Chapter 3. Line and surface integrals

Phan Quang Sang
sang.phanquang@phenikaa-uni.edu.vn

Department of Mathematics - PKA

November 23, 2022

Content

Parametric representation of a curve C:

$$I \ni t \mapsto \mathbf{r}(t) = [x_1(t), x_2(t), \ldots, x_n(t)] \in R^n.$$

Example:

$n = 2,\ r(t) = [x(t), y(t)] = x(t)\mathbf{i} + y(t)\mathbf{j}.$

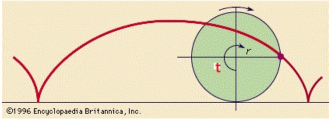
$n = 3,\ r(t) = [x(t), y(t), z(t)] = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$

Some examples:

- Parametric representation for a circle, an ellipse..

- Cycloid curve: a curve traced by a point on a circle being rolled along a straight line

$$x = a(t - \sin t),\ \ y = a(1 - \cos t),\ \ r > 0.$$



The curve C is called a smooth curve if it has at each point a unique tangent whose direction varies continuously as we move along C.

A curve is call piecewise smooth if it consists of finitely many smooth curves.

A (single) curve is a closed curve if its initial point coincides its terminal point.

An oriented curve is a curve where a consistent direction is defined along the curve.

For a simple closed curve, it is a positively oriented curve if when traveling on it one always has the curve interior to the left.

Tangent line to a curve (in parametric representation)

Let $P = \mathbf{r}(t_0)$ on C, then the *Tangent vector* of C at P is given by

$$\mathbf{r}\,'(t_0) = \frac{d\mathbf{r}}{dt}(t_0) = [x'(t_0), y'(t_0), z'(t_0)] \neq 0.$$

Then a para. repr. of the tangent line to C at P is

$$\mathbf{q}(w) = \mathbf{r}(t_0) + w\mathbf{r}'(t_0) \quad (w \text{ is parameter}). \tag{1}$$

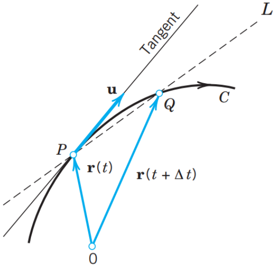
Example

Find a parametric representation of the tangent line to the ellipse (E) $\frac{x^2}{4} + y^2 = 1$ at the point $P(\sqrt{2}, 1/\sqrt{2})$.

Remark:

$$\mathbf{r}\,'(t_0) = \lim_{\Delta t \to 0} \frac{r(t_0 + \Delta t) - \mathbf{r}(t_0)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\overrightarrow{OQ} - \overrightarrow{OP}}{\Delta t},$$

where $Q = \ \mathbf{r}(t_0 + \Delta t)$.



Note that the curve C is called a smooth curve if $\mathbf{r}(t)$ is differentiable and its derivative $\mathbf{r}'(t)$ is continuous and different from the zero vector at every point of C.

Length of a Curve

If the curve C: $\mathbf{r} = \mathbf{r}(t) \in R^n$, $a \le t \le b$, then the length of C is given by

$$L = \int_a^b \sqrt{\mathbf{r}'(t) \bullet \mathbf{r}'(t)}dt.$$

For example, for $n = 3$ then

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}dt.$$

Note: Linear Element ds, for $n = 3$:
We write $d\mathbf{r} = r'(t)dt = [dx, dy, dz] = dx\ \mathbf{i} + dy\ \mathbf{j} + dz\ \mathbf{k}$, then

$$ds^2 = d\mathbf{r} \bullet d\mathbf{r}$$

$$ds = \sqrt{d\mathbf{r} \bullet d\mathbf{r}}$$

Example: Circular Helix, page 386

Work done by a force

The work W done by a constant force F in the displacement along a straight segment d is $W = F \bullet d$.

We try to define the work W done by a variable force F in the displacement along a directed curve C: as the limit of sums of works done in displacements along small chords of C.

This definition of W leads to a new concept-the line integral.

