CSCI 104
B-Trees (2-3, 2-3-4) and Red/Black Trees

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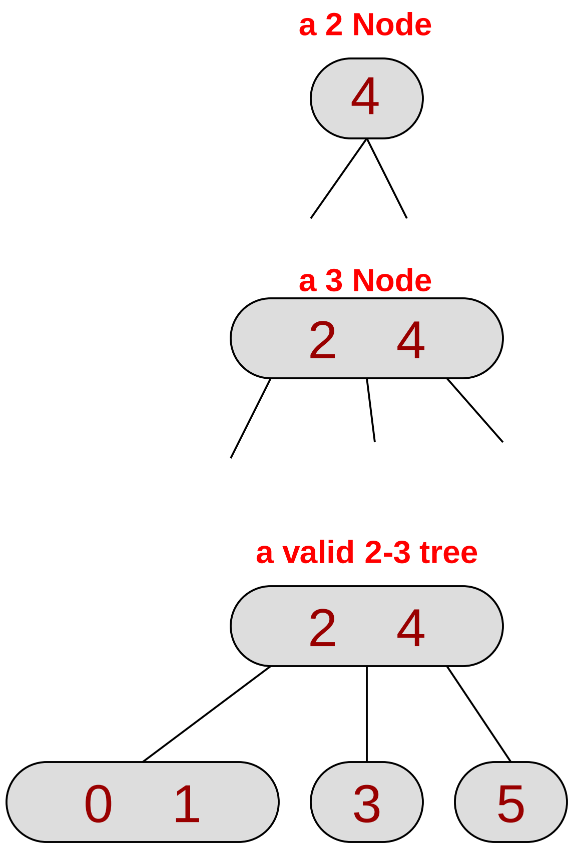
An example of B-Trees

2-3 TREES

Definition

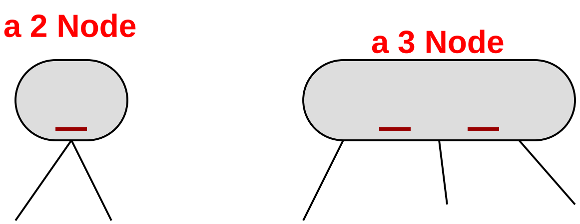
- 2-3 Tree is a tree where
  - Non-leaf nodes have 1 value & 2 children or 2 values and 3 children
  - All leaves are at the same level
- Following the line of reasoning…
  - All leaves at the same level with internal nodes having at least 2 children implies a (**full / complete**) tree
    - FULL (Recall complete just means the lower level is filled left to right but not full)
  - A full tree with n nodes implies…
    - Height that is bounded by $\log_2(n)$



Implementation of 2- & 3-Nodes

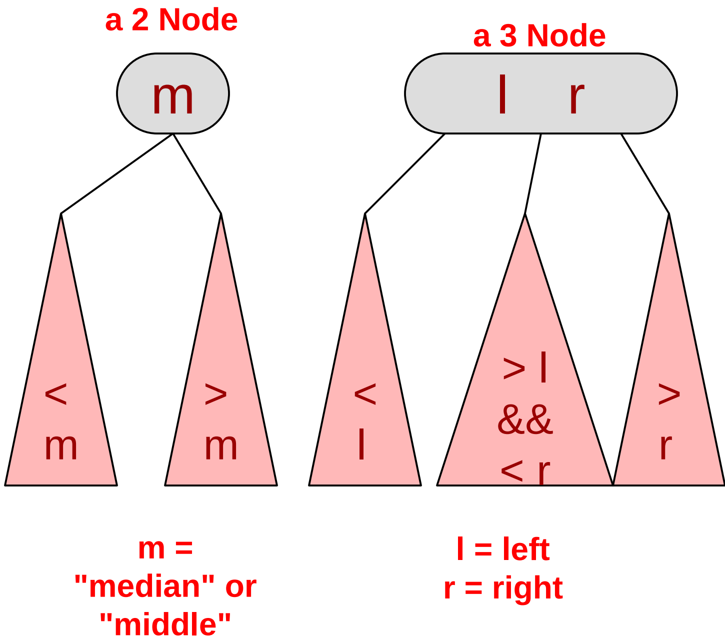
- You will see that at different times 2 nodes may have to be upgraded to 3 nodes
- To model these nodes we plan for the worst case…a 3 node
- This requires wasted storage for 2 nodes

```
template <typename T>
struct Item23 {
  T val1;
  T val2;
  Item23<T>* left;
  Item23<T>* mid;
  Item23<T>* right;
  bool twoNode;
};
```



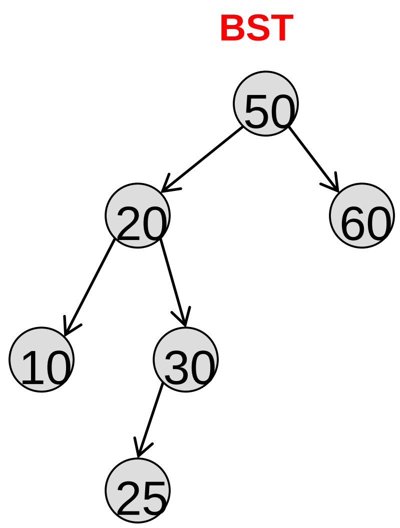
2-3 Search Trees

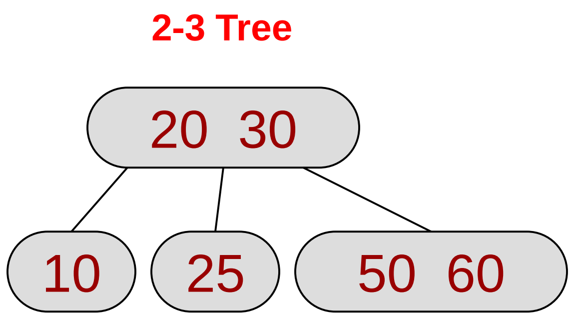
- Similar properties as a BST
- 2-3 Search Tree
  - If a 2 Node with value, *m*
    - Left subtree nodes are < node value
    - Right subtree nodes are > node value
  - If a 3 Node with value, *l* and *r*
    - Left subtree nodes are < *l*
    - Middle subtree > *l* and < *r*
    - Right subtree nodes are > *r*
- 2-3 Trees are almost always used as search trees, so from now on if we say 2-3 tree we mean 2-3 search tree



2-3 Search Tree

- Binary search tree compared to 2-3 tree
- Check if 55 is in the tree?

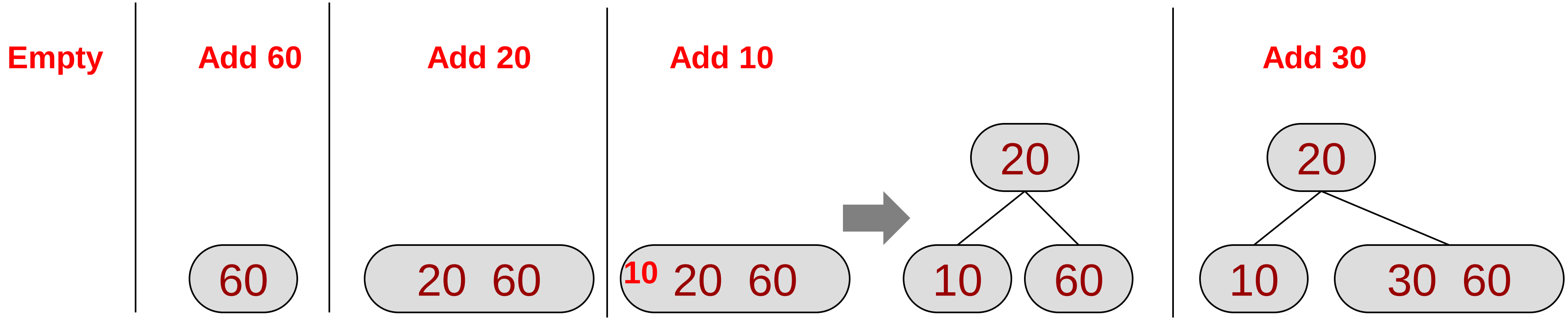




2-3 Insertion Algorithm

- Key: Since all leaves must be at the same level ("leaves always have their feet on the ground"), insertion causes the tree to "grow upward"
- To insert a value,
  - 1. walk the tree to a leaf using your search approach
  - 2a. If the leaf is a 2-node (i.e.1 value), add the new value to that node
  - 2b. Else break the 3-node into two 2-nodes with the smallest value as the left, biggest as the right, and median value promoted to the parent with smallest and biggest node added as children of the parent
  - Repeat step 2(a or b) for the parent
- Insert 60, 20, 10, 30, 25, 50, 80

Key: Any time a node accumulates 3 values, split it into single valued nodes (i.e. 2-nodes) and promote the median



