Chapter 1. Single-multivariable calculus

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Content

The derivative as an Instantaneous Rate of Change

Definition

The derivative of a function f at $x = c$, denoted by $f'(c)$, is

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h},$$

provided that the limit exists. In this case we say that f is differentiable at c.
Leibniz notation $\frac{df}{dx}(c)$.

Geometric interpretation of the derivative

Proposition

If f is differentiable at $x = c$, then $f'(c)$ is the slope of the tangent line at the point $(c, f(c))$ to the graph of f.
The equation of the tangent line is

$$y = f'(c)(x - c) + f(c).$$

Example: find the tangent line to $f(x) = \frac{1}{x}$ at $x = 2$.

Differentiation of fundamental functions

- $(x^n)' = nx^{n-1}$, for any $n \in \mathbb{R}$;
  For example: $\sqrt{x}' = \frac{1}{\sqrt{x}}$.
- $(e^x)' = e^x$;
  $(a^x)' = (\ln a)a^x$, for $0 < a \neq 1$;
- $(\ln x)' = \frac{1}{x}$;
  $(\log_a x)' = \frac{1}{x \ln a}$.
- $(\sin x)' = \cos x$;
- $(\cos x)' = -\sin x$;
- $(\tan x)' = \frac{1}{\cos^2 x}$;
- $(\cot x)' = -\frac{1}{\sin^2 x}$;

nth Derivative of a function (n times differentiable)

f': **the first derivative**; f'': **the second derivative**; f''': **the third derivative**

For higher derivatives we denote $f^{(4)}$, $f^{(5)}$ etc.

Example: Find some higher derivatives of $f(x) = \sqrt{x}, x \geq 0$.

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Solution: we rewrite $f(x) = x^{1/2}$, then

$$f'(x) = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2}$$

$$f''(x) = (\frac{1}{2})(-\frac{1}{2})x^{-1/2-1} = -\frac{1}{4}x^{-3/2},$$

$$f'''(x) = -\frac{1}{4}(-\frac{3}{2})x^{-3/2-1} = \frac{3}{8}x^{-5/2}, \ldots$$

$$f^{(n)}(x) = (-1)^{(n+1)}\frac{1 \cdot 3 \cdot \ldots (2n-3)}{2^n}x^{1/2-n}, n \in \mathbb{N}.$$

Global extrema and Local extrema

Definition of extrema

Let f be a function defined on the domain D.
Then f has a global (or absolute) maximum (respectively minimum), $f(c)$, at $x = c \in D$ if $f(c) \geq f(x)$ (respectively $f(c) \leq f(x)$) for all $x \in D$.

If the above inequality is right for only x in a neighborhood of c in D, we say that f has a local (or relative) maximum (respectively minimum) at c.

Remark: global extrema are points at which a function is either largest or smallest.

The global maximum (resp. minimum) corresponds to the highest (resp. lowest) point on the graph of f on D.

The local maximum (resp. minimum) is higher (resp. lower) than all nearby points.

The extreme value theorem-EVT

If f is continuous on a closed interval $[a, b]$, $a, b \in \mathbb{R}$, then f has a global max. and a global min. on $[a, b]$.

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Fermat's theorem

if f has a local extremum at an interior point $c \in (a, b)$ and $f'(c)$ exists, then $f'(c) = 0$.

In the case of the last theorem, we have a horizontal tangent line to the graph of f at the point $(c, f(c))$.

The Mean Value Theorem (MVT) (Lagrange's theorem)

If f is continuous on a closed interval $[a, b]$, $a, b \in \mathbb{R}$, and differentiable on the open interval (a, b), then there exists some point $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

A special case of the theorem, called **Roll's theorem**, when $f(b) = f(a)$, and then $f'(c) = 0$.