We divide the directed curve C by $n+1$ points A_i, $0 \leq i \leq n$. The the work ΔW_i done by F in the displacement on the chord $A_i A_{i+1}$ is approximated by

$$\Delta W_i \approx F(M_i) \bullet \overrightarrow{A_i A_{i+1}},$$

where M_i is some point on the chord $A_i A_{i+1}$. Then the work W is approximated by

$$W = \sum \Delta W_i \approx \sum F(M_i) \bullet \overrightarrow{A_i A_{i+1}}.$$

If when $n \to \infty$ such that all chords are very small, the above sum has a finite limit, then that limit is called the work done by F in the displacement along C.

In mathematics, such a limit is called the **line integral** of F along a directed curve C.

Definition of line integrals

Note that if $r = \mathbf{r}(t)$ is a parametric representation of C then

$$\Delta W_i \approx F(\mathbf{r}(t_i)) \bullet \mathbf{r}'(t_i)\Delta t_i, \quad \text{here} \quad \Delta t_i = t_{i+1} - t_i.$$

Definition

Let C be a piecewise smooth directed curve. Suppose that $r = \mathbf{r}(t)$, $a \leq t \leq b$ is a parametric representation of C. Then we define the line integral of a vector function $F = F(r(t))$ over C by

$$\int_C F(\mathbf{r}) \cdot d\mathbf{r} = \int_a^b F(\mathbf{r}(t)) \bullet \mathbf{r}'(t)dt \qquad (2)$$

Remark: one can show that if F is continuous on C (i.e. $F(\mathbf{r})$ is continuous on $[a, b]$) then the above line integral exists.

If C is a closed cuvre then the line integral is denoted by

$$\oint_C F(\mathbf{r}) \cdot d\mathbf{r}$$

For $n = 3$: $F = (F_1, F_2, F_3)$, $r(t) = [x(t), y(t), z(t)]$ and $d\mathbf{r} = [dx, dy, dz] = [x'(t), y'(t), z'(t)]dt$ so the line integral can be also written in the form

$$\int_C F(\mathbf{r}) \cdot d\mathbf{r} = \int_C F_1 dx + F_2 dy + F_3 dz$$

$$= \int_a^b [F_1 x'(t) + F_2 y'(t) + F_3 z'(t)]dt.$$

Some examples: page 415

Properties of line integrals

1. $\int_C k\mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = k \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$, here k is a constant.
2. $\int_C [\mathbf{F}(\mathbf{r}) + \mathbf{G}(\mathbf{r})] \cdot d\mathbf{r} = \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} + \int_C \mathbf{G}(\mathbf{r}) \cdot d\mathbf{r}$.
3. If the path C is subdivided into two arcs C_1 and C_2 and that have the same orientation as C then
$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_{C_1} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} + \int_{C_2} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}.$$
4. The line integral does not depend on parametric representation of C, i.e. any representations of C that give the same positive direction on C also yield the same value of the line integral.
5. The line integral over a closed curve with multiple boundary components.

- **Representation in *xyz* coordinates system**: a surface S is given by an equation

$$z = f(x, y) \quad \text{or} \quad g(x, y, z) = 0.$$

- **Parametric representation** of a surface:

$$\mathbf{r}(u, v) = [x(u, v), y(u, v), z(u, v)] = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k},$$

for $(u, v) \in D \subset \mathbb{R}^2$, where u and v are two parameters.

Some examples: pages 440-441

Tangent plane and normal vector of a surface

If $r = \mathbf{r}(u, v)$ is a parametric representation of a surface S, and a fixed point $P = \mathbf{r}(u_0, v_0) \in S$.

Then the partial derivatives $\mathbf{r}'_u(u_0, v_0)$ and $\mathbf{r}'_v(u_0, v_0)$ are two tangent vectors to S at P.