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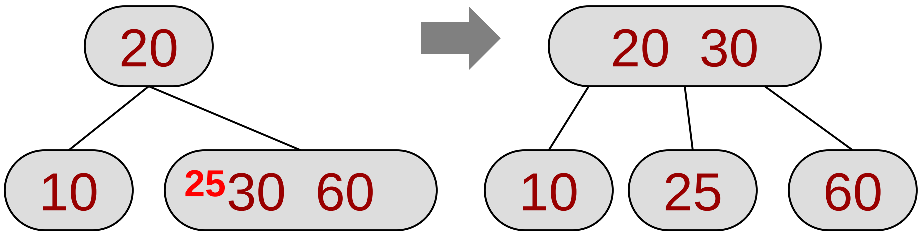
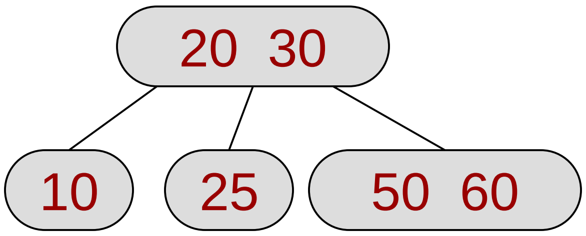
Add 25



Add 50



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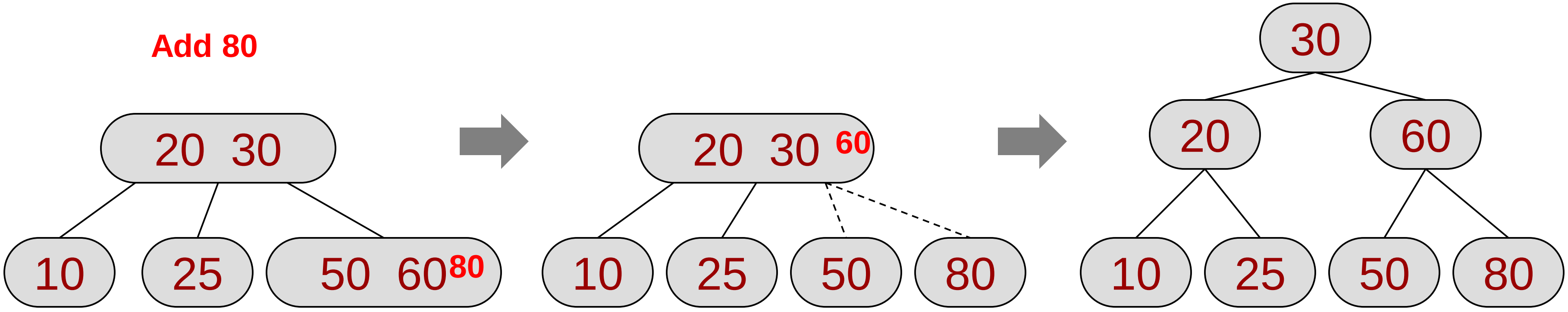

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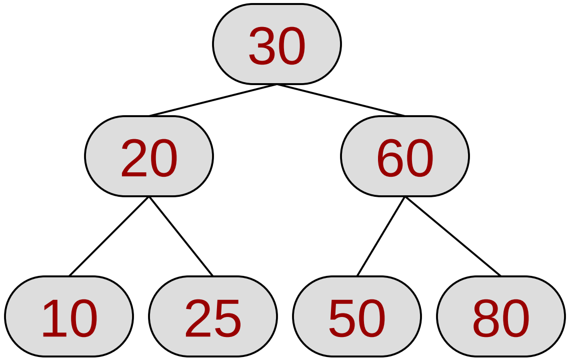
- Insert 90,91,92, 93

Add 90



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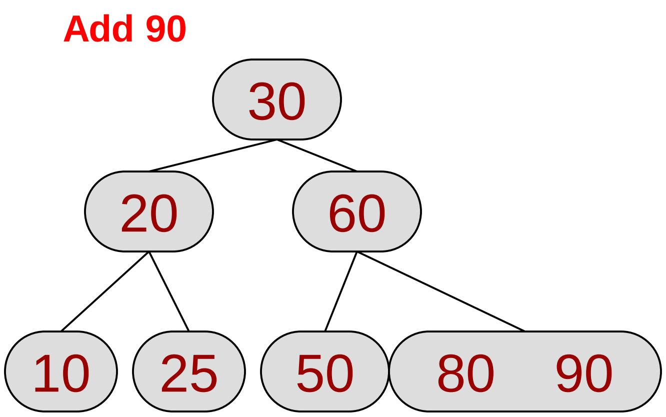

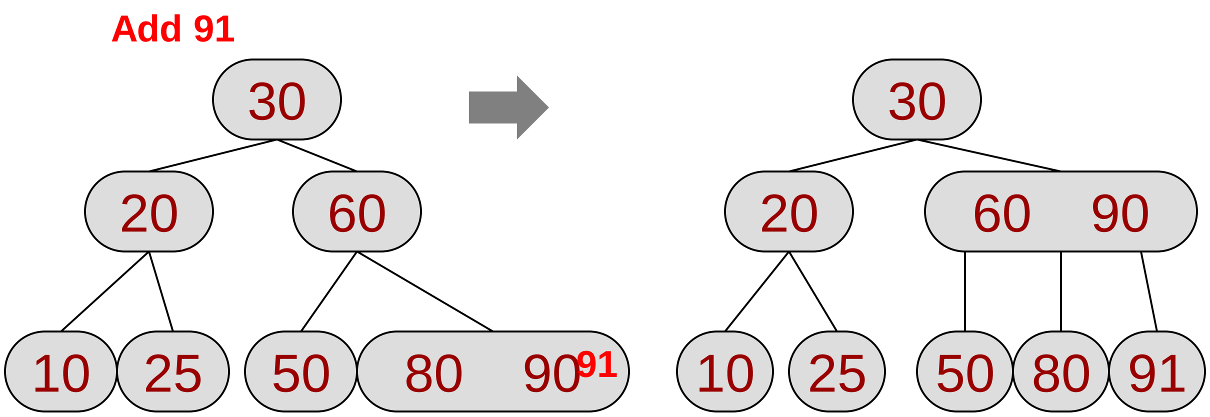

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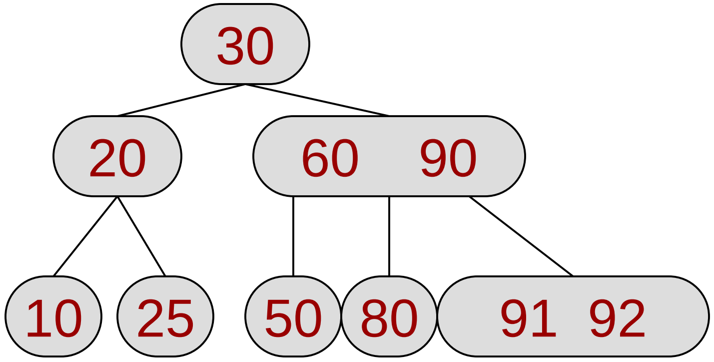

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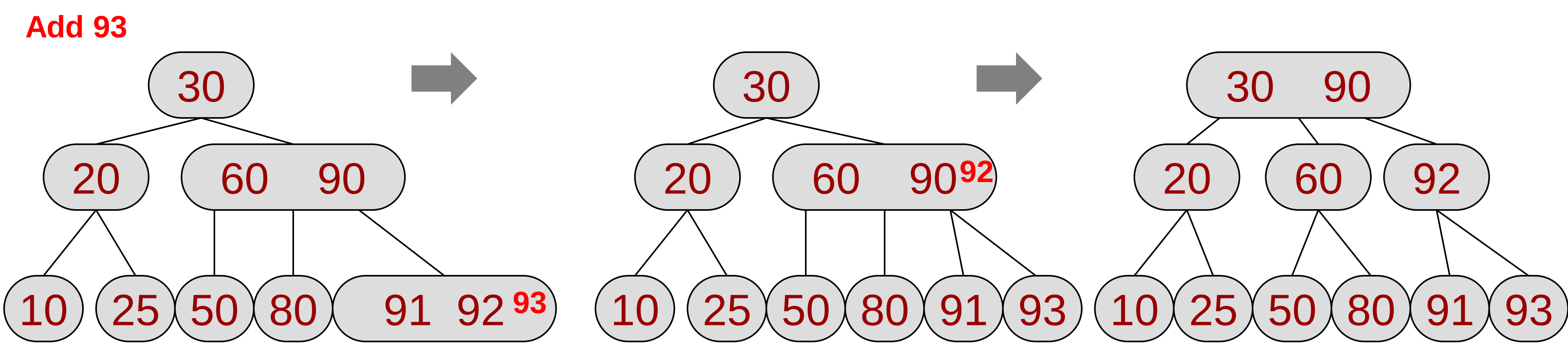


