A_{n-1}\}\\ &\subseteq \{x : 2 \nmid |\{A_k : x \in A_k, 1 \leq k \leq n\}|\}\\ &= (\{x : 2 \nmid |\{A_k : x \in A_k, 1 \leq k \leq n\}|\} \cap A_n) \cup (\{x : 2 \nmid |\{A_k : x \in A_k, 1 \leq k \leq n\}|\} - A_n)\\ &\subseteq \{x : x \in A_n, x \notin A_1\Delta ...\Delta A_{n-1}\} \cup \{x : x \in A_1\Delta ...\Delta A_{n-1}, x \notin A_n\}\\ &= A_1\Delta ...\Delta A_{n-1}\Delta A_n.\end{aligned}$$

□

Example 1.31. *Let f be a real-valued function on $\mathbb{R}$. If we have $M > 0$, for every group of finite many pairwise disjoint $x_1, ..., x_n \in \mathbb{R}$, satisfying*

$$|\sum_{k=1}^{n} f(x_k)| \leq M,$$

prove: $\{x : f(x) \neq 0\}$ is an at most countable set.

Figure 1: 1.33

Proof. $\{x : f(x) \neq 0\} = \{f(x) > 0\} \cup \{f(x) < 0\}$, consider $\{f(x) > 0\} = \bigcup_{n=1}^{\infty}\{f(x) > M/n\}$, then $|\{f(x) > M/n\}| < n$, if not, pick $x_1, ..., x_n \in \{f(x) > M/n\}$, then $|\sum_{k=1}^{n} f(x_k)| = \sum_{k=1}^{n} f(x_k) > n \cdot M/n = M$, a contradiction, then $\{f(x) > 0\}$ at most countable, similiarly, $\{f(x) < 0\}$ at most countable, then $\{f(x) \neq 0\}$ at most countable. □

Example 1.32. *If $A_2 \subseteq A_1, B_2 \subseteq B_1, A_2 \sim B_2, A_1 \sim B_1$, must there be $A_1 - A_2 \sim B_1 - B_2$?*

Proof. Not necessary. Let $A_1 = \mathbb{Q}, A_2 = \mathbb{N}, B_1 = \mathbb{N}, B_2 = \mathbb{N} \setminus \{1\}$, then $A_1 \sim B_1, A_2 \sim B_2$, but $|A_1 - A_2| = a \neq 1 = |\{1\}| = |B_1 - B_2|$. □

Example 1.33. *The direct product of finite many countable sets is countable.*

Proof. As shown in the figure 1. □

Example 1.34. *Let $E \subseteq \mathbb{R}$ uncountable, prove: $\exists x \in \mathbb{R}$ s.t. $\forall \delta > 0$, $E \cap (x-\delta, x+\delta)$ uncountable.*

Proof. Assume not, then $\forall x \in E, \exists \delta_x > 0$ s.t. $E \cap (x-\delta_x, x+\delta_x)$ countable, let $r_x, q_x \in \mathbb{Q}, x-\delta_x < r_x < x < q_x < x+\delta_x$, then $E = \bigcup_{x\in E}((r_x, q_x) \cap E) = E \cap (\bigcup_{x\in E}(r_x, q_x)) = E \cap (\bigcup_{n=1}^{\infty}(r_n, q_n)) = \bigcup_{n=1}^{\infty}((r_n, q_n) \cap E)$, a contradiction. □

Example 1.35. *Let $\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$, try to explicitly construct a strictly increasing function f whose set of discontinuity points is exactly $\{x_n\}_{n=1}^{\infty}$.*

Proof. Let $f_n(x) = \chi_{(x_n,\infty)}(x)/2^n, f(x) = \sum_{n=1}^{\infty} f_n(x)$ monotonically increases and uniformly converges, for every $a \notin \{x_n\}_{n=1}^{\infty}$, $f_n(x)$ continuous at $x = a$, $\forall n \geq 1$ so that $f(x)$ continuous at $x = a$; for every $x = x_n, n \geq 1$, $f(x) - f(x_n) \geq 1/2^n, \forall x > x_n$, $f(x)$ is not right continuous at $x = x_n$ so that not continuous at $x = x_n$. □

Example 1.36. *Prove that a real-valued function on $\mathbb{R}$ has at most countably many discontinuities of the first kind.*

Proof. Let $E_+ = \{|f(x) - f(x+)| > 0\}, E_- = \{|f(x) - f(x-)| > 0\}$, then $E_+ = \bigcup_{n=1}^{\infty}\{|f(x) - f(x_+)| > 1/n\}$, we only need to prove that $\forall n \geq 1, E_n^+ = \{|f(x) - f(x+)| > 1/n\}$ is countable. $\forall x \in E_n^+$, $\exists \delta_x > 0$ s.t. $\forall x < y < z < x + \delta_x, |f(y) - f(z)| < 1/(2n)$, then $\forall x' \in (x, x + \delta_x)$, if $x' \in E_n^+$, then $\exists \delta' > 0, x < x' < x' + \delta' < x + \delta_x$ s.t. $\forall x' < y' < x' + \delta', |f(y') - f(x'+)| < 1/(2n)$, then $|f(x') - f(y')| \geq |f(x') - f(x'+)| - |f(y') - f(x'+)| > 1/n - 1/(2n) = 1/(2n)$, a contradiction, so that $(x, x + \delta_x) \cap E_n^+ = \emptyset$, then $|E_n^+| = \sharp\{(x, x + \delta_x) : x \in E\} \leq a$ at most countable. □

Example 1.37. *Let $E \subseteq \mathbb{R}^3, \forall x, y \in E, |x - y| \in \mathbb{Q}$, prove that E at most countable.*

Proof. If all points in E are collinear on l, then pick $a \in E$, $\mathbb{Q} = \{q_n\}_{n=1}^{\infty}$, $E \subseteq \bigcup_{n=1}^{\infty}\{x : |x - a| = q_n, x \in l\} \sim \mathbb{N}$.

If $\exists a, b, c \in E$ non-collinear, then let $\mathbb{Q}^3 = \{(a_n, b_n, c_n)\}_{n=1}^{\infty}$, $E \subseteq \bigcup_{n=1}^{\infty}\{x : (|x - a|, |x - b|, |x - c|) = (a_n, b_n, c_n), x \in \mathbb{R}^3\} \sim \mathbb{N}$. □

Example 1.38. *Let $A \subseteq \mathbb{R}$ uncountable, prove: $\exists x \in A$ s.t. $\forall \delta > 0, (x-\delta, x) \cap A \neq \emptyset, (x, x+\delta) \cap A \neq \emptyset$, and the set E of all xs satisfying this condition is uncountable.*

Proof. Assume $E = \emptyset$, then $\forall x \in A, \exists \delta_x > 0$ s.t. $(x - \delta_x, x) \cap A = \emptyset$ or $(x, x + \delta_x) \cap A = \emptyset$, then $A = \{x : (x - \delta_x, x) \cap A = \emptyset\} \cup \{y : (y, y + \delta_y) \cap A = \emptyset\}$, where $\forall x, y \in \{x : (x - \delta_x, x) \cap A = \emptyset\}, x \neq y, (x-\delta_x, x) \cap (y-\delta_y, y) = \emptyset$ so that $\{x : (x-\delta_x, x) \cap A = \emptyset\} \preceq \mathbb{N}$, similarly, $\{y : (y, y+\delta_y) \cap A = \emptyset\} \preceq \mathbb{N}$, then $A \preceq \mathbb{N}$, a contradiction, then $E \neq \emptyset$. Note that $A = E \cup (A \setminus E)$, where $A \setminus E \preceq \mathbb{N}$, so that $|E| = |A| > a$, E is uncountable. □

Example 1.39. *Let $|A \cup B| = c$, prove: at least one of A and B has cardinality c.*

Proof. Since $|A|, |B| \leq |A \cup B| = c$, then assume that $|A| < c, |B| < c$, without loss of generality, we assume that $A \cup B = \mathbb{R}^2, A \subseteq \mathbb{R}^2, B \subseteq \mathbb{R}^2$, then $\exists x_0, y_0 \in \mathbb{R}$ s.t. $(x_0, y) \notin A, \forall y \in \mathbb{R}, (x, y_0) \notin B, \forall x \in \mathbb{R}$, so that $(x_0, y_0) \notin A \cup B$, a contradiction. □

Example 1.40. *If $A \cap B = \emptyset$, prove that $\overline{A} \cap B^\circ = \emptyset$.*

Proof. $B^\circ \cap A = \emptyset, A \subseteq X \setminus B^\circ$, $X \setminus B^\circ$ closed so that $\overline{A} \subseteq X \setminus B^\circ$, then $\overline{A} \cap B^\circ = \emptyset$. □

2 Lebesgue Measure

2.1 Outer Lebesgue measure

Definition 2.1.

$$\forall E \subseteq \mathbb{R} \quad , \quad m^*(E) = inf\{\sum_{n=1}^{\infty} l(I_n) : \{I_n\}_{n\geq 1}^{\infty} \ each\ I_n\ is\ an\ open\ interval\ and\ E \subseteq \bigcup_{n=1}^{\infty} I_n\}.$$

Lemma 2.2 (**Definition 1.1. validity assessment**).

Suppose $E \subseteq \mathbb{R}$, $if\ \exists\ a\ family\ interval\ \{I_\lambda\}_{\lambda\in\Lambda}\ s.t.\ E \subseteq \bigcup_{\lambda\in\Lambda} I_\lambda,\ then\ \exists\ \{I_n\}_{n=1}^{\infty} \subseteq \{I_\lambda\}_{\lambda\in\Lambda}\ covers\ E.$

Proof. If all I_λ are open intervals, $E \subseteq \bigcup_{\lambda\in\Lambda} I_\lambda = \bigcup_{\lambda\in\Lambda}\bigcup_{x\in I_\lambda}(r_x, R_x) = \bigcup_{n\geq 1}(r_n, R_n) \subseteq \bigcup_{n\geq 1} I_n$.

For general cases, let a_λ,b_λ denote the left and right boundary of I_λ respectively, and let $J_\lambda = (a_\lambda, b_\lambda)$, $A = \{a_\lambda : a_\lambda \in \bigcup_{\lambda\in\Lambda} I_\lambda\ and\ a_\lambda \notin \bigcup_{\lambda\in\Lambda} J_\lambda\}$, similarly $B = \{b_\lambda : b_\lambda \in \bigcup_{\lambda\in\Lambda} I_\lambda\ and\ b_\lambda \notin \bigcup_{\lambda\in\Lambda} J_\lambda\}$.

Note that each $a_\lambda \in A$ there exists a $J_{\lambda'} = (a_\lambda, *)$ and since $a_\lambda \notin \bigcup_{\lambda\in\Lambda} J_\lambda$ so $\forall a_\lambda, a_{\lambda'} \in A$, $(a_\lambda, *)\bigcap(a_{\lambda'}, *') = \emptyset$ then A is an at most countable set, similar to B.
$E \subseteq \bigcup_{\lambda\in\Lambda} I_\lambda = A\bigcup B\bigcup(\bigcup_{\lambda\in\Lambda} J_\lambda) = (\bigcup_{a_\lambda\in A, a_\lambda\in I_{\lambda'}} I_{\lambda'})\bigcup(\bigcup_{b_\lambda\in B, b_\lambda\in I_{\lambda^*}} I_{\lambda^*})\bigcup(\bigcup_{n\geq 1} I_n) = \bigcup_{k\geq 1} I_k^*$. □

Proposition 2.3. *If* $E_1 \subseteq E_2 \subseteq \mathbb{R}^1$*, then* $m^*(E_1) \leq m^*(E_2)$.

Proposition 2.4. *The outer measure is countably subadditive, that is, if* $\{E_k\}_{k\geq 1}$ *is any at most countable collection of sets, disjoint or not, then*

$$m^*(\bigcup_{k\geq 1} E_k) \leq \sum_{k\geq 1} m^*(E_k).$$

Proposition 2.5. *The outer measure of an interval is its length.*

Proof. Note that if $m^*([a,b]) = l([a,b]) = b - a$, then $\forall\ bounded\ interval\ I$, $\forall\ n \geq \frac{2}{b-a}$, $[a + \frac{1}{n}, b - \frac{1}{n}] \subseteq I \subseteq [a - \frac{1}{n}, b + \frac{1}{n}]$. By proposition 1.3. above, $b - a - \frac{2}{n} = m^*([a + \frac{1}{n}, b - \frac{1}{n}]) \subseteq m^*(I) \subseteq m^*([a - \frac{1}{n}, b + \frac{1}{n}]) = b - a + \frac{2}{n}$. Let $n \to \infty$, then $m^*(I) = b - a = l(I)$. The infinite interval case is trivial with the result of finite cases, we shall leave it to the readers.

Now we give the proof of a closed bounded interval $[a,b]$. We would like to show that $l([a,b])$ is the infimum of the set $I = \{x \in \mathbb{R}^1 : \exists\ open\ interval\ family\ \{I_n\}_{n\geq 1}\ ,\ x = \sum_{n\geq 1} l(I_n)\ ,\ [a,b] \subseteq$

$\bigcup_{n\geq 1} I_n\}$. For every $\varepsilon > 0$, $\exists\, 0 < \frac{1}{n} < \varepsilon$ *and* $[a,b] \subseteq (a-\frac{1}{2n}, b+\frac{1}{2n})$. So $l([a,b]) < l([a,b]) + \frac{1}{n} < l([a,b]) + \varepsilon$ and I are dense on the right side of $l([a,b])$. For every $y \in I$, $\exists$ *open interval family* $\{I_n\}_{n\geq 1}$, $s.t.\ y = \sum_{n\geq 1} l(I_n)$, $[a,b] \subseteq \bigcup_{n\geq 1} I_n$. By the Heine-Borel Theorem, any collection of open intervals covering $[a,b]$ has a finite subcollection that also covers $[a,b]$. We shall know that

$$[a,b] \subseteq \bigcup_{k=1}^{n} I_k (we\ denote\ I_k = (a_k, b_k)).$$

Consider picking out a branch from $\{I_k\}_{k=1}^n$. Since $[a,b] \subseteq \bigcup_{k=1}^n I_k$, $\exists\ k_1\ s.t.\ b \in I_{k_1}$ then $b_{k_1} > b$, $a_{k_1} < b$ if $a_{k_1} \leq a$ then we have done, if $a < a_{k_1} < b$ then $\exists\ k_2 \neq k_1\ s.t.\ a_{k_1} \in I_{k_2}$, continue this process until $a_{k_l} \leq a$ then we have a branch of $\{I_k\}_{k=1}^n$ and $y = \sum_{k=1}^n l(I_k) \geq \sum_{i=1}^l l(I_{k_i}) = b_{k_1} + (b_{k_2} - a_{k_1}) + \ldots + (b_{k_l} - a_{k_{l-1}}) - a_{k_l} \geq b_{k_1} - a_{k_l} > b - a = l([a,b])$. So, $l([a,b])$ is the infimum of the set I. $\square$

Proposition 2.6. *The outer measure is translation invariant, that is, for any set A and number y, $m^*(A+y) = m^*(A)$.*

Proof. Observe that if $\{I_k\}_{k\geq 1}$ is any countable collection of sets, then $\{I_k\}_{k\geq 1}$ covers A if and only if $\{I_k + y\}_{k\geq 1}$ covers $A + y$. Moreover, if each I_k is an open interval, then each $I_k + y$ is an open interval of the same length and so

$$\sum_{k\geq 1} l(I_k) = \sum_{k\geq 1} l(I_k + y).$$

The conclusion follows from these two observations. $\square$

2.2 Lebesgue measure on $\mathbb{R}^1$

Definition 2.7. *A set E is said to be* ***measurable*** *provided for any set A,*

$$m^*(A) = m^*(A \bigcap E) + m^*(A \bigcap E^c).$$

Remark 2.8. *It is equivalent to define the measurable set as $m^*(A) \geq m^*(A \bigcap E) + m^*(A \bigcap E^c)$.*

Proposition 2.9. *Any set of outer measures zero is measurable. Any countable set is measurable. A union of a countable collection of zero-measurable sets is zero measure. The complement set of any measurable set is measurable.*

Proposition 2.10. *The union of a finite collection of measurable sets is measurable.*

Proposition 2.11. *Let $\{E_k\}_{k=1}^{\infty}$ be a countable disjoint collection of measurable sets. Then*

$$m(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} m(E_k).$$

Proof. Let $E = \bigcup_{k=1}^{\infty} E_k$. We want to first show this inequality by induction.

$$m^*(A) \geq \sum_{k=1}^{\infty} m^*(A \bigcap E_k) + m^*(A \bigcap E^c) \geq m^*(A \bigcap E) + m^*(A \bigcap E^c).$$

When $n = 1$ we have $m^*(A) \geq m^*(A \bigcap E_1) + m^*(A \bigcap E_1^c) \geq \sum_{k=1}^{1} m^*(A \bigcap E_k) + m^*(A \bigcap E^c)$. Suppose $n \leq m$ is done, then for $n = m+1$,

$$m^*(A) \geq m^*(A \bigcap E_{m+1}) + m^*(A \bigcap E_{m+1}^c) \geq m^*(A \bigcap E_{m+1}) + \sum_{k=1}^{m} m^*(A \bigcap E_{m+1}^c \bigcap E_k) + m^*(A \bigcap E_{m+1}^c \bigcap E)$$

$$= \sum_{k=1}^{m+1} m^*(A \bigcap E_k) + m^*(A \bigcap E).$$

So $\bigcup_{k=1}^{\infty} E_k$ is measurable and Lebesgue measure has the countable additivity since

$$Let\ A = E\ and\ we\ have\ m(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} m(E_k).$$

□

Proposition 2.12. *Every interval is measurable.*

Proof. As we observed above, **the measurable sets are σ−algebra**. Therefore to show that every interval is measurable it suffices to show that every interval of the form (a, ∞) is measurable. Consider such an interval. Let A be any set. We assume a does not belong to A. We must show that

$$m^*(A \bigcap (a, \infty)) + m^*(A \bigcap (-\infty, a)) \leq m^*(A)$$

So we consider to show that for any countable collection $\{I_k\}_{k \geq 1}$ of open, bounded intervals that covers A,

$$m^*(A \bigcap (a, \infty)) + m^*(A \bigcap (-\infty, a)) \leq \sum_{k \geq 1} l(I_k).$$

Define $I_k^{'} = I_k \bigcap (a, \infty)$ and $I_k^{''} = I_k \bigcap (-\infty, a)$, so

$$m^*(A \bigcap (a, \infty)) + m^*(A \bigcap (-\infty, a)) \leq \sum_{k \geq 1} l(I_k^{'}) + \sum_{k \geq 1} l(I_k^{''}) = \sum_{k \geq 1} (l(I_k^{'}) + l(I_k^{''})) = \sum_{k \geq 1} l(I_k).$$

□

Definition 2.13. *A family of sets $\mathcal{F} \subseteq \mathcal{P}(\mathbb{R})$ is called **a σ−algebra** if*

(i)For every $F \in \mathcal{F}$, $F^c \in \mathcal{F}$;

(ii)For any $\{F_k\}_{k\geq 1} \subseteq \mathcal{F}$, $\bigcup_{k\geq 1} F_k \in \mathcal{F}$.

Definition 2.14. *The intersection of all the σ−algebras of subsets of $\mathbb{R}$ that contain the open sets is a σ−algebra called **the Borel σ−algebra**; members of this collection are called **Borel sets**.*

Theorem 2.15. *The collection $\mathcal{M}$ of measurable sets is a σ−algebra that contains the σ−algebra $\mathcal{B}$ of Borel sets. Each interval, each open set, each closed set, each G_δ set, and each F_σ set is measurable.*

Proposition 2.16. *The translate of a measurable set is measurable.*

Proof. Let E be a measurable set. Let A be any set and y be a real number. By the measurability of E and the translation invariance of outer measure,

$$m^*(A) = m^*(A-y) = m^*([A-y]\bigcap E) + m^*([A-y]\bigcap E^c) = m^*(A\bigcap[E+y]) + m^*(A\bigcap[E+y]^c).$$

□

2.3 Outer and inner approximation of Lebesgue measurable sets

Lemma 2.17 (Excision property). *If A is a measurable set of finite outer measure that is contained in B, then*

$$m^*(B-A) = m^*(B) - m^*(A).$$

Theorem 2.18. *Let E be any set of real numbers. Then each of the following five assertions is equivalent to the measurability of E.*

(i) *For each $\epsilon > 0$, there is **an open set O** containing E for which $m^*(O-E) < \epsilon$.*

(ii) *There is **a G_δ set** G containing E for which $m^*(G-E) = 0$.*

(iii) *For each $\epsilon > 0$, there is **a closed set** F contained in E for which $m^*(E-F) < \epsilon$.*

(iv) *There is **a F_σ set** F contained in E for which $m^*(E-F) = 0$.*

(v) *For each $\epsilon > 0$, there are **measurable sets G and F** for which $F \subseteq E \subseteq G$ and $m(G-F) < \epsilon$.*

The following property of measurable sets of finite outer measure asserts that such sets are "nearly" equal to the disjoint union of a finite number of open intervals.

Theorem 2.19. *Let E be a measurable set of finite outer measure. Then for each $\epsilon > 0$, there is a finite disjoint collection of open intervals $\{I_k\}_{k=1}^n$ for which if $O = \bigcup_{k=1}^n I_k$, then*

$$m(E\Delta O) = m(E - O) + m(O - E) < \epsilon.$$

Proof. According to assertion(i), for every $\epsilon > 0$, there is an open set $G \supseteq E$ and $m(G - E) < \frac{\epsilon}{2}$. Every open set of real numbers is the disjoint union of a countable collection of open intervals $\{I_k\}_{k=1}^\infty$. We have

$$\sum_{k=1}^n l(I_k) = m^*(\bigcup_{k=1}^n I_k) \leq m^*(G) < \infty$$

$$\sum_{k=1}^\infty l(I_k) < \infty$$

$$\exists\ n \quad s.t. \sum_{k=n+1}^\infty l(I_k) < \frac{\epsilon}{2}$$

Define $O = \bigcup_{k=1}^n I_k$, we have

$$m^*(O - E) + m^*(E - O) \leq m^*(G - E) + m^*(\bigcup_{k=n+1}^\infty I_k) \leq \frac{\epsilon}{2} + \sum_{k=n+1}^\infty m^*(I_k) \leq \epsilon.$$

□

Corollary 2.20 (**Littlewood's first principle**). *Let E be a measurable set of finite outer measure. Then for each $\epsilon > 0$, there is a finite disjoint collection of open rational intervals $\{I_k = (r_k, R_k)\}_{k=1}^n$ for which if $O = \bigcup_{k=1}^n I_k$, then*

$$m(E\Delta O) = m(E - O) + m(O - E) < \epsilon.$$

Proof. Note that if we have a finite disjoint collection of open intervals $\{I_k\}_{k=1}^n$ that satisfies the requirement with $\frac{\epsilon}{2}$, then every pair like $(a, b), (b, c)$ in $\{I_k\}_{k=1}^n$ we replace it to (a, c), and since $m((a, c)) = m((a, b) \bigcup (b, c))$ the modified $\{I_k'\}_{k=1}^{s\leq n}$ still reach the requirement, then we expand each I_k' to $J_k = (r_k, R_k)$ where $I_k' \subseteq J_k$ and $m(J_k - I_k') < \frac{\epsilon}{2s}$ and we set $O_1 = \bigcup_{k=1}^s J_k$ so that $m(E\Delta O_1) \leq m(E-O)+m(O_1-E) \leq m(E\Delta O)+m(O_1-O) \leq m(E\Delta O)+\sum_{k=1}^s m(J_k-I_k') < \epsilon$.

There was left to construct a finite disjoint collection of open intervals $\{I_k\}_{k=1}^n$ that satisfies the requirement. Since E is finite measurable, we have open set O so that $E \subseteq O$ and $m(O - E) < \frac{\epsilon}{2}$ also there is a sequence of disjoint open intervals s.t. $O = \bigcup_{k=1}^\infty I_k$ *with* $m(O) = \sum_{k=1}^\infty m(I_k) < \infty$. There is $N > 0$ *s.t.* $\sum_{k=1}^N m(I_k) > m(O) - \frac{\epsilon}{2}$. Let $O_1 = \bigcup_{k=1}^N I_k$, so finally we have:

$$m(O_1\Delta E) \leq m(O - E) + m(\bigcup_{k=N+1}^\infty I_k) < \epsilon.$$

□

2.4 Example and exercise

Example 2.21. *Let G_1, G_2 be two disjoint open sets, $E_1 \subseteq G_1$ and $E_2 \subseteq G_2$. Prove: $m^*(E_1 \bigcup E_2) = m^*(E_1) + m^*(E_2)$.*

Proof. (I)

$$m^*(E_1 \cup E_2) = m^*((E_1 \cup E_2) \cap G_1) + m^*((E_1 \cup E_2) \cap G_1^c) = m^*(E_1) + m^*(E_2).$$

□

Proof. (II) Since $m^*(E_1 \cup E_2) \leq m^*(E_1) + m^*(E_2)$, we only need to prove that $m^*(E_1) + m^*(E_2) \leq m^*(E_1 \cup E_2)$, note that $m^*(E_1 \cup E_2) = \inf\{\sum_{n=1}^{\infty} \ell(I_n) : E_1 \cup E_2 \subseteq \bigcup_{n=1}^{\infty} I_n, I_n$ open intervals$\}$, then we prove that $\forall \{I_n\}_{n=1}^{\infty}$, $m^*(E_1) + m^*(E_2) \leq \sum_{n=1}^{\infty} \ell(I_n)$. Since

$$m^*(E_1) \leq m(\bigcup_{n=1}^{\infty}(I_n \cap G_1)) \leq \sum_{n=1}^{\infty} m(I_n \cap G_1)$$
$$m^*(E_2) \leq m(\bigcup_{n=1}^{\infty}(I_n \cap G_2)) \leq \sum_{n=1}^{\infty} m(I_n \cap G_2),$$

then $m^*(E_1) + m^*(E_2) \leq \sum_{n=1}^{\infty}(m(I_n \cap G_1) + m(I_n \cap G_2)) \leq \sum_{n=1}^{\infty} \ell(I_n)$. □

Example 2.22. *If $d(E_1, E_2) = \inf\{d(x_1, x_2) : x_1 \in E_1, x_2 \in E_2\} > 0$, prove: $m^*(E_1 \bigcup E_2) = m^*(E_!) + m^*(E_2)$.*

Proof. Let $d(E_1, E_2) = d$, then $\forall x \in E_1, d(x, E_1) \geq d(E_1, E_2) = d > 0$, $\forall y \in E_2, d(x, y) \geq d(x, E_2) \geq d > 0$, then $(x - d/2, x + d/2) \cap E_2 = \emptyset$, let $G_1 = \bigcup_{x \in E_1}(x - d/2, x + d/2), E_1 \subseteq G_1, G_1 \cap E_2 = \emptyset$. Similarly, let $G_2 = \bigcup_{y \in E_2}(y - d/2, y + d/2), E_2 \subseteq G_2$, then $G_1 \cap G_2 = \bigcup_{x \in E_1} \bigcup_{y \in E_2}[(x - d/2, x + d/2) \cap (y - d/2, y + d/2)] = \emptyset$, by 2.21, $m^*(E_1 \cup E_2) = m^*(E_1) + m^*(E_2)$. □

Example 2.23. *Suppose $m^*(A) < \infty$, $m^*(B) < \infty$, prove: $|m^*(A) - m^*(B)| \leq m^*(A \Delta B)$.*

Proof.

$$m^*(A) \leq m^*((A - B) \bigcup B) \leq m^*(A - B) + m^*(B) \leq m^*(A \Delta B) + m^*(B)$$
$$\textit{similiarly we have another side.}$$

□

Example 2.24. *Suppose $\{E_n\}_{n \geq 1}$ increases, prove: $m^*(\lim_n E_n) = \lim_n m^*(E_n)$.*

Proof. When $\lim_n m^*(E_n) = \infty$ case is trivial so we only consider the case when $\lim_n m^*(E_n) < \infty$. Then for every $\epsilon > 0$ and every $n \geq 1$ we have an open set G_n s.t. $E_n \subseteq G_n$ and $m(G_n) < m^*(E_n)+\epsilon$. Note that $E_n \subseteq \bigcap_{k=n}^{\infty} G_n$, then we have:

$$m^*(\bigcup_{n=1}^{\infty} E_n) \leq m(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} G_k) = m(\lim_n \bigcap_{k=n}^{\infty} G_k) = \lim_n m(\bigcap_{k=n}^{\infty} G_k) \leq \lim_n m(G_n) < \lim_n m^*(E_n) + \epsilon.$$

□

Example 2.25. *Suppose $E \subseteq \mathbb{R}$, $0 < m^*(E) < \infty$. Prove: $f(x) = m^*((-\infty, x)\bigcap E)$ is a continuous function of x. Then prove $I = \{m^*(F) : F \subseteq E\}$ is a bounded closed interval.*

Proof. Randomly fix $x_0 \in \mathbb{R}$. Since $f(x)$ increases on $\mathbb{R}$ so $f(x^-)$ and $f(x^+)$ always exist. Then we have $x_n \uparrow x_0$ and $y_n \downarrow x_0$:

$$\lim_{x\to x_0^-} f(x) = \lim_{n\to\infty} m^*((-\infty, x_n)\bigcap E) = m^*((\bigcup_{n=1}^{\infty}(-\infty, x_n))\bigcap E) = m^*((-\infty, x_0)\bigcap E) = f(x_0);$$

$$\lim_{x\to x_0^+} f(x) = \lim_{n\to\infty} m^*((-\infty, y_n)\bigcap E) = \lim_{n\to\infty}(m^*(E) - m^*([y_n, \infty)\bigcap E))$$

$$= m^*(E) - m^*((\bigcup_{n=1}^{\infty}[y_n, \infty))\bigcap E) = m^*((-\infty, x_0]\bigcap E) = m^*((-\infty, x_0)\bigcap E) = f(x_0);$$

Note that $0, m^*(E) \in I$, since $f(x)$ is continuous and $f(-\infty) = 0$, $f(\infty) = m^*(E)$ we have $I = [0, m^*(E)]$. □

Example 2.26. *Let $E \subseteq \mathbb{R}$, $m(E) > 0$, $0 < \alpha < 1$. Prove: there exists an open interval s.t. $m(I\bigcap E) > \alpha \cdot m(I)$.*

Proof. For all $\epsilon > 0$ we have $E \subseteq \bigcup_{n=1}^{\infty} I_n$ and $\sum_{n=1}^{\infty} m(I_n) < m(E) + \epsilon$. Suppose there is a $0 < \alpha_0 < 1$ s.t. all open intervals have the property $m(E\bigcap I) \leq \alpha_0 \cdot m(I)$. Then there is a contradiction:

$$m(E) = m(\bigcup_{n=1}^{\infty}(I_n\bigcap E)) \leq \sum_{n=1}^{\infty} m(I_n\bigcap E) \leq \alpha_0 \sum_{n=1}^{\infty} m(I_n) < \alpha_0(m(E) + \epsilon).$$

□

Example 2.27. *Let $E \subseteq \mathbb{R}$, $m(E) > 0$, $[a, b]$ is a bounded closed interval. Prove: there exists finite many real number $\{x_k\}_{k=1}^n$ s.t. $m([a, b] - \bigcup_{k=1}^n E_{x_k}) < \epsilon$, where $E_{x_k} = \{x + x_k : x \in E\}$.*

Proof. Since $m(E) > 0$ there is an open interval $I = (c, d), c < d$ s.t. $m(E \bigcap I) > 1 - \frac{\epsilon}{2(b-a)}$ for every $\epsilon > 0$ and $m(I) < \frac{\epsilon}{2}$. Then we have:

$$[a,b] = \{ \bigcup_{k=1}^{\left\lfloor \frac{b-a}{m(I)} \right\rfloor} (a + (k-1)m(I), a + km(I))\} \bigcup \{a + km(I) : 0 \le k \le \left\lfloor \frac{b-a}{m(I)} \right\rfloor\} \bigcup (a + \left\lfloor \frac{b-a}{m(I)} \right\rfloor m(I), b]$$

$$= \{ \bigcup_{k=1}^{\left\lfloor \frac{b-a}{m(I)} \right\rfloor} I_{a+(k-1)m(I)-c}\} \bigcup \{a + km(I) : 0 \le k \le \left\lfloor \frac{b-a}{m(I)} \right\rfloor\} \bigcup (a + \left\lfloor \frac{b-a}{m(I)} \right\rfloor m(I), b]$$

$$m([a,b] - \bigcup_{k=1}^{\left\lfloor \frac{b-a}{m(I)} \right\rfloor} E_{a+(k-1)m(I)-c}) < b - a - \left\lfloor \frac{b-a}{m(I)} \right\rfloor m(I) + \sum_{k=1}^{\left\lfloor \frac{b-a}{m(I)} \right\rfloor} m(I - E)$$

$$< \frac{\epsilon}{2} + (1 - (1 - \frac{\epsilon}{2(b-a)}))m(I) \left\lfloor \frac{b-a}{m(I)} \right\rfloor \le \epsilon.$$

□

Example 2.28. *Suppose* $\{E_k\}_{k\ge 1}$ *is a sequence of measurable sets in* $[0, 1]$ *s.t.* $m(E_k) \to 1$ $(k \to \infty)$. *Prove: for every* $0 < \lambda < 1$, *there is a subsequence s.t.* $m(\bigcap_{n=1}^{\infty} E_{k_n}) > \lambda$.

