Иванцова Е.А. БФЗ-19-1 ДЗ-1

1. $\psi(x)=\dfrac{A}{x^2+a^2}$ $A-?$

$$\langle\psi|\psi\rangle=\int_{-\infty}^{+\infty}\frac{A}{x^2+a^2}\cdot\frac{A}{a^2-x^2}dx=A^2\int\frac{dx}{(a^2-x^2)(x^2+a^2)}\ominus$$

$$\frac{1}{(a-x)(a+x)(a^2+x^2)}=\frac{A}{a-x}+\frac{B}{a+x}+\frac{C}{a^2+x^2}$$

$$A(a+x)(a^2+x^2)+B(a-x)(a^2+x^2)+C(a-x)(a+x)=1$$

$x=0$ $Aa^3+Ba^3+Ca^2=1$

$x=-a$ $B4a^3=1$ $B=\frac{1}{4a^3}$

$x=a$ $A4a^3=1$ $A=\frac{1}{4a^3}$

$\frac{1}{4}+\frac{1}{4}+Ca^2=1$ $C=\frac{1}{2a^2}$

$$\ominus-\int_{-\infty}^{+\infty}\frac{A^2dx}{4a^3(x-a)}+\int_{-\infty}^{+\infty}\frac{A^2dx}{4a^3(x+a)}+\int_{-\infty}^{+\infty}\frac{A^2dx}{2a(a^2+x^2)}=-\frac{A^2}{4a^3}\ln|x-a|\Big|_{-\infty}^{+\infty}+$$

$$+\frac{A^2}{4a^3}\ln|x+a|\Big|_{-\infty}^{+\infty}+\frac{A^2}{2a^3}\operatorname{arctg}\frac{x}{a}\Big|_{-\infty}^{+\infty}=\frac{A^2}{4a^3}\lim_{x\to\infty}\ln\left|\frac{x+a}{x-a}\right|+$$

$$+\frac{A^2}{2a^3}\left(\frac{\pi}{2}+\frac{\pi}{2}\right)+\frac{A^2}{4a^3}\lim_{x\to-\infty}\ln\left|\frac{x+a}{x-a}\right|=\frac{\pi A^2}{2a^3}=1\qquad A=\sqrt{\frac{2a^3}{\pi}}$$

$$\lim_{x\to\infty}\frac{x+a}{x-a}=\frac{\infty}{\infty}\underset{\text{Лопиталь}}{=}\lim_{x\to\infty}\frac{1}{1}=1\qquad(\ln 1=0)$$

2. $\psi(x)=\dfrac{B}{x+ib}$ $B-?$

$$\langle\psi|\psi\rangle=\int_{-\infty}^{+\infty}\frac{B}{x+ib}\,\frac{B}{x-ib}dx=B^2\int_{-\infty}^{+\infty}\frac{dx}{x^2+b^2}=\frac{B^2}{b}\operatorname{arctg}\frac{x}{b}\Big|_{-\infty}^{+\infty}=$$

$$=\frac{B^2\pi}{b}=1\qquad B=\sqrt{\frac{b}{\pi}}$$

на простейшие разложим

3. $$\langle\psi|\psi\rangle=A\cdot B\int_{-\infty}^{+\infty}\frac{1}{(x-ib)(x^2+a^2)}dx=A\cdot B\left(\int_{-\infty}^{+\infty}\left(\frac{1}{(a^2-b^2)(x-ib)}-\frac{x+ib}{(a^2-b^2)(x^2+a^2)}\right)\right.$$

$$=\frac{AB}{a^2-b^2}\left(\int_{-\infty}^{+\infty}\frac{dx}{x-ib}-\int_{-\infty}^{+\infty}\frac{x+ib}{x^2+a^2}\right)dx=\frac{AB}{a^2-b^2}\left(\ln|x-ib|\Big|_{-\infty}^{+\infty}-\int_{-\infty}^{+\infty}\frac{xdx}{x^2+a^2}-\int_{-\infty}^{+\infty}\frac{ib}{x^2+a^2}dx\right)$$

$$=\frac{AB}{a^2-b^2}\left(\ln|x-ib|\Big|_{-\infty}^{+\infty}-\frac{\ln|x^2+a^2|}{2}\Big|_{-\infty}^{+\infty}-\frac{ib}{a}\operatorname{arctg}\left(\frac{x}{a}\right)\Big|_{-\infty}^{+\infty}\right)\ominus$$