2-3 Tree Removal

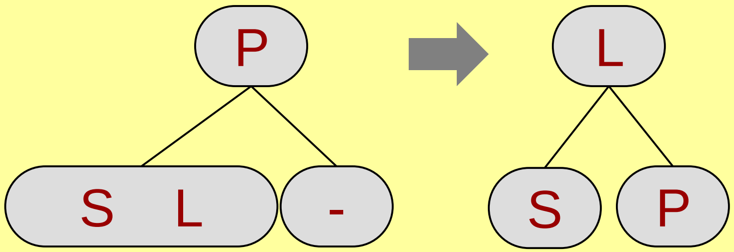
- Key: 2-3 Trees must remain "full" (leaf nodes all at the same level)
- Remove
  - 1. Find data item to remove
  - 2. If data item is not in a leaf node, find in-order successor (which is in a leaf node) and swap values (it's safe to put successor in your location)
  - 3. Remove item from the leaf node
  - 4. If leaf node is now empty, call fixTree(leafNode)

Another key: Want to get item to remove down to a leaf and then work up the tree

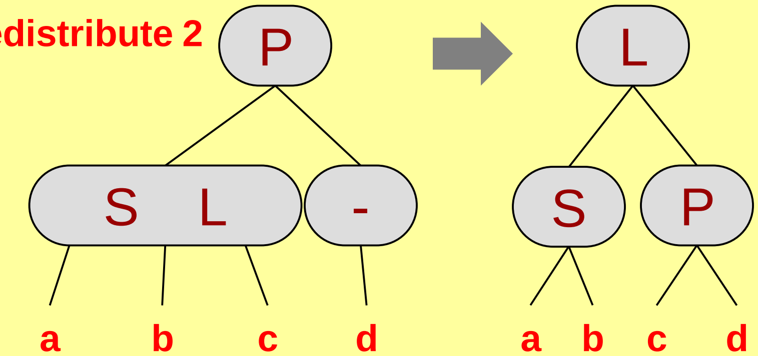
- fixTree(n)
  - If n is root, delete root and return
  - Let p be the parent of n
  - If a sibling of n has two items
    - Redistribute items between n, sibling, and p and move any appropriate child from sibling to n
  - Else
    - Choose a sibling, s, of n and bring an item from p into s redistributing any children of n to s
    - Remove node n
    - If parent is empty, fixTree(p)

Remove Cases

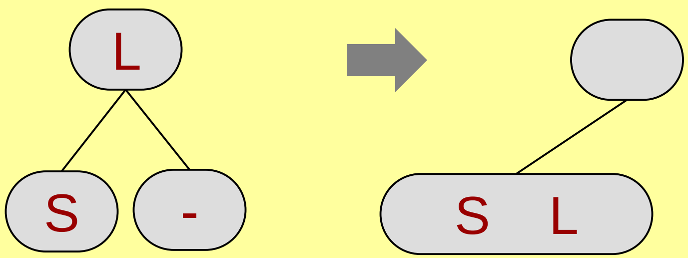
Redistribute 1



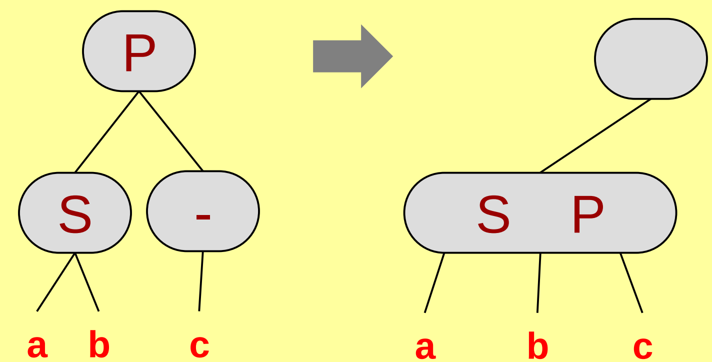
Redistribute 2



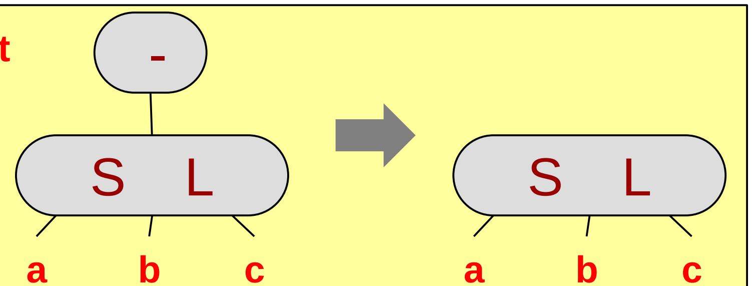
Merge 1



Merge 2



Empty root

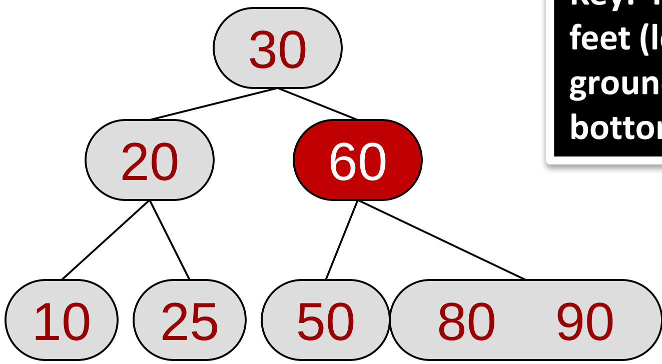


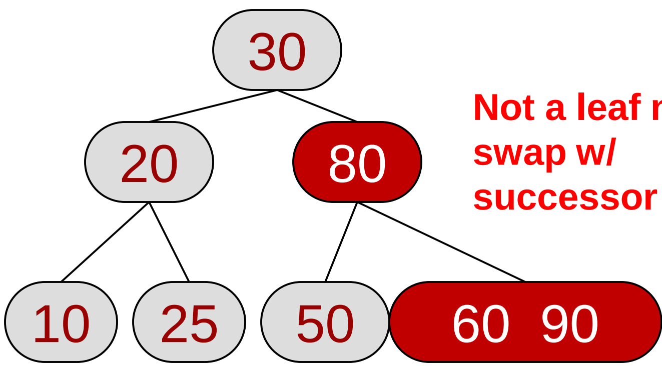
P = parent
S = smaller
L = larger

Remove Examples

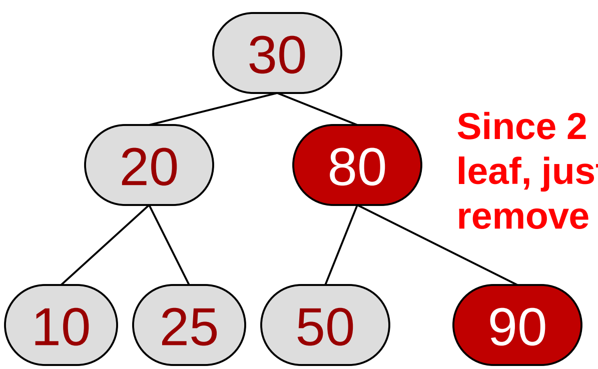
Key: Keep all your feet (leaves) on the ground (on the bottom row)

Remove 60



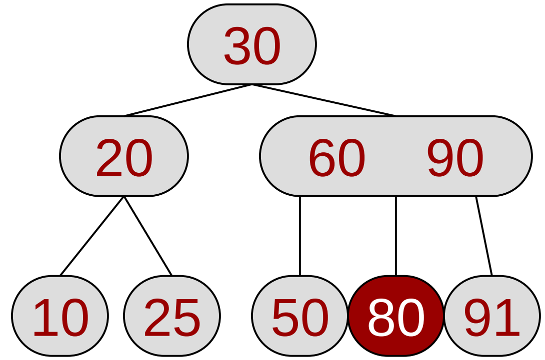


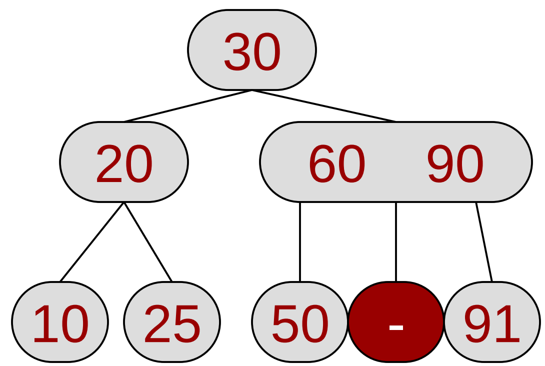
Not a leaf node so swap w/ successor at leaf