Proof. For every $0 < \lambda < 1$, we consider:

$$m([0,1] - \bigcap_{n=1}^{\infty} E_{k_n}) = m(\bigcup_{n=1}^{\infty}([0,1] - E_{k_n})) \le \sum_{n=1}^{\infty}(1 - m(E_{k_n}))$$

$$There\ exists\ such\ \{E_{k_n}\}\ s.t.\ m(E_{k_n}) > 1 - \frac{\lambda}{2^n}$$

$$so\ that\ \le \sum_{n=1}^{\infty} \frac{\lambda}{2^n} = \lambda.$$

□

Example 2.29. *Suppose* $m^*(E) < \infty$. *Prove those three statement are equivalent:*

(i) *E is measurable;*

(ii) *There exists a sequence of closed sets* $\{F_n\}$ *of* E *s.t.* $m(F_n) \to m^*(E)$*;*

(iii) *There exists a sequence of measurable sets* $\{E_n\}$ *of* E *s.t.* $m(E_n) \to m^*(E)$.

Proof. Since (i) → (ii) → (iii) is left to readers. we only prove (iii) → (i). Consider $K_n = \bigcup_{j=1}^{n} E_j$, $K_n \subseteq E$ and $m(E_n) \le m(K_n) \le m^*(E)$, then we have $m(K_n) \to m^*(E)$. Then:

$$0 = \lim_{n\to\infty}(m^*(E) - m(K_n)) = \lim_{n\to\infty} m^*(E - K_n) = m^*(E - \bigcup_{n=1}^{\infty} K_n)$$

$$So\ E = \bigcup_{n=1}^{\infty} K_n \ \sqcup\ E - \bigcup_{n=1}^{\infty} K_n\ is\ measurable.$$

□

Example 2.30. *Find sets A, B so that $A \bigcap B = \emptyset$ and $m^*(A \bigcup B) < m^*(A) + m^*(B)$.*

Solution 1. *There is a nonmeasurable set E so that there is a set D satisfies $m^*(D) < m^*(D \bigcap E) + m^*(D \bigcap E^c)$.*

Example 2.31. *Suppose $A \bigcup B$ is measurable and $m(A \bigcup B) = m^*(A) + m^*(B) < \infty$. Prove: A and B are both measurable sets.*

Proof. By Example 1.36, there is a G_δ set E s.t. $B \subseteq E$ and $m^*(B) = m(E)$. Then $(A \bigcup B) \bigcap E^c \subseteq A$. Since $m(A \bigcup B) = m((A \bigcup B) \bigcap E) + m((A \bigcup B) \bigcap E^c)$, $m((A \bigcup B) \bigcap E) \leq m(E) = m^*(B)$ we have $m((A \bigcup B) \bigcap E^c) = m^*(A)$ so that $m(A - ((A \bigcup B) \bigcap E^c)) = 0$ and A is measurable. □

Example 2.32. *Suppose E to be a measurable set, $a, b \in \mathbb{R}$. Prove: $F = \{ax + b : x \in E\}$ is measurable and $m(F) = |a| \cdot m(E)$.*

Proof. We first show that $m^*(a \cdot A) = |a| \cdot m^*(A)$ for every set $A \subseteq \mathbb{R}$. Since $m^*(A) = \inf\{\sum_{n=1}^{\infty} m(I_n)\}$, $m^*(a \cdot A) = \inf\{\sum_{k=1}^{\infty} m(J_k)\}$ and $a \cdot A \subseteq \bigcup_{n=1}^{\infty} a \cdot I_n$, $A \subseteq \bigcup_{k=1}^{\infty} a^{-1} \cdot J_k$ for $a \neq 0$ ($a = 0$ case is trivial), we have $a \cdot \{\sum_{n=1}^{\infty} m(I_n)\} = \{\sum_{k=1}^{\infty} m(J_k)\}$ so that $m^*(a \cdot A) = |a| \cdot m^*(A)$. Then we have the following two results:

$$a \cdot (A \bigcap B) = a \cdot A \bigcap a \cdot B$$
$$a \cdot A^c = (a \cdot A)^c \textit{ since } a \cdot A \sqcup a \cdot A^c = \Omega.$$

Finally for every set B we have $m^*(B) = |a| \cdot m^*(a^{-1} \cdot B)$ and $m^*(B \bigcap aE) + m^*(B \bigcap (aE)^c) = |a| \cdot m^*(a^{-1}B \bigcap E) + |a| \cdot m^*(a^{-1}B \bigcap E^c)$ and we have done. □

Example 2.33. *Suppose a measurable set $E \subseteq [0, \infty), \lambda > 0$. Prove: E^λ is measurable where $E^\lambda = \{x^\lambda : x \in E\}$.*

Proof. Consider $(a, b) \subseteq (0, \infty)$, $(a, b)^\lambda = (a^\lambda, b^\lambda)$ is measurable, let $F = E \cap (0, \infty)$, $\exists G_n$ open, $F \subseteq G_n \subseteq (0, \infty), m(G_n - F) < 1/n$, then $m(\bigcap_{n=1}^{\infty} G_n - F) = 0, F \subseteq \bigcap_{n=1}^{\infty} G_n \in G_\delta$, let $S = \bigcap_{n=1}^{\infty} G_n - F$, then $F = (\bigcap_{n=1}^{\infty} G_n) - S$, $F^\lambda = (\bigcap_{n=1}^{\infty} G_n^\lambda) - S^\lambda$ is measurable. □

Example 2.34. *Suppose $E \subseteq \mathbb{R}$ is measurable, $m(E) > 0$, prove: 0 is the inner point of $\{x - y : x, y \in E\}$.*

Proof. (Key observation: $z \in \{x - y : x, y \in E\}$ if and only if $E \bigcap E_z \neq \emptyset$.)

Let $F = \{x - y : x, y \in E\}$ and suppose $0 \notin Inn(F)$ so we have a sequence $z_n \to 0$ where $E \bigcap E_{z_n} = \emptyset$. There is an bounded open interval $I = (a, b)$ s.t. $m(E \bigcap (a, b)) > \delta(b - a)$, $0 < \delta = \frac{7}{8} < 1$. Given $\epsilon = \frac{1}{2}(b - a) > 0$ and let $F = E \bigcap (a, b)$, when n goes sufficiently large, we have $F \bigcap F_{z_n} \subseteq (a - \epsilon, b + \epsilon)$ so that $\frac{3}{2}(b - a) = b - a + 2\epsilon > m(F \bigcup F_{z_n}) = m(F) + m(F_{z_n}) > 2\delta(b - a) = \frac{7}{4}(b - a)$, contradiction. □

Lemma 2.35 (**Lemma for Example 1.43**). *Suppose A, B, C are measurable sets and $m(C) < \infty$, then if $m(A \bigcap C) + m(B \bigcap C) > m(C)$ we have $m(A \bigcap B) > 0$.*

Example 2.36. *Suppose $m(A) > 0, m(B) > 0$. Prove: $Inn(\{a - b : a \in A, b \in B\}) \neq \emptyset$, $Inn(\{a + b : a \in A, b \in B\}) \neq \emptyset$.*

Proof. Note that $z \in \{a - b : a \in A, b \in B\}$ if and only if $A \bigcap B_z \neq \emptyset$. First we try to find a z_0 s.t. $m(A \bigcap B_{z_0}) > 0$. By Example 1.30 we fix $0 < \alpha < 1$ and there exists an bounded closed interval $I = [a, b]$ s.t. $m(A \bigcap [a, b]) > \delta(b - a)$. Then by Example 1.32 we have $m([a, b] - \bigcup_{k=1}^{n} B_{z_k}) < \alpha(b - a)$ so that $m(\bigcup_{k=1}^{n} B_{z_k} \bigcap [a, b]) + m(A \bigcap [a, b]) > m([a, b]) = b - a$ i.e. $m(A \bigcap (\bigcup_{k=1}^{n} B_{z_k}) \bigcap [a, b]) + m((A - (\bigcup_{k=1}^{n} B_{z_k})) \bigcap [a, b]) \geq m(A \bigcap [a, b]) > m((\bigcup_{k=1}^{n} B_{z_k})^c \bigcap [a, b])$. So $m(A \bigcap (\bigcup_{k=1}^{n} B_{z_k})) > 0$ and there exists a k_0 s.t. $m(A \bigcap B_{z_{k_0}}) > 0$.

Finally, $0 \in Inn(\{x - y : x, y \in A \bigcap B_{z_{k_0}}\})$ so that $z_{k_0} \in Inn(\{a - b : a \in A, b \in B\})$. Let $B = -B$ and we have $Inn(\{a + b : a \in A, b \in B\}) \neq \emptyset$. □

Example 2.37. *Suppose $m(E) > 0$, for every $x, y \in E$, $\frac{x+y}{2} \in E$, then $Inn(E) \neq \emptyset$.*

Proof.

$$\{x + y : x, y \in \frac{E}{2}\} = \{\frac{x + y}{2} : x, y \in E\},\ m(\frac{E}{2}) > 0$$
$$\emptyset \neq Inn(\{x + y : x, y \in \frac{E}{2}\}) = Inn(\{\frac{x + y}{2} : x, y \in E\}) \subseteq Inn(E).$$

□

Example 2.38. *Adopt decimal representation on $(0, 1)$, x_n is the nth value. Let $A_9 = \{x \in (0, 1) : \max\{x_n\} = 9\}$. Prove: $m(A_9) = 1$.*

Proof. Consider $B_n = \{x : x \in A_9 \text{ and the first } 9 \text{ in } x \text{ is } x_n\}$, and we have $A_9 = \sqcup_{n=1}^{\infty} B_n$. Let $C_n = \{(x_1, ..., x_{n-1}, 9) : x_j \in \{0, ..., 8\}, \forall\ 1 \leq j \leq n - 1\}$, then each $B_n = \sqcup_{s \in C_n} \{(\overrightarrow{s}, x_{n+1}, ...) : x_k \in \{0, 1, ..., 9\} \text{for all } k > n\}$ and let $D_{\overrightarrow{s}} = \{(\overrightarrow{s}, x_{n+1}, ...) : x_k \in \{0, 1, ..., 9\} \text{for all } k > n\}$ so that $m(B_n) = \sum_{\overrightarrow{s} \in C_n} m(D_{\overrightarrow{s}}) = \sum_{\overrightarrow{s} \in C_n} \frac{1}{10^n} m(10^n \cdot D_{\overrightarrow{s}}) = \sum_{\overrightarrow{s} \in C_n} \frac{1}{10^n} = \frac{9^{n-1}}{10^n}$. Finally, $m(A_9) = \sum_{n=1}^{\infty} \frac{9^{n-1}}{10^n} = 1$. □

Example 2.39. *In Example 1.45, if* $A = \{x \in (0,1) : \{x_n\}\ has\ finitely\ many\ 9\}$*, prove:* $m(A) = 0$.

Proof. Let $S_0 = (0,1) - A_9$, $S_n = \{x \in A_9 : x_n = 9, 9 \notin \{x_k\}_{k=n+1}^{\infty}\}, \forall n \geq 1$, $V_n = \{(x_1, ..., x_{n-1}, 9) : x_i \in \{0, ..., 9\}, \forall 1 \leq i \leq n-1\}, \forall n \geq 1$, then $A = \bigcup_{n=0}^{\infty} S_n$, since $m(S_0) = 0$, $S_n = \bigcup_{\vec{s} \in V_n} P_{\vec{s}}, P_{\vec{s}} = \{(\vec{s}, x_{n+1}, ...) : x_j \neq 9, \forall j \geq n+1\}, \forall \vec{s} \in V_n$, then $m(P_{\vec{s}}) = \frac{1}{10^n} m(10^n P_{\vec{s}}) = \frac{1}{10^n} m((0,1) - A_9) = 0$ so that $m(S_n) = 0, \forall n \geq 0, m(A) = 0$. □

Lemma 2.40 (Lemma for Example 1.48). *Suppose* $A \subseteq \mathbb{R}$*, if every subset of* A *is measurable, then* $m(A) = 0$.

Proof. We first define an equivalence relation in $\mathbb{R}$, $x \sim y \quad \Leftrightarrow \quad x - y \in \mathbb{Q}$. Then we get an equivalence class F of $\mathbb{R}$ and a partition $\mathbb{R} = \sqcup_{x \in F} E(x)$ where $E(x) = \{y : y - x \in \mathbb{Q},\ y \in \mathbb{R}\}$. F has properties:

(i) $\forall\ r, s \in \mathbb{Q},\ r \neq s,\ (F + r) \bigcap (F + s) = \emptyset$
(ii) $\mathbb{R} = \sqcup_{r \in \mathbb{Q}} (F + r)$;

Then $A = A \bigcap \mathbb{R} = \bigcup_{r \in \mathbb{Q}} (A \bigcap (F + r)) = \bigcup_{j=1}^{\infty} (A \bigcap (F + r_j))$ where $\{r_j\}_{j \geq 1} = \mathbb{R} \bigcap \mathbb{Q}$. Let $S_n = \bigcup_{j=1}^{n} (A \bigcap (F + r_j)) \uparrow A$ and then $m(A) = m(\lim_n S_n) = \lim_n m(S_n)$. And $\forall\ n \geq 1, r \in \mathbb{Q}$, let $A_{n,r} = [-n, n] \bigcap A \bigcap (F + r) \uparrow A \bigcap (F + r)$ we have $m(A \bigcap (F + r)) = m(\lim_n A_{n,r}) = \lim_n m(A_{n,r})$. Finally it turns out that if there is $m(A_{n,r}) = 0$ then everything are done and $m(E) = 0$.

Now we try to proof $m(A_{n,r}) = 0$. Suppose $\{s_k\}_{k \geq 1} = [0,1] \bigcap \mathbb{Q}$. Consider $E =_{\Delta} \sqcup_{k=1}^{\infty} (A_{n,k} + s_k) \subseteq [-n, n+1]$. So E is measurable and $m(E) < \infty$. Since $(A_{n,r} + s_j) \bigcap (A_{n,r} + s_k) = \emptyset,\ j \neq k$, $m(E) = \sum_{k=1}^{\infty} m(A_{n,k} + s_k) = \sum_{k=1}^{\infty} m(A_{n,k}) < \infty$ so that $m(A_{n,r}) = 0$. □

Example 2.41. *Suppose* $m(E) > 0$*, prove: there exists a non-measurable set in* E.

Example 2.42. *Suppose* $F \subsetneq [0,1]$ *is a non-measurable set. Prove:* $\exists\ 0 < \epsilon < 1$ *s.t. every measurable set* $E \subseteq [0,1]$ *and* $m(E) \geq \epsilon$*, the set* $E \bigcap F$ *is non-measurable.*

Proof. We prove this statement by contradiction. There exists a sequence of measurable sets E_n in $[0,1]$ s.t. $m(E_n) > 1 - \frac{1}{n}$ and $E_n \bigcap F$ to be measurable. Then consider $E = \bigcup_{n=1}^{\infty} E_n$, $m(E) \geq m(E_n) = 1 - \frac{1}{n} \to 1$ so that $m(E) = 1$ and $E \bigcap F$ is measurable. Since $F = (E \bigcap F) \bigcup (E \bigcap F^c)$, F is then measurable for $E \bigcap F^c$ is zero-measurable, contradiction. □

Example 2.43. *Suppose* $f(x)$ *defined on* $\mathbb{R}$*, for every measurable set* E*,* $f(E)$ *is measurable. Prove: for every zero-measurable set* Z*,* $f(Z)$ *is also zero-measurable.*(Rink : use Example 1.48)

Proof. If $\exists Z \subseteq \mathbb{R}, m(Z) = 0, m(f(Z)) > 0$, then by 2.41, $\exists F \subseteq f(Z)$ not measurable, then $f^{-1}(F) \subseteq Z, m(f^{-1}(F)) = 0$, $f(f^{-1}(F)) = F$, a contradiction. $\square$

Example 2.44. *Suppose $f(x)$ is continuous on $\mathbb{R}$. Prove: f turns every measurable set to a measurable set $\Leftrightarrow$ f turns every zero-measurable set to a zero-measurable set.*

Proof. ($\Leftarrow$) For every measurable set E, there is a set $F \in F_\sigma$ and a zero-measurable set D s.t. $E = F \bigcup D$. $F = \bigcup_{n=1}^{\infty} F_n$ where F_n is closed in $\mathbb{R}$. If F_n is bounded, then $f(F_n)$ is closed so that measurable. If F_n is not bounded, then $F_n = \bigcup_{n=1}^{\infty}([-n, n] \bigcap F_n)$ so that again measurable. Finally, $f(E) = f(F \bigcup D) = f(E) \bigcup f(D) = \bigcup_{n=1}^{\infty} f(F_n) \bigcup f(D)$ is measurable. $\square$

3 Lebesgue Measurable Functions

3.1 Sums, products, and compositions

Definition 3.1. *An extended real-valued function f defined on a measurable set E is said to be **Lebesgue measurable**, if for each real number c, the set $\{f(x) > c\} = f^{-1}((c, \infty])$ is measurable.*

Remark 3.2. *Suppose f, g are both measurable functions, then $\{f > a\}$ is measurable implies that $\{f < a\}, \{f \geq a\}, \{a < f < b\}$ $\{f > g\}$ $etc.$ are all measurable.*

Proposition 3.3. *Suppose $\{f_n(x)\}_{n\geq 1}$ is a sequence of measurable functions, then*

$$\sup_{n\geq 1} f_n(x),\ \inf_{n\geq 1} f_n(x),\ \overline{lim}_{n\geq 1} f_n(x),\ \underline{lim}_{n\geq 1} f_n(x)$$

are all measurable.

Proof. Given any $\alpha > 0$, $\{\sup_{n\geq 1} f_n(x) > \alpha\} = \bigcup_{n=1}^{\infty}\{f_n(x) > \alpha\}$, $\{\inf_{n\geq 1} f_n(x) < \alpha\} = \bigcup_{n=1}^{\infty}\{f_n(x) < \alpha\}$, $\limsup_{n\to\infty} f_n(x) = \inf_{n\geq 1}\sup_{k\geq n} f_k(x)$, $\liminf_{n\to\infty} \sup_{n\geq 1}\inf_{k\geq n} f_k(x)$. □

Proposition 3.4. *Let $\{f_n\}$ be a sequence of measurable functions on a measurable set E that converges pointwise a.e. on E to the function f. Then f is measurable.*

Proof. $f = \limsup f_n(x) = \liminf f_n(x)$. □

Definition 3.5. *If A is any set, the **characteristic function** of A,χ_A, is the function on $\mathbb{R}$ defined by*

$$\chi_A(x) = \begin{cases} 1 & if\ x \in A \\ 0 & if\ x \notin A. \end{cases}$$

Definition 3.6. *A **real-valued** function φ defined on a measurable set E is called **simple** provided it is measurable and takes a finite number of values.*

Remark 3.7. *Suppose f, g are simple on E, $\lambda \in \mathbb{R}$, then $\lambda f, |f|, fg, f + g, f - g$ are all simple.*

Remark 3.8. *If φ is simple, has domain E and takes the distinct values $c_1, ..., c_n$, then*

$$\varphi = \sum_{k=1}^{n} c_k \cdot \chi_{E_k}\ on\ E,\ where\ E_k = \{\varphi = c_k\}$$

*This particular expression of φ as a linear combination of characteristic functions is called the **canonical representation** of the simple function φ.*

Lemma 3.9. *Let f be a measurable real-valued function on E. Assume f is bounded on E, that is, there is an $M \geq 0$ for which $|f| \leq M$ on E. Then for each $\epsilon > 0$, there are simple functions φ_ϵ and ψ_ϵ defined on E which have the following approximation properties:*

$$\varphi_\epsilon \leq f \leq \psi_\epsilon \text{ and } 0 \leq \varphi_\epsilon - \psi_\epsilon \leq \epsilon \text{ on } E.$$

Proof. Let (c, d) be an open, bounded interval that contains $f(E)$ and $c = y_0 < y_1 < ... < y_{n-1} < y_n = d$ be a partition of $[c, d]$ such that $y_k - y_{k-1} < \epsilon$ for $1 \leq k \leq n$.

Define $I_k = [y_{k-1}, y_k)$ and $E_k = f^{-1}(I_k)$ for $1 \leq k \leq n$. Define the simple functions φ_ϵ and ψ_ϵ on E by

$$\varphi_\epsilon = \sum_{k=1}^{n} y_{k-1} \cdot \chi_{E_k} \text{ and } \psi_\epsilon = \sum_{k=1}^{n} y_k \cdot \chi_{E_k}.$$

□

Theorem 3.10 (The Simple Approximation Theorem). *An extended real-valued function f on a measurable set E is measurable if and only if there is a sequence $\{\varphi_n\}$ of simple functions on E which converges pointwise on E to f and has the property that*

$$|\varphi_n| \leq |f| \text{ on } E \text{ for all } n$$

If f is nonnegative, we may choose $\{\varphi_n\}$ to be increasing.

Proof. Since each simple function is measurable, then $f = \lim_{n\to\infty}\varphi_n$ is measurable. It remains to prove the converse.

Assume f is measurable. We also assume $f \geq 0$ on E. If f is not nonnegative, there exist g, h nonnegative measurable on E satisfy $f = g - h$. This is because at least we have the special construction as follows:

$$g = f_+ = \begin{cases} f(x) & if\ f(x) \geq 0 \\ 0 & o.w. \end{cases} \qquad h = f_- = \begin{cases} -f(x) & if\ f(x) < 0 \\ 0 & o.w. \end{cases}$$

Define $E_n = \{f \leq n\}$. Then $f|_{E_n}$ is a nonnegative bounded measurable function. By Lemma2.9, applied to $f|_{E_n}$ with $\epsilon = \frac{1}{n}$ as follows:

$$0 \leq \varphi_n \leq f \leq \psi_n \text{ on } E_n \text{ and } 0 \leq \psi_n - \varphi_n \leq \frac{1}{n} \text{ on } E_n$$

Observe that

$$0 \leq f - \varphi_n \leq \psi_n - \varphi_n \leq \frac{1}{n} \text{ on } E_n$$

Extend φ_n to all of E by setting $\varphi_n(x) = n$ if $f(x) > n$. The function φ_n is a simple function defined on E and $0 \leq \varphi_n \leq f$ on E. We leave it to reader to check that the sequence $\{\varphi_n\}$ converges to f pointwise on E.

By replacing each φ_n with $max\{\varphi_1, ..., \varphi_n\}$ we have $\{\varphi_n\}$ increase. □

Proposition 3.11. *Suppose f, g are measurable functions on the measurable set D, $\lambda \in \mathbb{R}$, then $\lambda f, |f|, fg, f+g, f-g$ are all measurable functions.*

Proposition 3.12. *Let the function f be defined on a measurable set E. Then f is measurable **if and only if** for each open set O, the inverse image of O under f, $f^{-1}(O) = \{x \in E : f(x) \in O\}$, is measurable.*

Proof. If the inverse image of each open set is measurable, then since each interval (c, ∞) is open, the function f is measurable. Conversely, suppose f is measurable. Let O be open. Then we can express O as the union of a countable collection of open intervals $O = \bigcup_{k\geq 1} I_k$. Then we have $f^{-1}(O) = f^{-1}(\bigcup_{k\geq 1} I_k) = \bigcup_{k\geq 1} f^{-1}(I_k)$ is measurable. □

Proposition 3.13. *Let f be a function with real value defined on a set E of real numbers. Then f is continuous on E if and only if for each open set O, $f^{-1}(O) = E \bigcap \mathcal{U}$ where $\mathcal{U}$ is an open set.*

Proposition 3.14. *A real-valued function that is continuous on its measurable domain is measurable.*

Proposition 3.15. *A monotone function that is defined on an interval is measurable.*

Proposition 3.16. *Let f be an extended real-valued function on E.*

(i)If f is measurable on E and $f = g$ a.e. on E, then g is measurable on E.

*(ii)For **a measurable subset** D of E, f is measurable on E if and only if the restrictions of f to D and $E - D$ are measurable.*

3.2 Littlewood's three principles, Egoroff's theorem, and Lusin's theorem

Theorem 3.17 (Egoroff's theorem). *Assume E has finite measure. Let $\{f_n\}$ be a sequence of real-valued measurable functions on E that converges pointwise on E to the real-valued function f. Then for each $\epsilon > 0$, there is a closed set F contained in E for which*

$$\{f_n\} \to f \ uniformly \ on \ F \ and \ m(E - F) < \epsilon.$$

Proof. Consider $A_n^r = \{x : |f_m - f| < \frac{1}{r}, \forall m \geq n\}$, and when r is fixed we find that $A_n^r \uparrow E$ so that $m(E) = m(lim_{n\to\infty} A_n^r) = lim_{n\to\infty} m(A_n^r)$. Then $\exists n_r \ s.t. \ \forall m \geq n_r$, $0 \leq m(E) - m(A_m^r) = m(E - A_m^r) < \frac{\epsilon}{2^{n+1}}$.

Note that $\{f_n\}$ is converging uniformly on $E_1 = \bigcap_{r=1}^{\infty} A_{n_r}^r$ since for all $\epsilon > 0$ there exists $\frac{1}{r_0} < \epsilon$ *s.t.* $E_1 \subseteq A_{n_{r_0}}^{r_0}$ *and* $|f_m - f| < \frac{1}{r_0} < \epsilon$, $\forall m > n_{r_0}$ *and* $\forall x \in E_1$. Also, $m(E - E_1) = m(\bigcup_{r=1}^{\infty}(E - A_{n_r}^r)) \leq \sum_{r=1}^{\infty} m(E - A_{n_r}^r) \leq \sum_{r=1}^{\infty} \frac{\epsilon}{2^{r+1}} = \frac{\epsilon}{2}$. There exists a closed set $F \subseteq E_1$ and $m(E_1 - F) < \frac{\epsilon}{2}$ since E_1 is measurable. So in the end we have $m(E - F) \leq m((E - E_1) \bigcup (E_1 - F)) \leq m(E - E_1) + m(E_1 - F) < \epsilon$. □

Lemma 3.18. *Suppose F is closed in $\mathbb{R}$. f is continuous on F. Then there exists a extended continuous function g on $\mathbb{R}$ s.t. $g|_F = f$ and* $\sup|g| = \sup|f|$.

