Chapter 4. Differential equations

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Content

Model leads to a differential equation

Exponential growth: Suppose that the growth rate per capital is a constant,

$$\frac{1}{N}\frac{dN}{dt} = r,$$

where r is a constant.

Note: in the model, the rate of growth is directly proportional to the population size.

A quantity that continuously varies has a relationship with its rates of change and this relationship can be described by a differential equation.

Definition

An ODE is an equation of an unknown function, a variable and its derivatives. The highest order of the derivatives in the Eq. is called the order of the Eq.

Example: An ODE of order 1 is of the form

$$F(x, y, y') = 0, \tag{1}$$

where $y = y(x)$ is the unknown function of independent variable x.

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For a given couple (x_0, y_0), the initial condition is

$$y(x_0) = y_0.$$

Solution of ODE

Example: find the function y so that

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Definition

A function of the general form

$$y = \varphi(x, C),$$

where $C \in \mathbb{R}$ is an arbitrary constant satisfying Eq. (1) is called the general solution of Eq. (1).
Four each specific value $C = C_0$, the function

$$y = \varphi(x, C_0)$$

is called a proper solution of Eq (1).

General integral

Geometrically, each solution of the Eq. expresses a curve on the plan and is called an integral curve.

Definition

If the Eq. (1) can be transformed to an expression of the form

$$\Phi(x, y, C) = 0$$

(*"without derivative"*), here $C \in \mathbb{R}$ an arbitrary constant, then it is called the general integral of Eq. (1).
Similarly, the expression $\Phi(x, y, C_0) = 0$ is called a proper integral of Eq. (1).

Consider the ODE. (1),

$$F(x, y, y') = 0.$$

In particular, the explicit form is

$$y' = f(x, y), \tag{2}$$

or

$$M(x, y)dx + N(x, y)dy = 0.$$

(Note: $y' = \dfrac{dy}{dx}$)

Definition

A separable equation is an ODE. of the form

$$M(x)dx + N(y)dy = 0.$$

The general integral of the form

$$\int M(x)dx + \int N(y)dy = C, \ C \in \mathbb{R}.$$

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Example 1: solve $y'e^x = 1$.
Example 2: solve $(x+1)y' = 2y^2x^2$.
Example 3: solve $x(y^2-1)dx + (x^2-1)dy = 0$.

Definition

A function of two variables $f = f(x, y)$ is called a homogeneous function of order k ($k \in \mathbb{R}$) if

$$f(\lambda x, \lambda y) = \lambda^k f(x, y).$$

Some examples:

$$f = f(x, y) = x^3 - x^2 y + 2xy^2 - y^3.$$

$$g = g(x, y) = x^2(\ln x - \ln y) + 2xy.$$

$$z = f(\frac{y}{x}).$$

Definition

A homogeneous differential equations is an ODE of the form

$$M(x,y)dx + N(x,y)dy = 0,$$

where M, N are homogeneous functions of the same order.

Method: use the change of variable

$$y = xu(x)$$

then we can transform a homogeneous ODE to a separable equation.

Note: $y' = u + xu'$ and $dy = udx + xdu$

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Example 1: solve

$$(x^2 + y^2)dx - (x^2 + xy)dy = 0.$$

Example 2: solve

$$xy' = y \ln \frac{y}{x}.$$

Defition

A linear differential equation is an ODE. of the form

$$y' + p(x)y = q(x), \tag{3}$$

where p, q are functions of x.

Method: Look for solutions of the form $y = u(x)v(v)$, with $v \neq 0$.

$$\begin{aligned} u'v + uv' + p(x)uv &= q(x) \\ u'v + u[v' + p(x)v] &= q(x). \end{aligned} \tag{4}$$

Choose $v \neq 0$ s.th

$$v' + p(x)v = 0 \tag{5}$$

(It is called the corresponding homogeneous equation to Eq. (3), it can be separable)

The Eq. (4) becomes

$$u'v = q(x), \text{ hay}$$

$$u' = \frac{q(x)}{v(x)}.$$

Then,

$$u = \int \frac{q(x)}{v(x)} dx.$$

Example 1: solve

$$y' - 2\frac{y}{x} = x^2 \cos x.$$

Example 2: solve

$$xy' + \frac{1}{x}y = \frac{1}{x}.$$

Definition

Bernoulli equation is a ODE of the form

$$y' + p(x)y = q(x)y^\alpha, \tag{6}$$

here p, q are functions of x, the number $\alpha \neq 0$, $\alpha \neq 1$.

Method: (transform it to a linear equation by changing functions)

Remark:

1. If $\alpha > 0$ then $y = 0$ is a solution.
2. If $\alpha < 0$ then $y \neq 0$ (not a solution).

We look for $y \neq 0$: dividing both sides of (6) by y^{α}:

$$y^{-\alpha}y' + p(x)y^{1-\alpha} = q(x). \tag{7}$$

Let

$$z = z(x) = y^{1-\alpha}.$$

Note that

$$z' = (1-\alpha)y^{-\alpha}y'.$$

Then Eq. (7) becomes

$$\frac{z'}{1-\alpha} + p(x)z = q(x), \text{ or}$$

$$z' + (1-\alpha)p(x)z = (1-\alpha)q(x).$$

This is a linear Eq. we have studied.

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Example: solve

$$y' + 2\frac{y}{x} = \frac{y^3}{x^2}.$$