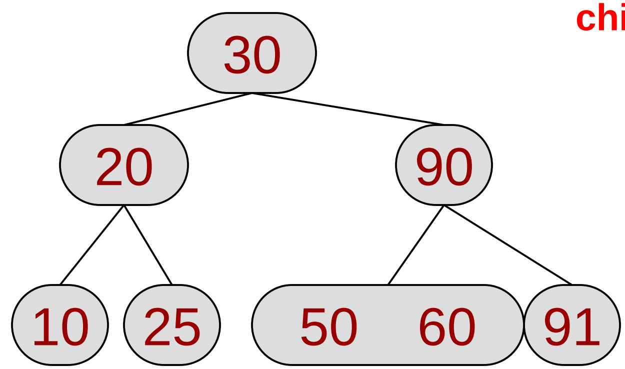
Since 2 items at leaf, just remove 60

Remove 80





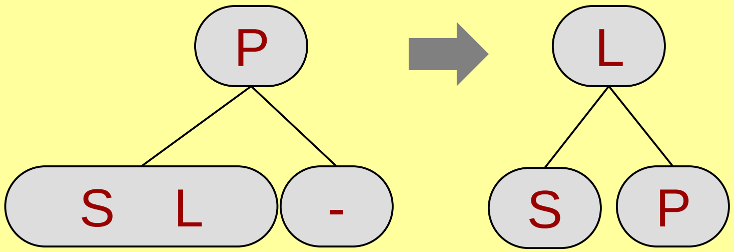
Can't just delete because a 3-node would have only 2 children



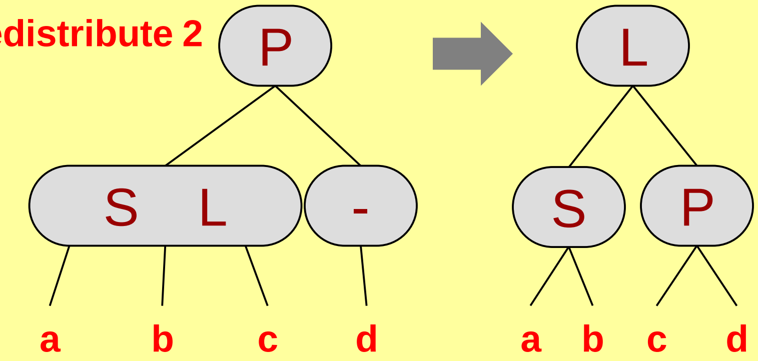
Rotate 60 down into 50 to make a 3-node at the leaf and 2-node parent

Remove Cases

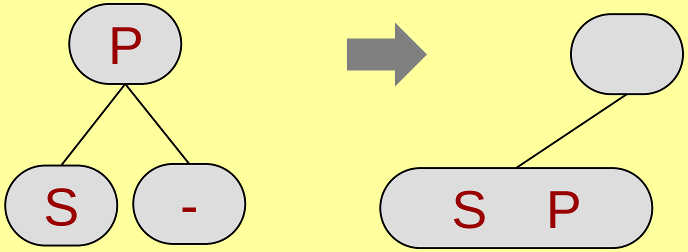
Redistribute 1



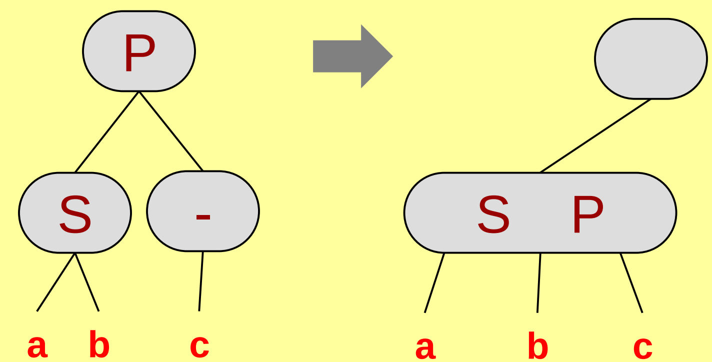
Redistribute 2



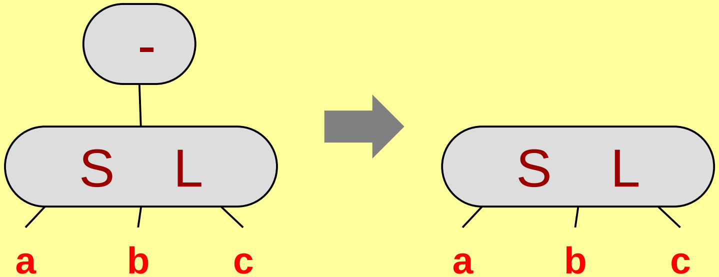
Merge 1



Merge 2

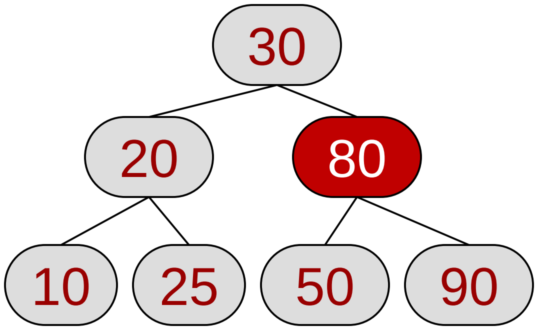


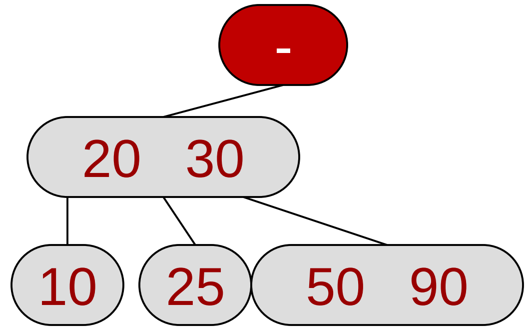
Empty root



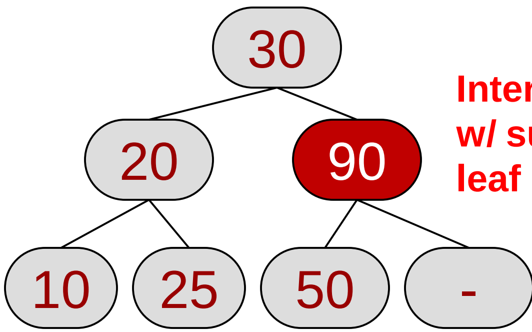
Remove Examples

Remove 80

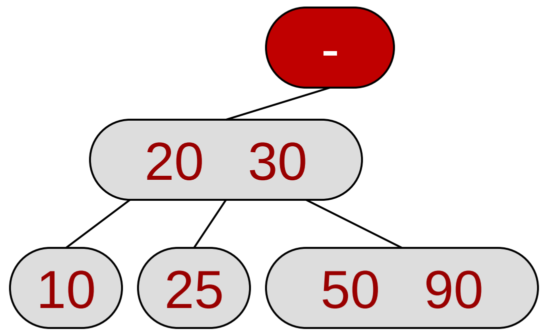




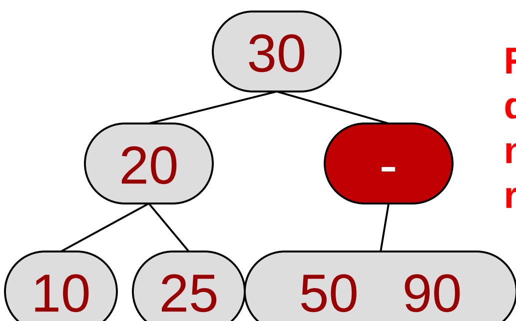
Rotate parent down and empty node up, then recurse



Internal so swap w/ successor at leaf



Remove root and thus height of tree decreases



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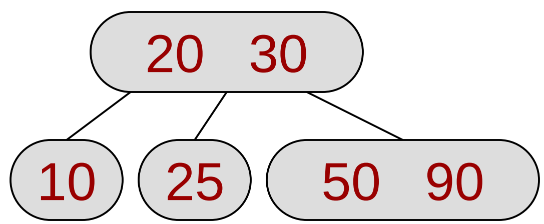