Lemma 3.19 (**Littlewood's second principle**). *Let f be a simple function defined on E. Then for each $\epsilon > 0$, there is a continuous function g on $\mathbb{R}$ and a closed set F contained in E for which*

$$f = g \text{ on } F \text{ and } m(E - F) < \epsilon.$$

Proof. We take the canonical representation of $f = \sum_{k=1}^{n} a_k \chi_{E_k}$. Then we choose closed sets $F_1, ..., F_n$ in $E_1, ..., E_n$ respectively so that $m(E_k - F_k) < \frac{\epsilon}{n}$. Then we take $F = \bigcup_{k=1}^{n} F_n$ and we have $m(E - F) = m(\bigcup_{k=1}^{n}(E_k - F_k)) \leq \sum_{k=1}^{n} m(E_k - F_k) < \epsilon$. Define $g = \sum_{k=1}^{n} a_k \chi_{F_k}$ on F. g is continuous on F since every $x \in F$, $x \in F_k$ *for some* k, then $x \in (\bigcup_{i \neq k} F_i)^c$, $\exists$ *open interval* $I_x \subseteq (\bigcup_{i \neq k} F_i)^c$ and it is surely that g has a linear extension on $\mathbb{R}$. □

Theorem 3.20 (**Lusin's theorem**). *Let f be a real-valued measurable function on E. Then for each $\epsilon > 0$, there is a continuous function g on $\mathbb{R}$ and a closed set F contained in E for which*

$$f = g \text{ on } F, \quad m(E - F) < \epsilon \text{ and } \sup_{x \in E}|g(x)| \leq \sup_{x \in E}|f(x)|.$$

Proof. We consider $m(E) < \infty$. The case $m(E) = \infty$ is left to readers. By SAT, there is a sequence of simple functions so that $f_n \to f$ on E. By LSP, for each f_n there is a continuous function on $\mathbb{R}$ s.t. $f_n = g_n$ on F_n and $m(E - F_n) < \frac{\epsilon}{2^{n+1}}$. Let $E_1 = \bigcap_{n=1}^{\infty} F_n$ and $m(E - E_1) = m(\bigcup_{n=1}^{\infty}(E - F_n)) \leq \sum_{n=1}^{\infty} m(E - F_n) < \frac{\epsilon}{2}$. Then $g_n \to f$ on E_1. By Egoroff's theorem, there is closed set $E_2 \subseteq E_1$ s.t. $g_n \to f$ uniformly on E_2 and $m(E_1 - E_2) < \frac{\epsilon}{2}$. So f is continuous on E_2 and can be extended to a continuous function on $\mathbb{R}$. Also we have $m(E - E_2) = m(E - E_1) + m(E_1 - E_2) < \epsilon$. □

3.3 Convergence in measure

Definition 3.21. *Let $\{f_n\}$ be a sequence of measurable functions on E and f a measurable function on E for which f and each f_n is finite a.e. on E. The sequence $\{f_n\}$ is said to **converge in measure** on E to f provided for each $\eta > 0$,*

$$\lim_{n \to \infty} m(\{|f_n(x) - f(x)| > \eta\}) = 0.$$

Remark 3.22. *Convergence in measure $f_n \Rightarrow f$ neither implies nor is implied by pointwise convergence $f_n \to f$.*

Example 3.23 ($f_n \Rightarrow f$ not implies $f_n \to f$)**.** *Let $f_{n,k} = \chi_{[\frac{k-1}{n},\frac{k}{n}]}$, where $1 \le k \le n$, then consider $\{g_n\}$ where $g_1 = f_{1,1}$, $g_2 = f_{2,1}$, $g_3 = f_{2,2}, \ldots$. $\{g_n\}$ converges in measure to $g = 0$ since for each $\eta > 0$ and $\epsilon > 0$, $\forall n \ge N_\eta = \frac{t(1+t)}{2} + 1$, where $t > \frac{1}{\epsilon}$, $m(\{|g_n - g| > \eta\}) \le \frac{1}{t} < \epsilon$. However, $\{g_n\}$ not pointswise converges to g.*

Example 3.24 ($f_n \to f$ not implies $f_n \Rightarrow f$)**.** *Let $f_n = \chi_{(n,\infty)}$ and $f_n \to f = 0$. However, there is no $f_n \Rightarrow f$ since $m(\{|f_n - f| > \eta\}) = m((n,\infty)) = \infty$, $\forall 0 < \eta < 1$.*

Proposition 3.25. *Assume $m(E) < \infty$. $f_n \to f$ a.e. on E and f is finite a.e. on E. Then $f_n \Rightarrow f$ on E.*

Proof. Given $\eta > 0$ and E_1 that $f_n \to f$ on E_1 and $m(E - E_1) = 0$, consider $A_n = \bigcup_{k=n}^{\infty}\{x \in E_1 : |f_n(x) - f(x)| > \eta\}$. Note that $A_n \downarrow \limsup A_n = \emptyset$. With $m(A_1) < m(E) < \infty$, we have:

$$0 \le m(\{|f_n - f| > \eta\}) \le m(A_n \bigcup (E - E_1)) = m(A_n) \to \lim_{n\to\infty} m(A_n) = m(\bigcap_{n=1}^{\infty} A_n) = 0.$$

□

Theorem 3.26 (**Riesz**)**.** *If $f_n \Rightarrow f$ on E, then there is a subsequence $f_{n_k} \to f$ a.e. on E.*

Proof. There exists a n_k $m(\{|f_{n_k} - f| > \frac{1}{2^k}\}) < \frac{1}{2^k}$. Then consider:

$$m(F = \limsup\{|f_{n_k} - f| > \frac{1}{2^k}\}) \le m(\bigcup_{k=m}^{\infty}\{|f_{n_k} - f| > \frac{1}{2^k}\}) < \sum_{k=m}^{\infty} \frac{1}{2^k} \to 0,\ m \to \infty.$$

So $f_{n_k} \to f$ on $E - F$, i.e. $f_{n_k} \to f$ *a.e.* on E. □

Definition 3.27. *A measurable function sequence finite a.e. on D is called **a Cauchy series in measure**, if for all $\eta > 0$ and $\epsilon > 0$ there exists a $N > 0$, s.t. every $m, n > N$, there is:*

$$m(\{|f_n - f_m| > \eta\}) < \epsilon.$$

Proposition 3.28. *Suppose f_n, f are finite a.e. on D, then $f_n \Rightarrow f$ on D if and only if $\{f_n\}$ is a Cauchy series in measure.*

Proof. "$\Rightarrow$": Note that $\{|f_n - f_m| \geq \eta\} \subseteq \{|f_n - f| \geq \frac{\eta}{2}\} \bigcup \{|f_m - f| \geq \frac{\eta}{2}\}$. So

$$m(\{|f_n - f_m| > \eta\}) \leq m(\{|f_n - f| \geq \frac{\eta}{2}\}) + m(\{|f_m - f| \geq \frac{\eta}{2}\}) < \epsilon, \ for \ all \ n > N$$

$$where \ N \ let \ m(\{|f_n - f| > \frac{\eta}{4}\}) < \frac{\epsilon}{2}, \ \forall n > N.$$

"$\Leftarrow$": Since $\{f_n\}$ is a Cauchy series in measure, there exists a subsequence $\{f_{n_k}\}$ s.t. $m(\{|f_{n_{k+1}} - f_{n_k}| \geq \frac{1}{2^k}\}) < \frac{1}{2^k}$. Let $E = \limsup \{|\mathrm{f_{n_{k+1}}} - \mathrm{f_{n_k}}| \geq \frac{1}{2^k}\}$, one can check $m(E) = 0$. For every $x \in D - E$, $\{f_{n_k}(x)\}$ is a Cauchy series in $\mathbb{R}$. Then we have $f(x) = \lim f_{n_k}(x)$ formed by every $x \in D - E$ and $f(x)$ is finite a.e. on D. For fixed $\eta > 0$, $\exists k \ s.t. \ \frac{1}{2^{k-1}}$:

$$|f_{n_k} - f| \leq |f_{n_{k+t}} - f| + \sum_{j=k}^{k+t-1} |f_{n_{j+1}} - f_{n_j}|$$

$$let \ t \to \infty, \ we \ have \ \frac{1}{2^{k-1}} < \eta \leq |f_{n_k} - f| \leq \sum_{j=k}^{\infty} |f_{n_{j+1}} - f_{n_j}|$$

$$(D - E) \bigcap \{|f_{n_k} - f| \geq \eta\} \subseteq \bigcup_{p=k}^{\infty} \{|f_{n_{p+1}} - f_{n_p}| \geq \frac{1}{2^p}\}$$

$$m(\{|f_{n_k} - f| \geq \eta\}) = m((D - E) \bigcap \{|f_{n_k} - f| \geq \eta\}) \leq \sum_{p=k}^{\infty} \frac{1}{2^p} = \frac{1}{2^{k-1}}$$

So $f_{n_k} \Rightarrow f$ on D. At last, we have:

$$(D - E) \bigcap \{|f_n - f| \geq \delta\} \subseteq \{|f_n - f_{n_k}| \geq \frac{\delta}{2}\} \bigcup \{|f_{n_k} - f| \geq \frac{\delta}{2}\}$$

$$\exists N, \ when \ n, n_k > N, \ we \ have \ m(\{|f_n - f_{n_k}| \geq \frac{\delta}{2}\}) < \frac{\epsilon}{2} \ and \ m(\{|f_{n_k} - f| \geq \frac{\delta}{2}\}) < \frac{\epsilon}{2}$$

$$m(\{|f_n - f| \geq \delta\}) \leq m(\{|f_n - f_{n_k}| \geq \frac{\delta}{2}\}) + m(\{|f_{n_k} - f| \geq \frac{\delta}{2}\}) < \epsilon.$$

□

3.4 Example and exercise

Example 3.29. *f is finite a.e. measurable function on $[a, b]$. $\{f > \alpha\}, \{f \leq \alpha\}$ right-continuous, $\{f < \alpha\}, \{f \geq \alpha\}$ left-continuous.*

Example 3.30. *Suppose $\{f_\lambda(x)\}_{\lambda \in \Lambda}$ is a family of measurable functions on $[a, b]$. Whether $f(x) = \sup\{f_\lambda(x) : \lambda \in \Lambda\}$ measurable? If all $f_\lambda(x)$ are continuous on $[a, b]$?*

Solution 2. *Since $m([a, b]) > 0$, there is a nonmeasurable set $E \subseteq [a, b]$. Let $\{f_\lambda = \chi_{\{\lambda\}} : \lambda \in E\}$ on $[a, b]$. Then $\sup f_\lambda = \chi_E$ which is nonmeasurable.*

If every f_λ is continuous on $[a, b]$, then

$$\{\sup f_\lambda \leq \alpha\} = \bigcap_{\lambda \in \Lambda} \{f_\lambda \leq \alpha\}$$

where $\{f_\lambda \leq \alpha\}$ is closed in $[a, b]$ so that closed in $\mathbb{R}$, $\sup f_\lambda$ is measurable.

Example 3.31. *Suppose $\{f_n(x)\}_{n\geq 1}$ to be a sequence of measurable functions on D. Then the set E of all $x \in D$ that converge to extended real-valued is measurable.*

Proof.

$$E = \{\limsup f_n = \liminf f_n\}.$$

□

Example 3.32. *Suppose $f(x) \in C^1(\mathbb{R})$. Then $f^{'}(x)$ is measurable.*

Proof.

$$f^{'}(x) = \lim \frac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n}} =_{\Delta} g_n(x),\ g_n(x) \to f^{'}(x).$$

□

Example 3.33. *To make finite a.e. measurable function f on $\mathbb{R}$ to be a constant a.e. if and only if for any $\lambda \in \mathbb{R}$, we have at least one zero-measure in $\{f > \lambda\}$ and $\{f < \lambda\}$.*

Proof. Suppose $\exists \lambda,\ s.t.\ m(\{f > \lambda\}) = 0$, let $\lambda_0 = \inf\{\lambda \in \mathbb{R} : m(\{f > \lambda\}) = 0\}$ with $m(\{f > \alpha\})$ right-continuous we have $\lambda_0 \in \{\lambda \in \mathbb{R} : m(\{f > \lambda\}) = 0\}$ so that $m(\{f > \lambda_0\}) = 0$ and therefore $\lambda_0 > -\infty$, or $f = -\infty$ a.e. on $\mathbb{R}$.

Note that $\forall \lambda < \lambda_0,\ m(\{f > \lambda\}) > 0$ so $m(\{f < \lambda\}) = 0$, by $m(\{f < \alpha\})$ left-continous we have $m(\{f < \lambda_0\}) = \lim_{\lambda \to \lambda_0^-} m(\{f < \lambda\}) = 0$. At last we have $f = \lambda_0\ a.e\ on\ \mathbb{R}$. □

Example 3.34. *In 3.17, let $f(x) = \infty$ a.e., then what is the new conclusion and proof?*

Solution 3. *For every $\epsilon > 0$, $\exists F \subseteq D$ a closed subset s.t. $m(D - F) < \epsilon$, and $f_n \to \infty$ uniformly on F.*

Proof. Let $D_1 = \{x \in D : |f_n(x)| < \infty, f(x) = \infty, \lim_{n\to\infty} f_n(x) = f(x)\}$, then $m(D_1) = m(D)$, let $A_n^{(N)} = D_1 \cap [\bigcap_{k=n}^{\infty}\{|f_k(x)| > N\}], N, n = 1, 2, ...$, if we fix N, then $\{A_n^{(N)}\}_{n=1}^{\infty} \uparrow$, $\bigcup_{n=1}^{\infty} A_n^{(N)} = D_1$, $\lim_{n\to\infty} m(A_n^{(N)}) = m(D_1)$, given $\epsilon/2^{N+1}$, then $\exists n_N \geq 1$ s.t. $m(D - A_{n_N}^{(N)}) = m(D_1 - A_{n_N}^{(N)}) < \epsilon/2^{N+1}$. Let $E = \bigcap_{N=1}^{\infty} A_{n_N}^{(N)}$, then $m(D_1 - E) = m(\bigcup_{N=1}^{\infty}(D_1 - A_{n_N}^{(N)})) \leq \sum_{N=1}^{\infty} m(D_1 - A_{n_N}^{(N)}) < \sum_{N=1}^{\infty} \epsilon/2^{N+1} = \epsilon/2$. Then take $F \subseteq E$ closed s.t. $m(E - F) < \epsilon/2$, then $f_n \to \infty$ uniformly on F, where $m(D - F) = m(D_1 - F) \leq m(D_1 - E) + m(E - F) < \epsilon$. □

Example 3.35. *Suppose f is a real-valued measurable function on $[a, b]$, prove that there exists a sequence $\{h_k\}_{k=1}^{\infty}$ and $0 < h_k \to 0$ s.t. $f(x + h_k) \to f(x)$ a.e.*

Proof. By 3.20, for $\epsilon_k = \frac{b-a}{2^{k+1}}$, $k \geq 1$, there exists a closed set $F_k \subseteq (a, b)$ s.t. $m([a, b] - F_k) < \epsilon_k$ and f is continuous so that uniformly on F_k. Then there exists a $h_k > 0$ s.t. $F_{k(-h_k)} \subseteq [a, b]$ and every $|x-y| = h_k$. $x, y \in F_k$ we have $|f(x)-f(y)| < \frac{1}{k}$ i.e. $|f(x+h_k)-f(x)| < \frac{1}{k}$, $for\ all\ x \in E_k =_{\Delta} F_k \bigcap F_{k\cdot(-h_k)}$. Consider $E = \liminf E_k$ so that every $x \in E$ has property $\lim_{k\to\infty} f(x+h_k) = f(x)$. Finally we check for $m(E) = b - a$:

$$Since\ m([a, b] - E_n) = m([a, b] - F_k) + m([a, b] - F_{k(-h_k)}) < 2\frac{b-a}{2^{k+1}} = \frac{b-a}{2^k}$$
$$we\ have$$

$$m([a, b] - E) = m([a, b] - \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} E_k) = m(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}([a, b] - E_k)) \leq m(\bigcup_{k=n}^{\infty}([a, b] - E_k))(\forall n \geq 1)$$
$$\leq \sum_{k=n}^{\infty} m([a, b] - E_k) < (b - a)\sum_{k=n}^{\infty}\frac{1}{2^k} \to 0,\ n \to \infty.$$

□

Example 3.36. *Suppose f is finite a.e. and measurable on $[a, b]$. Prove that there is $\{g_n(x)\}$ a sequence of continuous functions on $[a, b]$ s.t. $g_n(x) \to f(x)$, a.e. and* $\max_{x\in[a,b]}|g_n(x)| \leq \sup_{x\in[a,b]}|f(x)|$.

Proof. We wish to construct a sequence of measurable functions that satisfy $\max_{x\in[a,b]}|g_n(x)| \leq \sup_{x\in[a,b]}|f(x)|$ and converge to $f(x)$ in measure, so that we can then use Riesz theorem to find a subsequence converge pointwise a.e. to $f(x)$.

By Lusin theorem, we have continuous function $g_n(x) = f(x)$ on F_n satisfies $\max_{x\in[a,b]}|g_n(x)| \leq \sup_{x\in[a,b]}|f(x)|$ and $m([a, b] - F_n) < \frac{1}{n}$, so for every $\eta > 0$, $m(\{|g_n(x) - f(x)| > \eta\}) \leq m(\{g_n \neq f\}) < \frac{1}{n} \to 0$, $n \to \infty$ ao that $g_n \rightrightarrows f$ on $[a, b]$. □

Example 3.37. *Suppose $\{f_n\}$ to be a sequence of real-valued measurable functions on $[a, b]$. Prove that $f_k \to 0$, a.e. if and only if $\forall\epsilon > 0$, $m(\{\mathrm{sup_{p\geq k}|f_p(x)|} > \epsilon\}) \to 0$, $k \to 0$.*

Proof. All x that fail to satisfy the limit 0 form the set $E = \bigcup_{r=1}^{\infty} \limsup\ \{|f_n(x)| > \frac{1}{r}\}$, so

$f_k \to 0,\ a.e.$ now is equivalent to $m(E) = 0$.

$$m(E) = m(\bigcup_{r=1}^{\infty} \text{lim sup } \{|f_n(x)| > \frac{1}{r}\}) \ \Leftrightarrow\ \forall r \geq 1,\ m(\text{lim sup } \{|f_n(x)| > \frac{1}{r}\}) = 0$$

$$Note\ that\ \text{lim sup } \{|f_n(x)| > \frac{1}{r}\} = \bigcap_{k=1}^{\infty}\{\sup_{p\geq k}|f_p(x)| > \frac{1}{r}\}$$

$$Then\ \Leftrightarrow\ m(\bigcap_{k=1}^{\infty}\{\sup_{p\geq k}|f_p(x)| > \frac{1}{r}\}) = \lim_{k\to\infty} m(\{\sup_{p\geq k}|f_p(x)| > \frac{1}{r}\}) = 0$$

$$\forall \epsilon > 0,\ m(\{\mathrm{sup_{p\geq k}|f_p(x)|} > \epsilon\}) \to 0,\ k \to 0.$$

□

Example 3.38. *Suppose $\{f_k(x)\}$ to be a sequence of real-valued measurable functions on $[a, b]$. Prove that there exists a positive number sequence $\{a_k\}$ s.t. $a_k f_k(x) \to 0,\ a.e.$*

Proof. Note that $[a, b] = \bigcup_{n=1}^{\infty}\{f_k(x) < n\}$, $\forall k \geq 1$, so $b - a = m(\bigcup_{n=1}^{\infty}\{f_k(x) < n\}) = \lim_{n\to\infty} m(\{f_k(x) < n\})$ so that there exists a $n_k > 0$ s.t. $m(\{f_k(x) < n_k\}) > b - a - \frac{1}{2^k}$. Set $a_k = \frac{1}{k\cdot n_k}$ and we immediately have $|a_k f_k(x)| < \frac{1}{k}$, $\forall x \in \{f_k(x) < n_k\}$. Let $E = \text{lim sup } \{f_k(x) \geq n_k\}$, then every $x \in E^c$ has $a_k f_k(x) \to 0$ and for E, $m(E) \leq m(\bigcup_{k=n}^{\infty}\{f_k(x) \geq n_k\}) \leq \sum_{k=n}^{\infty} \frac{1}{2^k} \to 0,\ n \to \infty$ so that $m(E) = 0$. □

Example 3.39. *For $x \in (0, 1)$ we use decimal representation, x_k is the kth value of it. Now let $f(x) = max\{x_k : k \geq 1\}$. Prove: f is measurable on $(0, 1)$.*

Proof. By Example 1.45, $m(\{f(x) = 9\}) = 1$ i.e. $f(x) = 9$ a.e. so that f is measurable on $(0, 1)$. □

Example 3.40. *Suppose f is an real-valued measurable function on $\mathbb{R}$ and $f(x) = f(x + 1)$ a.e. Find $g(x)$ s.t. $g(x) = f(x)$ a.e. and $g(x) = g(x + 1)$, $\forall x \in \mathbb{R}$.*

Proof. Consider $A_x = \{x + k : k \in \mathbb{Z}\}$ then we have either $A_x = A_y$ or $A_x \bigcap A_y = \emptyset$. Let D denotes the zero-measured set s.t. $f(x) \neq f(x + 1)$ and $A = \{x : A_x \bigcap D \neq \emptyset\}$. Then we define $g(x)$ as follows:

$$g(x) = \begin{cases} f(x) & x \in \mathbb{R} - A \\ 0 & x \in A \end{cases}$$

It is easy to check for every $x \in \mathbb{R}$, $g(x) = g(y)$, $\forall x, y \in A_x$ so $g(x) = g(x + 1)$, $\forall x \in \mathbb{R}$. Additionally, $m(A) = m(\bigcup_{x\in D} A_x) = m(\bigcup_{n=-\infty}^{\infty} D_n) = 0$. □

Example 3.41. *Suppose f, g are measurable functions on $(0,1)$ and both are monotonically decreasing and left-continuous and for every $\lambda \in \mathbb{R}$ there is $m(\{f \geq \lambda\}) = m(\{g \geq \lambda\})$. Prove: $f(x) = g(x)$, $\forall x \in (0,1)$.*

Proof. Assume for contradiction that there is an $x_0 \in (0,1)$ s.t. $f(x_0) \neq g(x_0)$, suppose $f(x_0) > g(x_0)$. Since g is left-continuous, we have $\delta > 0$ s.t. every $y \in (x_0 - \delta, x_0]$ satisfies $g(y) < f(x_0)$. Then:

$$x_0 = m((0,x_0]) \leq m(\{f \geq f(x_0)\}) = m(\{g \geq f(x_0)\}) \leq m((0, x_0 - \delta]) = x_0 - \delta, \; \textit{contradiction}.$$

□

Example 3.42. *Suppose on the measurable set D we have $f_k \Rightarrow f$ and $g_k \Rightarrow g$. Prove:*

(i) $f_k \pm g_k \Rightarrow f \pm g$
(ii) $|f_k| \Rightarrow |f|$
(iii) $\min\{f_k, g_k\} \Rightarrow \min\{f, g\}$ *and* $\max\{f_k, g_k\} \Rightarrow \max\{f, g\}$
(iv) *When $m(D) < \infty$, $f_k g_k \Rightarrow fg$.*

Proof. We only prove (iv). Assume for contradiction that there is $\epsilon_0 > 0, \delta_0 > 0$ and a subsequence $\{f_{n_k} g_{n_k}\}$ of $\{f_k g_k\}$ s.t. $m(\{|f_{n_k} g_{n_k} - fg| \geq \delta_0\}) \geq \epsilon_0$. Without loss of generality, assume $f_{n_k} g_{n_k} \to fg$ a.e. Since $m(D) < \infty$, we have $f_{n_k} g_{n_k} \Rightarrow fg$, contradiction.

When $m(D) = \infty$, A counterexample is given below:

$$D = \mathbb{R}, f_k = x + \frac{1}{k} \Rightarrow f = x \; on \; D, while \; {f_k}^2 = x^2 + \frac{2}{k}x + \frac{1}{k^2} \; not \; measure-converges \; to \; f^2 = x^2.$$

□

Example 3.43. *Suppose $f(x_1, x_2) \in C(\mathbb{R}^2)$ and $g_1(t), g_2(t)$ to be real-valued measurable functions on $[a,b]$. Prove: $f(g_1(t), g_2(t))$ is measurable on $[a,b]$.*

Proof. (I) For every given open set O on $\mathbb{R}$, $f^{-1}(O)$ is an open set in $\mathbb{R}^2$. So $f^{-1}(O) = \sqcup_{n=1}^{\infty}([a_n, b_n) \times [c_n, d_n))$ and let $g(t) = (g_1(t), g_2(t))$. Then $g^{-1}(\sqcup_{n=1}^{\infty}([a_n, b_n) \times [c_n, d_n))) = \bigcup_{n=1}^{\infty} g^{-1}([a_n, b_n) \times [c_n, d_n)) = \bigcup_{n=1}^{\infty}(g_1^{-1}([a_n, b_n)) \bigcap g_2^{-1}([c_n, d_n)))$ is measurable. □

Proof. (II) Since $g_1(t), g_2(t)$ are real-valued measurable functions on $[a,b]$, there exists sequences of simple functions s.t. $h_k^1 \to g_1$ and $h_k^2 \to g_2$. Then $f(h_k^1, h_k^2)$ is a simple function so that $f(g_1, g_2) \leftarrow f(h_k^1, h_k^2), k \to \infty$ is measurable. □

Example 3.44. *Suppose f, g are measurable on $\mathbb{R}$. Prove: $f(x)g(y)$ is measurable on $\mathbb{R}^2$.*

Proof. Consider $p(x,y) = f(x)$ and $q(x,y) = g(x)$, both are measurable functions on $\mathbb{R}^2$. So that $f(x)g(y) = p(x,y)q(x,y)$ is measurable on $\mathbb{R}^2$. □

Example 3.45. *Suppose f is a finite a.e. measurable function on $[0,1]$. Prove: there is a decreasing g on $[0,1]$, s.t. for every $\lambda \in \mathbb{R}$, $m(\{g > \lambda\}) = m(\{f > \lambda\})$.*

Proof. We construct $g(x)$ by **generalized inverse function**. Consider the function $h(\lambda) = m(\{f > \lambda\}), \lambda \in \mathbb{R}$. $h(x)$ is decreasing, right-continuous and $h(-\infty) = 1, h(\infty) = 0$. Let $g(x) = \inf\{\lambda : m(\{f > \lambda\}) \leq x\}$, then:

$$m(\{g(x) > \lambda\}) = m(\{x < m(\{f(x) > \lambda\})\} \bigcap [0,1]) = m([0, m\{f(x) > \lambda\})) = m(\{f(x) > \lambda\}).$$

□

Example 3.46. *Suppose f is continuous on $[a,b]$. For every $y \in \mathbb{R}$, let $\eta(y) = \sharp solutions\ of\ f(x) = y$ on $[a,b]$. Prove: $\eta(y)$ is measurable.*

Proof. We construct a sequence of simple functions converges to $\eta(y)$. We give a unit partition of $[a,b]$ into 2^n pieces i.e. $a = x_0 < x_1 < \ldots < x_{2^n-1} < x_{2^n} = b$ and let $\chi_i = \chi_{f([x_i-1, x_i))}$ then construct $\eta_n(y) = \sum_{i=1}^{2^n} \chi_i(y) \to \eta(y)$ so that $\eta(y)$ is measurable. □

4 Lebesgue integration

4.1 The general Lebesgue integral

4.1.1 Lebesgue integral on non-negative simple function

Definition 4.1. *Let f be a non-negative simple function, then we denote $f = \sum_{i=1}^{s} a_i \chi_{D_i}(x)$, where $0 < s < \infty, a_i \geq 0, \forall 1 \leq i \leq s$ and $\{D_1, ..., D_s\}$ is a partition of the measurable set $D = \sqcup_{i=1}^{s} D_i$ which is the definition domain of f. Observe that in Riemann integral we have*

$$\int_D f(x)\mathrm{dx} = \int_D \sum_{i=1}^{s} a_i \chi_{D_i}(x)\mathrm{dx} = \sum_{i=1}^{s} a_i \int_D \chi_{D_i}(x)\mathrm{dx}$$

So we define

$$\int_D \chi_{D_i}(x)\mathrm{dx} = m(D_i)$$

Then

$$\int_D f(x)\mathrm{dx} = \sum_{i=1}^{s} a_i \int_D \chi_{D_i}(x)\mathrm{dx} = \sum_{i=1}^{s} a_i m(D_i).$$

When

$$\int_D f(x)\mathrm{dx} < \infty$$

*we say $f(x)$ is **integrable on D**, denoted as $f(x) \in L(D)$.*

Proposition 4.2. *Let f and g be non-negative simple functions defined on a measurable set D, and suppose $f = g$ a.e. on D. Then we have*

$$\int_D f(x)\mathrm{dx} = \int_D g(x)\mathrm{dx}.$$

Proof. Let $f(x) = \sum_{i=1}^{s} a_i \chi_{E_i}$ and $g(x) = \sum_{j=1}^{t} b_j \chi_{F_j}$, where $D = \bigsqcup_{i=1}^{s} E_i = \bigsqcup_{j=1}^{t} F_j$. Then

$$D = \bigsqcup_{i=1}^{s} \bigsqcup_{j=1}^{t} (E_i \bigcap F_j),$$

if $m(E_i \bigcap F_j) > 0$, then $f(x) = g(x)$ a.e. on $E_i \bigcap F_j$ so that $a_i = b_j$. Then we have

$$\begin{aligned} \int_D f(x)\mathrm{dx} = \sum_{i=1}^{s} a_i m(E_i) &= \sum_{i=1}^{s} \sum_{j=1}^{t} a_i m(E_i \bigcap F_j) = \sum_{i=1}^{s} \sum_{j=1}^{t} b_j m(E_i \bigcap F_j) \\ &= \sum_{j=1}^{t} b_j \sum_{i=1}^{s} m(E_i \bigcap F_j) = \sum_{j=1}^{t} b_j m(F_j) = \int_D g(x)\mathrm{dx}. \end{aligned}$$

□

Proposition 4.3. *Let f and g be non-negative simple functions defined on a measurable set D.*
(i) If $f(x) \leq g(x)$ a.e. on D, then

$$\int_D f(x)\,\mathrm{dx} \leq \int_D g(x)\,\mathrm{dx}.$$

(ii)

$$\int_D f(x)\mathrm{dx} \leq \max_{x\in D}\{f(x)\} \cdot m(D).$$

In particular, if $m(D) = 0$, then

$$\int_D f(x)\mathrm{dx} = 0.$$

(iii) If λ and μ are two non-negative real numbers, then

$$\int_D (\lambda f(x) + \mu g(x))\mathrm{dx} = \lambda \int_D f(x)\mathrm{dx} + \mu \int_D g(x)\mathrm{dx}.$$

(iv) If A and B are two disjoint measurable subsets of D, then

$$\int_{A\cup B} f(x)\,\mathrm{dx} = \int_A f(x)\,\mathrm{dx} + \int_B f(x)\,\mathrm{dx}.$$

Proof. (i) Similar to 4.2, for every $m(E_i \bigcap F_j) > 0$, we have $a_i \leq b_j$ so that

$$\int_D f(x)\mathrm{dx} = \sum_{i=1}^{s}\sum_{j=1}^{t} a_i m(E_i \bigcap F_j) \leq \sum_{i=1}^{s}\sum_{j=1}^{t} b_j m(E_i \bigcap F_j) = \int_D g(x)\mathrm{dx}.$$

(ii)

$$\int_D f(x)\mathrm{dx} = \sum_{i=1}^{s} a_i m(E_i) \leq \max_{1\leq i\leq s}\{a_i\} \cdot \sum_{i=1}^{s} m(E_i) = \max_{x\in D}\{f(x)\} \cdot m(D).$$

(iii) Since $\lambda f(x) + \mu g(x) = \sum_{i=1}^{s}\sum_{j=1}^{t}(\lambda a_i + \mu b_j)\chi_{E_i\bigcap F_j}$, then

$$\begin{aligned}
\int_D (\lambda f(x) + \mu g(x))\mathrm{dx} &= \sum_{i=1}^{s}\sum_{j=1}^{t}(\lambda a_i + \mu b_j) m(E_i \bigcap F_j)\\
&= \lambda \sum_{i=1}^{s}\sum_{j=1}^{t} a_i m(E_i \bigcap F_j) + \mu \sum_{j=1}^{t}\sum_{i=1}^{s} b_j m(E_i \bigcap F_j)\\
&= \lambda \int_D f(x)\mathrm{dx} + \mu \int_D g(x)\mathrm{dx}.
\end{aligned}$$

(iv)

$$\int_{A\bigcup B} f(x)\mathrm{dx} = \sum_{i=1}^{s} a_i m(E_i) = \sum_{i=1}^{s} a_i m(E_i \bigcap A) + \sum_{i=1}^{s} a_i m(E_i \bigcap B) = \int_A f(x)\mathrm{dx} + \int_B f(x)\mathrm{dx}.$$

□

4.1.2 Lebesgue integral on non-negative measurable function

We wish for every monotone increasing non-negative simple function $f_n \to f$ and $g_n \to f$, we have

$$\lim_{n\to\infty} \int_D f_n(x)\mathrm{dx} = \lim_{n\to\infty} \int_D g_n(x)\mathrm{dx}$$

So that we can then define Lebesgue integral on non-negative measurable function f as

$$\int_D f(x)\mathrm{dx} = \lim_{n\to\infty} \int_D f_n(x)\mathrm{dx}.$$

To achieve this, we have the following two statements:

Lemma 4.4. *Let g and f_n $(n \geq 1)$ be non-negative simple functions defined on a measurable set D, satisfying the following two conditions:*

(i) The sequence $\{f_n(x)\}_{n\geq 1}$ is monotonically increasing a.e. on D.