$x \to \infty$ $x(1+\frac{\ }{x^2})$

$\ominus \frac{AB\, ib\pi}{a(b^2-a^2)} = \left| \begin{array}{l} A = \sqrt{\frac{2a^3}{\pi}} \text{ из 1 номера} \\ B = \sqrt{\frac{b}{\pi}} \text{ из 2 номера} \end{array} \right| = \sqrt{\frac{2a^3 b \pi^2}{\pi a^2 \pi}} \frac{ib}{(b^2-a^2)} = \frac{\sqrt{2ab^3}\, i}{b^2-a^2}$

4. $\delta(f(x)) = \sum_{x_i} \frac{1}{|f'(x_i)|}\delta(x-x_i)$ x_i — нули первого порядка $f(x)$

$f(x) = \sum_{x_i}(\underbrace{f(x_i)}_{\to 0} + f'(x_i)(x-x_i))$

$\delta(f'(x_i)(x-x_i)) = \sum_{x_i}\frac{1}{|f'(x_i)|}\delta(x-x_i)$

$f(x_i) > 0$ $\int_{cf'(x_i)}^{df'(x_i)} \delta(s)\frac{ds}{f'(x_i)} = \frac{1}{f'(x_i)}$ $s = \sum_{x_i} f'(x_i)(x-x_i)$

$f(x_i) < 0$ $-\int_{df'(x_i)}^{cf'(x_i)} \delta(s)\frac{ds}{f'(x_i)} = -\frac{1}{f'(x_i)}$

5. $f_1(x) = \alpha_1 e^{\frac{i\pi x}{a}}$ $f_2(x) = \alpha_2 e^{-\frac{i\pi x}{a}}$ $x \in [0, a]$

а) $\langle f_1|f_2\rangle = \int_0^a \alpha_1 e^{-\frac{i\pi x}{a}} \alpha_2 e^{-\frac{i\pi x}{a}} dx = -\alpha_1\alpha_2 \frac{a}{2i\pi} e^{-\frac{2i\pi x}{a}}\Big|_0^a =$

$= -\frac{\alpha_1\alpha_2 a}{2i\pi}(\overbrace{e^{-2i\pi} - 1}^{=0}) = 0$

$e^{-2i\pi} = \cos(2\pi) - i\underbrace{\sin(2\pi)}_{\to 0} = 1$

б) $\langle f_1|f_1\rangle = \int_0^a \alpha_1 e^{\frac{i\pi x}{a}} \alpha_1 e^{-\frac{i\pi x}{a}} dx = \alpha_1^2 x\Big|_0^a = \alpha_1^2 a = 1$ $\alpha_1 = \sqrt{\frac{1}{a}}$

$\langle f_2|f_2\rangle = \int_0^a \alpha_2 e^{-i\frac{\pi x}{a}} \alpha_2 e^{\frac{i\pi x}{a}} dx = \alpha_2^2 x\Big|_0^a = \alpha_2^2 a = 1$ $\alpha_2 = \sqrt{\frac{1}{a}}$

в) $\psi(x) = \sqrt{\frac{2}{a}} \sin\frac{\pi x}{a}$ $|\psi\rangle = c_1|f_1\rangle + c_2|f_2\rangle$

① $\langle f_1|\psi\rangle = \int_0^a \sqrt{\frac{1}{a}} e^{\frac{i\pi x}{a}} \sqrt{\frac{2}{a}} \sin\frac{\pi x}{a} dx$ = сложно

2 путь

$\sqrt{\frac{2}{a}}\sin\frac{\pi x}{a} = \sqrt{\frac{2}{a}}\frac{e^{\frac{i\pi x}{a}}}{2i} - \sqrt{\frac{2}{a}}\frac{e^{-\frac{i\pi x}{a}}}{2i}$ $f_1(x) = \sqrt{\frac{1}{a}} e^{\frac{i\pi x}{a}}$ $f_2(x) = \sqrt{\frac{1}{a}} e^{-\frac{i\pi x}{a}}$

$\frac{c_1}{\sqrt{a}} = \frac{1}{\sqrt{a}\sqrt{2}}$ $c_1 = \frac{1}{i\sqrt{2}}$

$$\frac{c_2}{\sqrt{a}} = -\frac{1}{\sqrt{2}\, i \sqrt{a}}$$

$$c_2 = -\frac{1}{i\sqrt{2}}$$

$$\psi = \frac{1}{i\sqrt{2}} |f_1\rangle - \frac{1}{i\sqrt{2}} |f_2\rangle$$