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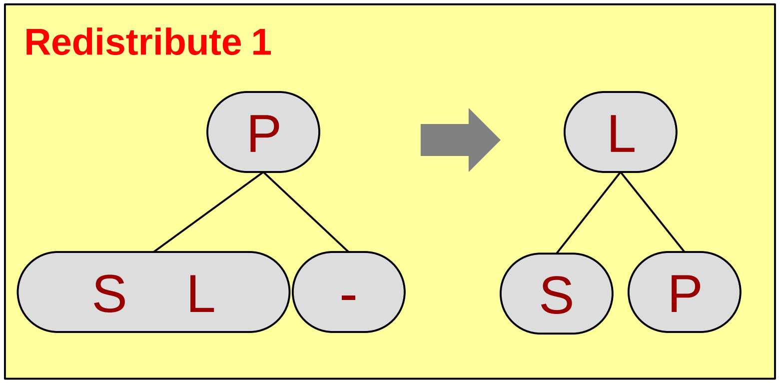
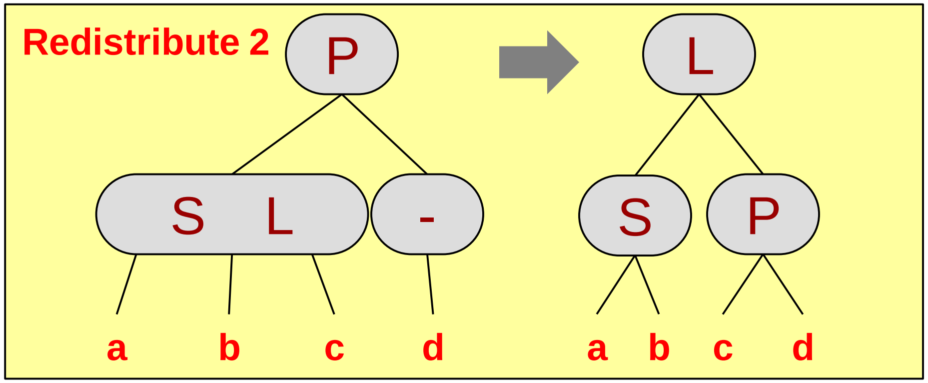
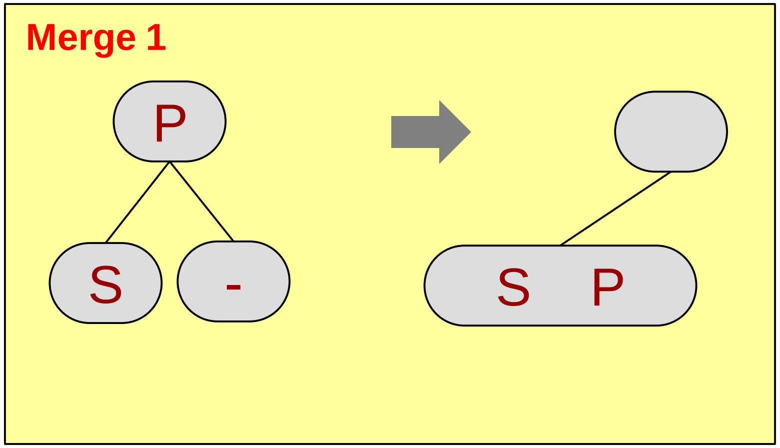
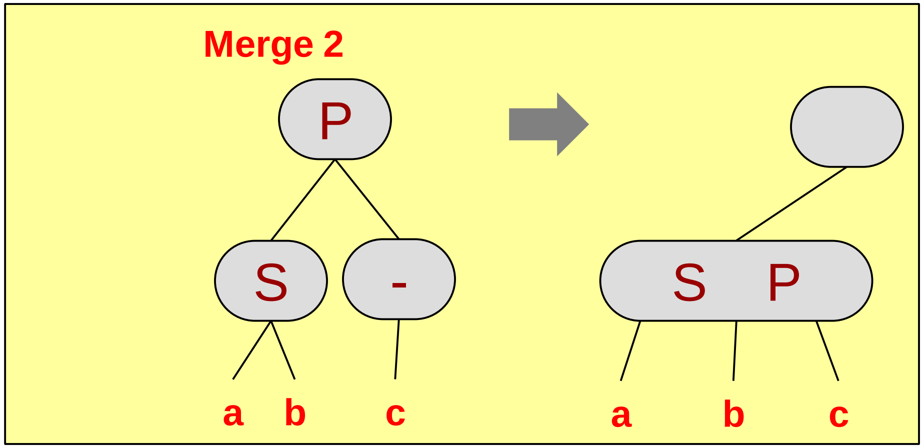
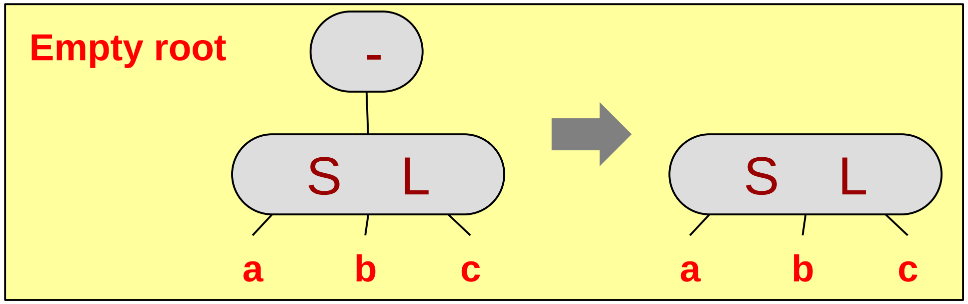
Remove Cases











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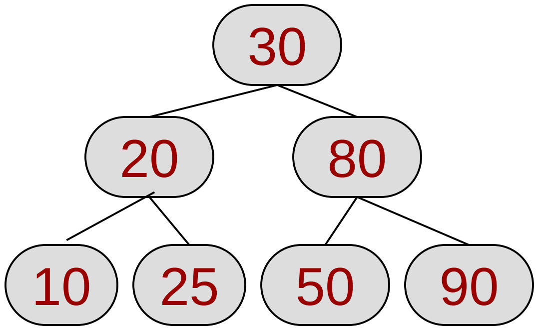
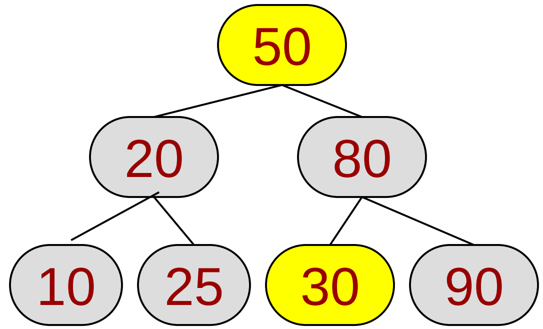
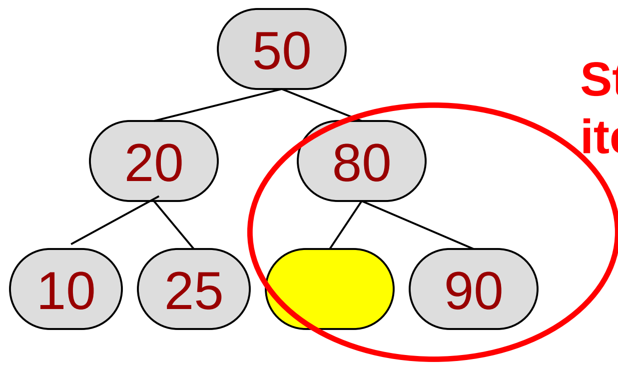
Remove Exercise 1

Remove 30

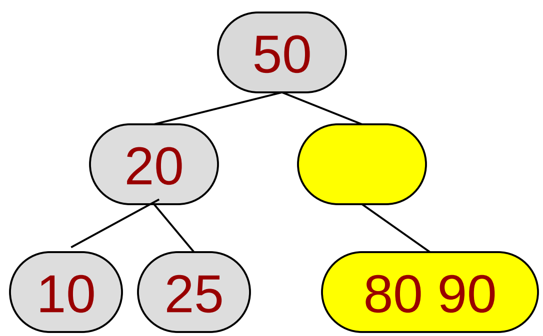




Step 1: Not a leaf, so swap with successor

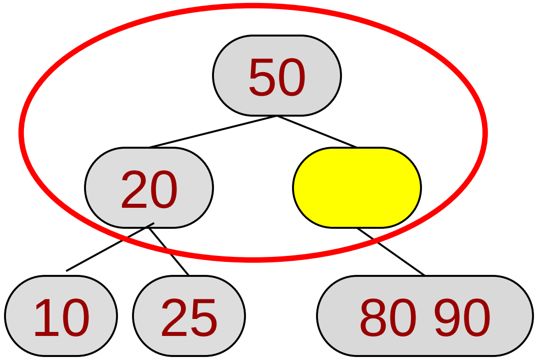


Step 2: Remove item from node

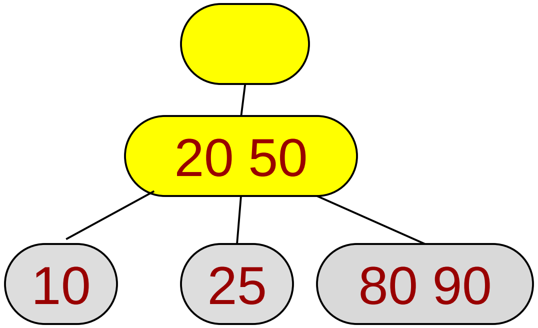


Step 3: Two values and 3 nodes, so merge. Must maintain levels.

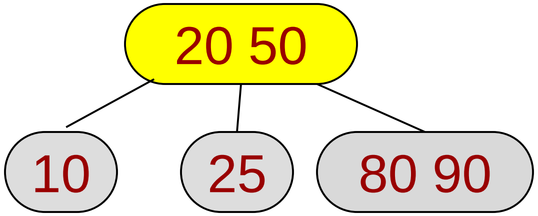
Remove Exercise 1 (cont.)