(ii) $0 \leq g(x) \leq \lim_{n\to\infty} f_n(x)$ a.e. on D.

Then we have

$$\int_D g(x)\mathrm{dx} \leq \lim_{n\to\infty} \int_D f_n(x)\mathrm{dx}.$$

Proof. We first suppose $m(D) < \infty$.

Since we have

$$g(x) \leq \lim_{n\to\infty} f_n(x) \ a.e.\ on\ D$$

By Erogoff Theorem, for every $\epsilon > 0$, there is $D_1 \subseteq D$ s.t. $f_n(x) \to \lim f_n$ uniformly on D_1 with $m(D - D_1) < \epsilon$. Then we have $N > 0$ s.t. every $n > N$ there is $\lim_{n\to\infty} f_n(x) < f_n(x) + \epsilon$, $\forall x \in D_1$. We now have

$$\begin{aligned}\int_D g(x)\mathrm{dx} = \int_{D_1 \bigcup (D \setminus D_1)} g(x)\mathrm{dx} &= \int_{D_1} g(x)\mathrm{dx} + \int_{D\setminus D_1} g(x)\mathrm{dx} \\ &\leq \int_{D_1} f_n(x)\mathrm{dx} + \epsilon \cdot m(D_1) + \max_{x\in D\setminus D_1}\{g(x)\} \cdot m(D \setminus D_1), \forall n > N\end{aligned}$$

let $n \to \infty$, we have

$$\begin{aligned}\int_D g(x)\mathrm{dx} &\leq \lim_{n\to\infty} \int_{D_1} f_n(x)\mathrm{dx} + \epsilon \cdot m(D_1) + \max_{x\in D\setminus D_1}\{g(x)\} \cdot m(D \setminus D_1) \\ &\leq \lim_{n\to\infty} \int_D f_n(x)\mathrm{dx} + \epsilon \cdot (m(D) + \max_{x\in D\setminus D_1}\{g(x)\}) \\ &\to \lim_{n\to\infty} \int_D f_n(x)\mathrm{dx}, \epsilon \to 0^+.\end{aligned}$$

Suppose $m(D) = \infty$.

Let $D_k = D \bigcap [-k, k]$, then we have

$$\int_{D_k} g(x)\mathrm{dx} \le \lim_{n\to\infty} \int_{D_k} f_n(x)\mathrm{dx} \le \lim_{n\to\infty} \int_D f_n(x)\mathrm{dx}$$

Let $g(x) = \sum_{i=1}^{s} a_i \chi_{E_i}$, then since $E_i \bigcap D_k \uparrow E_i$ then $\lim_{k\to\infty} m(E_i \bigcap D_k) = m(\bigcup_{k=1}^{\infty}(E_i \bigcap D_k)) = m(E_i)$, we have

$$\int_{D_k} g(x)\mathrm{dx} = \sum_{i=1}^{s} a_i m(E_i \bigcap D_k) \to \sum_{i=1}^{s} a_i m(E_i) = \int_D g(x)\mathrm{dx}, k \to \infty$$

so that finally we have

$$\int_D g(x)\mathrm{dx} = \lim_{k\to\infty} \int_{D_k} g(x)\mathrm{dx} \le \lim_{n\to\infty} \int_D f_n(x)\mathrm{dx}.$$

□

Theorem 4.5. *Let $\{f_n\}$ and $\{g_n\}$ be two sequences of non-negative simple functions defined on a measurable set D. Suppose that for almost all $x \in D$, both $\{f_n(x)\}$ and $\{g_n(x)\}$ are monotonically increasing and converge to the same limit. Then,*

$$\lim_{n\to\infty} \int_D f_n(x)\mathrm{dx} = \lim_{n\to\infty} \int_D g_n(x)\mathrm{dx}.$$

Proof. For every fixed $n \ge 1$, for almost all $x \in D$ we have

$$0 \le f_n(x) \le \lim_{k\to\infty} g_k(x)$$

and $g_k(x)$ are monotonically increasing a.e. on D. Then by 4.4, we have

$$\int_D f(x)\mathrm{dx} \le \lim_{k\to\infty} \int_D g_k(x)\mathrm{dx}$$

let $n \to \infty$ and we have

$$\lim_{n\to\infty} \int_D f_n(x)\mathrm{dx} \le \lim_{k\to\infty} \int_D g_k(x)\mathrm{dx}$$

Similarly, we have

$$\lim_{k\to\infty} \int_D g_k(x)\mathrm{dx} \le \lim_{n\to\infty} \int_D f_n(x)\mathrm{dx},$$

so that finally we have

$$\lim_{n\to\infty} \int_D f_n(x)\mathrm{dx} = \lim_{k\to\infty} \int_D g_k(x)\mathrm{dx}.$$

□

Proposition 4.6. *Let f and g be non-negative measurable functions on a measurable set D.*

(i) If λ and μ are two non-negative real numbers, then

$$\int_D (\lambda f(x) + \mu g(x))\,\mathrm{dx} = \lambda \int_D f(x)\,\mathrm{dx} + \mu \int_D g(x)\,\mathrm{dx}.$$

(ii) If A and B are two disjoint measurable subsets of D, then

$$\int_{A\cup B} f(x)\,\mathrm{dx} = \int_A f(x)\,\mathrm{dx} + \int_B f(x)\,\mathrm{dx}.$$

(iii) If $f(x) = g(x)$ a.e. on D, then

$$\int_D f(x)\mathrm{dx} = \int_D g(x)\mathrm{dx}.$$

Proof. (i) There are sequences of non-negative simple functions s.t. $f_n(x) \uparrow f(x)$ and $g_n(x) \uparrow g(x)$ a.e. on D. Then $\lambda f_n(x) + \mu g_n(x) \uparrow \lambda f(x) + \mu g(x)$ a.e. on D. We have

$$\begin{aligned}\int_D (\lambda f(x) + \mu g(x))\mathrm{dx} &= \lim_{n\to\infty} \int_D (\lambda f_n(x) + \mu g_n(x))\mathrm{dx}\\ &= \lim_{n\to\infty} (\lambda \int_D f_n(x)\mathrm{dx} + \mu \int_D g_n(x)\mathrm{dx})\\ &= \lambda \lim_{n\to\infty} \int_D f_n(x)\mathrm{dx} + \mu \lim_{n\to\infty} \int_D g_n(x)\mathrm{dx}\\ &= \lambda \int_D f(x)\mathrm{dx} + \mu \int_D g(x)\mathrm{dx}.\end{aligned}$$

(ii)

$$\begin{aligned}\int_{A\bigcup B} f(x)\mathrm{dx} &= \lim_{n\to\infty} \int_{A\bigcup B} f_n(x)\mathrm{dx} = \lim_{n\to\infty} (\int_A f_n(x)\mathrm{dx} + \int_B f_n(x)\mathrm{dx})\\ &= \lim_{n\to\infty} \int_A f_n(x)\mathrm{dx} + \lim_{n\to\infty} \int_B f_n(x) = \int_A f(x)\mathrm{dx} + \int_B f(x)\mathrm{dx}.\end{aligned}$$

(iii) Since $f(x) = g(x)$ a.e. on D, then $f_n(x) \uparrow g(x)$ a.e. on D, we have

$$\int_D f(x)\mathrm{dx} = \lim_{n\to\infty} \int_D f_n(x)\mathrm{dx} = \lim_{n\to\infty} \int_D g_n(x)\mathrm{dx} = \int_D g(x)\mathrm{dx}.$$

□

Remark 4.7. *Let $f(x), g(x)$ be non-negative measurable functions on D, if $f(x) \le g(x)$ a.e. on D, then*

$$\int_D g(x)\mathrm{dx} = \int_D (g(x) - f(x))\mathrm{dx} + \int_D f(x)\mathrm{dx} \ge \int_D f(x)\mathrm{dx}.$$

4.1.3 Levi and Fatou

Theorem 4.8 (**Levi Monotone Convergence Theorem**). *Let f and f_n $(n \geq 1)$ be non-negative measurable functions on a measurable set $\boldsymbol{D}$. If $\{f_n(x)\}$ converges monotonically increasing to $f(x)$ a.e. on $\boldsymbol{D}$, then*

$$\int_D f(x)\,\mathrm{dx} = \lim_{n\to\infty} \int_D f_n(x)\,\mathrm{dx}.$$

Proof. For each $n \geq 1$, there is a sequence of non-negative simple measurable functions $f_n^k(x) \uparrow f_n(x), k \to \infty$ a.e. on $\boldsymbol{D}$. In Figure 2, we have $\varphi_n(x) \uparrow f(x)$, since $f_n^n(x) \leq \varphi_n(x) \leq f_n(x)$ and let $n \to \infty$ we have $f(x) \leq \varphi_n(x) \leq f(x)$. Since $\{\varphi_n(x)\}_{n=1}^{\infty}$ is a sequence of non-negative simple measurable functions s.t. $\varphi_n(x) \uparrow \varphi(x)$, then by definition we have

$$\int_D f(x)\mathrm{dx} = \lim_{n\to\infty} \int_D \varphi_n(x)\mathrm{dx}$$

then since $f_k^n(x) \leq \varphi_n(x) \leq f_n(x)$, we have

$$\int_D f_k^n(x)\mathrm{dx} \leq \int_D \varphi_n(x)\mathrm{dx} \leq \int_D f_n(x)\mathrm{dx}$$

let $n \to \infty$, we have

$$\int_D f_k(x)\mathrm{dx} \leq \lim_{n\to\infty} \int_D \varphi_n(x)\mathrm{dx} \leq \liminf_{n\to\infty} \int_D f_n(x)\mathrm{dx}$$

then let $k \to \infty$, we have

$$\limsup_{k\to\infty} \int_D f_k(x)\mathrm{dx} \leq \lim_{n\to\infty} \int_D \varphi_n(x)\mathrm{dx} \leq \liminf_{n\to\infty} \int_D f_n(x)\mathrm{dx}$$

so that

$$\lim_{k\to\infty} \int_D f_k(x)\mathrm{dx} = \lim_{n\to\infty} \int_D \varphi_n(x)\mathrm{dx} = \int_D f(x)\mathrm{dx}.$$

□

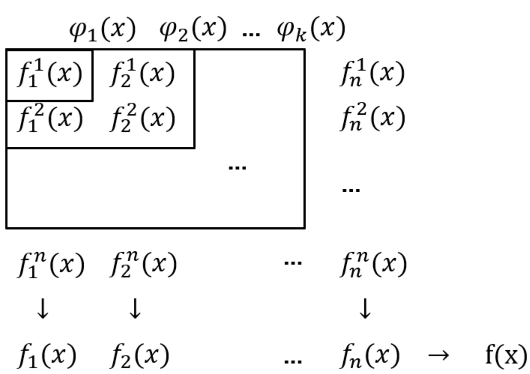

Figure 2: $\varphi_n(x) = \max\{f_p^q(x) : 1 \leq p, q \leq n\}$

Corollary 4.9. *Let $\{u_k(x)\}_{k\geq 1}$ be a sequence of non-negative measurable functions on a measurable set D. Then,*

$$\int_D \left(\sum_{k=1}^{\infty} u_k(x)\right) \mathrm{dx} = \sum_{k=1}^{\infty} \int_D u_k(x)\, \mathrm{dx}.$$

Proof. Let $S_n(x) = \sum_{k=1}^{n} u_k(x)$ and $S(x) = \sum_{k=1}^{\infty} u_k(x)$, then by definition we have $S_n(x) \uparrow S(x)$ on D. So that

$$\begin{aligned}\int_D (\sum_{k=1}^{\infty} u_k(x))\mathrm{dx} &= \int_D S(x)\mathrm{dx} = \lim_{n\to\infty} \int_D S_n(x)\mathrm{dx} = \lim_{n\to\infty} \int_D \sum_{k=1}^{n} u_k(x)\mathrm{dx} \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \int_D u_k(x)\mathrm{dx} = \sum_{k=1}^{\infty} \int_D u_k(x)\mathrm{dx}.\end{aligned}$$

□

Theorem 4.10 (Fatou Lemma). *Let f_n $(n \geq 1)$ be a sequence of non-negative measurable functions on a measurable set D. Then*

$$\int_D \left(\liminf_{n\to\infty} f_n(x)\right) \mathrm{dx} \leq \liminf_{n\to\infty} \int_D f_n(x)\, \mathrm{dx}.$$

Proof. Let

$$g_n(x) = \inf_{k\geq n} f_k(x) \uparrow \liminf_{m\to\infty} f_m(x)$$

then

$$\int_D \left(\liminf_{n\to\infty} f_n(x)\right) \mathrm{dx} = \lim_{n\to\infty} \int_D g_n(x)\mathrm{dx} = \lim_{n\to\infty} \int_D \inf_{k\geq n} f_k(x)\mathrm{dx} \leq \liminf_{n\to\infty} \int_D f_n(x)\mathrm{dx}.$$

□

4.1.4 Lebesgue integral on general measurable function

Let f be a measurable function defined on a measurable set D. For each $x \in D$, define

$$f_+(x) = \max_{x\in D}\{0, f(x)\}, \quad f_-(x) = \max_{x\in D}\{0, -f(x)\},$$

Then $f_+(x), f_-(x)$ both are non-negative measurable functions and

$$f(x) = f_+(x) - f_-(x), \quad |f(x)| = f_+(x) + f_-(x).$$

If $\int_D f_+(x)\mathrm{dx}$ and $\int_D f_-(x)\mathrm{dx}$ are not simultaneously ∞, then the Lebesgue integral of f over D is defined as

$$\int_D f(x)\mathrm{dx} = \int_D f_+(x)\mathrm{dx} - \int_D f_-(x)\mathrm{dx}.$$

Furthermore, when $\int_D f(x)\mathrm{dx}$ is finite, f is said to be Lebesgue integrable on D, denoted by $f \in L(D)$. Finally we have $\int_D f(x)\mathrm{dx} < \infty \Leftrightarrow \int_D f(x)\mathrm{dx} \in L(D) \Leftrightarrow \int_D f_+(x)\mathrm{dx} < \infty$ and $\int_D f_-(x)\mathrm{dx} < \infty \Leftrightarrow f_+(x), f_-(x) \in L(D)$.

Proposition 4.11. *Let f be a measurable function on a measurable set D.*

(i) $f \in L(D)$ if and only if

$$|f| \in L(D),$$

and $f(x) \in L(D)$, we have

$$\left|\int_D f\,dx\right| \leq \int_D |f|\,dx.$$

(ii) If $f \in L(D)$, then f is finite a.e. on D.

(iii) If g is also a measurable function on D and $f(x) = g(x)$ a.e. on D, then if $f(x) \in L(D)$, we have $g(x) \in L(D)$ and

$$\int_D f(x)\mathrm{dx} = \int_D g(x)\mathrm{dx}.$$

Proof. (i) $f(x) \in L(D)$ if and only if $\int_D f_+(x)\mathrm{dx} < \infty$ and $\int_D f_-(x)\mathrm{dx} < \infty$ if and only if $\int_D f_+(x)\mathrm{dx} + \int_D f_-(x)\mathrm{dx} = \int_D (f_+(x) + f_-(x))\mathrm{dx} = \int_D |f(x)|\mathrm{dx} < \infty$. Now when $f(x) \in L(D)$, we have $|f(x)| \in L(D)$ and

$$|\int_D f(x)\mathrm{dx}| = |\int_D f_+(x)\mathrm{dx} - \int_D f_-(x)\mathrm{dx}| \leq |\int_D f_+(x)\mathrm{dx}| + |\int_D f_-(x)\mathrm{dx}| = \int_D |f(x)|\mathrm{dx}.$$

(ii) Since $f \in L(D)$ implies $f_+ \in L(D)$, then

$$\infty > \int_D f_+(x)\mathrm{dx} \geq \int_{\{f \geq n\}} f(x)\mathrm{dx} \geq n \cdot m(\{f \geq n\})$$

$$n \cdot m(\{f(x) \geq n\}) \leq \int_{\{f(x) \geq n\}} f_+(x)\mathrm{dx} \leq \int_D f_+(x)\mathrm{dx} < \infty,$$

So that

$$m(\{f(x) \geq n\}) \leq \frac{1}{n}\int_D f_+(x)\mathrm{dx} < \infty,$$

then

$$0 \leq \lim_{n\to\infty} m(\{f \geq n\}) \leq \lim_{n\to\infty} \frac{1}{n}\int_D f_+(x)\mathrm{dx} = 0,$$

also since $m(\{f(x) \geq 1\}) < \infty$, we have

$$\lim_{n\to\infty} m(\{f \geq n\}) = m(\bigcap_{n=k}^{\infty}\{f \geq n\}) = m(\{f = \infty\})$$

Finally, we have $m(\{f(x) = \infty\}) = 0$. Similarly $m(\{f = -\infty\}) = 0$ then $|f(x)| < \infty$ a.e. on D.

(iii) Since $f(x) = g(x)$ a.e. on D, then $f_+(x) = g_+(x)$ and $f_-(x) = g_-(x)$ a.e. on D so that

$$\int_D g(x)\mathrm{dx} = \int_D g_+(x)\mathrm{dx} - \int_D g_-(x)\mathrm{dx} = \int_D f_+(x)\mathrm{dx} - \int_D f_-(x)\mathrm{dx} = \int_D f(x)\mathrm{dx} < \infty.$$

□

Corollary 4.12. *If $f(x) \in L(D)$, then for any measurable $H \subseteq D$ satisfies $f(x) \in L(H)$.*

Proof. Since $|f| \in L(D)$, then

$$\int_D |f(x)|\mathrm{dx} = \int_A |f(x)|\mathrm{dx} + \int_{D\setminus A} |f(x)|\mathrm{dx} < \infty,$$

so that $|f(x)| \in L(A)$ i.e. $f(x) \in L(A)$. □

Corollary 4.13. *If f is bounded on D, where $m(D) < \infty$, then $f \in L(D)$. In particular, if f is continuous on $[a,b]$, then $f \in L([a,b])$.*

Proof. Let $|f(x)| \le M, \forall x \in D$, then we have

$$\int_D |f(x)|\mathrm{dx} \le \int_D M\mathrm{dx} = M \cdot m(D) < \infty.$$

□

Proposition 4.14. *Let $f(x), g(x) \in L(D)$. Then we have*

(i) $f(x) + g(x) \in L(D)$ and

$$\int_D (f(x) + g(x))\,\mathrm{dx} = \int_D f(x)\,\mathrm{dx} + \int_D g(x)\,\mathrm{dx}.$$

(ii) If $\lambda \in \mathbb{R}$, then $\lambda f(x) \in L(D)$, and

$$\int_D \lambda f(x)\,\mathrm{dx} = \lambda \int_D f(x)\,\mathrm{dx}.$$

(iii) If A and B are two disjoint measurable subsets of D, then

$$\int_{A\cup B} f(x)\,\mathrm{dx} = \int_A f(x)\,\mathrm{dx} + \int_B f(x)\,\mathrm{dx}.$$

(iv) For any $\varepsilon > 0$, there exists a simple function $h(x)$ on D taking rational values such that

$$\int_D |f(x) - h(x)|\,\mathrm{dx} < \varepsilon.$$

Proof. (i) Since

$$\int_D |f(x) + g(x)|\mathrm{dx} \le \int_D (|f(x)| + |g(x)|)\mathrm{dx} = \int_D |f(x)|\mathrm{dx} + \int_D |g(x)|\mathrm{dx} < \infty,$$

then $f(x)+g(x) \in L(D)$. Since $(f_+(x)-f_-(x))+(g_+(x)-g_-(x)) = f(x)+g(x) = (f+g)_+(x)-(f+g)_-(x)$ on D and $f_-(x), g_-(x), (f+g)_-(x)$ are finite a.e. on D, then we have

$$f_+(x)+g_+(x)+(f+g)_-(x) = f_-(x)+g_-(x)+(f+g)_+(x) \text{ a.e. on } D.$$

So that

$$\int_D f_+(x)\mathrm{dx} + \int_D g_+(x)\mathrm{dx} + \int_D (f+g)_-(x)\mathrm{dx} = \int_D f_-(x)\mathrm{dx} + \int_D g_-(x)\mathrm{dx} + \int_D (f+g)_+(x)\mathrm{dx}$$

then finally we have

$$\begin{aligned}\int_D (f(x)+g(x))\mathrm{dx} &= \int_D (f+g)_+(x)\mathrm{dx} - \int_D (f+g)_-(x)\mathrm{dx} \\ &= \int_D (f_+(x)-f_-(x))\mathrm{dx} + \int_D (g_+(x)-g_-(x))\mathrm{dx} = \int_D f(x)\mathrm{dx} + \int_D g(x)\mathrm{dx}.\end{aligned}$$

(ii) When $\lambda \geq 0$, we have $(\lambda f(x))_+ = \lambda f_+(x)$ and $(\lambda f(x))_- = \lambda f_-(x)$, when $\lambda < 0$, we have $(\lambda f(x))_+ = (-\lambda) f_-(x)$ and $(\lambda f(x))_- = (-\lambda) f_+(x)$, so that

$$\int_D \lambda f(x)\mathrm{dx} = \int_D (\lambda f(x))_+\mathrm{dx} - \int_D (\lambda f(x))_-\mathrm{dx} = \int_D (-\lambda) f_-(x)\mathrm{dx} - \int_D (-\lambda) f_+(x)\mathrm{dx} = \lambda \int_D f(x)\mathrm{dx}.$$

(iv) First suppose $f(x)$ be non-negative, then there is a sequence of rational-valued non-negative simple measurable functions $h_n(x) \uparrow f(x)$ a.e. on D. Then we have

$$\int_D |f(x)-h_n(x)|\mathrm{dx} = \int_D (f(x)-h_n(x))\mathrm{dx} = \int_D f(x)\mathrm{dx} - \int_D h_n(x)\mathrm{dx} \to 0, n \to \infty,$$

so for every $\epsilon > 0$, there is $N_\epsilon > 0$ s.t.

$$\int_D |f(x) - h_{N_\epsilon}(x)|\mathrm{dx} < \epsilon,$$

Given a general $f(x) \in L(D)$, we have

$$\int_D |f_+(x)-h_1(x)|\mathrm{dx} < \frac{\epsilon}{2}, \ \int_D |f_-(x)-h_2(x)|\mathrm{dx} < \frac{\epsilon}{2},$$

then let $h(x) = h_1(x) - h_2(x)$ we finally have

$$\int_D |f(x)-h(x)|\mathrm{dx} \leq \int_D |f_+(x)-h_1(x)|\mathrm{dx} + \int_D |f_-(x)-h_2(x)|\mathrm{dx} < \epsilon.$$

□

Remark 4.15. *Let $f(x), g(x) \in L(D)$ and $f(x) \leq g(x)$ a.e. on D, then*

$$\int_D g(x)\mathrm{dx} = \int_D (g(x)-f(x))\mathrm{dx} + \int_D f(x)\mathrm{dx} \geq \int_D f(x)\mathrm{dx}.$$

Theorem 4.16 (**Control Convergence Theorem**). *Let f and f_n be measurable functions on a measurable set D. If the following two conditions are satisfied:*

(i) There exists $g \in L(D)$ such that for every $n \geq 1$, $|f_n(x)| \leq g(x)$ a.e. on D;

(ii) $f_n(x) \to f(x)$ a.e. on D.

then $f(x) \in L(D)$, $f_n(x) \in L(D), \forall n \geq 1$ and

$$\int_D f(x)\mathrm{dx} = \lim_{n\to\infty} \int_D f_n(x)\mathrm{dx}.$$

Proof. Since $|f_n(x)| \leq g(x)$ a.e. on D, then $|f(x)| \leq g(x)$ a.e. on D, so that $f_n(x), f(x) \in L(D)$. Note that $g(x) \pm f_n(x) \geq 0$, by 4.10, we have

$$\int_D \liminf_{n\to\infty}(g(x) \pm f_n(x))\mathrm{dx} \leq \liminf_{n\to\infty} \int_D (g(x) \pm f_n(x))\mathrm{dx}$$

then

$$\int_D g(x)\mathrm{dx} \pm \int_D f(x)\mathrm{dx} \leq \liminf_{n\to\infty}(\int_D g(x)\mathrm{dx} \pm \int_D f_n(x)\mathrm{dx}) = \int_D g(x)\mathrm{dx} + \liminf_{n\to\infty}(\pm \int_D f_n(x)\mathrm{dx})$$

so that

$$\pm \int_D f(x)\mathrm{dx} \leq \liminf_{n\to\infty}(\pm \int_D f_n(x)\mathrm{dx})$$

then we have

$$\int_D f(x)\mathrm{dx} \leq \liminf_{n\to\infty} \int_D f_n(x)\mathrm{dx}$$

and

$$\int_D f(x)\mathrm{dx} \geq \limsup_{n\to\infty} \int_D f_n(x)\mathrm{dx}$$

so we have

$$\limsup_{n\to\infty} \int_D f_n(x)\mathrm{dx} \leq \int_D f(x)\mathrm{dx} \leq \liminf_{n\to\infty} \int_D f_n(x)\mathrm{dx}$$

finally we have

$$\int_D f(x)\mathrm{dx} = \lim_{n\to\infty} \int_D f_n(x)\mathrm{dx}.$$

□

Example 4.17. *Let $f \in L([a,b])$, prove:*

$$\frac{\mathrm{d}}{\mathrm{dx}} \int_a^b f(y)sin(xy)\mathrm{dy} = \int_a^b yf(y)cos(xy)\mathrm{dy}.$$

Proof. For each $x \in \mathbb{R}$, $f(y)sin(xy) \in L(D)$ and given $\Delta_n \to 0$ we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^b f(y)sin(xy)\mathrm{dy} = \lim_{n\to\infty}\frac{1}{\Delta_n}\int_a^b f(y)[sin(x+\Delta_n)y - sin(xy)]\mathrm{dy}$$

let

$$f_n(y) = \frac{1}{\Delta_n}f(y)[sin(x+\Delta_n)y - sin(xy)] \to yf(y)cos(xy),\ n\to\infty$$

since $|f_n(y)| \le |yf(y)| \in L(D)$, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^b f(y)sin(xy)\mathrm{dy} = \lim_{n\to\infty}\int_a^b f_n(y)\mathrm{dy} = \int_a^b yf(y)cos(xy)\mathrm{dy}.$$

□

Theorem 4.18 (**Countable Additivity of Integration**). *Let $f \in L(D)$ and let $\{E_k\}_{k\ge1}$ be a partition of D. Then we have*

$$\int_D f(x)\,\mathrm{dx} = \sum_{k=1}^{\infty}\int_{E_k} f(x)\,\mathrm{dx}.$$

Proof. Let $f_n(x) = \sum_{k=1}^n f(x)\chi_{E_k}(x) \to f(x)$ on D and $|f_n(x)| \le \sum_{k=1}^n |f(x)|\chi_{E_k}(x) \le |f(x)|$ for every $n \ge 1$, then by 4.16, we have

$$\int_D f(x)\mathrm{dx} = \lim_{n\to\infty}\int_D f_n(x)\mathrm{dx} = \lim_{n\to\infty}\sum_{k=1}^n\int_D f(x)\chi_{E_k}\mathrm{dx} = \lim_{n\to\infty}\sum_{k=1}^n\int_{E_k} f(x)\mathrm{dx} = \sum_{k=1}^{\infty}\int_{E_k} f(x)\mathrm{dx}.$$

□

Theorem 4.19 (**Absolute Continuity of the Integral**). *Let $f \in L(D)$. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any measurable subset $A \subseteq D$ with $m(A) < \delta$, we have*

$$\left|\int_A f(x)\,\mathrm{dx}\right| < \varepsilon.$$

Proof. Since

$$|\int_D f(x)\mathrm{dx}| \le \int_D |f(x)|\mathrm{dx}$$

We only consider the case when $f \in L(D)$ and non-negative. Let $f_n(x) = f(x)\cdot\chi_{\{f(x)\le n\}} \uparrow f(x)$, by 4.8, we have

$$\lim_{n\to\infty}\int_D f_n(x)\mathrm{dx} = \int_D f(x)\mathrm{dx}$$

so that

$$\lim_{n\to\infty}\int_D (f(x) - f_n(x))\mathrm{dx} = 0$$

For every $\epsilon > 0$, there exists $N > 0$, every $n > N$ we have

$$\int_D (f(x) - f_n(x))\mathrm{dx} < \frac{\epsilon}{2}$$

Let $m(A) < \eta = \frac{\epsilon}{2N}$, then

$$\int_A f(x)\mathrm{dx} = \int_A (f(x) - f_N(x))\mathrm{dx} + \int_A f_N(x)\mathrm{dx} \le \int_D (f(x) - f_n(x))\mathrm{dx} + N \cdot m(A) < \frac{\epsilon}{2} + N \cdot \eta = \epsilon.$$

□

Corollary 4.20. *Let $f \in L((a,b))$, then*

$$F(x) = \int_a^x f(t)\mathrm{dt}$$

is uniformly continuous on (a,b).