(Geometrical significance of the theorem)

Another form:

$$f(x + h) = f(x) + f'(c)h, \; c = x + \theta h, \quad \text{for } 0 < \theta < 1$$

Proof ideas:

Prove the Roll's theorem then apply it by considering the function

$$g(x) := f(x) - \frac{f(b) - f(a)}{b - a} x$$

$$\text{(or } \; g(x) := f(x) - f(a) - \frac{f(b) - f(a)}{b - a}[x - a] \text{)}$$

Cauchy's theorem

Let two functions f and g be continuous on a closed interval $[a, b]$ $(a, b \in \mathbb{R})$, with $g(a) \neq g(b)$ and differentiable on the open interval (a, b) with $g'(x) \neq 0$, for every $x \in (a, b)$.
Then there exists some point $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof ideas: Apply the Roll's theorem for the function

$$h(x) := f(x) - \frac{f(b) - f(a)}{g(b) - g(a)} g(x)$$

$$(\text{or} \quad h(x) := f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)} [g(x) - g(a)] \ )$$

L'Hospital's Rule

for limits the **indeterminate form** $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

L'Hospital's Rule

Suppose that f and g are differentiable functions near the point a such that

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \text{ (or } \infty), \text{ and}$$

$$g'(x) \neq 0 \text{ for } x \text{ near } a.$$

If $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$ then $\lim_{x\to a} \frac{f(x)}{g(x)} = L$.

The rule works for $a = \pm\infty$ as well, also for one-sided limits.

Discuss about limits of the form $0 \cdot \infty$, $\infty - \infty$, $0 \cdot \infty$, 0^∞, ∞^0, 0^0, 1^∞.

Some examples

$$\lim_{x\to 0}\frac{\sin x}{x} = 1,$$

$$\lim_{x\to 0}\frac{e^x - 1}{x} = 1,$$

$$\lim_{u\to 0}\frac{ln(u+1)}{u} = 1,$$

$$\lim_{x\to\infty}\frac{\ln x}{x}$$

$$\lim_{x\to\infty}\frac{e^x}{x}$$

Taylor polynomial

Let the function f be continuous on $[a, b]$ $(a, b \in \mathbb{R})$, $(n+1)$ times differentiable on (a, b), and $c \in (a, b)$ be a fixed point.

Problem: find a polynomial P_n with degree not exceeding n so that

$$f(c) = P_n(c),\ f'(c) = P_n'(c), \ldots, f^{(n)}(c) = P_n^{(n)}(c)$$

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Find $P_n(x)$ of the form

$$P_n(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n.$$

Result:

$$a_0 = f(c),\ \ a_1 = \frac{f'(c)}{1!}, \dots, a_n = \frac{f^{(n)}(c)}{n!}.$$

$P_n(x)$ is called the n-th order **Taylor polynomial** of f at the point c.

Consider the difference

$$R_n(x) = f(x) - P_n(x), \ x \in (a, b),$$

and let $G(x) = (x - c)^{n+1}$. Then we have some remarks:

1. $R_n(c) = R_n'(c) = \ldots = R_n^{(n)}(c) = 0;$
2. $G(c) = G'(c) = \ldots = G^{(n)}(c) = 0;$
3. $R_n^{(n+1)}(x) = f^{(n+1)}(x), \ G^{(n+1)}(x) = (n+1)!$

Proposition

For $x \in (a, b)$ and $x \neq x$, there exists $\overline{c}$ between x and c so that

$$R_n(x) = \frac{f^{(n+1)}(\overline{c})}{(n+1)!}(x - c)^{n+1}.$$

Proof ideas: Apply successively the Cauchy's theorem $(n+1)$ times for the functions R_n and G (with c has the same role as a in the theorem).

Taylor's formula

Taylor's theorem

If the function f is continuous on $[a, b]$, $(n+1)$ times differentiable on (a, b), then for any fixed point $c \in (a, b)$ we have

$$f(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x),$$

where

$$R_n(x) = \frac{f^{(n+1)}(\overline{c})}{(n+1)!}(x-c)^{n+1},$$

for some point $\overline{c}$ between x and c.