Start over with the empty parent. Do another merge



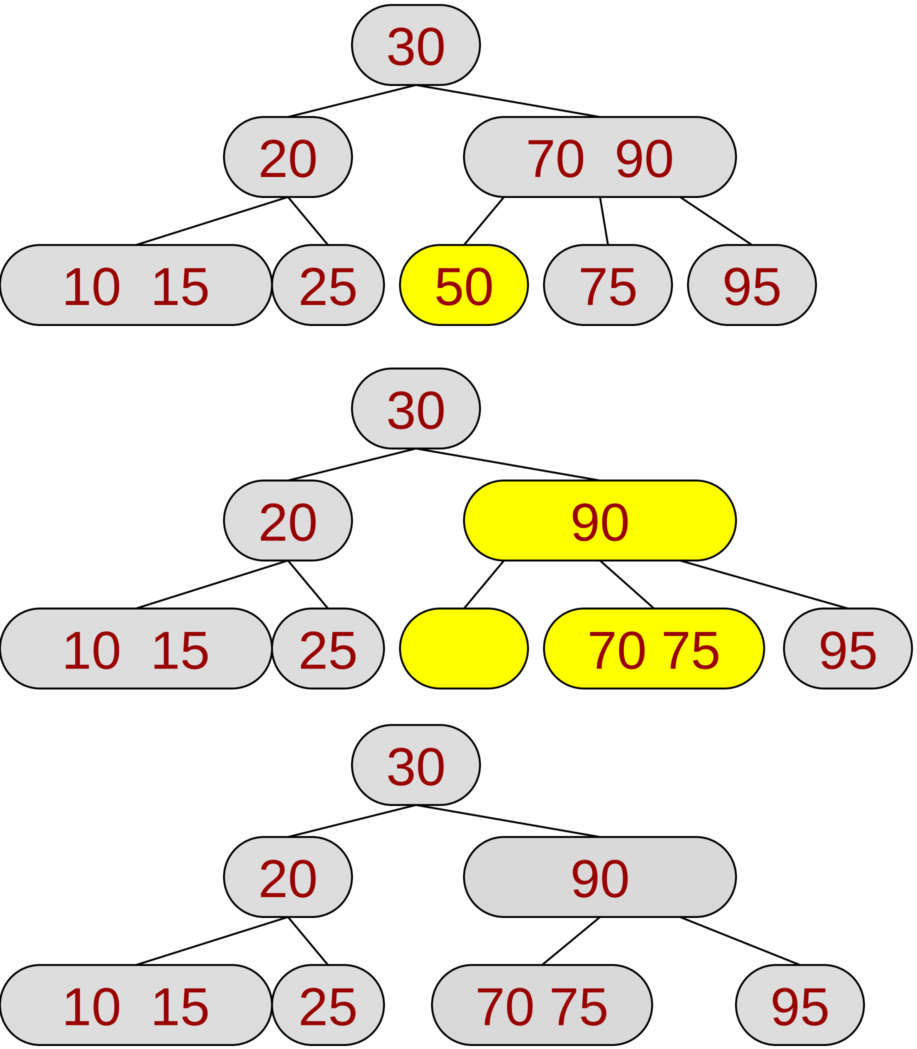
Step 4: Merge values



Step 5: Can delete the empty root node.

Remove Exercise 2

Remove 50



Step 1: It's a leaf node, so no need to find successor. Remove the item from node.

Step 2: Since no 3-node children, push a value of parent into a child.

Step 3: Delete the node.

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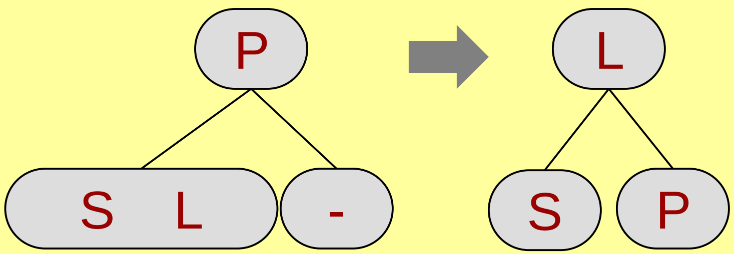
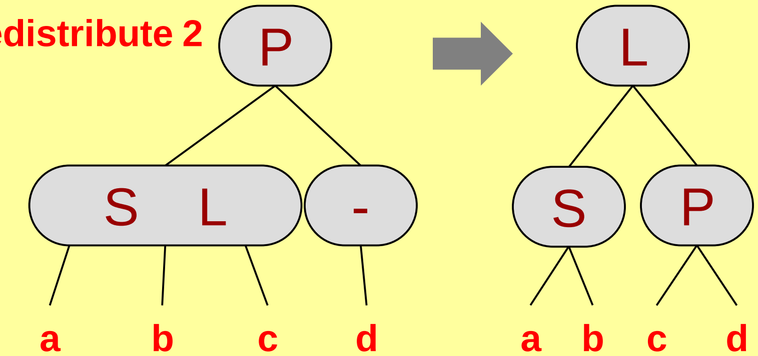
Remove Cases