Suppose that they are linearly independents then they span the tangent plane and a normal vector of S at P can be given by their cross product

$$\mathbf{N}(u_0, v_0) = \mathbf{r}'_u(u_0, v_0) \times \mathbf{r}'_u(u_0, v_0) \neq \mathbf{0}.$$

Remark: If S is given by the equation $g(x, y, z) = 0$ then by Theo. 2 Sec. 9.7 a normal vector of S at P is

$$\mathbf{N}(P) = \mathrm{grad(g)(P)} = \nabla \mathrm{g(P)}$$

($\nabla g \perp r'(t)$ if $r = r(t)$ is an any curve C on S through P)

Remark: a unit normal vector can be taken by

$$\mathbf{n} = \frac{\mathbf{N}}{|\mathbf{N}|}$$

The surface S is called to be smooth if the two tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ are linearly independents at any point $P \in S$ and the unit normal vector $\mathbf{n}$ depends continuously on P.

A surface S is called piecewise smooth if it consists of finitely many smooth surfaces.

Orientation of Surfaces

A smooth surface is said to be **orientable** if it is possible to make a consistent choice of surface normal vector at every point, i.e. there exists a unit normal normal $\mathbf{n}$ that continuously varies over S.

The direction of S is then defined by the direction of the unit normal vector, called the **positive direction**. The opposite direction is called the negative direction

Suppose that the surface S is oriented with $C = \partial S$.
We say that the positive direction on C is suitable to the direction on the surface S if if a persons stands upright in the direction of the normal vector and goes along the positive direction of the boundary C then the domain S is to the left of that person.

Flux through a Surface

In flow problems, the flux across a surface S of a vector field $F = \rho v$ is the mass of fluid crossing S per unit time:

$$\Phi = F \bullet n \, |S|,$$

where ρ is the density of the fluid and v is the velocity vector of the flow.

This flux leads to the definition of surface integrals.
It is a algebraic quantity: it can be positive or negative, depending on the direction of v and n.

Definition of surface integrals

Let S be a (positively) directed surface by a normal vector $\widetilde{\mathbf{n}}$. We say that a param. repr. $\mathbf{r}(u, v)$, $(u, v) \in D \subset R^2$, is **suitable to the positive direction of** S if the normal vector $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$ has the same direc. as $\widetilde{\mathbf{n}}$.

Definition

Let $\mathbf{N}$ as above, and $\mathbf{n} = \frac{1}{|\mathbf{N}|}\mathbf{N}$. We define and denote the surface integral (of second type) of a vector function $\mathbf{F} = \mathbf{F}(\mathbf{r})$ over S by

$$\iint_S \mathbf{F} \cdot \mathbf{n} dA = \iint_D \mathbf{F}(\mathbf{r}(u, v)) \bullet \mathbf{N}(u, v) du dv. \qquad (3)$$

Remark: $dA = |N| du dv = |\mathbf{r}_u \times \mathbf{r}_v| du dv$ is called the **area element** on S, and we have

$$\mathbf{n} dA = \mathbf{n}|N| du dv = \mathbf{N} du dv.$$

Let $\mathbf{F} = [F_1, F_2, F_3]$, $\mathbf{N} = [N_1, N_2, N_3]$ then the surface integral

$$\iint_S \mathbf{F} \cdot \mathbf{n} dA = \iint_D (F_1 N_1 + F_2 N_2 + F_3 N_3) du dv. \quad (4)$$

Note that

$$\mathbf{n} = (\cos\alpha, \cos\beta, \cos\gamma),$$

where α, β, γ are the angles between $\mathbf{n}$ and the coord. axes:

$$|\mathbf{n}|^2 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

$$\iint_S \mathbf{F} \cdot \mathbf{n} dA = \iint_S (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma) dA \quad (5)$$

Therefore the surface integral can be also denoted in the form

$$\iint_S (F_1 dydz + F_2 dzdx + F_3 dxdy). \quad (6)$$

Remark: the surface integral has similar properties as double integrals. However if we change the direction of S then the surface integral will change the sign.