The above formula is called the finite Taylor expansion of order n of f at the point c.
In particular if $c = 0$, it is called the Maclaurin expansion.
$R_n(x)$ is called the approximation error, or the remainder.

Little-o notation

It is a notation representing the asymptotic behavior of a function at a given point. The statement

$$f(x) = o(g(x)) \text{ as } x \to a,$$

can be intuitively interpreted as saying that $f(x)$ goes to zero much faster than $f(x)$ at a, or, more mathematically,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 0.$$

Write $f(x) = h(x) + o(g(x))$ as $x \to a$ means

$$\lim_{x \to a} \frac{f(x) - h(x)}{g(x)} = 0.$$

Some examples:

Using the little-o notation, the statement in Taylor's theorem reads as

$$f(x) = P_n(x) + o((x-c)^n), \text{ as } x \to c.$$

Some examples: write Taylor expansions of the following functions at 0:

$$\frac{1}{x+1} = 1 - x + x^2 - \cdots + (-1)^n x^n + \frac{(-1)^{n+1}}{(1+\theta x)^{n+1}} x^{n+1}$$

$$\frac{1}{x-1} = 1 + x + x^2 + \cdots + x^n + o(x^n), \;\; x \to 0$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{e^{\theta x}}{(n+1)!} x^{n+1}$$

$$\ln(x+1) = x - \frac{x^2}{2} + \cdots + (-1)^{n-1} \frac{x^n}{n} + o(x^n), \;\; x \to 0$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + o(x^2), \;\; x \to 0$$

Example: consider the infinite sequence (progression)

$S_0 = 1,$

$S_1 = 1 + \frac{1}{2},$

$S_2 = 1 + \frac{1}{2} + \frac{1}{2^2},$

$\ldots\ldots$

$S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}, \cdots$

What happens for S_n as n is very large (i.e. $n \to \infty$)?

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. . . . . .
$S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}, \cdots$

What happens for S_n as n is very large (i.e. $n \to \infty$)?
We say that the infinite sum

$$1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \cdots$$

is a series and its value is the limit of S_n if it exists.
Here

$$\lim_{n\to\infty} S_n = 2$$

Definition: A series is an infinite sum, represented by an infinite expression of the form

$$a_0 + a_1 + a_2 + \cdots \,, \text{ denoted by } \sum_{n=0}^{\infty} a_n,$$

where (a_n) is any ordered sequence of terms, such as numbers, functions, or anything else that can be added.
The nth partial sums of the series:

$$S_n = a_0 + a_1 + a_2 + \cdots a_n.$$

The value of the series is the limit of S_n as $n \to \infty$ (if the limit exists).

$$\sum_{n=0}^{\infty} a_n = \lim_{n\to\infty} S_n \quad \text{(convergent, or divergent)}$$

Geometric series: for $a \in \mathbb{R}$ and $q \neq 1$,

$$\sum_{n=0}^{\infty} aq^n$$

If $a \neq 0$, then the series is convergent iff $-1 < q < 1$, and

$$\sum_{n=0}^{\infty} aq^n = \frac{a}{1-q}.$$

Remark:

$$S_n = a + aq + aq^2 + \dots + aq^{n-1} = a\left(\frac{q^n - 1}{q - 1}\right) = \frac{u_{n+1} - u_1}{q - 1}$$

Absolute convergence

A series converges absolutely if the series of absolute values

$$\sum_{n=0}^{\infty} |a_n|$$

converges.

A series converges absolutely then converges.

Taylor/ Maclaurin series

The function f is said to be infinitely differentiable, smooth, or of class C^∞, if it has derivatives of all orders.

Taylor's theorem

Let f be an infinitely differentiable function for x near a fixed point $c \in \mathbb{R}$.
Then the Taylor series of f at c is the power series

$$f(c) + \frac{f'(c)}{1!}(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots, \text{ or}$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

The Taylor series at $c = 0$ is also called a Maclaurin series.