4.2 Riemann and Lebesgue integral

In this section, we denote the Riemann integral and the Lebesgue integral of f on $[a,b]$ as

$$(R)\int_a^b f(x)\,\mathrm{dx} \text{ and } (L)\int_a^b f(x)\,\mathrm{dx}.$$

Let f be bounded on $[a,b]$. For each $x \in [a,b]$ and $\delta > 0$, define

$$M_\delta(x) = \sup\{f(y) : y \in (x-\delta, x+\delta) \cap [a,b]\},$$

$$m_\delta(x) = \inf\{f(y) : y \in (x-\delta, x+\delta) \cap [a,b]\}.$$

Here, for a fixed x, $M_\delta(x)$ is monotonically increasing with respect to δ, and $m_\delta(x)$ is monotonically decreasing with respect to δ. Thus, the following two finite limits exist:

$$M_0(x) = \lim_{\delta\to 0^+} M_\delta(x), \quad m_0(x) = \lim_{\delta\to 0^+} m_\delta(x), \quad x \in [a,b].$$

$M_0(x)$ and $m_0(x)$ are called the **Baire upper function** and **Baire lower function** of f, respectively. Clearly, for any $x \in [a,b]$ and $\delta > 0$,

$$m_\delta(x) \le m_0(x) \le f(x) \le M_0(x) \le M_\delta(x).$$

Proposition 4.21 (**Baire**)**.** *Let f be bounded on $[a,b]$, $x_0 \in [a,b]$. Then f be continuous at x_0 if and only if $M_0(x_0) = m_0(x_0)$.*

Proof. ($\Rightarrow$) Since for every $\epsilon > 0$ there exists $\delta > 0$ s.t.

$$f(x_0) - \epsilon \leq m_\delta(x_0) \leq m_0(x_0) \leq f(x_0) \leq M_0(x_0) \leq M_\delta(x_0) \leq f(x_0) + \epsilon,$$

so that

$$f(x_0) - \epsilon \leq m_0(x_0) \leq f(x_0) \leq M_0(x_0) \leq f(x_0) + \epsilon, \forall \epsilon > 0.$$

Let $\epsilon \to 0^+$, we have

$$M_0(x_0) = m_0(x_0) = f(x_0).$$

($\Leftarrow$) Since $m_0(x_0) = f(x_0) = M_0(x_0)$, for every $\epsilon > 0$, there are $\delta_1 > 0$ s.t. $m_0(x_0) - \epsilon < m_{\delta_1}(x_0) \leq m_0(x_0)$ and $\delta_2 > 0$ s.t. $M_0(x_0) \leq M_{\delta_2}(x_0) < M_0(x_0) + \epsilon$, we pick $\delta = min\{\delta_1, \delta_2\} > 0$, so that every $x \in B_\delta(x_0)$ we have

$$0 \leq |f(x) - f(x_0)| \leq max\{|m_\delta(x_0) - f(x_0)|, |M_\delta(x_0) - f(x_0)|\} < \epsilon.$$

$\square$

Proposition 4.22. *Let f be bounded on $[a,b]$. Then $M_0(x), m_0(x)$ are bounded measurable functions so that $M_0(x), m_0(x) \in L[a,b]$.*

Proof. Explicitly $M_0(x), m_0(x)$ are bounded on $[a,b]$. Now we prove they are measurable functions.

We first give a bunch of definition as follows:

$$[a,b] = \bigcup_{k=1}^{2^n} [x_{k-1}^{(n)}, x_k^{(n)}]$$

where

$$x_k^{(n)} = a + \frac{k}{2^n}(b-a), \ 0 \leq k \leq 2^n$$

then let

$$M_k^{(n)} = \sup\{f(x) : x \in [x_{k-1}^{(n)}, x_k^{(n)}]\}$$
$$m_k^{(n)} = \inf\{f(x) : x \in [x_{k-1}^{(n)}, x_k^{(n)}]\}$$
$$\chi_{n,k}(x) = \chi_{[x_{k-1}^{(n)}, x_k^{(n)}]}$$

and

$$U_n(x) = \sum_{k=1}^{2^n} M_k^{(n)} \chi_{n,k}(x)$$

$$L_n(x) = \sum_{k=1}^{2^n} m_k^{(n)} \chi_{n,k}(x)$$

and Daboux sum in Riemann integral,

$$S_n = \frac{b-a}{2^n} \sum_{k=1}^{2^n} M_k^{(n)}$$

$$s_n = \frac{b-a}{2^n} \sum_{k=1}^{2^n} m_k^{(n)}.$$

Now we have the following claim:

Claim 1: $L_n(x) \le f(x) \le U_n(x)$, where $L_n(x), U_n(x)$ are simple functions.
Claim 2: for every $x \in [a,b]$, $\{L_n(x)\}$ monotone increases and $\{U_n(x)\}$ monotone decreases.
Claim 3:

$$s_n = (R)\int_D L_n(x)\mathrm{dx} = (L)\int_D L_n(x)\mathrm{dx} \le (L)\int_D U_n(x)\mathrm{dx} = (R)\int_D U_n(x)\mathrm{dx} = S_n.$$

Claim 4: let $\lim_{n\to\infty} U_n(x) = U(x)$ and $\lim_{n\to\infty} L_n(x) = L(x)$ we have $U(x), L(x)$ to be bounded and measurable functions, so that $U(x), L(x) \in L([a,b])$ and $L(x) \le f(x) \le U(x)$.

Claim 5:

$$\lim_{n\to\infty}(L)\int_a^b L_n(x)\mathrm{dx} = (L)\int_a^b L(x)\mathrm{dx} \le (L)\int_a^b U(x)\mathrm{dx} = \lim_{n\to\infty}(L)\int_a^b U_n(x)\mathrm{dx}.$$

Claim 6: $M_0(x) = U(x)$ and $m_0(x) = L(x)$ a.e. on $[a,b]$.

This is because let $A = \{x_k^{(n)} : 0 \le k \le 2^n \textit{ and } n \ge 1\}, m(A) = 0$, every $x \in [a,b] - A$ satisfies $L(x) = m_0(x)$ and $U(x) = M_0(x)$. □

Lemma 4.23. *Let $m(A) > 0$ and $f \in L(A)$ with $f > 0$ on A, then*

$$(L)\int_A f(x)\mathrm{dx} > 0.$$

Proof. Since $f > 0$ on A, then

$$A = \bigcup_{n=1}^{\infty}\{f(x) \ge \frac{1}{n}\}$$

then there exists n_0 s.t. $m(\{f(x) \ge \frac{1}{n_0}\}) > 0$ finally we have

$$(L)\int_A f(x)\mathrm{dx} \ge (L)\int_{\{f(x)\ge\frac{1}{n_0}\}} f(x)\mathrm{dx} \ge \frac{1}{n_0}\cdot m(\{f(x) \ge \frac{1}{n_0}\}) > 0.$$

□

Theorem 4.24. *Let f be a bounded function on $[a,b]$. Then*

(i) $f \in R[a,b]$ if and only if f is continuous a.e. on $[a,b]$.
(ii) If $f \in R[a,b]$, then $f \in L[a,b]$ and

$$(R)\int_a^b f(x)\mathrm{dx} = (L)\int_a^b f(x)\mathrm{dx}.$$

Proof. (i) ($\Rightarrow$) Note that

$$0 = \lim_{n\to\infty}(S_n - s_n) = \lim_{n\to\infty}(L)\int_a^b (U_n(x) - L_n(x))\mathrm{dx} = (L)\int_a^b U(x)\mathrm{dx} - (L)\int_a^b L(x)\mathrm{dx}.$$

Since $M_0(x) = U(x), m_0(x) = L(x)$ a.e. on $[a,b]$, so that

$$(L)\int_a^b (M_0(x) - m_0(x))\mathrm{dx} = 0$$

Since $M_0(x) - m_0(x) \geq 0$, by 4.23, we have $M_0(x) - m_0(x) = 0$ a.e. on $[a,b]$ so that $f(x)$ is continuous a.e. on $[a,b]$.

($\Leftarrow$) We have $U(x) = M_0(x) = m_0(x) = L(x)$ a.e. on $[a,b]$, then

$$\lim_{n\to\infty}(S_n - s_n) = \lim_{n\to\infty}(L)\int_a^b (U_n(x) - L_n(x))\mathrm{dx} = (L)\int_a^b (U(x) - L(x))\mathrm{dx} = 0, f \in R[a,b].$$

(ii) Since $f \in R[a,b]$, we have

$$\begin{aligned}(R)\int_a^b f(x)\mathrm{dx} &= \lim_{n\to\infty} S_n = \lim_{n\to\infty}(L)\int_a^b L_n(x)\mathrm{dx} = (L)\int_a^b L(x)\mathrm{dx} \leq (L)\int_a^b f(x)\mathrm{dx} \\ &\leq (L)\int_a^b U(x)\mathrm{dx} = \lim_{n\to\infty}(L)\int_a^b U_n(x)\mathrm{dx} = \lim_{n\to\infty} s_n = (R)\int_a^b f(x)\mathrm{dx}.\end{aligned}$$

so that $f \in L[a,b]$ and

$$(R)\int_a^b f(x)\mathrm{dx} = (L)\int_a^b f(x)\mathrm{dx}.$$

□

Example 4.25. *Calculate*

$$(L)\int_0^\infty e^{-x}\mathrm{dx}.$$

Consider $f_n(x) = e^{-x}\chi_{[0,n]} \uparrow e^{-x}$, *by 4.8, we have*

$$(L)\int_0^\infty e^{-x}\mathrm{dx} = \lim_{n\to\infty}(L)\int_0^\infty e^{-x}\chi_{[0,n]}\mathrm{dx} = \lim_{n\to\infty}(R)\int_0^\infty e^{-x}\chi_{[0,n]}\mathrm{dx} = \lim_{n\to\infty}(1 - e^{-n}) = 1.$$

4.3 Example and exercise

Example 4.26. *Let* $m(E) > 0$, $f \in L(E)$, f *be non-negative, and*

$$\int_E f(x)\,dx = 0.$$

Then

$$f(x) = 0 \quad a.e. \;\; on\ E.$$

Proof. We try to prove $m(\{f(x) > 0\}) = 0$, note that

$$\{f(x) > 0\} = \bigcup_{n=1}^{\infty}\{f(x) \geq \frac{1}{n}\}$$

So it is equivalent to prove for every $n \geq 1$ there is $m(\{f \geq \frac{1}{n}\}) = 0$. Since

$$0 = \int_E f(x)\mathrm{dx} \geq \int_{\{f \geq \frac{1}{n}\}} f(x)\mathrm{dx} \geq \frac{1}{n} \cdot m(\{f \geq \frac{1}{n}\}) \geq 0.$$

□

Example 4.27. *Let $f \in L(E)$. Prove that:*

$$k \cdot m(\{|f| > k\}) \to 0 \quad \textit{as } k \to \infty.$$

Proof.

$$k \cdot m(\{|f| > k\}) = \int_{\{|f|>k\}} k \ \mathrm{dx} < \int_{\{|f|>k\}} |f|\mathrm{dx} \leq \int_E |f|\mathrm{dx}.$$

□

Example 4.28. *Let $m(E) < \infty$, and $\{f_k\}$ be a sequence of measurable functions finite a.e. on E. Prove $f_k \Rightarrow 0$ if and only if*

$$\int_E \frac{|f_k(x)|}{1 + |f_k(x)|}\,\mathrm{dx} \to 0 \quad \textit{as } k \to \infty.$$

Proof. ($\Rightarrow$)

For every $\delta > 0$, we have

$$\begin{aligned}\int_E \frac{|f_k(x)|}{1 + |f_k(x)|}\,\mathrm{dx} &= \int_{\{|f_k| \geq \delta\}} \frac{|f_k(x)|}{1 + |f_k(x)|}\,\mathrm{dx} + \int_{\{|f_k|<\delta\}} \frac{|f_k(x)|}{1 + |f_k(x)|}\,\mathrm{dx}\\ &\leq m(\{|f_k| \geq \delta\}) + \delta \cdot m(E) \to \delta \cdot m(E)\ (k \to \infty) \to 0\ (\delta \to 0^+);\end{aligned}$$

($\Leftarrow$)

For every $\delta > 0$, we have

$$\int_E \frac{|f_k(x)|}{1 + |f_k(x)|}\,\mathrm{dx} \geq \int_{\{|f_k| \geq \delta\}} \frac{|f_k(x)|}{1 + |f_k(x)|}\,\mathrm{dx} \geq \frac{\delta}{1+\delta} \cdot m(\{|f_k| \geq \delta\})$$

$$\textit{then } m(\{|f_k| \geq \delta\}) \leq \frac{1+\delta}{\delta} \int_E \frac{|f_k(x)|}{1 + |f_k(x)|}\,\mathrm{dx} \to 0\ (k \to \infty).$$

□

Example 4.29. *Let $f \in L([a,b])$ and $\varepsilon > 0$, where $[a,b]$ is a finite interval. Prove that:*

(i) *There exists a bounded measurable function g such that*

$$\int_a^b |f(x) - g(x)| \,\mathrm{dx} < \varepsilon;$$

(ii) *There exists a continuous function h such that*

$$\int_a^b |f(x) - h(x)| \,\mathrm{dx} < \varepsilon.$$

(iii) *There exists a polynomial P such that*

$$\int_a^b |f(x) - P(x)| \,\mathrm{dx} < \varepsilon;$$

(iv) *There exists a step function S such that*

$$\int_a^b |f(x) - S(x)| \,\mathrm{dx} < \varepsilon.$$

Proof. (i)

Let $f_n(x) = f(x) \cdot \chi_{\{|f| \le n\}}$ then $f_n(x) \to f(x)$ a.e. on $[a,b]$. Since $|f - f_n| \le |f| \in L([a,b])$, we have

$$\lim_{n\to\infty} \int_a^b |f - f_n| \mathrm{dx} = 0;$$

(ii)

For every $\epsilon > 0$, we have a bounded measurable function g s.t.

$$\int_a^b |f - g| \mathrm{dx} < \frac{\epsilon}{2}$$

then since there is $N > 0$ s.t. $|g| < N$, by Lusin theorem, there is $h \in C([a,b])$ with $|h| < N$ and $m(\{h \ne g\}) < \frac{\epsilon}{4N}$, then we have

$$\int_a^b |f - h| \mathrm{dx} \le \int_a^b |f - g| \mathrm{dx} + \int_a^b |g - h| \mathrm{dx} < \frac{\epsilon}{2} + 2N \cdot \frac{\epsilon}{4N} = \epsilon;$$

(iii)

For every $\epsilon > 0$, we have $h \in C([a,b])$ s.t.

$$\int_a^b |f - h| \mathrm{dx} < \frac{\epsilon}{2}$$

by Weierstrass approximation theorem, we have a polynomial P s.t. $||P-h|| < \frac{\epsilon}{2(b-a)}$. Thus,

$$\int_a^b |f-P|\mathrm{dx} \leq \int_a^b |f-h|\mathrm{dx} + \int_a^b |h-P|\mathrm{dx} < \frac{\epsilon}{2} + \int_a^b ||h-P||\mathrm{dx} < \epsilon;$$

(iv)

For every $\epsilon > 0$, we have $h \in C([a,b])$ s.t.

$$\int_a^b |f-h|\mathrm{dx} < \frac{\epsilon}{2}$$

Since $[a,b]$ is bounded, h is uniformly continuous on $[a,b]$, there exists $\delta > 0$ s.t. when $|x-y| < \delta$ then $|h(x)-h(y)| < \frac{\epsilon}{2(b-a)}$. We choose n sufficiently large s.t. $\frac{b-a}{n} < \delta$ then partite $[a,b]$ uniformly into n cuts, where $x_k = a + \frac{k}{n}(b-a),\ 0 \leq k \leq n$. Let $S(x) = \sum_{k=1}^n h(x_k) \cdot \chi_{[x_{k-1},x_k]}$, then

$$\int_a^b |f-S|\mathrm{dx} \leq \int_a^b |f-h|\mathrm{dx} + \int_a^b |h-S|\mathrm{dx} < \frac{\epsilon}{2} + \int_a^b \frac{\epsilon}{2(b-a)}\mathrm{dx} = \epsilon.$$

□

Example 4.30. *Let $f \in L([a,b])$ and $k \in \mathbb{R}$. Prove that as $k \to \infty$:*

(i) ***Riemann-Lebesgue Lemma***

$$\int_a^b f(x)\cos kx\,\mathrm{dx} \to 0, \quad \int_a^b f(x)\sin kx\,\mathrm{dx} \to 0;$$

(ii)

$$\int_a^b f(x)|\cos kx|\,\mathrm{dx} \to \frac{2}{\pi}\int_a^b f(x)\,\mathrm{dx}.$$

Proof. (i)

When $[a,b]$ is bounded. We first suppose f is a polynomial function on $[a,b]$. Then when $\deg f = 0$ i.e. $f \equiv c$, we have

$$\int_a^b c \cdot \cos kx\mathrm{dx} = \frac{c(\sin kb - \sin ka)}{k} \to 0,\ k \to \infty$$

and similarly $\int_a^b c \cdot \sin kx\mathrm{dx} \to 0,\ k \to \infty$. Now we suppose $\deg f \leq n$ satisfy the condition, when $\deg f = n+1$, we have

$$\int_a^b f(x)\cos kx\mathrm{dx} = \frac{f(b)\sin kb - f(a)\sin ka}{k} - \frac{1}{k}\int_a^b f'(x)\sin kx\mathrm{dx} \to 0,\ k \to \infty$$

For $f \in L([a,b])$, for every $\epsilon > 0$, there exists a polynomial P s.t. $\int_a^b |f-P|\mathrm{dx} < \epsilon$. Thus

$$\begin{aligned}|\int_a^b f(x)\cos kx\mathrm{dx}| &\le |\int_a^b (f(x)-P(x))\cos kx\mathrm{dx}| + |\int_a^b P(x)\cos kx\mathrm{dx}| \\ &\le \int_a^b |f(x)-P(x)|\mathrm{dx} + |\int_a^b P(x)\cos kx\mathrm{dx}| < \epsilon \ (for\ some\ k);\end{aligned}$$

When $a = -\infty, b = \infty$, there exists a continuous function g of compactly supported set s.t. $\int_a^b |f-g|\mathrm{dx} < \frac{\epsilon}{2}$. Let $|x| > A$, $g(x) = 0$. Since $g \in L([-A,A])$, there exists an M s.t. when $k > M$ there is $|\int_{-A}^A g(x)\cos kx\mathrm{dx}| < \frac{\epsilon}{2}$. Then when $k > M$, we have

$$|\int_{\mathbb{R}} f(x)\cos kx\mathrm{dx}| \le |\int_{\mathbb{R}}(f(x)-g(x))\cos kx\mathrm{dx}| + |\int_{-A}^A g(x)\cos kx\mathrm{dx}| < \epsilon.$$

(ii)

By 4.29, we only need to consider the case when f is a step function $S(x)$, i,e, there is a partition of $[a,b] : a = x_0 < x_1 < \dots < x_n = b$, s.t. for every $0 \le i \le n-1$, $S(x)|_{(x_i,x_{i+1})} = a_i \in \mathbb{R}$. Note that if $S(x) = 1, x \in [a,b]$ satisfies

$$\int_a^b S(x)|\cos kx|\,\mathrm{dx} \to \frac{2}{\pi}\int_a^b S(x)\,\mathrm{dx}$$

then we have

$$\int_a^b S(x)|\cos kx|\,\mathrm{dx} = \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} a_i|\cos kx|\mathrm{dx} \to \sum_{i=0}^{n-1}\frac{2}{\pi}\int_{x_i}^{x_{i+1}} a_i\mathrm{dx} = \frac{2}{\pi}\int_a^b S(x)\mathrm{dx}, k \to \infty$$

Now we prove

$$\int_a^b |\cos kx|\mathrm{dx} \to \frac{2}{\pi}(b-a), k \to \infty$$

since

$$\lim_{k\to\infty}\int_a^b |\cos kx|\mathrm{dx} = \lim_{k\to\infty}\frac{1}{k}\int_{ka}^{kb}|\cos x|\mathrm{dx} = \lim_{k\to\infty}\frac{1}{k}\frac{k(b-a)}{\pi}\int_0^{\pi}|\cos x|\mathrm{dx} = \frac{2}{\pi}(b-a).$$

□

Example 4.31. *Let $0 < \alpha < 1$. Prove that $x^{-\alpha} \in L([0,1])$ and compute its integral.*

Proof. Since $f(x) = x^{-\alpha} \in C((0,1])$ is a measurable function, $f(x) \in L([0,1])$ if and only if

$$(L)\int_0^1 f(x)\mathrm{dx} < \infty.$$

Consider a sequence $f_k(x) = x^{-\alpha} \cdot \chi_{[\frac{1}{k},1]}(x) \uparrow f(x)$ for every $k \geq 1$. Note that every $f_k(x)$ is continuous on $[0,1]$ except $\{\frac{1}{k}\}$, then

$$(L)\int_0^1 f_k(x)\mathrm{dx} = (R)\int_0^1 f_k(x)\mathrm{dx} = \frac{1}{1-\alpha}(1-(\frac{1}{k})^{1-\alpha}) < \infty$$

By 4.16

$$(L)\int_0^1 f(x)\mathrm{dx} = \lim_{k\to\infty}(L)\int_0^1 f_k(x)\mathrm{dx} = \lim_{k\to\infty}\frac{1}{1-\alpha}(1-(\frac{1}{k})^{1-\alpha}) = \frac{1}{1-\alpha} < \infty.$$

□

Example 4.32. *Let $f \in L(\mathbb{R})$, $f(0) = 0$, and $f'(0)$ exist finitely. Prove that $\frac{f(x)}{x} \in L(\mathbb{R})$.*

Proof. Since $\frac{f(x)}{x}$ is a measurable function on $\mathbb{R}$, $\frac{f(x)}{x} \in L(\mathbb{R})$ if and only if

$$(L)\int_{\mathbb{R}} |\frac{f(x)}{x}|\mathrm{dx} < \infty.$$

Note that

$$f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{f(x)}{x} < \infty$$

then there exists $\delta > 0$ s.t. $|\frac{f(x)}{x} - f'(0)| < 1, \forall 0 < |x| < \delta$, then

$$\begin{aligned}(L)\int_{\mathbb{R}} |\frac{f(x)}{x}|\mathrm{dx} &= (L)\int_{|x|<\delta} |\frac{f(x)}{x}|\mathrm{dx} + (L)\int_{|x|\geq\delta} |\frac{f(x)}{x}|\mathrm{dx} \\ &\leq (1+|f'(0)|)\cdot 2\delta + \frac{1}{\delta}(L)\int_{\mathbb{R}} |f(x)|\mathrm{dx} < \infty.\end{aligned}$$

□

Example 4.33. *Let f be non-negative and measurable on $[0,1]$. If there exists $p > 0$ such that $f^p \in L([0,1])$, prove that for any $q \in (0,p)$, $f^q \in L([0,1])$.*

Proof.

$$\begin{aligned}\int_0^1 f^q(x)\mathrm{dx} &= \int_{f(x)\geq 1} f^q(x)\mathrm{dx} + \int_{f(x)<1} f^q(x)\mathrm{dx} \\ &\leq \int_0^1 f^p(x)\mathrm{dx} + m(\{f(x) < 1\}) < \infty.\end{aligned}$$

□

Example 4.34. *Suppose for every $\lambda \in (a,b)$, $f \in L((a,\lambda))$. Prove that $f \in L((a,b))$ if and only if the limit*

$$\lim_{\lambda \to b^-} \int_a^\lambda |f(x)| dx$$

exists and is finite. Moreover, when this condition holds,

$$\int_a^b f(x) dx = \lim_{\lambda \to b^-} \int_a^\lambda f(x) dx.$$

Proof. For every $\lambda \in (a,b)$, let $h_\lambda(x) = |f(x)| \cdot \chi_{(a,\lambda)}(x) \uparrow |f(x)|,\ \lambda \to b^-$. Then by **??**

$$(L) \int_a^b |f(x)| \mathrm{dx} = \lim_{\lambda \to b^-} \int_a^b |h_\lambda(x)| \mathrm{dx} = \lim_{\lambda \to b^-} \int_a^\lambda |f(x)| \mathrm{dx}.$$

□

Example 4.35. *Let f and f_k (for $k \geq 1$) belong to $L(\mathbb{R})$, with $|f_k(x)| \leq f(x)$. Prove:*

$$\begin{aligned} \int_{\mathbb{R}} \liminf_{k \to \infty} f_k(x) dx &\leq \liminf_{k \to \infty} \int_{\mathbb{R}} f_k(x) dx \\ &\leq \limsup_{k \to \infty} \int_{\mathbb{R}} f_k(x) dx \leq \int_{\mathbb{R}} \limsup_{k \to \infty} f_k(x) dx. \end{aligned}$$

Proof. Let $f_{k+}(x) = \max\{f_k(x), 0\}$ and $f_{k-}(x) = \max\{-f_k(x), 0\}$, we have $f_{k+}(x) \to f_+(x), f_{k-}(x) \to f_-(x), \forall x \in \mathbb{R}$. We now suppose $f_k(x), f(x)$ to be non-negative. By 4.10, we have

$$\int_{\mathbb{R}} \liminf_{k \to \infty} f_k(x) dx \leq \liminf_{k \to \infty} \int_{\mathbb{R}} f_k(x) dx$$

and consider

$$\limsup_{k \to \infty} f_k(x) = \lim_{k \to \infty} \sup_{n \geq k} f_n(x),\ \ \sup_{n \geq k} f_n(x) \leq f(x)$$

by 4.16, we have

$$\int_{\mathbb{R}} \limsup_{k \to \infty} f_k(x) \mathrm{dx} = \lim_{k \to \infty} \int_{\mathbb{R}} \sup_{n \geq k} f_n(x) \mathrm{dx} = \limsup_{k \to \infty} \int_{\mathbb{R}} \sup_{n \geq k} f_n(x) \mathrm{dx} \geq \limsup_{k \to \infty} \int_{\mathbb{R}} f_k(x) \mathrm{dx}.$$

□

Example 4.36. *Let $f \in L(E)$ and $E \subseteq \mathbb{R}$. Prove:*

(i) *The function*

$$F(x) = \int_{(-\infty,x) \cap E} f(t) \mathrm{dt}$$

is uniformly continuous in $x \in \mathbb{R}$;

(ii) *The set*

$$I = \left\{ \int_e f(x) \mathrm{dx} : e \text{ is a measurable subset of } E \right\}$$

is a closed interval, and describe its endpoints.