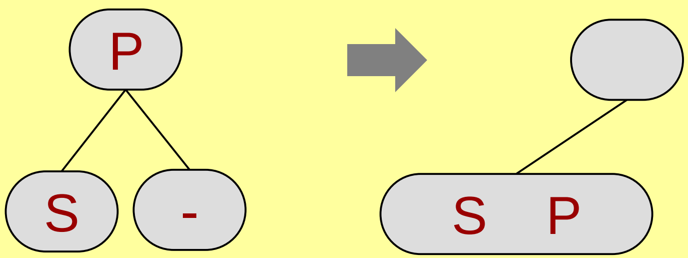
Redistribute 1



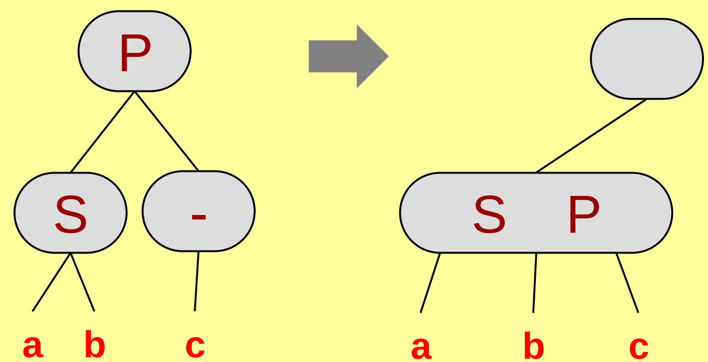
Redistribute 2



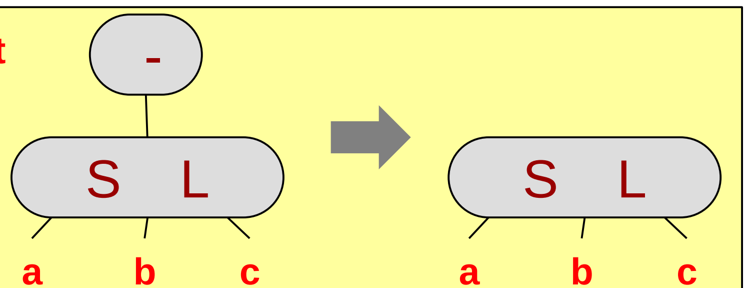
Merge 1



Merge 2

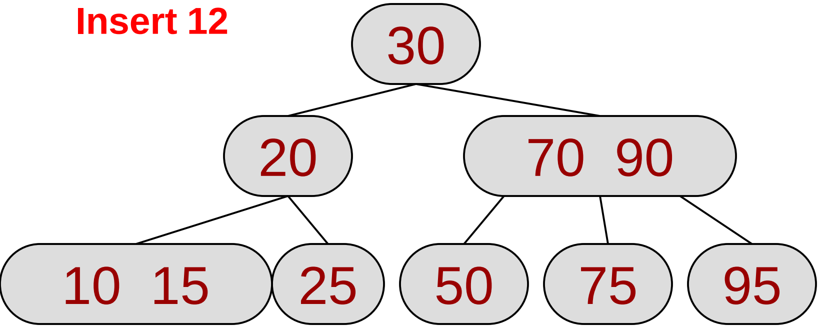


Empty root



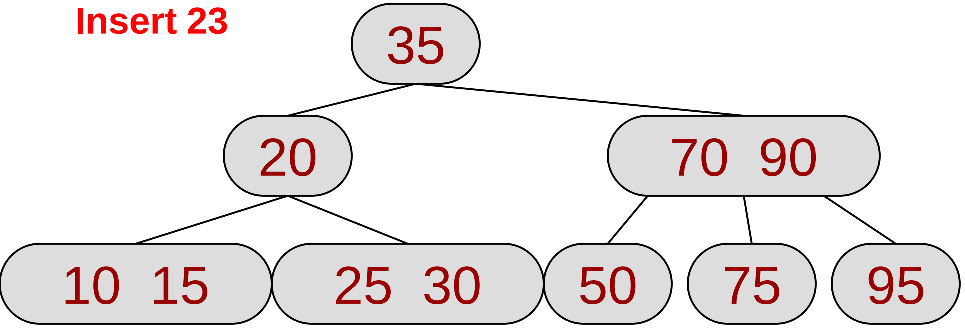
Insertion Exercise 1

Insert 12



Insertion Exercise 2

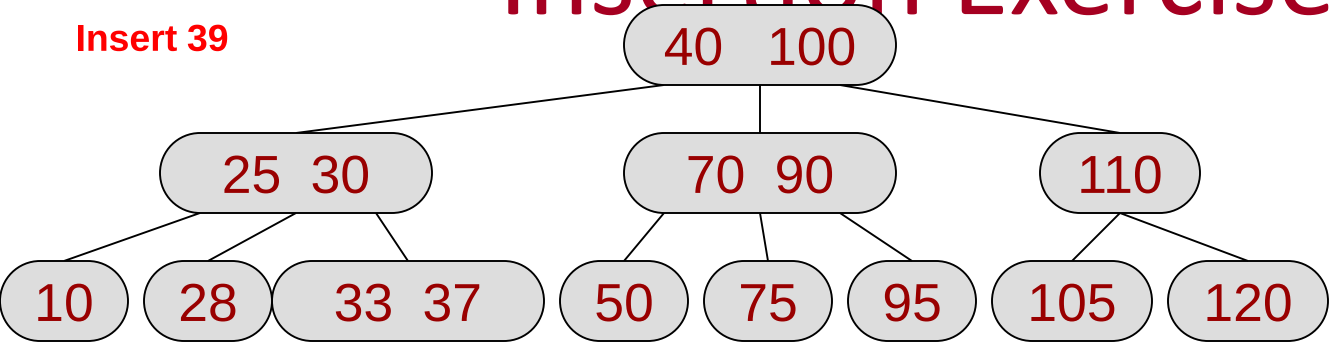
Insert 23



USC Viterbi School of Engineering

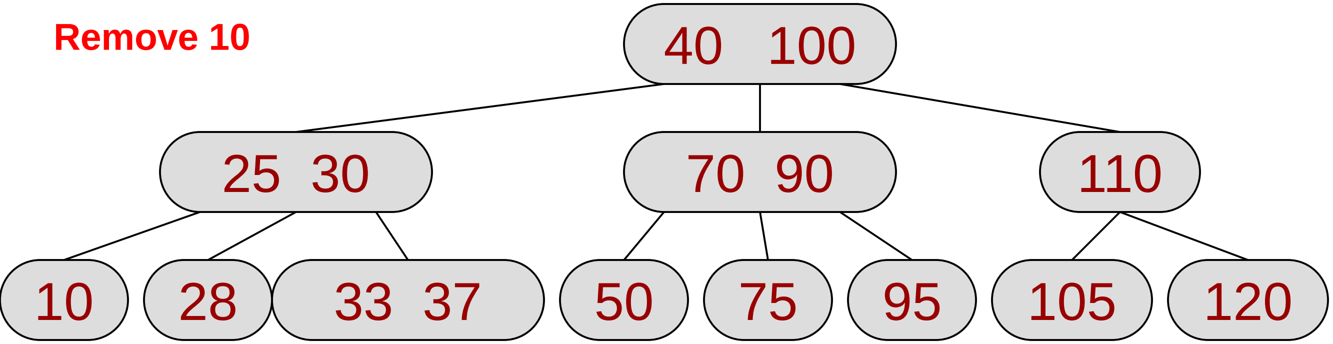
Insertion Exercise 3

Insert 39



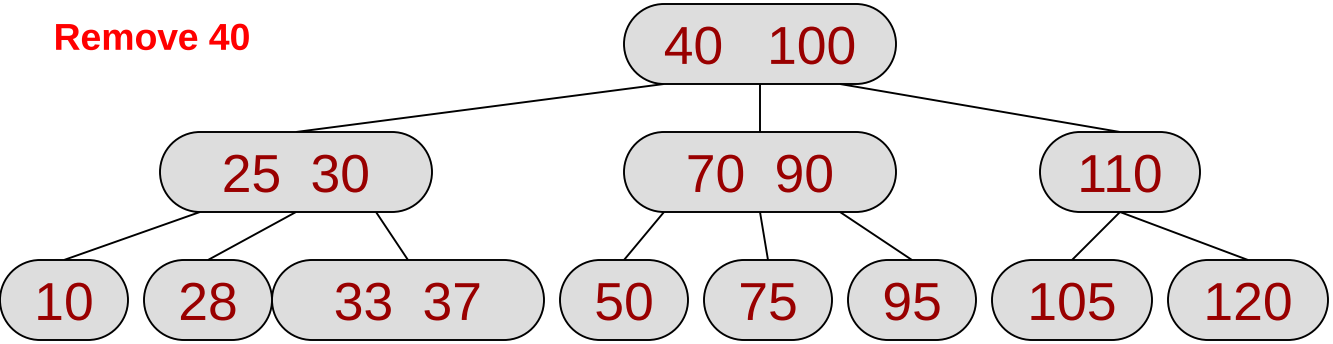
Removal Exercise 4

Remove 10



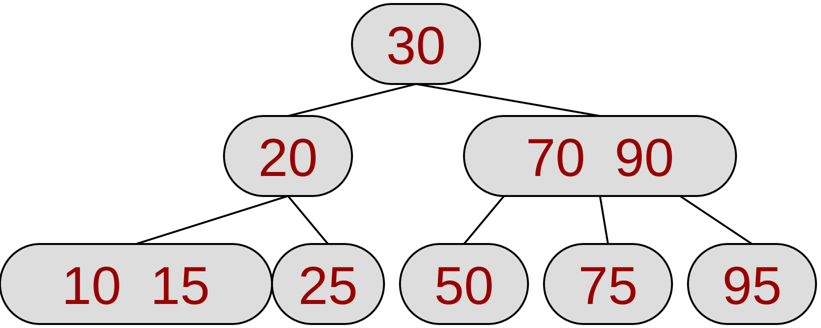
Removal Exercise 5

Remove 40



Removal Exercise 6

Remove 30

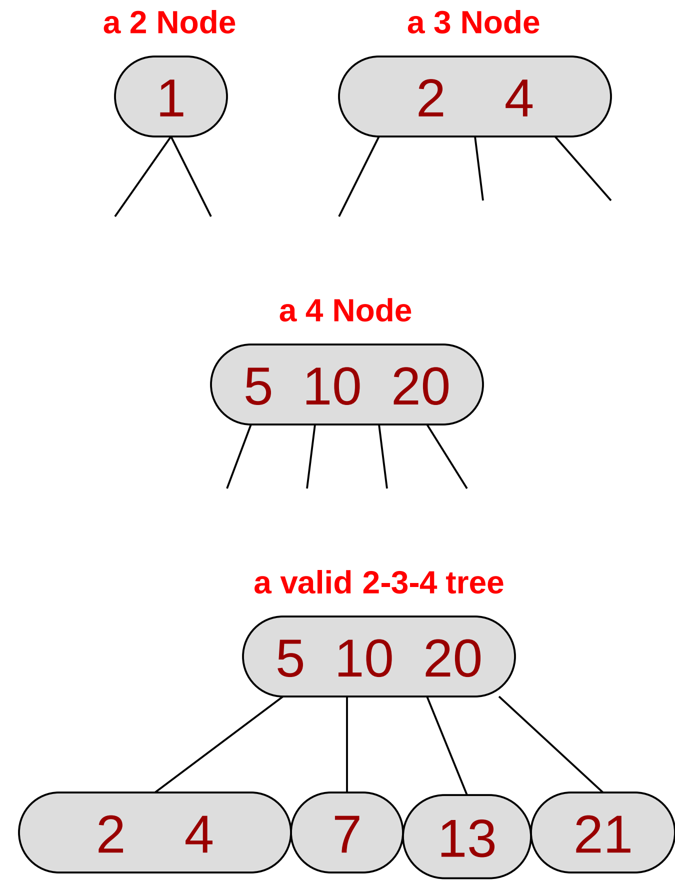


Other Resources

- http://www.cs.usfca.edu/~galles/visualization/BTree.html

Definition

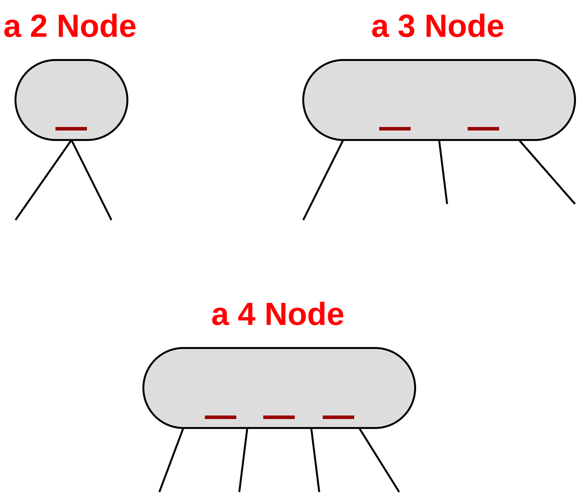
- 2-3-4 trees are very much like 2-3 trees but form the basis of a balanced, **binary** tree representation called Red-Black (RB) trees which are commonly used [used in C++ STL map & set]
  - We study them mainly to ease understanding of RB trees
- 2-3-4 Tree is a tree where
  - Non-leaf nodes have 1 value & 2 children or 2 values & 3 children or 3 values & 4 children
  - All leaves are at the same level
- Like 2-3 trees, 2-3-4 trees are always full and thus have an upper bound on their height of $\log_2(n)$



