Chapter 2. Multiple integrals

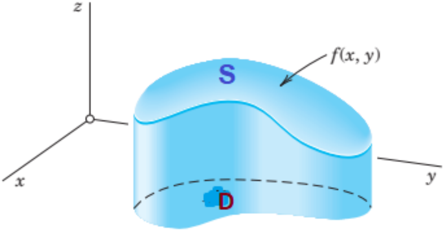
Phan Quang Sang
sang.phanquang@phenikaa-uni.edu.vn

Department of Mathematics - PKA

March 31, 2021

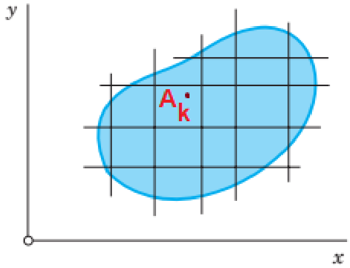
Content

What the volume V of the region under a surface S (and above the xy-plane) is.



A surface can be given as the graph of a function $z = f(x, y)$ over a domain D

We subdivide the region D by drawing parallels to the x- and y-axes into a series of n small rectangles, assumed of area A_k.



Subdivision of a region R

From each of these rectangles we will choose a point (x_k, y_k).

Now, over each of these smaller rectangles we will construct a box whose height is given by $z_k = f(x_k, y_k)$.

The volume of each of these boxes is

$$\Delta V_k = f(x_k, y_k) A_k$$

The volume V is then approximately,

$$V \approx \sum_{k=1}^{n} f(x_k, y_k) A_k$$

In general, the approximation is more explicit (better) if we will take n larger and larger;

and to get the exact volume we will need to take the limit as both n go to infinity, provided that all rectangles are very small.

In other word,

$$V = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k, y_k) A_k$$

In fact, if the function $z = f(x, y)$ is continuous on D, then the above limit exits and does not depend on the subdivision of D and the choice of the points $M_k(x_k, y_k)$.

Definition of double (multiple) integrals

Let the function $z = f(x, y)$ defined on a closed and bounded domain D of $\mathbb{R}^m$, $m = 2$.

1. Partition: we subdivide D into a series of n small regions A_k, assumed also of area A_k, so that $\max_k d_k \to 0$
2. Choosing arbitrary sampling points $M_k(x_k, y_k) \in A_k$, for each $k = 1, 2, \cdots, n$
3. Make the Riemann sum defined by $\sum_{k=1}^{n} f(x_k, y_k)A_k$
4. The the double integral of f over D is defined by the limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k)A_k, \text{ denoted by } \iint_D f(x, y)\, dA$$

if it exists, and does not depend on the subdivision of D and the choice of the sampling points $M_k(x_k, y_k)$.

If the double integral exists, then we can subdivide the region D by drawing parallels to the x- and y-axes.

The area element then can be written by $dA = dxdy$, and therefore the double integral is also denoted by

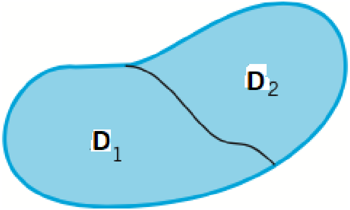
$$\iint_D f(x, y)\, dxdy.$$

Properties of double integrals

* Double integrals have properties similar to those of definite integrals: see Ref. [1], p.427.

$$\iint_D kf(x,y)dxdy = k\iint_D f(x,y)dxdy,\ \ k\in\mathbb{R}$$

$$\iint_D [f(x,y)\pm g(x,y)]dxdy = \iint_D f(x,y)dxdy \pm \iint_D g(x,y)dxdy.$$



If $D = D_1 \cup D_2$ such that the interiors of D_1 and D_2 has not common point then

$$\iint_D f(x,y)dxdy = \iint_{D_1} f(x,y)dxdy + \iint_{D_2} f(x,y)dxdy$$

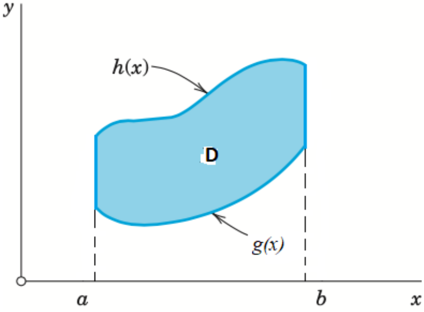
The mean value theorem for double integrals

If the function $f(x, y)$ is continuous on the closed and bounded domain R, then there exists at least one point $(x_0, y_0) \in D$ such that we have

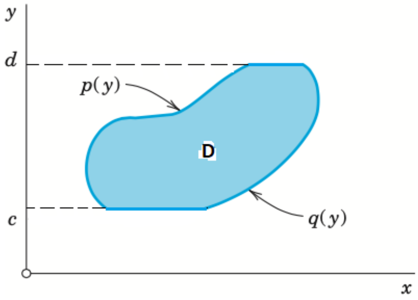
$$\iint_D f(x, y)\ dxdy = f(x_0, y_0)A,$$

where A is the area of D.