Proof. (i) For every $\epsilon > 0$, since $f \in L(E)$, by 4.19, there is $\delta > 0$ s.t. $\int_e f(t)\mathrm{dt} < \epsilon$ for every $e \subseteq E$ and $m(e) < \delta$. Then for every $x_1, x_2 \in \mathbb{R}, |x_1 - x_2| < \delta$, suppose $x_1 < x_2$, we have $m((x_1, x_2) \bigcap E) \leq m((x_1, x_2)) = x_2 - x_1 < \delta$, then

$$|F(x_1) - F(x_2)| = |\int_{(x_1,x_2)\bigcap E} f(t)\mathrm{dt}| < \epsilon;$$

(ii) Note that if $f(x)$ is non-negative on E, then by 4.16, we have

$$F(x) = \int_{(-\infty,x)\bigcap E} f(t)\mathrm{dt} \to \begin{cases} \int_E f(t)\mathrm{dt}, x \to \infty \\ 0, x \to -\infty \end{cases}$$

let $E_+ = \{f > 0\}, E_- = \{f < 0\}$ and $F_{E_+}(x), F_{E_-}(x)$, note that

$$I_{E_+} = [0, \int_{E_+} f(t)\mathrm{dt}],\ I_{E_-} = [\int_{E_-} f(t)\mathrm{dt}, 0],\ I_{E_+}, I_{E_-} \subseteq I$$

and for every measurable set $e \subseteq E$, we have

$$\int_{E_-} f(t)\mathrm{dt} \leq \int_e f(t)\mathrm{dt} = (\int_{e\bigcap E_+} + \int_{e\bigcap E_-})f(t)\mathrm{dt} \leq \int_{E_+} f(t)\mathrm{dt}$$

finally, we have

$$I = [\int_{E_-} f(t)\mathrm{dt}, \int_{E_+} f(t)\mathrm{dt}].$$

□

Example 4.37. *Let $\{f_k\}$ be a sequence of non-negative measurable functions on a measurable set E s.t. $f_k \rightrightarrows f$. Prove:*

$$\int_E f(x)\mathrm{dx} \leq \liminf_{k\to\infty} \int_E f_k(x)\mathrm{dx}.$$

Proof. There exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ s.t.

$$\lim_{k\to\infty} \int_E f_{n_k}(x)\mathrm{dx} = \liminf_{k\to\infty} \int_E f_k(x)\mathrm{dx}$$

and

$$f(x) = \lim_{k\to\infty} f_{n_k}(x) \text{ a.e. on } E$$

then we have

$$\int_E f(x)\mathrm{dx} = \int_E \lim_{k\to\infty} f_{n_k}(x)\mathrm{dx} \leq \liminf_{k\to\infty} \int_E f_{n_k}(x)\mathrm{dx} = \lim_{k\to\infty} \int_E f_{n_k}(x)\mathrm{dx} = \liminf_{k\to\infty} \int_E f_k(x)\mathrm{dx}.$$

□

Example 4.38. *Let $\{n_k\}_{k\geq 1}$ be a strictly increasing sequence of positive integers. Prove: the set of points x for which $\{\sin n_k x\}_{k\geq 1}$ converges has measure zero.*

Proof. Since $\sin n_k x$ is a periodic function, we have

$$\{x : \limsup_{k\to\infty} \sin n_k x = \liminf_{k\to\infty} \sin n_k x, x \in \mathbb{R}\} = \bigcup_{l=-\infty}^{\infty} \{\limsup_{k\to\infty} \sin n_k x = \liminf_{k\to\infty} \sin n_k x, x \in [2l\pi, 2(l+1)\pi)\}$$

so now we prove that

$$m(E := \{x : \limsup_{k\to\infty} \sin n_k x = \liminf_{k\to\infty} \sin n_k x, x \in [0, 2\pi)\}) = 0$$

consider

$$f(x) := \begin{cases} \lim_{k\to\infty} \sin n_k x, x \in E \\ 0, x \notin E \end{cases}$$

since $|f(x)| \leq 1, m([0, 2\pi)]) < \infty$, we have $f(x) \in L([0, 2\pi))$, and $f(x)\sin n_k x \to f^2(x), k \to \infty$, then by 4.16 and 4.30 we have

$$\lim_{k\to\infty} \int_0^{2\pi} f(x) \sin n_k x \mathrm{dx} = \int_0^{2\pi} f^2(x) \mathrm{dx} \to 0$$

then there exists zero-measurable set E_0 s.t. $\forall x \in F = E \setminus E_0$, $f(x) = 0$ i.e. $\sin n_k x \to 0$ and $\cos 2n_k x = 1 - 2\sin^2 n_k x \to 1$, then consider $g(x) := \chi_F$, by 4.16 we have

$$0 = \lim_{k\to\infty} \int_0^{2\pi} g(x) \cos 2n_k x \mathrm{dx} = \lim_{k\to\infty} \int_F \cos 2n_k x \mathrm{dx} = \int_F 1 \mathrm{dx} = m(F)$$

finally, we have $m(E) = m(E_0 \bigcup F) = 0$. □

Example 4.39. *Let $m(E) > 0$.*

(i) *If for every $x \in E$, $a_n \cos nx + b_n \sin nx \to 0$ as $n \to \infty$, prove that:*

$$a_n \to 0 \quad \textit{and} \quad b_n \to 0.$$

(ii) *If the series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

converges absolutely on E, prove that:

$$\sum_{n=1}^{\infty} (|a_n| + |b_n|) < \infty.$$

Hint:

$$a_n \cos nx + b_n \sin nx = r_n \cos(nx + \theta_n),$$

where $r_n = \sqrt{a_n^2 + b_n^2}$.

Proof. (i)

Let $f_n(x) = a_n \cos nx + b_n \sin nx$ on E, then $f_n \to 0$ on E. For every $x_1, x_2 \in E$, we have

$$a_n \sin n(x_2 - x_1) = f_n(x_1) \sin nx_2 - f_n(x_1) \sin nx_1 \to 0,\ n \to \infty$$

Since we have $\delta > 0$ s.t. $(-\delta, \delta) \subseteq F = \{x - y : x, y \in E\}$ so that every $x \in (-\delta, \delta)$ has $a_n \sin nx \to 0$, if $a_n \not\to 0$ then we have $\{n_k\}$ s.t. $|a_{n_k}| \geq \epsilon_0 > 0$ so $\sin n_k x \to 0$ for every $x \in (-\delta, \delta)$, by 4.38, contradiction.

□

Example 4.40. *Let $f \in L([a, b])$, and suppose that for every non-negative integer k,*

$$\int_a^b x^k f(x) \mathrm{dx} = 0.$$

Prove:

$$f(x) = 0 \quad a.e.$$

Proof. For every polynomial function $P(x)$, we have $\int_a^b P(x)f(x)\mathrm{dx} = 0$. Given any bounded measurable function $g(x)$, $g(x) \in L([a, b])$ and by 3.20, there exists $h_n(x) \in C([a, b])$ s.t. $m(\{h_n(x) \neq g(x)\}) < \frac{1}{n}$ and $\sup|h_n(x)| \leq \sup|g(x)| < M$. There is a sequence of polynomial functions s.t. $||P_n(x) - h_n(x)|| < \frac{1}{n}$. Now we have

$$\begin{aligned}
|\int_a^b g(x)f(x)\mathrm{dx}| &\leq \int_a^b |P_n(x) - h_n(x)||f(x)|\mathrm{dx} + \int_a^b |h_n(x) - g(x)||f(x)|\mathrm{dx} \\
&\leq \frac{1}{n}\int_a^b |f(x)|\mathrm{dx} + 2M\int_{\{h_n \neq g\}} |f(x)|\mathrm{dx} \to 0, n \to \infty
\end{aligned}$$

now let $f_k(x) = f(x)\chi_{\{|f| \leq k\}} \to f(x)$, then $f_k(x)f(x) = f_k^2(x) \uparrow\to f^2(x)$ and

$$\int_a^b f^2(x)\mathrm{dx} = \lim_{k \to \infty} \int_a^b f_k(x)f(x)\mathrm{dx} = 0$$

then we have

$$f(x) = 0 \text{ a.e. on } [a, b].$$

□

Example 4.41. *Let g be a bounded measurable function on $[-1, 1]$. If for every even continuous function f on $[-1, 1]$,*

$$\int_{-1}^1 f(x)g(x)\mathrm{dx} = 0,$$

prove:

$$g(-x) = -g(x) \quad a.e.$$

Proof. Let $y = -x$ we have $\int_{-1}^{1} f(x)g(-x)\mathrm{dx}$ so that

$$\int_{-1}^{1} f(x)(g(x) + g(-x))\mathrm{dx} = 0, \ \ h(x) := g(x) + g(-x)$$

then by 3.20, there exists $i_k(x) \in C[0,1]$ s.t. $m(\{h \neq i_k\}) < \frac{1}{2k}$ and $\sup|i_k(x)| \leq \sup|h(x)| < M$, and let

$$h_k(x) := \begin{cases} i_k(x), x \in [0,1] \\ i_k(-x), x \in [-1,0) \end{cases}$$

then $h_k(x) \in C[-1,1]$, $m(\{h_k \neq h\}) < \frac{1}{k}$ and $\sup|h_k(x)| \leq \sup|h(x)| < M$. Now we have

$$|\int_{-1}^{1} h^2(x)\mathrm{dx}| \leq \int_{-1}^{1} |h_k(x) - h(x)||h(x)|\mathrm{dx} \leq 2M \int_{\{h_k \neq h\}} |h(x)| \to 0, k \to \infty$$

then we have

$$\int_{-1}^{1} h^2(x)\mathrm{dx} = 0 \text{ i.e. } h(x) = g(x) + g(-x) = 0 \text{ a.e. on } [-1,1].$$

□

Example 4.42. *Let $f \in L(\mathbb{R})$. If for every continuous function g with compact support,*

$$\int_{\mathbb{R}} f(x)g(x)\mathrm{dx} = 0,$$

prove:

$$f(x) = 0 \ \textit{a.e. on } \mathbb{R}.$$

Proof. $\forall k \geq 1$, let

$$f_k(x) = f(x) \cdot \chi_{\{|f| \leq k\}} \cdot \chi_{\{|x| \leq k\}}$$

then $f_k(x)$ is bounded and measurable, by 3.20, for every $n \geq 1$ and $f_k(x)$, there exists a continuous function $g_n(x)$ on $|x| \leq k$ s.t. $|g_n(x)| \leq k, |x| \leq k$ and $m(\{x : |x| \leq k, g_k \neq f_k\}) < \frac{1}{n}$. When $|x| \geq k + \frac{1}{n}$, let $g_n(x) = 0$, then there is a continuous function $g_n(x)$ s.t. $|g_n(x)| \leq k$ and is a continuation of $g_n(x)$ on the closed set $\{|x| \leq k\} \bigcup \{|x| \geq k + \frac{1}{n}\}$ to $\mathbb{R}$. Since $f_k(x)f(x) = f_k^2(x) \uparrow\to f^2(x)$ we have

$$\begin{aligned}
&\int_{\mathbb{R}} f(x)g_n(x)\mathrm{dx} = 0, n \geq 1 \\
&\int_{\mathbb{R}} f^2(x)\mathrm{dx} = \lim_{k\to\infty} \int_{\mathbb{R}} f_k(x)f(x)\mathrm{dx} \\
&|\int_{\mathbb{R}} f_k(x)f(x)\mathrm{dx}| = |\int_{\mathbb{R}} (g_n(x) + f_k(x) - g_n(x))f(x)\mathrm{dx}| \leq \int_{\mathbb{R}} |f_k(x) - g_n(x)||f(x)|\mathrm{dx} \\
&= (\int_{\{k<|x|<k+\frac{1}{n}\}} + \int_{\{|x| \leq k, f_k \neq g_n\}})|f_k(x) - g_n(x)||f(x)|\mathrm{dx} \\
&\leq 2k \int_{\{k<|x|<k+\frac{1}{n}\}} |f(x)|\mathrm{dx} + 2k \int_{\{|x| \leq k, f_k \neq g_n\}} |f(x)|\mathrm{dx} \to 0, n \to \infty.
\end{aligned}$$

□

Example 4.43. *Let $f \in L(\mathbb{R})$. Prove:*

$$\int_{x_1}^{x_2} f(ax+b)dx = \frac{1}{a}\int_{ax_1+b}^{ax_2+b} f(x)dx,$$

where $x_1 < x_2$ and $a \neq 0$.

Proof. We only prove the case when $a > 0$, and $a < 0$ is similar. When $f(x) = \chi_D(x)$, we have

$$\begin{aligned} m((x_1,x_2)\bigcap \frac{1}{a}(D_{-b})) &= \int_{x_1}^{x_2} \chi_{\frac{1}{a}(D_{-b})}(x)\mathrm{dx} = \int_{x_1}^{x_2} \chi_D(ax+b)\mathrm{dx} \\ &= \frac{1}{a}\int_{ax_1+b}^{ax_2+b} \chi_D(x)\mathrm{dx} = \frac{1}{a}m(D\bigcap(ax_1+b, ax_2+b)) \end{aligned}$$

Then for every $|f| \in L(\mathbb{R})$ we have

$$\begin{aligned} \int_{x_1}^{x_2} f(ax+b)dx &= \lim_{n\to\infty}\int_{x_1}^{x_2} f_n(ax+b)\mathrm{dx} \\ &= \frac{1}{a}\lim_{n\to\infty}\int_{ax_1+b}^{ax_2+b} f_n(x)\mathrm{dx} = \frac{1}{a}\int_{ax_1+b}^{ax_2+b} f(x)dx. \end{aligned}$$

□

Example 4.44. *Let $f \in L(\mathbb{R})$. Prove:*

$$\sum_{n=-\infty}^{\infty} f(x+n)$$

converges absolutely almost everywhere.

Proof. Let

$$S(x) = \sum_{n=-\infty}^{\infty} f(x+n)$$

and note that $S(x) = S(x+1)$ for every $x \in \mathbb{R}$. So we only need to prove that

$$\sum_{n=-\infty}^{\infty} |f(x+n)| < \infty, \ a.e. \ on \ [0,1]$$

Consider

$$\int_0^1 \sum_{n=-\infty}^{\infty} |f(x+n)|\mathrm{dx} = \sum_{n=-\infty}^{\infty}\int_0^1 |f(x+n)|\mathrm{dx} = \sum_{n=-\infty}^{\infty}\int_n^{n+1} |f(y)|\mathrm{dy} = \int_{\mathbb{R}} |f(y)|\mathrm{dy} < \infty$$

So

$$\sum_{n=-\infty}^{\infty} |f(x+n)| < \infty, \ a.e. \ on \ [0,1].$$

□

Example 4.45. *Let f be a measurable periodic function on $\mathbb{R}$ with positive period T, and $f \in L([0,T])$. Prove:*

$$\frac{1}{x}\int_0^x f(t)\mathrm{dt} \to \frac{1}{T}\int_0^T f(t)\mathrm{dt},\ x \to \infty.$$

Proof. For every $x > 0$, we have $a_x \in \mathbb{N}, b_x \in [0,T)$ s.t. $x = a_x T + b_x$ where $a_x \uparrow\to \infty$ as $x \to \infty$, then

$$\frac{1}{x}\int_0^x f(t)\mathrm{dt} = \frac{1}{a_x T + b_x}(a_x\int_0^T + \int_0^{b_x})f(t)\mathrm{dt} = \frac{\int_0^T f(t)\mathrm{dt}}{T + \frac{b_x}{a_x}} + \frac{\int_0^{b_x} f(t)\mathrm{dt}}{a_x T + b_x} \to \frac{1}{T}\int_0^T f(t)\mathrm{dt},\ x \to \infty.$$

□

Example 4.46. *Let f be continuous on $\mathbb{R}$, and define*

$$\Delta_n(x) = n\left[f\left(x + \frac{1}{n}\right) - f(x)\right].$$

If for every $x \in \mathbb{R}$, $\Delta_n(x) \to 0$, and there exists a constant M such that $|\Delta_n(x)| \leq M$, prove that f is a constant function.

Proof. By 4.16, for any $a, b \in \mathbb{R}$, we have

$$\lim_{n\to\infty}\int_a^b \Delta_n(x)\mathrm{dx} = \lim_{n\to\infty}\int_a^b n[f(x + \frac{1}{n}) - f(x)]\mathrm{dx} = 0$$

On the other hand, we have

$$\int_a^b n[f(x+\frac{1}{n}) - f(x)]\mathrm{dx} = n\int_a^b f(x+\frac{1}{n})\mathrm{dx} - n\int_a^b f(x)\mathrm{dx} = n\int_{a+\frac{1}{n}}^{b+\frac{1}{n}} f(x)\mathrm{dx} - n\int_a^b f(x)\mathrm{dx}$$
$$= n(\int_b^{b+\frac{1}{n}} - \int_a^{a+\frac{1}{n}})f(x)\mathrm{dx} \to f(b) - f(a),\ n \to \infty.$$

then we have $f(a) = f(b)$ for every $a, b \in$ R. □

Example 4.47. *Let f be defined on a measurable set E. If for every $\varepsilon > 0$, there exist $g_\varepsilon, h_\varepsilon \in L(E)$ such that*

$$g_\varepsilon(x) \leq f(x) \leq h_\varepsilon(x) \quad \textit{and} \quad \int_E [h_\varepsilon(x) - g_\varepsilon(x)]\mathrm{dx} < \varepsilon,$$

prove that $f \in L(E)$.

Proof. Let $r_n(x) = h_n(x) - g_n(x) \geq 0, \forall n \geq 1$, $g_n(x) \leq f(x) \leq h_n(x)$ and $\int_E r_n(x)\mathrm{dx} < 1/n$. Then

$$\int_E \liminf_{n\to\infty} r_n(x)\mathrm{dx} \leq \lim_{n\to\infty} \int_E r_n(x)\mathrm{dx} = 0,$$

then $\liminf_{n\to\infty} r_n(x) = 0$ a.e. on E so that $\liminf_{n\to\infty} h_n(x) = f(x) = \limsup_{n\to\infty} g_n(x)$ a.e. on E. Then $f(x)$ is a measurable function on E. Since $|f(x)| \leq \max\{|g_1(x)|, |h_1(x)|\} \leq |g_1(x)| + |h_1(x)|$, then

$$\int_E |f(x)|\mathrm{dx} \leq \int_E |g_1(x)|\mathrm{dx} + \int_E |h_1(x)|\mathrm{dx} < \infty,$$

finally $f \in L(E)$. □

Example 4.48. *Let $\{f_k\}$ and $\{g_k\}$ be two sequences of measurable functions on a measurable set E with $|f_k(x)| \leq g_k(x)$. Suppose $f_k(x) \to f(x)$ and $g_k(x) \to g(x)$ a.e., and*

$$\int_E g_k(x)\mathrm{dx} \to \int_E g(x)\mathrm{dx} < \infty.$$

Prove:

$$\int_E f_k(x)\mathrm{dx} \to \int_E f(x)\mathrm{dx}.$$

Proof. Since $|f_k(x)| \leq g_k(x), |f(x)| \leq g(x)$ a.e. on E, then $f_k, f \in L(E)$. Since $g_k(x) \pm f_k(x) \geq 0$, by 4.8, we have

$$\begin{aligned}\int_E g(x)\mathrm{dx} + \liminf_{k\to\infty} \pm \int_E f_k(x)\mathrm{dx} &= \liminf_{k\to\infty} \int_E (g_k(x) \pm f_k(x))\mathrm{dx} \\ &\geq \int_E \liminf_{k\to\infty}(g_k(x) \pm f_k(x))\mathrm{dx} = \int_E g(x)\mathrm{dx} + \int_E \liminf_{k\to\infty}(\pm f(x))\mathrm{dx}\end{aligned}$$

then

$$\limsup_{k\to\infty} \int_E f_k(x)\mathrm{dx} \leq \int_E f(x)\mathrm{dx} \leq \liminf_{k\to\infty} \int_E f_k(x)\mathrm{dx}.$$

□

Example 4.49. *Let $\{r_k\}_{k\geq 1} = [0,1] \bigcap \mathbb{Q}$. Prove:*

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k^2\sqrt{|x - r_k|}}$$

converges almost everywhere on $[0,1]$.

Proof.

$$\int_0^1 f(x)\mathrm{dx} = \int_0^1 \sum_{k=1}^{\infty} \frac{1}{k^2\sqrt{|x-r_k|}}\mathrm{dx} = \sum_{k=1}^{\infty} \frac{1}{k^2}\int_0^1 \frac{1}{\sqrt{|x-r_k|}}\mathrm{dx}$$
$$= \sum_{k=1}^{\infty} \frac{2(\sqrt{r_k}+\sqrt{1-r_k})}{k^2} < \infty.$$

□

Example 4.50. *Let $f \in L(\mathbb{R})$, $a > 0$. Prove:*

$$n^{-a} f(nx) \to 0, \ \ a.e.$$

Proof.

$$\int_{\mathbb{R}} \sum_{n=1}^{\infty} \frac{|f(nx)|}{n^a}\mathrm{dx} = \sum_{n=1}^{\infty} \int_{\mathbb{R}} \frac{|f(nx)|}{n^a}\mathrm{dx} = \sum_{n=1}^{\infty} \frac{1}{n^{1+a}} \int_{\mathbb{R}} |f(y)|\mathrm{dy} < \infty.$$

□

Example 4.51. *Let $f \in L([0,1])$, $\int_0^1 f(x)\mathrm{dx} = a$. Prove: $\forall n \in \mathbb{N}^+$, there exists a measurable set $E \subset [0,1]$, $m(E) = \frac{1}{n}$ satisfying*

$$\int_E f(x)\mathrm{dx} = \frac{a}{n}.$$

Proof. Consider

$$F(x) = \int_x^{x+\frac{1}{n}} f(x)\mathrm{dx} \in C([0, 1-\frac{1}{n}])$$

then since

$$F(0) + F(\frac{1}{n}) + ... + F(\frac{n-1}{n}) = \int_0^1 f(x)\mathrm{dx} = a$$

there exist x_1, x_2 s.t. $F(x_1) \le \frac{a}{n}$ and $F(x_2) \ge \frac{a}{n}$, so that there is x_3 s.t. $F(x_3) = \frac{a}{n}$ i.e.

$$\int_{x_3}^{x_3+\frac{1}{n}} f(x)\mathrm{dx} = \frac{a}{n}.$$

□

Example 4.52. *Let $\{E_k\}_{1\le k\le n}$ be n measurable subsets of $[0,1]$. If every point in $[0,1]$ belongs to at least q of these sets, prove: $\exists E_{k_0} \in \{E_k\}_{1\le k\le n}$ satisfies $m(E_{k_0}) \ge q/n$.*

Proof.

$$m(E_1) + ... + m(E_n) = \int_0^1 \chi_{E_1} + ... + \chi_{E_n} \mathrm{dx} \geq \int_0^1 q\mathrm{dx} = q.$$

□

Example 4.53. *Let f be non-negative and measurable on $[0,1]$. If there exists a non-negative integer k such that*

$$\int_0^1 f^k(x)\mathrm{dx} = \int_0^1 f^{k+1}(x)\mathrm{dx} = \int_0^1 f^{k+2}(x)\mathrm{dx} < \infty,$$

prove: there exists a measurable set $E \subset [0,1]$ such that $f(x) = \chi_E(x)$ a.e.

Proof. Consider

$$\int_0^1 f^k(x)(f(x)-1)^2\mathrm{dx} = \int_0^1 (f^{k+2}(x) - 2f^{k+1}(x) + f^k(x))\mathrm{dx} = 0,$$

then $f^k(x)(f(x)-1) = 0$ a.e. on $[0,1]$, let $E = \{f = 1\}$, then $f(x) = \chi_E(x)$ a.e. on $[0,1]$. □

Example 4.54. *Let $f \in L([0,1])$ be strictly positive, $0 < \lambda < 1$. Prove:*

$$\inf\left\{\int_E f(x)\mathrm{dx} : E \subseteq [0,1], m(E) \geq \lambda\right\} > 0.$$

Proof. Since $f(x) > 0, \forall x \in [0,1]$, $\{f = 0\} = \bigcap_{n=1}^{\infty}\{f < 1/n\}$ then $0 = m(\bigcap_{n=1}^{\infty}\{f < 1/n\}) = \lim_{n\to\infty} m(\{f < 1/n\})$, then $\exists n_0$ s.t. $m(\{f < 1/n_0\}) < \lambda/2$ so that for every $E \subseteq [0,1], m(E) \geq \lambda$, $m(E \setminus \{f < 1/n_0\}) > \lambda/2$. Then

$$\int_E f(x)\mathrm{dx} \geq \int_{E\setminus\{f<1/n_0\}} f(x)\mathrm{dx} \geq \frac{1}{n_0} \cdot m(E \setminus \{f < 1/n_0\}) > \frac{\lambda}{2n_0}.$$

□

Example 4.55. *Let f be a non-negative integrable function on $[a,b]$, $\{F_\lambda\}_{\lambda\in\Lambda}$ be a family of closed subsets of $[a,b]$ s.t. for any $\lambda_1, \lambda_2 \in \Lambda$, either $F_{\lambda_1} \subseteq F_{\lambda_2}$ or $F_{\lambda_1} \supseteq F_{\lambda_2}$. If for every $\lambda \in \Lambda$,*

$$\int_{F_\lambda} f(x)\mathrm{dx} \geq 1,$$

prove:

$$\int_{\bigcap_{\lambda\in\Lambda} F_\lambda} f(x)\mathrm{dx} \geq 1.$$

Proof. Suppose $\int_{\bigcap_{\lambda\in\Lambda} F_\lambda} f(x)\mathrm{dx} < 1$, then by 4.19, $\exists G \supseteq \bigcap_{\lambda\in\Lambda} F_\lambda$ open in $[a,b]$ s.t. $\int_G f(x)\mathrm{dx} < 1$. Since $\bigcup_{\lambda\in\Lambda}([a,b]\setminus F_\lambda) \supseteq [a,b]\setminus G$ and $([a,b],\mathbb{R}_{\mathbb{B}})$ is a compact space, then $\exists \bigcup_{k=1}^{s}([a,b]\setminus F) \supseteq [a,b]\setminus G$ so that $\bigcap_{k=1}^{s} F_k \subseteq G$, $\exists i, \bigcap_{k=1}^{s} F_k = F_i$ then $\int_{F_i} f(x)\mathrm{dx} < 1$, a contradiction. □

Example 4.56. *Let $f \in L([0,1])$ and $0 < \lambda < 1$. If for every measurable subset $E \subset [0,1]$ with $m(E) = \lambda$, we have*

$$\int_E f(x)\mathrm{dx} = 0,$$

prove:

$$f(x) = 0 \quad a.e. \ on \ [0,1].$$

Proof. Suppose $m(\{f > 0\}) > 0$ then $m(\{f < 0\}) > 0$. Let $A_1 \subseteq \{f > 0\}, A_2 \subseteq \{f < 0\}$ and $0 < m(A_1) = m(A_2) < \min\{\lambda, 1-\lambda\}$, then take $S \subseteq [0,1]\setminus(A_1 \bigcup A_2)$ s.t. $m(S)+m(A_i) = \lambda, i = 1,2$. Now we have

$$0 = \int_{S\bigcup A_1} f(x)\mathrm{dx} - \int_{S\bigcup A_2} f(x)\mathrm{dx} = \int_{A_1} f(x)\mathrm{dx} - \int_{A_2} f(x)\mathrm{dx} > 0.$$

□

Example 4.57. *In 4.56, if the condition "any set E with measure λ" is replaced with "any open set E with measure λ", what happens to the conclusion?*

Proof. The conclusion still holds. For every $m(E) = \lambda$, we have a sequence of open sets G_n s.t. $m(G_n) = \lambda$ and $m(G_n \Delta E) < \frac{1}{n}$. Then, we have

$$\int_E f(x)\mathrm{dx} = \int_{G_n} f(x)\mathrm{dx} + \int_{E\setminus G_n} f(x)\mathrm{dx} - \int_{G_n\setminus E} f(x)\mathrm{dx} \to 0, \ n \to \infty.$$

□

Example 4.58. *Let $m(E) < \infty$, f be non-negative and measurable on E. Prove that the following three propositions are equivalent:*

(i) $f \in L(E)$;
(ii) $\sum_{k=1}^{\infty} 2^k m(\{f \geq 2^k\}) < \infty$;
(iii) $\sum_{k=1}^{\infty} m(\{f \geq k\}) < \infty$.