Some examples...

HW: write the Taylor series of the following function at 0:

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0. \end{cases}$$

Let $\mathbb{R}^n$ be the set of all $n-$ tuples $(x_1, x_2, \ldots, x_n)$,

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_1 \in \mathbb{R}, x_2 \in \mathbb{R}, \ldots, x_n \in \mathbb{R}\}.$$

Definition

Suppose $D \subseteq \mathbb{R}^n$. We write a real-valued function f of n independent variables on D as

$$\begin{aligned} f : D &\longrightarrow \mathbb{R} \\ (x_1, x_2, \ldots, x_n) &\mapsto f(x_1, x_2, \ldots, x_n) \end{aligned}$$

D is called the domain of f, and the set

$$\{w \in \mathbb{R} : w = f(x_1, x_2, \ldots, x_n), (x_1, x_2, \ldots, x_n) \in D\}$$

is the range of f.

Example 1: Give the domain of the function

$$f(x, y, z) = \frac{x + y}{z^2}.$$

Evaluate the function at the point $(1, 2, 3)$ and $(-1, 2, -3)$.

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Example 2: considering the function

$$f(x, y) = \sqrt{4 - x^2 - y^2}.$$

Give and then graph the domain of f in $x - y$ plane. Find the range of f.

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Example 3 The same above question for the function

$$f = \sqrt{y^2 - x}.$$

The graph of a function of two variables

Let $z = f(x, y)$, $(x, y) \in D \subseteq \mathbb{R}^2$. The graph of f is a 3D-set (a surface) given by

$$G(f) = \{(x, y, z) \in \mathbb{R}^3 \mid z = f(x, y),\ (x, y) \in D\}$$

Example 1: $z = x + y - 1$.

Example 2: $z = \sqrt{4 - x^2 - y^2}$. (an upper hemisphere)

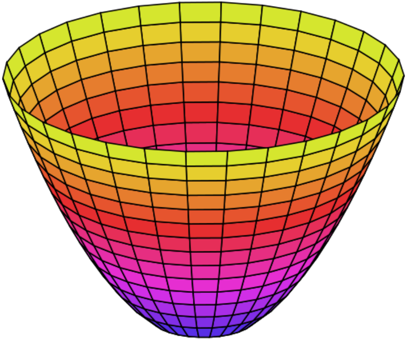
Example 3: $z = 4x^2 + y^2$. (an elliptic paraboloid)

Example 4: $z = 2x^2 - y^2$. (a hyperbolic paraboloid)

Example 5: $z = \sqrt{4 - 2x^2 - y^2}$. (an hemi-ellipsoid)

a hemisphere

$$z = \sqrt{4 - x^2 - y^2}$$



an elliptic paraboloid

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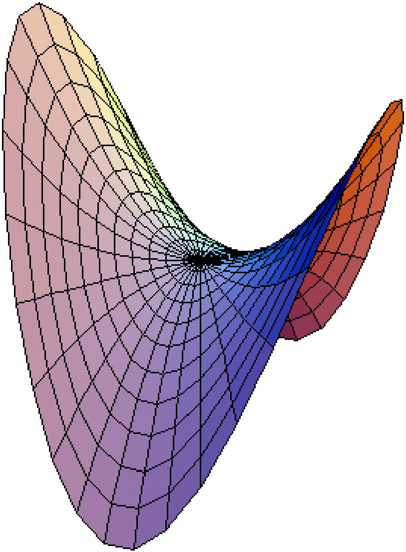

a hyperbolic paraboloid

$$z = 2x^2 - y^2$$

Informal definition of limits

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L.$$

Remark (x,y) can approach (x_0,y_0) along any path that ends up at the point (x_0,y_0).