Suppose that S is given by the equation

$$z = f(x, y), \ (x, y) \in D.$$

If $\cos\gamma > 0$ then

$$\iint_S F_3 dxdy = \iint_D F_3(x, y, f(x, y))dxdy \tag{7}$$

If $\cos\gamma < 0$ then

$$\iint_S F_3 dxdy = -\iint_D F_3(x, y, f(x, y))dxdy \tag{8}$$

Some examples

Example 1: (Ex. 1, p.444) Compute the flux of water through the parabolic cylinder $S: y = x^2, 0 \le x \le 2, 0 \le z \le 3$ if the velocity vector is $\mathbf{v} = \mathbf{F} = [3z^2, 6, 6xz]$, speed being measured in meters/ sec. (Generally $F = \rho v$, but water has the density $\rho = 1g/cm^3 = 1ton/m^3$.)

Example 2: (Ex. 2, p.445) Evaluate the surface integral for $F = [x^2, 0, 3y^2]$ and S is the portion of the plane $x + y + z = 1$ in the first octant.

Let the function $F = F(x, y, z) : \mathbb{R}^3 \to \mathbb{R}$ is continuous on the smooth (or piecewise smooth) surface S, which is parameterized by $\mathbf{r} = \mathbf{r}(u, v)$, $(u, v) \in D \subset R^2$.

Definition

We denote and define the surface integral of type 1 of F over S by

$$\iint_S F(x, y, z)dA = \iint_D F(\mathbf{r}(u, v))|\mathbf{N}(u, v)|dudv, \qquad (9)$$

where $\mathbf{N}(u, v) = \mathbf{r}'_u \times \mathbf{r}'_v$.

We recall that the area element of the surface is

$$dA = |\mathbf{N}|dudv = |\mathbf{r}'_u \times \mathbf{r}'_v|dudv$$

Remark: here we disregard the orientation of S.

Application of surface integrals of type 1

- **Total mass of of a surface**

If F is the mass density of S, then the above integral (9) gives the total mass of S:

$$M(S) = \iint_S F dA. \tag{10}$$

- **Area of a surface**

If $G \equiv 1$, then (9) gives the surface of S:

$$A(S) = |S| = \iint_S dA = \iint_D |\mathbf{r}'_u \times \mathbf{r}'_v| du dv. \tag{11}$$

Some examples

Example 1: (Ex. 4, p.448) **Area of a Sphere**

Example 2: (Ex. 5, p.448) **Torus surface**
A parametric representation of a torus is

$$\mathbf{r}(u, v) = (a + b\cos v)\cos u\ \mathbf{i} + (a + b\cos v)\sin u\ \ \mathbf{j} + b\sin v\ \mathbf{k},$$

for $u, v \in [0, 2\pi]$.

Suppose that S: $z = f(x, y)$, $(x, y) \in D$. Then we take

$$\mathbf{r}(u, v) = [u, v, f(u, v)].$$

We compute

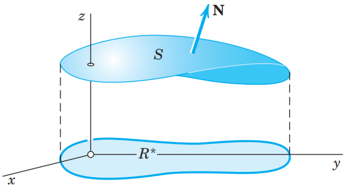
$$\mathbf{r}'_u = [1, 0, f'_u], \quad \mathbf{r}'_v = [0, 1, f'_v],$$

$$\mathbf{N} = \mathbf{r}'_u \times \mathbf{r}'_v = [-f'_u, -f'_v, 1], \quad |\mathbf{N}| = \sqrt{1 + (f'_u)^2 + (f'_v)^2}.$$

The surface integral (9) becomes

$$\iint_S F dA = \iint_D F(x, y, f(x, y)) \sqrt{1 + (f'_x)^2 + (f'_y)^2} dxdy,$$

here D is the is the projection of S into the xy-plane and the normal vector $\mathbf{N}$ on S points up.



If it points down, the integral on the right is preceded by a minus sign.