Proof. (i) → (ii) Let $E_n := \{2^n \le f(x) < 2^{n+1}\}, n \ge 1$, then

$$\sum_{k=1}^{\infty} 2^k m(\{f \ge 2^k\}) = \sum_{k=1}^{\infty}\sum_{l=k}^{\infty} 2^k m(E_l) = \sum_{l=1}^{\infty}\sum_{k=1}^{l} 2^k m(E_l) \le \sum_{l=1}^{\infty} 2^{l+1} m(E_l)$$
$$\le 2\sum_{l=1}^{\infty}\int_{E_l} f(x)\mathrm{dx} = 2\int_E f(x)\mathrm{dx} < \infty;$$

(ii) → (iii) Since $m(\{f \ge 1\}) \le m(E) < \infty$, we now consider

$$\sum_{k=2}^{\infty} m(\{f \ge k\}) \le \sum_{l=1}^{\infty}\sum_{k=2^l}^{2^{l+1}-1} m(\{f \ge 2^l\}) = \sum_{l=1}^{\infty} 2^l m(\{f \ge 2^l\}) < \infty;$$

(iii) → (i) Let $F_k := \{k \le f(x) < k+1\}, k \ge 0$, then we have

$$\int_E f(x)\mathrm{dx} = \sum_{k=0}^{\infty}\int_{F_k} f(x)\mathrm{dx} \le \sum_{k=0}^{\infty}(k+1)m(F_k) = \sum_{k=0}^{\infty} k\cdot m(F_k) + m(E) = \sum_{k=1}^{\infty} m(\{f \ge k\}) + m(E) < \infty.$$

□

Example 4.59. *Let f be non-negative and measurable on a measurable set E with $m(E) = \infty$. Prove that the following two conditions are necessary for $f \in L(E)$:*

(i) $\sum_{k=1}^{\infty} m(\{f \ge k\}) < \infty$;
(ii) $\sum_{k=1}^{\infty} \frac{1}{2^k} m\left(\left\{f \ge \frac{1}{2^k}\right\}\right) < \infty$.
Is (i) *or* (ii) *a sufficient condition for $f \in L(E)$?*

Proof. (i) Let $F_k := \{k \le f(x) < k+1\}, k \ge 0$, then we have

$$\sum_{k=1}^{\infty} m(\{f \ge k\}) = \sum_{k=1}^{\infty}\sum_{l=k}^{\infty} m(F_l) = \sum_{k=1}^{\infty} k\cdot m(F_k) = \sum_{k=1}^{\infty}\int_{F_k} k\mathrm{dx} \le \sum_{k=1}^{\infty}\int_{F_k} f(x)\mathrm{dx} \le \int_E f(x)\mathrm{dx} < \infty$$

then let $g \equiv \frac{1}{2}$ on E, then

$$\int_E g(x)\mathrm{dx} = \int_E \frac{1}{2}\mathrm{dx} = \frac{1}{2}m(E) = \infty$$

so that $g(x) \notin L(E)$, but

$$\sum_{k=1}^{\infty} m(\{f \ge k\}) = 0 < \infty;$$

(ii) Let $E_k := \{2^k \le f(x) < 2^{k+1}\}, -\infty < k < \infty$, then we have

$$\sum_{k=1}^{\infty}\frac{1}{2^k}m(\{f \ge \frac{1}{2^k}\}) = \sum_{k=1}^{\infty}\frac{1}{2^k}m(\{f \ge \frac{1}{2}\}) + \sum_{k=2}^{\infty}\frac{1}{2^k}\sum_{l=-k}^{-2} m(E_l) = m(\{f \ge \frac{1}{2}\}) + \sum_{k=2}^{\infty}\frac{1}{2^{k-1}}m(E_{-k})$$
$$= 2\cdot\frac{1}{2}m(\{f \ge \frac{1}{2}\}) + 2\cdot\sum_{k=2}^{\infty}\frac{1}{2^k}m(E_{-k})$$
$$\le 2\int_{\{f \ge \frac{1}{2}\}} f(x)\mathrm{dx} + 2\int_{\{0<f<\frac{1}{2}\}} f(x)\mathrm{dx} = 2\int_E f(x)\mathrm{dx} < \infty$$

then let $h(x) = \frac{1}{x} \cdot \chi_{(0,1]}$, then

$$\int_{\mathbb{R}} h(x)\mathrm{dx} = \int_0^1 \frac{1}{x}\mathrm{dx} = \infty$$

but we also have

$$\sum_{k=1}^{\infty} \frac{1}{2^k} m(\{f \geq \frac{1}{2^k}\}) = m(\{f \geq \frac{1}{2}\}) + \sum_{k=2}^{\infty} \frac{1}{2^{k-1}} m(E_{-k}) = m(\{f \geq \frac{1}{2}\}) = 1 < \infty.$$

□

Example 4.60. *Let f and $f_k(k \geq 1)$ be non-negative functions in $L(E)$, and further, $f_k(x) \to f(x)$ a.e. and $\int_E f_k(x)dx \to \int_E f(x)dx$. Prove: for any measurable subset $e \subseteq E$, $\int_e f_k(x)dx \to \int_e f(x)dx$.*

Proof. Consider $e_k(x) = f_k(x) \cdot \chi_e(x)$ for every $k \geq 1$, then $f_k(x) \cdot \chi_e(x) \to f(x) \cdot \chi_e(x)$ on E. Since $|e_k(x)| \leq f_k(x)$ for every $k \geq 1$, by 4.48 we have

$$\int_e f_k(x)\mathrm{dx} = \int_E e_k(x)\mathrm{dx} \to \int_E f(x) \cdot \chi_e(x)\mathrm{dx} = \int_e f(x)\mathrm{dx},\ k \to \infty.$$

□

Example 4.61. *Let f and $f_k(k \geq 1)$ be functions in $L(R)$, and for any measurable set E, $\left\{\int_E f_k(x)dx\right\}_{k\geq 1}$ converges monotonically and increasingly to $\int_E f(x)dx$. Prove: $f_k(x) \to f(x)$ a.e.*

Proof. Suppose there is a positive measurable set D s.t. $f_k(x) > f_{k+1}(x)$ for every $x \in D$ and some $k \geq 1$. Then

$$\int_D f_k(x)\mathrm{dx} > \int_D f_{k+1}(x)\mathrm{dx}$$

contradiction. So $f_k(x) \uparrow g(x) \leq f(x)$ for a.e. on $\mathbb{R}$. If $m(E := \{g(x) < f(x)\}) > 0$, then

$$\int_E f_k(x)\mathrm{dx} \leq \int_E g(x)\mathrm{dx} < \int_E f(x)\mathrm{dx}$$

contradiction. Finally, we have $f_k(x) \to f(x)$ a.e. on $\mathbb{R}$. □

Example 4.62. *Let $\delta > 0$, $\{\alpha_k\}$ be a sequence of non-negative real numbers, $E_k \subseteq [a,b]$, $m(E_k) \geq \delta$, $k = 1, 2, \cdots$. If $\sum_k \alpha_k \chi_{E_k}(x) < \infty$ a.e., prove: $\sum_k \alpha_k < \infty$.*

Proof. Let $f(x) := \sum_k \alpha_k \chi_{E_k}(x)$, since $f(x) < \infty$ a.e. on $[a,b]$, then $m(\{f \geq n\}) \to 0, n \to \infty$, there is M s.t. $m(\{f \geq M\}) < \frac{\delta}{2}$, we take $E = \{f \geq M\}$. Now we have $f(x) < M$ in $[a,b] \setminus E$ and $m(E) < \frac{\delta}{2}$. Then

$$\infty > \int_{[a,b]\setminus E} f(x)\mathrm{dx} = \int_{[a,b]\setminus E} \sum_k \alpha_k \chi_{E_k}(x)\mathrm{dx} = \sum_k \int_{[a,b]\setminus E} \alpha_k \chi_{E_k}(x)\mathrm{dx}$$
$$= \sum_k \alpha_k \cdot m(E_k \bigcap([a,b] \setminus E)) \geq \frac{\delta}{2} \sum_k \alpha_k.$$

□

Example 4.63. *Let $f, g \in R[a,b]$ and equal on a dense subset of $[a,b]$. Prove: their integrals over $[a,b]$ are equal, and $f(x) = g(x)$ a.e. on $[a,b]$.*

Proof. Since $f, g \in R[a,b]$, there is a zero-measurable set $E_0 \subseteq [a,b]$ s.t. $f, g \in C([a,b] \setminus E_0)$. Let S be the dense subset, then for every $x \in [a,b] \setminus E_0$, there is a sequence $s_n \to x$ in S,so that $f(x) = \lim_{n\to\infty} f(s_n) = \lim_{n\to\infty} g(s_n) = g(x)$ i.e. we have $f(x) = g(x)$ a.e. on $[a,b]$ and then

$$\int_a^b f(x)\mathrm{dx} = \int_a^b g(x)\mathrm{dx}.$$

□

Example 4.64. *For each $x \in \mathbb{R}$, let $f(x,y)$ be integrable as a function of y on $[a,b]$; and for each $y \in [a,b]$, let $f(x,y)$ be differentiable as a function of x on $\mathbb{R}$. Further, suppose there exists $g \in L([a,b])$ such that for any $x \in \mathbb{R}$ and $y \in [a,b]$,*

$$\left| \frac{\mathrm{d}}{\mathrm{dx}} f(x,y) \right| \leq g(y).$$

Prove: for any $x \in \mathbb{R}$,

$$\frac{\mathrm{d}}{\mathrm{dx}} \int_a^b f(x,y)\mathrm{dy} = \int_a^b \frac{\mathrm{d}}{\mathrm{dx}} f(x,y)\mathrm{dy}.$$

Proof. For every $x_0 \in \mathbb{R}$, let $f_n(y) = n(f(x_0 + \frac{1}{n}, y) - f(x_0, y)), n \geq 1$, then $f_n(y) \to \frac{\mathrm{d}}{\mathrm{dx}} f(x_0, y)$. Since $|f_n(y)| = |n \cdot \frac{\mathrm{d}}{\mathrm{dx}} f(x_0 + \frac{\theta_n}{n}, y) \cdot \frac{1}{n}| \leq g(y) \in L([a,b])$, then by 4.16, we have

$$\frac{\mathrm{d}}{\mathrm{dx}} \int_a^b f(x_0,y)\mathrm{dy} = \lim_{n\to\infty} \int_a^b f_n(y)\mathrm{dy} = \int_a^b \lim_{n\to\infty} f_n(y)\mathrm{dy} = \int_a^b \frac{\mathrm{d}}{\mathrm{dx}} f(x_0,y)\mathrm{dy}, \forall x_0 \in \mathbb{R}.$$

□

Example 4.65. *Let f and g be non-negative measurable functions on a measurable set E, both finite almost everywhere. Define*

$$E_y = \{g(x) \geq y\}, \quad F(y) = \int_{E_y} f(x)\mathrm{dx}$$

Prove:

$$\int_0^\infty F(y)\mathrm{dy} = \int_E f(x)g(x)\mathrm{dx}.$$

Proof. Since $f(x) \cdot \chi_{\{g \geq y\}}(x)$ is non-negative measurable function on E^2, then by Fubini theorem, we have

$$\begin{aligned}\int_0^\infty F(y)\mathrm{dy} &= \int_0^\infty \int_{E_y} f(x)\mathrm{dxdy} = \int_0^\infty \int_E f(x) \cdot \chi_{\{g \geq y\}}(x)\mathrm{dxdy} \\ &= \int_E f(x) \int_0^\infty \chi_{\{g \geq y\}}(x)\mathrm{dydx} = \int_E f(x) \int_0^{g(x)} 1\mathrm{dydx} = \int_E f(x)g(x)\mathrm{dx}.\end{aligned}$$

□

Example 4.66. *Let $f(x)$ and $g(x)$ be continuous on $[a,b]$ with $f(x) \leq g(x)$. Define $E = \{(x,y) : x \in [a,b], f(x) \leq y \leq g(x)\}$, and let $h(x,y)$ be integrable on E. Prove:*

$$\int_E h(x,y)\mathrm{dxdy} = \int_a^b \mathrm{dx} \int_{f(x)}^{g(x)} h(x,y)\mathrm{dy}.$$

Proof. Since $h(x,y) \in L(E)$, by Fubini theorem, we have

$$\int_E h(x,y)\mathrm{dxdy} = \int_a^b \mathrm{dx} \int_{f(x)}^{g(x)} h(x,y)\mathrm{dy}.$$

□

Example 4.67. *Let $f \in L(R)$. Prove: if either of the following two conditions is satisfied, then $f(x) = 0$ a.e.*

(i) For any open set G with measure 1, $\int_G f(x)\mathrm{dx} = 0$.
(ii) For any open set G, $\int_G f(x)\mathrm{dx} = \int_{\overline{G}} f(x)\mathrm{dx}$.

Proof. (i) For any $x \in \mathbb{R}$,

$$\int_{-\infty}^x f(x)\mathrm{dx} = \sum_{n=1}^\infty \int_{x-n}^{x-(n-1)} f(x)\mathrm{dx} = 0,$$

then $\int_G f(x)\mathrm{dx} = 0$, G open. Given any measurable set E, $\exists G_n, n \geq 1$, $G_n \subseteq G_{n-1} \subseteq E, m(G_n \setminus E) < \frac{1}{n}$, $m(\bigcap_{n=1}^{\infty} G_n) = m(E)$. $f_n(x) = f(x)\chi_{G_n} \to f(x)\chi_{\bigcap G_n}$, then

$$\int_E f(x)\mathrm{dx} = \int_{\bigcap G_n} f(x)\mathrm{dx} = \int_{\mathbb{R}} \lim_{n\to\infty} f_n(x)\mathrm{dx} = \lim_{n\to\infty} \int_{G_n} f(x)\mathrm{dx} = 0.$$

(ii) Given any open set G, $\overline{G} = \overline{G \bigcap \mathbb{Q}}$, $m(G \bigcap \mathbb{Q}) = 0$, for any $\epsilon > 0$, $\exists \delta > 0$, $\exists G \supseteq G' \supseteq G \bigcap \mathbb{Q}$ open, $m(G') < \delta$, so that

$$|\int_G f(x)\mathrm{dx}| = |\int_{\overline{G}} f(x)\mathrm{dx}| = |\int_{\overline{G \bigcap \mathbb{Q}}} f(x)\mathrm{dx}| = |\int_{\overline{G'}} f(x)\mathrm{dx}| = |\int_{G'} f(x)\mathrm{dx}| < \epsilon.$$

□

Example 4.68. *Let $\{A_n\}_{n\geq 1}$ be a sequence of measurable sets with $\sum_{n=1}^{\infty} m(A_n) < \infty$. Define*

$$G_k = \{x : \textit{exactly } k \textit{ sets in } \{A_n\}_{n\geq 1} \textit{ contain } x\}, \quad k = 1, 2, \cdots.$$

Prove: G_k is measurable and

$$\sum_{k=1}^{\infty} k \cdot m(G_k) = \sum_{n=1}^{\infty} m(A_n).$$

Proof. When $k = 1$, we have

$$G_1 = \bigcup_{n=1}^{\infty} A_n - \bigcup_{1\leq i_1 < i_2} (A_{i_1} \bigcap A_{i_2})$$

so that G_1 is measurable. Suppose G_k is measurable for all $k \leq n-1$. Then $k = n$ we have

$$G_n = \bigcup_{k=1}^{\infty} A_k - \bigcup_{l=1}^{n-1} G_l - \bigcup_{1\leq i_1 < \ldots < i_{n+1}} (A_{i_1} \bigcap \ldots \bigcap A_{i_{n+1}})$$

so that G_n is measurable. Now we have G_k be a measurable set for all $k \geq 1$. Note that $m(A_n) \to 0, n \to \infty$ so that $m(\bigcap_{n=1}^{\infty} A_n) = 0$ and

$$\bigcup_{n=1}^{\infty} A_n = \sum_{n=1}^{\infty} G_n + \bigcap_{n=1}^{\infty} A_n$$

then we have

$$\sum_{n=1}^{\infty} m(A_n) = \sum_{n=1}^{\infty} m(A_n - \bigcap_{l=1}^{\infty} A_l) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} m(A_n \bigcap G_k) = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} m(A_n \bigcap G_k) = \sum_{k=1}^{\infty} k \cdot m(G_k).$$

□

Example 4.69. *Let $f \in L([0,1])$, $E = \{x \in [0,1] : f(x) \textit{ is an integer}\}$. Prove:*

$$\int_0^1 |\cos[\pi f(x)]|^n \mathrm{dx} \to m(E), \ n \to \infty.$$

Proof. Let $g_n(x) = |\cos[\pi f(x)]|^n, n \geq 1$, we have $g_n(x) \to \chi_E$ on $[0,1]$ and $|g_n(x)| \leq 1 \in L([0,1]), n \geq 1$, then by 4.16 we have

$$\lim_{n\to\infty} \int_0^1 g_n(x)\mathrm{dx} = \int_0^1 \lim_{n\to\infty} g_n(x)\mathrm{dx} = \int_0^1 \chi_E(x)\mathrm{dx} = m(E).$$

□

Example 4.70. *Let $f \in L([0,1])$ be non-negative, $\int_0^1 f(x)\mathrm{dx} > 0$, $0 < \alpha \leq 1$. Find the limit*

$$\lim_{n\to\infty} \int_0^1 n \ln\left[1 + \left(\frac{f(x)}{n}\right)^\alpha\right] \mathrm{dx}.$$

Proof. When $\alpha = 1$, let $f_n(x) = n\ln[1 + \frac{f(x)}{n}] \to f(x), n \to \infty$, $|f_n(x)| \leq |f(x)| \in L([0,1])$, then by 4.16, we have

$$\lim_{n\to\infty} \int_0^1 f_n(x)\mathrm{dx} = \int_0^1 f(x)\mathrm{dx}.$$

When $0 < \alpha < 1$, let $g_n(x) = n^\alpha \ln[1 + (\frac{f(x)}{n})^\alpha] \to f^\alpha(x)$, $|g_n(x)| \leq f^\alpha(x)$, then by 4.16, we have

$$\lim_{n\to\infty} \int_0^1 g_n(x)\mathrm{dx} = \int_0^1 f^\alpha(x)\mathrm{dx}$$

so that

$$\lim_{n\to\infty} n^{1-\alpha} \int_0^1 g_n(x)\mathrm{dx} = \infty.$$

□

Example 4.71. *Let $f \in L([0,1])$. Does the limit*

$$\lim_{n\to\infty} \frac{1}{n} \int_0^1 \ln(1 + e^{nf(x)})\mathrm{dx}$$

exist? If it exists, find this limit.

Proof. Let $g_n(x) = \frac{1}{n}\ln(1 + e^{nf(x)}), n \geq 1$. When $x \in \{f(x) \leq 0\}$, we have $g_n(x) \to 0$, when $x \in \{f(x) > 0\}$, we have

$$f(x) = \frac{1}{n}\ln(e^{nf(x)}) \leq g_n(x) \leq \frac{1}{n}\ln(2e^{nf(x)}) = f(x) + \frac{\ln 2}{n}, n > N$$

so that $g_n(x) \to f(x), n \to \infty$. Also, for every $n \geq 1$, $|g_n(x)| \leq \frac{1}{n}\ln 2 \vee \frac{1}{n}\ln(2e^{nf(x)}) \leq \frac{1}{n}\ln 2 + |f(x)| \in L([0,1])$. Then by 4.16, we have

$$\lim_{n\to\infty} \int_0^1 g_n(x)\mathrm{dx} = \int_0^1 \lim_{n\to\infty} g_n(x)\mathrm{dx} = \int_{\{f>0\}} f(x)\mathrm{dx}.$$

□

5 Differentiation and integration

5.1 Monotonic function

Definition 5.1. *We say $E \subseteq \mathbb{R}$ is **Vitali covered** by a family of intervals Λ if for every $x \in E$ and $\epsilon > 0$, there exists $I \in \Lambda$ s.t. $x \in I$ and $m(I) < \epsilon$. ($\{x\}$ is not regarded as interval here.)*

Lemma 5.2 (Vitali). *If $E \subseteq \mathbb{R}$ with $m^*(E) < \infty$ is Vitali covered by Λ, then for every $\epsilon > 0$, there are finite many pairwise disjoint intervals $\{I_k\}_{1 \le k \le n}$ s.t.*

$$m^*(E - \bigcup_{k=1}^{n} I_k) < \epsilon.$$

Proof. Since $m^*(E) < \infty$, we have an open set $G \subseteq \mathbb{R}$ s.t. $E \subseteq G$ and $m(G) < \infty$. Let

$$\Lambda_0 := \{I \in \Lambda : \overline{I} \subseteq G\}$$

We claim that Λ_0 is also a Vitali cover of E. For every $x \in E \subseteq G$, there exists $\delta_x > 0$ s.t. $[x - \delta_x, x + \delta_x] \subseteq G$, then for any $\epsilon > 0$, since Λ is a Vitali cover of E, there is $I \in \Lambda$ satisfying $x \in I$ and $m(I) < \min\{\delta_x, \epsilon\} \le \epsilon$ so that $\overline{I} \subseteq [x - \delta_x, x + \delta_x] \subseteq G$ then $I \in \Lambda_0$. We have $\Lambda_0 \subseteq \Lambda$ be a Vitali cover of E.

Case 1: There is pairwise disjoint $\{I_k\}_{1 \le k \le n}$ in Λ_0 s.t. $E - \bigcup_{k=1}^{n} \overline{I}_k = \emptyset$.

Case 2: Any finite pairwise disjoint $\{I_k\}_{1 \le k \le n}$ in Λ_0 s.t.$E - \bigcup_{k=1}^{n} \overline{I}_k \neq \emptyset$. We pick $I_1 \in \Lambda_0$ and $E - \overline{I_1} \neq \emptyset$. Let $\Lambda_1 := \{I \in \Lambda_0 : \overline{I} \bigcap \overline{I_1} = \emptyset\} \neq \emptyset$. Consider $0 < M := \sup\{\ell(I) : I \in \Lambda_1\} \le m(G) < \infty$, then there is $I_2 \in \Lambda_1$ s.t. $\ell(I_2) > M/2$ so that$\ell(I) < 2\ell(I_2),\ \forall I \in \Lambda_1$. Then $E - \bigcup_{k=1}^{2} \overline{I}_k \neq \emptyset$, let $\Lambda_2 := \{I \in \Lambda_1 : \overline{I} \bigcap \overline{I_2} = \emptyset\} \neq \emptyset$, similarly we have $I_3 \in \Lambda_2$ s.t. $\ell(I) < 2\ell(I_3),\ \forall I \in \Lambda_2$. Follow this process, finally we gain a sequence of pairwise disjoint intervals $\{I_i\}_{i=1}^{\infty}$ and each $i \ge 1$ satisfies $\ell(I) < \ell(I_{i+1}),\ \forall I \in \Lambda_i$. Since $\sum_{i=1}^{\infty} \ell(I_i) = m(\bigsqcup_{i=1}^{\infty} I_i) \le m(Q) < \infty$, there is $N > 0$ s.t.

$$m(\bigcup_{i=N+1}^{\infty} [x_i - r_i, x_i + r_i]) =: m(\bigsqcup_{i=N+1}^{\infty} I_i) = \sum_{i=N+1}^{\infty} \ell(I_i) < \frac{\epsilon}{5}.$$

We wish to prove $m^*(E - \bigcup_{i=1}^{N} I_i) < \epsilon$, then we turn to prove

$$E - \bigcup_{i=1}^{N} \overline{I}_i \subseteq \bigcup_{i=N+1}^{\infty} [x_i - 5r_i, x_i + 5r_i],$$

where $m(\bigcup_{i=N+1}^{\infty} [x_i - 5r_i, x_i + 5r_i]) < \epsilon/5$.

For every $x \in E - \bigcup_{i=1}^{N} \overline{I}_i$, since Λ_0 Vitali covers E, there is $I \in \Lambda_0$ s.t. $x \in I$ and $\overline{I} \bigcap \overline{I_i} = \emptyset,\ \forall 1 \le i \le N$. If for every $i \ge 1$, $\overline{I} \bigcap \overline{I_i} = \emptyset$, then $\ell(I) < 2\ell(I_i) \to 0, i \to \infty$, a contradiction. Then there exists $k \ge N + 1$ s.t. $\overline{I} \bigcap \overline{I_k} \neq \emptyset$. Let $k_0 := \min\{k : \overline{I} \bigcap \overline{I_k} \neq \emptyset\}$, then $I \in \Lambda_{k_0 - 1}$ and $\ell(I) < 2\ell(I_{k_0})$, so that $\mathrm{d}(x, x_{k_0}) \le \ell(I) + r_{k_0} \le 2\ell(I_{k_0}) + r_{k_0} = 5r_{k_0}$ i.e. $x \in [x_{k_0} - 5r_{k_0}, x_{k_0} + 5r_{k_0}]$. ◻

Theorem 5.3 (Lebesgue). *Let f be a monotonic increasing real-valued function on $[a, b]$, then f is differentiable on $[a, b]$ a.e. and $f'(x) \in L([a, b])$ satisfies*

$$\int_a^b f'(x) \mathrm{d}x \le f(b) - f(a).$$

Proof. Let

$$(D^+f)(x) = \limsup_{h\to 0^+} \frac{f(x+h)-f(x)}{h}, \ (D_+f)(x) = \liminf_{h\to 0^+} \frac{f(x+h)-f(x)}{h}$$
$$(D^-f)(x) = \limsup_{h\to 0^-} \frac{f(x+h)-f(x)}{h}, \ (D_-f)(x) = \liminf_{h\to 0^-} \frac{f(x+h)-f(x)}{h}$$

(P_1) $m(\{D^+f > D_-f\}) = 0$.

(P_2) $D^+f = D_+f = D^-f = D_-f$ a.e. on $\mathbb{R}$.

Let $g(x) = -f(-x)$ is a monotonic increasing real-valued function and $(D^+g)(-x) = (D^-f)(x)$, $(D_-g)(-x) = (D_+f)(x)$. Since $(D^+g)(-x) \le (D_-g)(-x)$ a.e. on $\mathbb{R}$, then $(D^-f)(x) \le (D_+f)(x)$, together with (P_1) we have $(D^+f)(x) \le (D_-f)(x) \le (D^-f)(x) \le (D_+f)(x) \le (D^+f)(x)$ a.e. on $\mathbb{R}$.

Now let f be the monotonic increasing real-valued function on $[a,b]$. Let

$$f(x) = \begin{cases} f(b), x > b \\ f(x), a \le x \le b \\ f(a), x < a. \end{cases}$$

then f is a monotonic increasing real-valued function on $\mathbb{R}$. By (P_2), $f'(x)$ exists a.e. on $\mathbb{R}$. Let $g_n(x) = n(f(x+1/n) - f(x)), \forall n \ge 1$, then $g_n \to f'$ a.e. on $\mathbb{R}$, where g_n is non-negative and measurable, then f' is non-negative and measurable. By 4.10,

$$\int_a^b f'(x)\mathrm{dx} = \int_a^b \liminf_{n\to\infty} g_n(x)\mathrm{dx} \le \liminf_{n\to\infty} \int_a^b g_n(x)\mathrm{dx},$$

also

$$\begin{aligned}\int_a^b g_n(x)\mathrm{dx} &= n\int_a^b [f(x+\frac{1}{n}) - f(x)]\mathrm{dx} \\ &= n[\int_{a+\frac{1}{n}}^{b+\frac{1}{n}} f(x)\mathrm{dx} - \int_a^b f(x)\mathrm{dx}] \\ &= n[\int_b^{b+\frac{1}{n}} f(x)\mathrm{dx} - \int_a^{a+\frac{1}{n}} f(x)\mathrm{dx}] \\ &= f(b) - n\int_a^{a+\frac{1}{n}} f(x)\mathrm{dx} \\ &\le f(b) - f(a).\end{aligned}$$

□

5.2 Bounded variation function

Definition 5.4. *Let f be a real-valued function on $[a,b]$, $a = x_0 < x_1 < \ldots < x_n = b$ is a partition of $[a,b]$, $X = \{x_k\}_{0\le k\le n}$. Then*

*(i) The **variation** of f on X,*

$$V(X) = \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|.$$

(ii) The ***total variation*** *of f on $[a,b]$,*

$$T_a^b(f) = \sup\{V(X) : X \textit{ is a partition of } [a,b]\}.$$

(iii) If $T_a^b(f) < \infty$, then we say f is a ***bounded variation function*** *on $[a,b]$.*

Example 5.5. *Monotonic real-valued functions on $[a,b]$ are bounded-variation.*

Proof. Let $X = \{x_k\}_{k=0}^n$ be a partition of $[a,b]$, then

$$V(X) = \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| = |f(b) - f(a)| < \infty,$$

so that $T_a^b(f) = |f(b) - f(a)| < \infty$. □

Example 5.6. *Continuous functions are not necessarily bounded-variation.*

Proof. Consider $f(x) = x\cos\frac{\pi}{2x}(x \neq 0)$ and $f(0) = 0$, then $f \in C[0,1]$, but for every $n \geq 1$, let $X_n = \{0, \frac{1}{2n}, \frac{1}{2n-1}, ..., \frac{1}{3}, \frac{1}{2}, 1\}$, then $V(X_n) = 1 + \frac{1}{2} + ... + \frac{1}{n} \to \infty, n \to \infty$. □

Theorem 5.7. *(i) Any f bounded-variation on $[a,b]$ is bounded on $[a,b]$.*

(ii) If f, g are bounded-variation on $[a,b]$, then $f \pm g, fg$ are also bounded-variation on $[a,b]$ and $T_a^b(f \pm g) \leq T_a^b(f) + T_a^b(g)$. If $\exists \lambda > 0$ s.t. $|g(x)| \geq \lambda > 0$, then f/g is also bounded-variation on $[a,b]$.

(iii) Let f be a real-valued function on $[a,b]$, then for any $c \in [a,b]$, $T_a^b(f) = T_a^c(f) + T_c^b(f)$.

Proof. (i) Consider $X = \{a, x, b\}, x \in [a,b]$, then $|f(x) - f(a)| + |f(b) - f(x)| \leq T_a^b(f)$ so that $|f(x)| - |f(a)| \leq |f(x) - f(a)| \leq T_a^b(f)$ i.e. $|f(x)| \leq T_a^b(f) + |f(a)|, \forall x \in [a,b]$, f is bounded on $[a,b]$.

(ii) Let $X = \{x_k\}_{k=0}^n$ be a partition of $[a,b]$, then $V_{f\pm g}(X) = \sum_{k=1}^n |(f(x_k) - f(x_{k-1})) \pm (g(x_k) \pm g(x_{k-1}))| \leq \sum_{k=1}^n |f(x_k) - f(x_{k-1})| + \sum_{k=1}^n |g(x_k) - g(x_{k-1})| = V_f(X) + V_g(X) \leq T_a^b(f) + T_a^b(g) < \infty$, then $T_a^b(f \pm g) \leq T_a^b(f) + T_a^b(g)$. By (i), $\exists M, N > 0$ s.t. $|f(x)| \leq M, |g(x)| \leq N, \forall x \in [a,b]$, then $V_{fg}(X) = \sum_{k=1}^n |f(x_k)g(x_k) - f(x_{k-1})g(x_{k-1})| \leq NV_f(X) + MV_g(X) \leq NT_a^b(f) + MT_a^b(g) < \infty$. If $|g(x)| \geq \lambda > 0, \forall x \in [a,b]$, then $V_{f/g}(X) = \sum_{k=1}^n |f(x_k)/g(x_k) - f(x_{k-1})/g(x_{k-1})| \leq (MT_a^b(g) + NT_a^b(f))/\lambda^2$.