Example 1:

$$\lim_{(x,y)\to(1,1)} (2x+y) = 2+1 = 3.$$

Example 2:

$$\lim_{(x,y)\to(0,1)} (x^2-y^2) = 0^2-1^2 = -1.$$

* Limit laws (similarly as of function of single variable)

Remark If (x, y) approaches (x_0, y_0) along two paths, however $f(x, y)$ approaches two different values, then there is not limit.

Example: show that the following limits don't exist

1. $$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}.$$

2. $$\lim_{(x,y)\to(0,0)} \frac{xy}{xy + y^2}.$$

3. $$\lim_{(x,y)\to(0,0)} \frac{4xy}{xy + y^3}.$$

Continuity

A function $f(x, y)$ is continuous at (x_0, y_0) if the following hold,

1. $f(x, y)$ is defined at (x_0, y_0);
2. $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists;
3. $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

Example 1: The function $f(x, y) = x^2 + 2x + y^2 - 1$ is continuous at $(0, 0)$.
Example 2: The function

$$f(x, y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & \text{for} \quad (x, y) \neq (0, 0) \\ 0 & \text{for} \quad (x, y) = (0, 0) \end{cases}$$

continuous at any point except $(0, 0)$.
Remark The composition of continuous functions is also continuous. Example $f(x, y) = e^{x^2+y^2}$. ($f = e^u$, with $u = x^2 + y^2$)

Partial derivatives

Definition

Let f is a function of x, y. Then its partial derivatives at (x_0, y_0) are

$$\frac{d}{dx} f(x, y_0)|_{x=x_0} := f_x(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0) \ (\text{w.r.t. } x),$$

$$\frac{d}{dy} f(x_0, y)|_{y=y_0} := f_y(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) \ (\text{w.r.t. } y)$$

Example: find the partial derivatives of

$$f(x, y) = x^3 + 2x^2 + xy^2 + 3y - 1$$

Remark To compute f_x, we treat y as constant, and differentiate f with respect to x.

Higher-order partial derivatives

We can define the second partial derivatives

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}, f_{yy} = \frac{\partial^2 f}{\partial y^2}, f_{xy} = \frac{\partial^2 f}{\partial y \partial x}, f_{yx} = \frac{\partial^2 f}{\partial x \partial y}.$$

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Example 1: $f(x, y) = x^3 + 2x^2 + xy^2 + 3y - 1$

The mixed derivative theorem

If the function $f(x, y)$ has the mixed derivatives f_{xy}, f_{yx} which are both continuous in a neighborhood of (x_0, y_0), then we have

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0).$$

Differentiability

Consider the function $z = f(x, y)$ and the $(x_0, y_0) \in Df$.
Let $\triangle x = x - x_0$, $\triangle y = y - y_0$ and
$\triangle z = f(x, y) - f(x_0, y_0) = f(x_0 + \triangle x, y_0 + \triangle y) - f(x_0, y_0)$.

Definition

The function $z = f(x, y)$ is called differentiable at (x_0, y_0) if $\triangle z$ can be expressed in the form

$$\triangle z = [A\triangle x + B\triangle y] + \varepsilon_1 \triangle x + \varepsilon_2 \triangle y,$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\triangle x, \triangle y) \to (0, 0)$.
Then, the **differential** (also called the total differential) of f is denoted by dz, or df, and defined by

$$dz(x_0, y_0) = [A\triangle x + B\triangle y].$$

Remark: $\triangle z \cong dz(x_0, y_0)$ as $\triangle x, \triangle y$ small.

Theorem

If $z = f(x, y)$ has the derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ which are defined in a neighborhood of (x_0, y_0) and continuous at (x_0, y_0), then f is differentiable at (x_0, y_0).
Moreover $A = \frac{\partial f}{\partial x}(x_0, y_0)$, $B = \frac{\partial f}{\partial y}(x_0, y_0)$, and so

$$dz(x_0, y_0) = \left[\frac{\partial f}{\partial x}(x_0, y_0)\triangle x + \frac{\partial f}{\partial y}(x_0, y_0)\triangle y\right].$$

Remark: if x and y are independent variable then $dx = \triangle x$, $dy = \triangle y$, so we get the small increments formula

$$\triangle z \cong dz = f_x dx + f_y dy.$$

Example : Give the total differential of the function

$$f(x, y) = x^2 + 2xy + xy^2 + 1$$

at the point $(1, 1)$, and then at any point.