By Two Successive Integrations



$$\iint_D f(x,y)dxdy = \int_a^b \left[\int_{g(x)}^{h(x)} f(x,y)dy \right] dx. \qquad (1)$$



$$\iint_D f(x,y)\,dxdy = \int_c^d \left[\int_{p(y)}^{q(y)} f(x,y)\,dx \right] dy. \qquad (2)$$

Example 1: Evaluate the following double integral of $f(x, y) = x^2 + y$ over the domain bounded by the curves $x = 2$, $y = \frac{1}{x}$, $y = x$.

Example 1: Evaluate the following double integral of $f(x, y) = x^2 + y$ over the domain bounded by the curves $x = 2$, $y = \frac{1}{x}$, $y = x$.

$$D = \{(x, y) \in \mathbb{R}^2 |\ 1 \leq x \leq 2,\ \frac{1}{x} \leq y \leq x\}.$$

Then

$$\iint_D f(x, y) dxdy = \int_1^2 \left[\int_{\frac{1}{x}}^{x} (x^2 + y) dy \right] dx.$$

Example 2: Evaluate the double integral

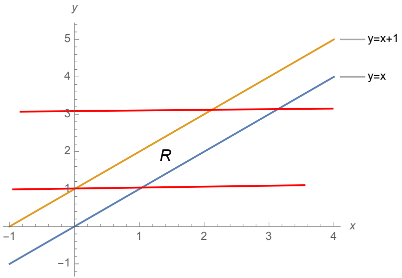
$$I = \iint_D xy \; dxdy,$$

where D is the domain bounded by the curves $y = x$, $y = x + 1$, $y = 1$, and $y = 3$.

Example 2: Evaluate the double integral

$$I = \iint_D xy \ dxdy,$$

where D is the domain bounded by the curves $y = x$, $y = x + 1$, $y = 1$, and $y = 3$.



$$I = \int_1^3 \left[\int_{y-1}^{y} xy \ dx \right] dy = \frac{20}{3}.$$

$$I = \int_1^3 \left(y \frac{x^2}{2} \Big|_{(y-1)}^{y} \right) dy = \int_1^3 \left(y^2 - \frac{y}{2} \right) dy = \frac{20}{3}.$$

Suppose that $(u, v) \mapsto (x, y)$, where $x = x(u, v), y = y(u, v)$ is a bijective from D^* to D satisfying the Jacobian determinant

$$J \neq 0,$$

here J is the Jacobian determinant given by

$$J = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

Then we have the formula for a change of variables

$$\int_D f(x, y) dxdy = \iint_{D^*} f(x(u, v), y(u, v)) \left|J\right| dudv. \qquad (3)$$

Note*: J can be equal to 0 at a finite number of points.

Example 3: Evaluate the double integral

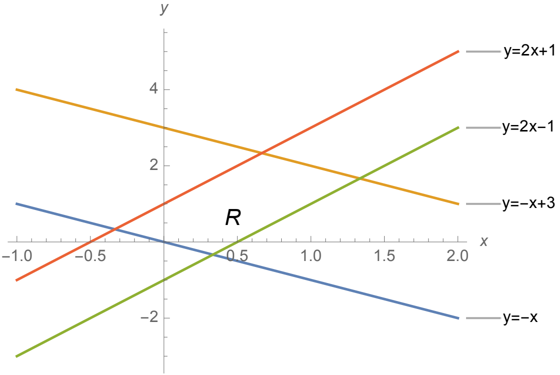
$$I = \iint_D (x + y)\; dxdy,$$

where D is the domain bounded by the curves $y = -x$, $y = -x + 3$, $y = 2x - 1$, and $y = 2x + 1$.

Example 3: Evaluate the double integral

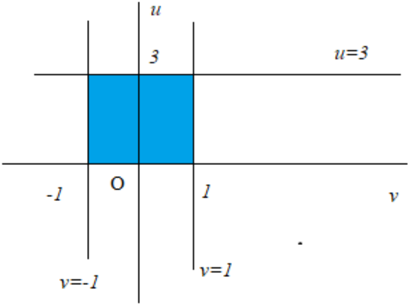
$$I = \iint_D (x+y)\ dxdy,$$

where D is the domain bounded by the curves $y = -x$, $y = -x + 3$, $y = 2x - 1$, and $y = 2x + 1$.



$$u = x + y, v = -2x + y \;\Leftrightarrow\; x = \frac{1}{3}(u - v),\; y = \frac{1}{3}(2u + v)$$

$J = \frac{1}{3} \neq 0$



$$I = \iint_{D^*} u\frac{1}{3}dudv = \frac{1}{3}\int_0^3 udu \int_{-1}^1 dv = 3.$$

Polar coordinates

$$\begin{cases} x & = & r\cos\varphi \\ y & = & r\sin\varphi \end{cases}, \quad r \geq 0,\ 0 \leq \varphi \leq 2\pi.$$

Then $J = r$, and

$$\int_D f(x,y)dxdy = \iint_{D^*} f(r\cos\varphi, r\sin\varphi)r\ drd\varphi. \qquad (4)$$