(iii) Let X_0, X_1, X_2 on $[a,b], [a,c], [c,b]$ respectively. Then $V(X_1) + V(X_2) = V(X_1 \bigcup X_2) \leq T_a^b(f)$, then $T_a^c(f) + T_c^b(f) \leq T_a^b(f)$. Also, $V(X_0) \leq V(X_0 \bigcup \{c\}) \leq T_a^c(f) + T_c^b(f)$, then $T_a^b(f) \leq T_a^c(f) + T_c^b(f)$. Finally we have $T_a^b(f) = T_a^c(f) + T_c^b(f)$. □

Definition 5.8. *Let f be a bounded-variation function on $[a,b]$, then $T_a^x(f) < \infty, \forall x \in [a,b]$ and $T_a^x(f)$ is a non-negative monotonic increasing real-valued function on $[a,b]$, we say it the **indefinite total variation** of f.*

Theorem 5.9. *If f is bounded-variation on $[a,b]$, then $f(x), T_a^x(f)$ share the same sets of right-continuous points and left-continuous points.*

Proof. Let $x_0 \in [a,b)$ be a right-continuous point of $T_a^x(f)$. Then $|f(x_0+h)-f(x_0)| \leq T_{x_0}^{x_0+h}(f) = T_a^{x_0+h}(f) - T_a^{x_0}(f) \to 0, h \to 0^+$, so that x_0 is also a right-continuous point of $f(x)$.

Let $x_1 \in [a,b)$ be a right-continuous point of $f(x)$. For every $\epsilon > 0$, $\exists \delta_1 > 0$ s.t. $|f(x_1) - f(x_1+h)| < \epsilon/2, \forall 0 \leq h < \delta_1$. Also $\lim_{h\to 0^+} T_a^{x_0+h}(f)$ exists so that $\delta_2 > 0$ s.t. $|T_a^{y_1}(f) - T_a^{y_2}(f)| < \epsilon/2, \forall x_1 < y_1, y_2 \leq x_1 + \delta_2$. Then for any partition $X = \{z_k\}_{k=0}^n$ of $[x_1, x_1+\delta_2]$. Let $X' = X \bigcup \{z_{1/2}\}$, where $x_1 < z_{1/2} < \min\{z_1, x_1+\delta_1\}$, then $V(X) \leq V(X') = |f(z_{1/2}) - f(z_0)| + (|f(z_1) - f(z_{1/2})| + \sum_{k=2}^n |f(z_k) - f(z_{k-1})|) \leq |f(z_{1/2}) - f(z_0)| + |T_a^{x_1+\delta_2}(f) - T_a^{z_{1/2}}(f)| < \epsilon$. Then $T_{x_1}^x(f) \leq T_{x_1}^{x_1+\delta_2}(f) \leq \epsilon, \forall x \in (x_1, x_1+\delta_2)$ so that x_1 is a right-continuous point of $T_a^x(f)$. □

Corollary 5.10. *If f is bounded-variation on $[a,b]$, then the set of discontinuous points of f is at most countable.*

Theorem 5.11. *Let f be a real-valued function on $[a,b]$, then f is bounded-variation on $[a,b]$ if and only if $\exists f_1, f_2$ real-valued monotonic increasing functions on $[a,b]$ s.t. $f = f_1 - f_2$.*

Proof. ($\Rightarrow$) Consider $f(x) = T_a^x(f) - (T_a^x(f) - f(x))$, for $a \leq x_1 < x_2 \leq b$, $f(x_2) - f(x_1) \leq |f(x_2) - f(x_1)| \leq T_{x_1}^{x_2}(f) = T_a^{x_2}(f) - T_a^{x_1}(f)$, then $T_a^{x_1}(f) - f(x_1) \leq T_a^{x_2}(f) - f(x_2)$ so that $T_a^x(f) - f(x)$ is a real-valued monotonic increasing function.

($\Leftarrow$)

$$T_a^b(f) = T_a^b(f_1 - f_2) \leq T_a^b(f_1) + T_a^b(f_2) < \infty.$$

□

Corollary 5.12. *If f is bounded-variation on $[a,b]$, then f' exists a.e. on $[a,b]$ and $f' \in L[a,b]$.*

5.3 Indefinite integral

Definition 5.13. *If $f \in L[a,b]$, then*

$$F(x) = \int_a^x f(t)\mathrm{dt}, x \in [a,b],$$

*is called the **indefinite integral** of f.*

Theorem 5.14. *If $f \in L[a,b]$, then the indefinite integral $F(x)$ is a continuous bounded-variation function and*

$$T_a^b(F) = \int_a^b |f(t)|\mathrm{dt}.$$

Proof. By 4.19, $F(x)$ is continuous on $[a,b]$. For any partition $X = \{x_k\}_{k=0}^n$ on $[a,b]$,

$$\sum_{k=1}^n |F(x_k) - F(x_{k-1})| = \sum_{k=1}^n |\int_{x_{k-1}}^{x_k} f(t)\mathrm{dt}| \le \sum_{k=1}^n \int_{x_{k-1}}^{x_k} |f(t)|\mathrm{dt} = \int_a^b |f(t)|\mathrm{dt}.$$

Then $T_a^b(F) \le \int_a^b |f(t)|\mathrm{dt}$ so that $F(x)$ is bounded-variation on $[a,b]$. Now we prove $\int_a^b |f(t)|\mathrm{dt} \le T_a^b(F)$ so that finally we will have $T_a^b(F) = \int_a^b |f(t)|\mathrm{dt}$.

Suppose $f \in C[a,b]$, then consider

$$Q_1 = \{f > 0\} \bigcap (a,b), Q_2 = \{f < 0\} \bigcap (a,b),$$

Q_1, Q_2 are open sets, then let $Q_1 = \bigsqcup_{n=1}^\infty (a_n, b_n), Q_2 = \bigsqcup_{n=1}^\infty (c_n, d_n)$, then

$$\begin{aligned}\int_a^b |f(t)|\mathrm{dt} &= \int_{Q_1} f(t)\mathrm{dt} - \int_{Q_2} f(t)\mathrm{dt} = \sum_{k=1}^\infty \{\int_{a_k}^{b_k} f(t)\mathrm{dt} + \int_{c_k}^{d_k} [-f(t)]\mathrm{dt}\} \\ &= \lim_{n\to\infty} [\sum_{k=1}^n |F(b_k) - F(a_k)| + \sum_{k=1}^n |F(d_k) - F(c_k)|] \le T_a^b(F).\end{aligned}$$

Now consider a general $f \in L[a,b]$, by 4.29, $\forall \epsilon > 0$, $\exists g \in C[a,b]$ s.t. $\int_a^b |f(t) - g(t)|\mathrm{dt} < \epsilon$, then

$$\begin{aligned}\int_a^b |f(t)|\mathrm{dt} - \epsilon &\le \int_a^b |g(t)|\mathrm{dt} = T_a^b(\int_a^x g(t)\mathrm{dt}) = T_a^b(F(x) + \int_a^x [g(t) - f(t)]\mathrm{dt}) \\ &\le T_a^b(F(x)) + T_a^b(\int_a^x [g(t) - f(t)]\mathrm{dt}) \\ &\le T_a^b(F(x)) + \int_a^b |g(t) - f(t)|\mathrm{dt} < T_a^b(F(x)) + \epsilon.\end{aligned}$$

□

Lemma 5.15. *If $f \in L[a,b]$ and $F(x) \equiv 0$, then $f = 0$ a.e. on $[a,b]$.*

Proof.

$$\int_a^b |f(t)|\mathrm{dt} = T_a^b(F) = 0.$$

□

Theorem 5.16. *Let $f \in L[a,b]$ and $F(x) = \int_a^x f(t)\mathrm{dt}$, then $F'(x) = f(x)$ a.e. on $[a,b]$.*

Proof. Suppose f is bounded, $|f(x)| \le M$ and $F(x) = \int_a^x f(t)\mathrm{dt}$ is differentiable a.e. on $[a,b]$, $F'(x) \in L[a,b]$. Then consider $g_n(x) = n[F(x+1/n) - F(x)] \to F'(x)$ a.e. on $[a,b]$. Since $|g_n(x)| = |n\int_x^{x+1/n} f(t)\mathrm{dt}| \le M$. So that by 4.16, for $\forall x \in [a,b]$, we have

$$\begin{aligned}
\int_a^x F'(t)\mathrm{dt} &= \lim_{n\to\infty}\int_a^x g_n(t)\mathrm{dt} \\
&= \lim_{n\to\infty} n\int_a^x [F(t+\frac{1}{n}) - F(t)]\mathrm{dt} \\
&= \lim_{n\to\infty} [n\int_x^{x+\frac{1}{n}} F(t)\mathrm{dt} - n\int_a^{a+\frac{1}{n}} F(t)\mathrm{dt}] \\
&= \lim_{n\to\infty} (F(x+\frac{\theta_n}{n}) - F(a+\frac{\eta_n}{n})) \\
&= F(x) - F(a) = \int_a^x f(t)\mathrm{dt}.
\end{aligned}$$

Then $\int_a^x (F'(t) - f(t))\mathrm{dt} \equiv 0, \forall x \in [a,b]$, so that $F'(t) = f(t)$ a.e. on $[a,b]$.

For a general $f \in L[a,b]$, $f = \max\{0, f(x)\} - \max\{0, -f(x)\}$. Suppose f is non-negative, let $f_n(x) = f(x)\chi_{\{f\le n\}}$ and $G_n(x) = F(x) - \int_a^x f_n(t)\mathrm{dt}$, then f_n is bounded, $f - f_n \ge 0$, $G_n(x)$ is a monotonic increasing function so that $G_n'(x) = F'(x) - f_n(x)$ a.e. on $[a,b]$, then $F'(x) \ge f(x)$ a.e. on $[a,b]$ and $\int_a^b F'(x)\mathrm{dx} \ge \int_a^b f(x)\mathrm{dx}$. Since $F(x)$ is monotonic increasing, $\int_a^b F'(x)\mathrm{dx} \le F(b) - F(a) = \int_a^b f(x)\mathrm{dx}$. Then $\int_a^b (F'(x) - f(x))\mathrm{dx} = 0$, $F'(x) = f(x)$ a.e. on $[a,b]$. □

Definition 5.17. *Let $f \in L[a,b]$, $x \in (a,b)$ is called a* ***Lebesgue point*** *of f if*

$$\lim_{h\to 0^+}\frac{1}{2h}\int_{x-h}^{x+h}|f(t) - f(x)|\mathrm{dt} = 0.$$

Theorem 5.18. *If $f \in L[a,b]$, then except a zero-measured set Z, all points in $(a,b)\setminus Z$ are Lebesgue points and $F'(x) = f(x), \forall x \in (a,b)\setminus Z$, where $F(x) = \int_a^x f(t)\mathrm{dt}$.*

5.4 Example and exercise

Example 5.19. *Let f be a bounded increasing function on $\mathbb{R}$. Prove: $f' \in L(\mathbb{R})$.*

Proof. f' exists a.e. on $\mathbb{R}$, $f' \ge 0$, $\exists M > 0$ s.t. $|f| < M$, consider $f_n(x) = f'(x)\chi_{[-n,n]}, n \ge 1$, then $f_n \uparrow f$ so that

$$\int_{\mathbb{R}} f'(x)\mathrm{dx} = \lim_{n\to\infty}\int_{\mathbb{R}} f_n(x)\mathrm{dx} = \lim_{n\to\infty}\int_{-n}^{n} f'(x)\mathrm{dx} \le \lim_{n\to\infty}(f(n) - f(-n)) < 2M < \infty.$$

□

Example 5.20. *Let $\{f_n\}_{n\geq 1}$ be a sequence of real-valued increasing functions on $[a,b]$, suppose for every $x \in [a,b]$, the series $\sum_{n=1}^{\infty} f_n(x)$ converges to $s(x)$. Prove:*

$$s'(x) = \sum_{n=1}^{\infty} f_n'(x), \ \ a.e.$$

Proof. First we prove $\sum_{n=1}^{\infty} fn'(x) < \infty$ a.e. on $[a,b]$. Let $E = \{x \in [a,b] : f'(x), f_n'(x) \text{ exist}, \forall n \geq 1\}$, $m(E) = b-a$, for every $n \geq 1$, we have $s(x) = \sum_{k=1}^{n} f_k(x) + \sum_{k=n+1}^{\infty} f_k(x)$ so that $s'(x) = \sum_{k=1}^{n} f_k'(x) + (\sum_{k=n+1}^{\infty} f_k(x))' \geq \sum_{k=1}^{n} f_k'(x)$, $n \to \infty$ then $\sum_{k=1}^{\infty} f_k'(x) \leq s'(x) < \infty$ on E.

Now we prove $s'(x) = \sum_{k=1}^{\infty} f_k'(x)$ a.e. on $[a,b]$. Without loss of generality, assume $f_n(a) = 0, \forall n \geq 1$, let $s_n(x) = \sum_{k=1}^{n} f_k(x)$, $s(x) - s_n(x) = \sum_{k=n+1}^{\infty} f_k(x)$ so that $s(x) - s_n(x)$ monotonic increasing, $0 \leq s(x) - s_n(x) \leq s(b) - s_n(b) \to 0, n \to \infty$, then $\exists n_k \uparrow \infty$ s.t. $0 \leq s(x) - s_{n_k}(x) < 1/2^k$, then $\sum_{k=1}^{\infty}(s(x) - s_{n_k}(x)) < \infty$ on $[a,b]$. Then $\sum_{k=1}^{\infty}(s(x) - s_{n_k}(x))' < \infty$ so that $s_k'(x) \to s'(x)$ a.e. on $[a,b]$. □

Example 5.21. *Let $\{r_n\}_{n=1}^{\infty} = \mathbb{Q} \bigcap (0,1)$, define*

$$f(x) = \sum_{r_n < x} 2^{-n}, \ \ x \in (0,1], \ \ f(0) = 0.$$

Prove: $f(x)$ is strictly increasing on $[0,1]$ and $f'(x) = 0$ almost everywhere.

Proof. Let $x, y \in [0,1], x < y$, then $\{n : r_n < x\} \subsetneq \{n : r_n < y\}$ so that $f(x) < f(y)$, f strictly increasing on $[0,1]$. Note that $f(x) = \sum_{n=1}^{\infty} 2^{-n}\chi_{(r_n,\infty)}, \forall x \in [0,1]$, $f(x) \leq \sum_{n=1}^{\infty} 2^{-n} < \infty$, since $2^{-n}\chi_{(r_n,\infty)}$ real-valued monotonic increasing on $[0,1]$, by 5.20, $f'(x) = \sum_{n=1}^{\infty}(2^{-n}\chi_{(r_n,\infty)})' = 0, \forall x \in [0,1] \setminus \{r_n\}_n$ so that a.e. on $[0,1]$. □

Example 5.22. *If f is of bounded variation on $[0,a]$, prove:*

$$F(x) = \frac{1}{x}\int_0^x f(t)\,\mathrm{d}t, (F(0) = 0)$$

is also of bounded variation on $[0,a]$.

Proof. First we assume f is a real-valued monotonic increasing function on $[0,a]$. Then let $0 < x_1 < x_2 \leq a$, $x_1 \int_{x_1}^{x_2} f(t)\mathrm{d}t - (x_2 - x_1)\int_0^{x_1} f(t)\mathrm{d}t \geq x_1(x_2 - x_1)f(x_1) - (x_2 - x_1)x_1 f(x_1) = 0$ i.e. $F(x_2) \geq F(x_1)$, $F(x)$ is monotonic increasing so that of bounded variation.

Now $\exists f_1, f_2$ real-valued monotonic increasing on $[0,a]$ s.t. $f = f_1 - f_2$, then $F(x) = \frac{1}{x}\int_0^x f_1(t)\mathrm{d}t - \frac{1}{x}\int_0^x f_2(t)\mathrm{d}t$ so that of bounded variation. □

Example 5.23. *Let $\{f_k\}_{k\geq 1}$ be a sequence of functions of bounded variation on $[a,b]$, and suppose $f_k(x) \to f(x)$ for all $x \in [a,b]$. Additionally, assume there exists $M > 0$ such that $T_a^b(f_k) \leq M$ for all $k \geq 1$. Prove: $T_a^b(f) \leq M$.*

Proof. Given any net X on $[a,b]$, $\sum_{k=1}^{m}|f_n(x_k)-f_n(x_{k-1})| \le M, \forall n \ge 1$, let $n \to \infty$ we have $V_f(X) \le M$, finally $T_a^b(f) \le M$. $\square$

Example 5.24. *Let f be a real-valued function on $[a,b]$, $\exists M > 0$ s.t. for any $\epsilon > 0$, $T_{a+\epsilon}^b(f) \le M$. Prove: f is of bounded variation on $[a,b]$.*

Proof. Since $|f(b)-f(x)| \le T_x^b(f) \le M$ so that $|f(x)| \le |f(b)| + M, \forall x \in (a,b]$. Then given any net $X = \{x_k\}_{k=0}^n$ on $[a,b]$, we have

$$V_f(X) = \sum_{k=1}^{n}|f(x_k)-f(x_{k-1})| \le |f(a)-f(x_1)| + T_{x_1}^b(f) \le |f(a)| + |f(b)| + 2M.$$

$\square$

Example 5.25. *Let f, g be of bounded variation on $[a,b]$, $f(a) = g(a) = 0$, prove: $T_a^b(fg) \le T_a^b(f)T_a^b(g)$.*

Proof. Given any net $X = \{x_k\}_{k=0}^n$ on $[a,b]$, then

$$\begin{aligned}
V_{fg}(X) &= \sum_{k=1}^{n}|f(x_k)g(x_k)-f(x_{k-1})g(x_{k-1})| \le \sum_{k=1}^{n}(|f(x_k)||g(x_k)-g(x_{k-1})| + |g(x_{k-1})||f(x_k)-f(x_{k-1})|) \\
&= \sum_{k=1}^{n}|f(x_k)-f(a)||g(x_k)-g(x_{k-1})| + \sum_{k=1}^{n}|g(x_{k-1})-g(a)||f(x_k)-f(x_{k-1})| \\
&\le \sum_{k=1}^{n}\sum_{i=1}^{k}|f(x_i)-f(x_{i-1})||g(x_k)-g(x_{k-1})| + \sum_{k=1}^{n}\sum_{j=1}^{k-1}|f(x_k)-f(x_{k-1})||g(x_j)-g(x_{j-1})| \\
&= \sum_{k=1}^{n}\sum_{i=1}^{n}|f(x_k)-f(x_{k-1})||g(x_i)-g(x_{i-1})| \\
&= (\sum_{k=1}^{n}|f(x_k)-f(x_{k-1})|)(\sum_{i=1}^{n}|g(x_i)-g(x_{i-1})|) = V_f(X)V_g(X) \le T_a^b(f)T_a^b(g).
\end{aligned}$$

Finally we have $T_a^b(fg) \le T_a^b(f)T_a^b(g)$. $\square$

Example 5.26. *Let f be of bounded variation on $[a-1,b+1]$, prove: $\exists M > 0$ s.t. when $|h|$ sufficiently small,*

$$\int_a^b |f(x+h)-f(x)|\mathrm{d}x \le M|h|.$$

Proof. First assume f is a monotonic increasing function on $[a-1, b+1]$, then when $h > 0$ we have

$$\int_a^b |f(x+h) - f(x)|\mathrm{dx} = \int_a^b (f(x+h) - f(x))\mathrm{dx} = \int_b^{b+h} f(x)\mathrm{dx} - \int_a^{a+h} f(x)\mathrm{dx}$$
$$\leq hf(b+h) - hf(a) \leq h(f(b+1) - f(a-1)).$$

□

Example 5.27. *Suppose f is differentiable on $[a, b]$, f' is of bounded variation. Prove: f' is continuous.*

Proof. $\exists f_1, f_2$ monotonic increasing on $[a, b]$ s.t. $f = f_1 - f_2$, for every $x_0 \in [a, b]$, $A = \liminf_{x \to x_0^-} f(x)$, $B = \limsup_{x \to x_0^+} f(x)$, assume $A < f'(x_0)$, then $\exists \delta > 0$ s.t. $f'(x) < A_1 < f'(x_0), \forall x \in (x_0 - \delta, x_0)$, $f'(x) \notin (A_1, f'(x_0)), \forall x \in (x_0 - \delta, x_0)$, a contradiction. □

Example 5.28. *Let $f \in L([0, 1])$, g be a real-valued, monotonically increasing function on $[0, 1]$. If for any $[a, b] \subset [0, 1]$,*

$$(\int_a^b f(x)\mathrm{dx})^2 \leq [g(b) - g(a)](b - a),$$

prove: $f^2 \in L([0, 1])$.

Proof. Let $b = x + h, a = x$, $h > 0$, then we have

$$(\int_x^{x+h} f(t)\mathrm{dt})^2 \leq (g(x+h) - g(x))h$$
$$(\frac{1}{h}\int_x^{x+h} f(t)\mathrm{dt})^2 \leq \frac{g(x+h) - g(x)}{h}$$
$$h \to 0^+, \text{ then we have}$$
$$f^2(x) \leq g'(x), \text{ a.e. on } [0, 1].$$

□

Example 5.29. *Let f be a bounded measurable function on $\mathbb{R}$. If there exist $0 < \lambda < 1$ and $1 \leq p < \infty$ such that for any bounded interval $[a, b]$,*

$$\left(\int_a^b |f(x)|\mathrm{dx}\right)^p \leq \lambda(b - a)^{p-1} \int_a^b |f(x)|^p \mathrm{dx}.$$

Prove: $f(x) = 0$, a.e.

Proof. Since f is bounded on $[a,b]$, then $|f|, |f|^p \in L[a,b]$. Let $a = x, b = x+h, h > 0$, then we have

$$(\frac{1}{h}\int_x^{x+h} |f(t)|\mathrm{dt})^p \le \frac{\lambda}{h}\int_x^{x+h} |f(t)|^p \mathrm{dt},$$

let $h \to 0^+$, we have $|f(x)|^p \le \lambda |f(x)|^p, 0 < \lambda < 1$ a.e. on $\mathbb{R}$ so that $f(x) = 0$ a.e. on $\mathbb{R}$. □

Example 5.30. *If $f \in AC[a,b]$, prove: $|f(x)|^p \in AC[a,b]$, $p \ge 1$. Does this proposition hold for $0 < p < 1$? Investigate the function $f(x) = x^2 \sin^2 \frac{1}{x}$, $f(0) = 0$, $x \in [0,1]$, $p = \frac{1}{2}$.*

Proof. Since $f \in AC[a,b]$, $\exists M > 0$ s.t. $|f| < M, \forall x \in [a,b]$, then

$$||f(x_1)|^p - |f(x_2)|^p| = p|f(x_1) + \theta f(x_2)|^{p-1}|f(x_1) - f(x_2)| \le pM^{p-1}|f(x_1) - f(x_2)|.$$

□

Example 5.31. *For any $0 < \lambda < 1$, prove: there exists a strictly increasing, absolutely continuous function $f(x)$ on $[0,1]$ such that $m(E) \ge \lambda$, where $E = \{x \in (0,1) : f'(x) = 0\}$.*

Proof. Let E be a complete nowhere dense set in $[0,1]$ s.t. $m(E) = \lambda$, let $f(x) = \int_0^x \chi_{[0,1]\setminus E}(t)\mathrm{dt}$, then $f \in AC[0,1]$ and monotonic increases on $[0,1]$. For any $0 \le x_1 < x_2 \le 1$, $\exists (x_1', x_2') \subseteq (x_1, x_2)$ s.t. $(x_1', x_2') \bigcap E = \emptyset$, $f(x_2) - f(x_1) \ge f(x_2') - f(x_1') = \int_{x_1'}^{x_2'} \chi_{[0,1]\setminus E}(t)\mathrm{dt} = x_2' - x_1' > 0$, so that f is strictly increasing on $[0,1]$. □

Example 5.32. *Let f be differentiable on $\mathbb{R}$, $f, f' \in L(\mathbb{R})$. Prove: $\int_{\mathbb{R}} f'(x)\mathrm{dx} = 0$.*

Proof. Since $\int_{\mathbb{R}} |f(t)|\mathrm{dt} < \infty$, then $\liminf_{x\to\pm\infty} |f(x)| = 0$, $\exists a_n \uparrow \infty, b_n \downarrow -\infty$ s.t. $\lim_{n\to\infty} f(a_n) = 0, \lim_{n\to\infty} f(b_n) = 0$, let $f_n'(x) = f'(x)\chi_{[b_n,a_n]} \to f'(x)$, where $|f_n'(x)| \le |f'(x)| \in L(\mathbb{R})$, then we have

$$\int_{\mathbb{R}} f'(x)\mathrm{dx} = \lim_{n\to\infty}\int_{\mathbb{R}} f_n'(x)\mathrm{dx} = \lim_{n\to\infty}\int_{b_n}^{a_n} f'(x)\mathrm{dx} = \lim_{n\to\infty}(f(a_n) - f(b_n)) = 0.$$

□

Example 5.33. *Let $f \in AC[a,b]$, $\forall a, b \in \mathbb{R}, a < b$, prove: for any $x \in \mathbb{R}$,*

$$\frac{\mathrm{d}}{\mathrm{dx}}\int_a^b f(x+y)\mathrm{dy} = \int_a^b f'(x+y)\mathrm{dy}.$$

Proof.

$$\frac{\mathrm{d}}{\mathrm{dx}}\int_a^b f(x+y)\mathrm{dy} = \frac{\mathrm{d}}{\mathrm{dx}}\int_{a+x}^{b+x} f(t)\mathrm{dt} = \frac{\mathrm{d}}{\mathrm{dx}}\int_c^{b+x} f(t)\mathrm{dt} - \frac{\mathrm{d}}{\mathrm{dx}}\int_c^{a+x} f(t)\mathrm{dt}$$
$$= f(b+x) - f(a+x) = \int_{a+x}^{b+x} f'(t)\mathrm{dt} = \int_a^b f'(x+y)\mathrm{dy}.$$

□

Example 5.34. *Let f_n $(n \geq 1)$ be monotonically increasing, absolutely continuous functions on $[a,b]$, suppose $\sum_{n=1}^{\infty} f_n(x)$ converges everywhere on $[a,b]$ to $s(x)$. Prove: $s(x)$ is also monotonically increasing and absolutely continuous.*

Proof. By 5.20, $s'(x) = \sum_{n=1}^{\infty} f_n'(x)$ a.e. on $[a,b]$, then

$$\int_a^x s'(x)\mathrm{dx} = \int_a^x \sum_{n=1}^{\infty} f_n'(x)\mathrm{dx} = \sum_{n=1}^{\infty}\int_a^x f_n'(x)\mathrm{dx} = \sum_{n=1}^{\infty}(f_n(x) - f_n(a)) = s(x) - s(a).$$

□

Example 5.35. *Let $\alpha > 0$, $\beta > 0$, $f(x) = x^{\alpha}\sin x^{-\beta}$, $0 < x \leq 1$, $f(0) = 0$. Prove: $f(x) \in AC((0,1])$ if and only if $\alpha > \beta$.*

Proof. $\alpha > \beta$, $f'(x) = \alpha x^{\alpha-1}\sin x^{-\beta} - \beta x^{\alpha-\beta-1}\cos x^{-\beta}$, then $|f'(x)| \leq \alpha x^{\alpha-1} + \beta x^{\alpha-\beta-1} \in L([0,1])$ □

Example 5.36. *Let f be a monotonically increasing real-valued function on $[a,b]$. Prove: $f = g + h$, where g is a monotonically increasing, absolutely continuous function, and h is a monotonically increasing real-valued function with $h'(x) = 0$, a.e.*

Proof.

$$f(x) = \int_a^x f'(t)\mathrm{dt} + f(x) - \int_a^x f'(t)\mathrm{dt}.$$

□

Example 5.37. *Let $f \in C[a,b]$, differentiable a.e. and $f' \in L([a,b])$. If for any $\varepsilon > 0$, $f \in AC[a+\varepsilon, b]$, prove: $f \in AC[a,b]$.*

Proof.

$$\int_a^x f(t)\mathrm{dt} = \int_{a+\epsilon}^x f(t)\mathrm{dt} + \int_a^{a+\epsilon} f(t)\mathrm{dt} = f(x) - f(a+\epsilon) + \int_{[a,a+\epsilon]} f(t)\mathrm{dt} \to f(x) - f(a), \epsilon \to 0^+.$$

$\square$

References

[1] H. L. Royden and P. M. Fitzpatrick. *Real Analysis*. English. 4th. Classic Original Edition Book Library. Reprint. Originally published: Upper Saddle River, NJ : Pearson Prentice Hall, ©2010. ISBN 9780131437470. Text in English; with Chinese introduction. Beijing: China Machine Press, 2010. ISBN: 9787111313052.

[2] Walter Rudin. *Real and Complex Analysis*. English. 3rd ed. Huazhang Mathematics Original Classic Series. English reprint edition. Originally published: New York : McGraw-Hill, 1987. ISBN 0-07-054234-1. Beijing: China Machine Press, 2019. ISBN: 978-7-111-61955-0.

[3] Xingwei Zhou and Wenchang Sun. *Real Variable Functions*. Chinese. 3rd ed. Twelfth Five-Year Plan National Level Planning Textbook for Regular Higher Education and Nankai University Mathematics Teaching Series. In Chinese. Beijing: Science Press, 2014. ISBN: 978-7-03-040451-0.