Tangent plane

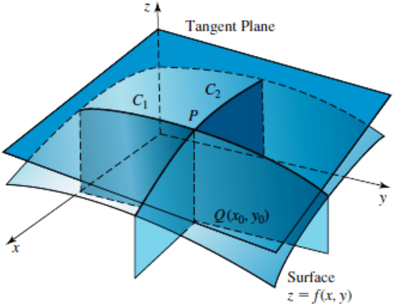

Figure 10.37 The surface $z = f(x, y)$ and its tangent plane at $P = (x_0, y_0, z_0)$.

If there exists the partial derivative $f_x(x_0, y_0)$, $f_y(x_0, y_0)$, then the tangent plane to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0), where $z_0 = f(x_0, y_0)$, has the equation

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Example: find the tangent plane to the surface

$$z = f(x, y) = 4x^2 + y^2$$

at the point $(1, 2, 8)$.

Solution: $z'_x = 8x$, $z'y = 2y$ then $z(1, 2) = 8$, $z_x(1, 2) = 8$, $z_y(1, 2) = 4$.
The equation of the tangent plane at the point $(1, 2, 8)$ is

$$z = 8 + 8(x - 1) + 4(y - 2),$$

or

$$z = 8x + 4y - 8.$$

Linearization of $z = f(x, y)$

Since $\triangle z \cong dz(x_0, y_0)$ then

$$f(x, y) \approx f(x_0, y_0) + dfx_0, y_0),$$

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$

which is the **standard linear approximation** of f at the point (x_0, y_0) by its tangent plane

Example: find the linearization of $f(x, y) = \ln(x - 2y^2)$ at $(3, 1)$ and use it to find an approximation for $f(3.05, 0.95)$.

Partial derivatives of composite function

Example: $z = e^u \ln v,\ u = x^2 y,\ v = x^2 + y^2$

Generalizing ...

Implicit differentiation

Example: consider the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Under certain assumption, the equation

$$F(x,y) = c, \text{ for a constant } c \in \mathbb{R},$$

can determine an unique **implicit function** $y = y(x)$ on some domain and we can express a **implicit derivative**

$$y'(x) = \frac{dy}{dx} = -\frac{F_x}{F_y}.$$

Similarly, the equation $F(x,y,z) = c, c \in \mathbb{R}$, can determine an unique **implicit function** $z = z(x,y)$ on some domain and

$$z_x = \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \; z_y = \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

A vector-valued function is a function from domain $\mathbb{R}^n$ to codomain $\mathbb{R}^m$,

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

$$(x_1, x_2, \ldots, x_n) \mapsto \begin{bmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_m(x_1, x_2, \ldots, x_n) \end{bmatrix}$$

Examples:

$$f(x,y) = \begin{bmatrix} 4x^2 + y^2 \\ 2x + y \end{bmatrix}, \text{ or also } f(x,y) = (4x^2 + y^2, 2x + y);$$

$$g(x,y) = \begin{bmatrix} x \sin y \\ \cos(xy) \\ \sin(x+y) \end{bmatrix}; \quad h(x,y,z) = \begin{bmatrix} \frac{x}{yz} \\ x + y + z \end{bmatrix}.$$

Jacobi matrix

The Jacobi matrix or the derivative matrix of f at $x_0 \in \mathbb{R}^n$ is

$$Df(x_0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} (x_0) \tag{1}$$

Example: find the Jacobi matrix of the function

$$f(x, y) = (4x^2 + y^2, 2x + y).$$