Example 4: Evaluate the double integral

$$I = \iint_D e^{x^2+y^2}\ dxdy,$$

where

$$D = \{(x,y) \in \mathbb{R}^2 |\ x^2 + y^2 \leq 1\}.$$

Example 5: Evaluate the double integral

$$I = \iint_D \sqrt{x^2 + y^2}\, dxdy,$$

where D is determined by

$$x^2 + y^2 - 2 \geq 0,\ \ x^2 + y^2 - 1 \leq 0,\ \ x \geq 0, y \geq 0.$$

Example 5: Evaluate the double integral

$$I = \iint_D \sqrt{x^2 + y^2}\ dxdy,$$

where D is determined by

$$x^2 + y^2 - 2 \geq 0,\ \ x^2 + y^2 - 1 \leq 0,\ \ x \geq 0, y \geq 0.$$

$$I = \int_0^{\pi/6} d\varphi \int_{2\sin\varphi}^{1} r^2\ dr = \frac{1}{3}(\frac{\pi}{6} + 3\sqrt{3} - \frac{16}{3}).$$

1. Geometry: The area of a flat region, the volume of an object
2. Mechanic: the total mass, the center of gravity of a mass, the moments of inertia, the polar moment of inertia (see Ref, p. 429)

The area of a flat region D in $xy-$plan is given by:

$$A = \iint_D dxdy. \tag{5}$$

Example: find the area bounded by the curve

$$(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$$

The area of a flat region D in $xy-$plan is given by:

$$A = \iint_D dxdy. \tag{5}$$

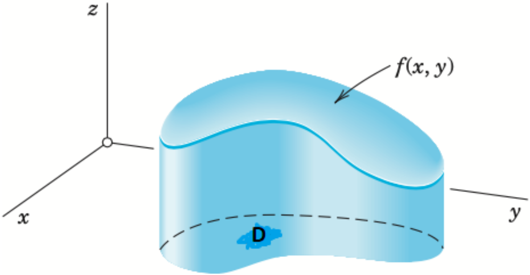
Example: find the area bounded by the curve

$$(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$$

In pol.coord. $r^2 = 2a^2 \cos 2\varphi$.
$D: \ \varphi \in [0; \pi/4], \ 0 \leq r \leq a\sqrt{\cos 2\varphi}$

$$A = 4\int_0^{\pi/4} d\varphi \int_0^{a\sqrt{\cos 2\varphi}} r \ dr = 4a^2 \int_0^{\pi/4} \cos 2\varphi d\varphi = 2a^2$$

The **volume** V beneath the surface $z = f(, y)$ and above a region D in $xy-$plan is



$$V = \iint_D f(x, y)dxdy. \qquad (6)$$

Example: find the volume of the region bounded by the cylinder $x^2 + y^2 = 2x$ and inside the sphere $x^2 + y^2 + z^2 = 4$.

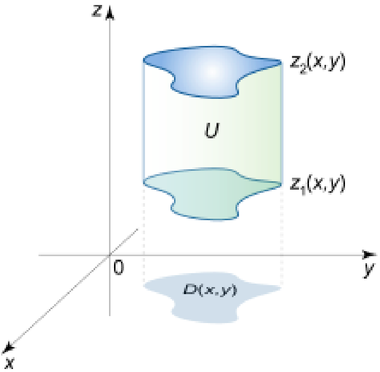
$$V = 4\int_0^{\pi/4} d\varphi \int_0^{2\cos\varphi} r\sqrt{4-r^2}\ dr = \frac{8}{3}(\frac{\pi}{2} - \frac{2}{3}).$$

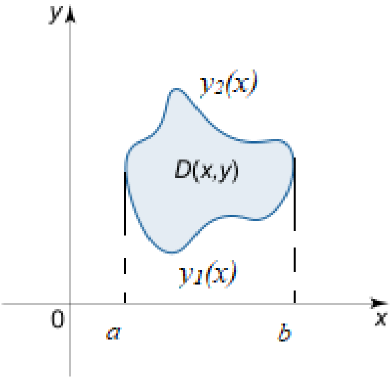
$$I = \iiint_T f(x, y, z)dV, \tag{7}$$

or

$$I = \iiint_T f(x, y, z)dxdydz. \tag{8}$$

Evaluation of Triple Integrals





$$I = \iint_R \left[\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz \right] dxdy. \tag{9}$$

$$= \int_a^b \left\{ \int_{y_1(x)}^{y_2(x)} \left[\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz \right] dy \right\} dx. \tag{10}$$

Example: compute

$$I = \iiint_T x x dx dy dz, \tag{11}$$

where

$$T = \{(x, y, z) \in \mathbb{R}^3 | x, y, z \geq 0,\ \ x + y + z \ \ \leq 1\}.$$

Example: compute

$$I = \iiint_T x x dx dy dz, \tag{11}$$

where

$$T = \{(x, y, z) \in \mathbb{R}^3 | x, y, z \geq 0,\ x + y + z \ \leq 1\}.$$

$$I = \int_0^1 \left\{ \int_0^{1-x} \left[\int_0^{1-x-y} x dz \right] dy \right\} dx = \frac{1}{24}.$$