The area of S is given by the formula

$$A(S) = \iint_D \sqrt{1 + (f_x)^2 + (f_y)^2}\, dxdy. \qquad (12)$$

Let $\mathbf{F} = [F_1, F_2, F_3]$ be the vector function, we recall that

$$div(F) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}. \tag{13}$$

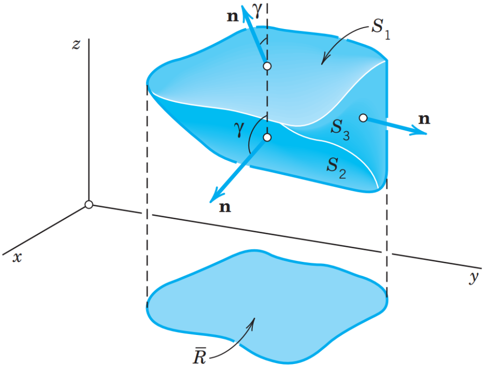
Theorem (Transf. Between Triple and Surface Integrals)

Suppose that $\mathbf{F}$ *is continuous and has continuous first partial derivatives in a closed bounded region* T *in space* $\mathbb{R}^3$*, whose boundary is a piecewise smooth surface* $S = \partial T$*, oriented outwards. Then*

$$\iint_S \mathbf{F} \cdot \mathbf{n} dA = \iiint_T div(F) dV \tag{14}$$

The above formula can be written in the form:

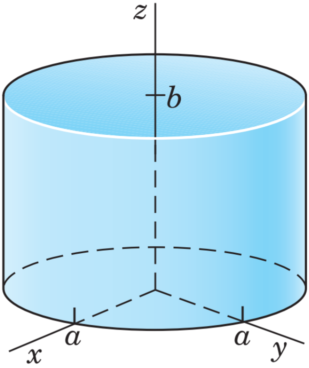
$$\iint_S (F_1 dydz + F_2 dzdx + F_3 dxdy) = \iiint_T \left[\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right] dxdyd \qquad (15)$$



Example: (Ex. 1, p.454) Evaluate

$$I = \iint_S (x^3 dydz + x^2 ydzdx + x^2 zdxdy),$$

where S is the outward closed surface consisting of the cylinder $x^2 + y^2 = a^2$, $0 \leq z \leq b$ and the two circular disks (bases) on $z = 0$ và $z = b$.



Stokes's Theorem

Let $\mathbf{F} = [F_1, F_2, F_3]$ be the vector function, we recall that *curl* (*torsion*) of $\mathbf{F}$ is defined by:

$$\text{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right]. \tag{16}$$

Theorem (Transf. Between Surface and Line Integrals)

Let S be a piecewise smooth orientable surface in space with piecewise smooth boundary $C = \partial S$, oriented suitably to the direction of S. Let $\mathbf{F}$ be a continuous vector function that has continuous first partial derivatives in a domain in space containing S. Then

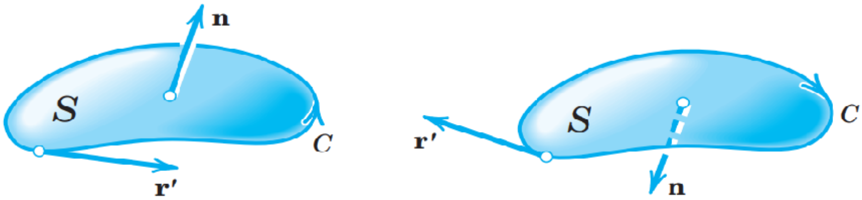
$$\oint_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \iint_S (curl\ \mathbf{F}) \cdot \mathbf{n}\, dA. \tag{17}$$

The line integral of a vector field over a loop is equal to the flux of its curl through the enclosed surface.

The above formula can be written in the form:

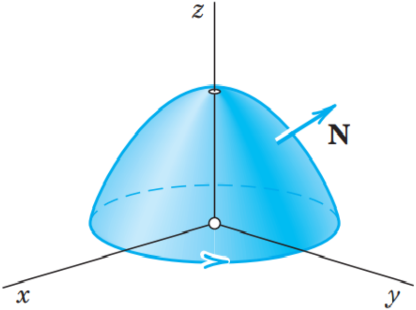
$$\iint_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) dydz + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) dzdx + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \right.$$

$$= \oint_C (F_1 dx + F_2 dy + F_3 dz). \qquad (18)$$



Example: (Ex. 1, p.464) Verify the Stokes's Theorem for F
$S = [y, z, x]$ and S the paraboloid:

$$z = f(x, y) = 1 - x^2 - y^2, \ z \geq 0.$$



Examples: Ex. 2, 3, 5 pages. 466-467