2-, 3-, & 4-Nodes

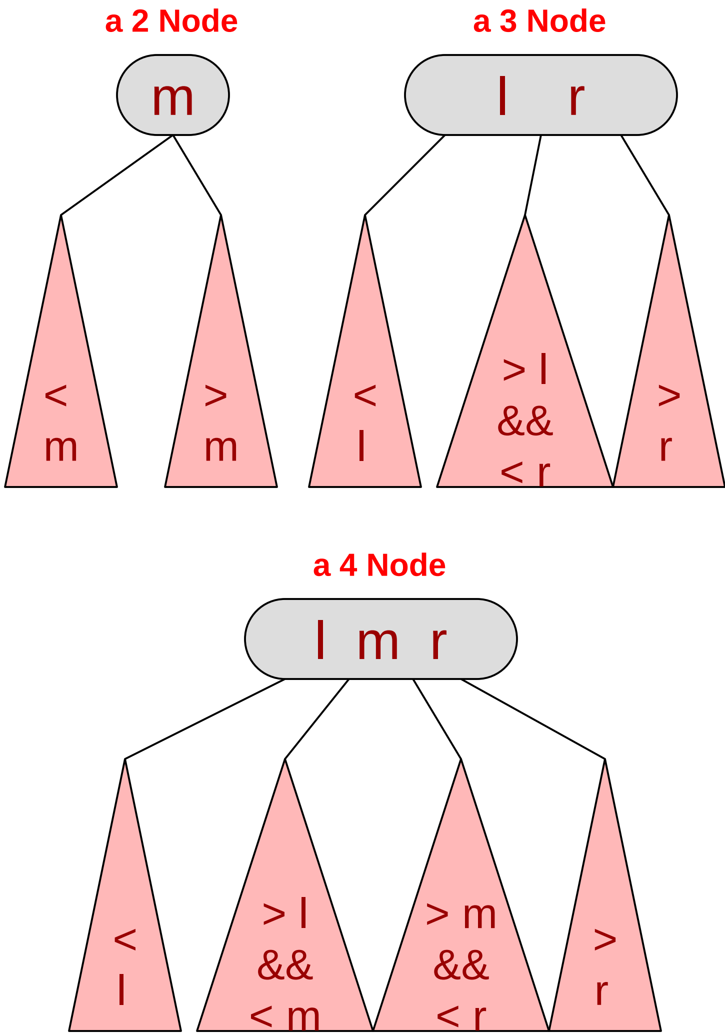
- 4-nodes require more memory and can be inefficient when the tree actually has many 2 nodes

```
template <typename T>
struct Item234 {
  T val1;
  T val2;
  T val3;
  Item234<T>* left;
  Item234<T>* midleft;
  Item234<T>* midright;
  Item234<T>* right;
  int nodeType;
};
```



2-3-4 Search Trees

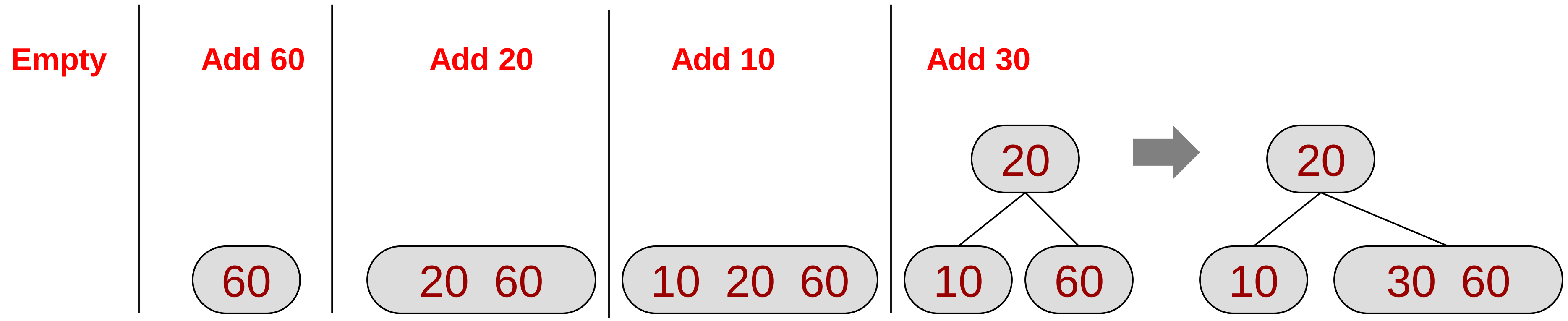
- Similar properties as a 2-3 Search Tree
- 4 Node:
  - Left subtree nodes are < *l*
  - Middle-left subtree > *l* and < *r*
  - Right subtree nodes are > *r*



2-3-4 Insertion Algorithm

- Key: Rather than search down the tree and then possibly promote and break up 4-nodes on the way back up, split 4 nodes on the way down
- To insert a value,
  - 1. If node is a 4-node
    - Split the 3 values into a left 2-node, a right 2-node, and promote the middle element to the parent of the node (which definitely has room) attaching children appropriately
    - Continue on to next node in search order
  - 2a. If node is a leaf, insert the value
  - 2b. Else continue on to the next node in search tree order
- Insert 60, 20, 10, 30, 25, 50, 80

Key: 4-nodes get split as you walk down thus, a leaf will always have room for a value



2-3-4 Insertion Algorithm

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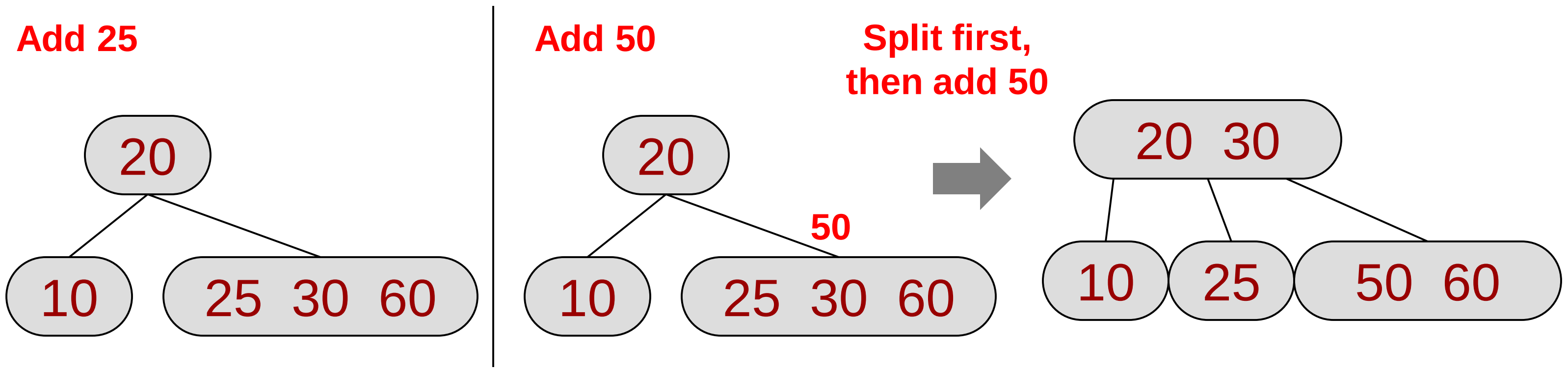
Key: 4-nodes get split as you walk down thus, a leaf will always have room for a value

- Insert 60, 20, 10, 30, 25, 50, 80



2-3-4 Insertion Algorithm

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  - 1. If node is a 4-node
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  - 2a. If node is a leaf, insert the value
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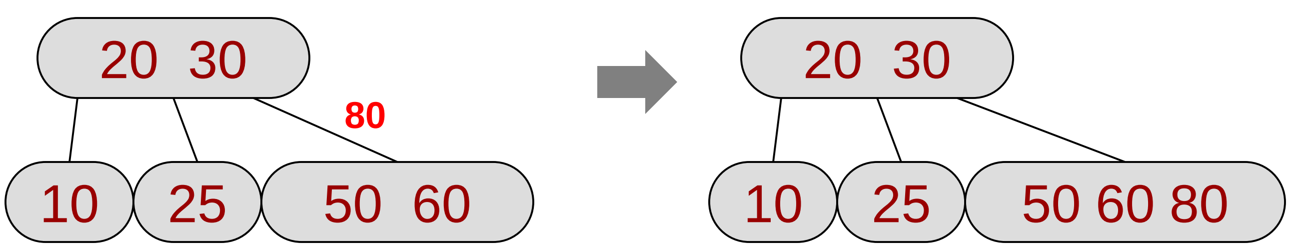
- Insert 60, 20, 10, 30, 25, 50, 80

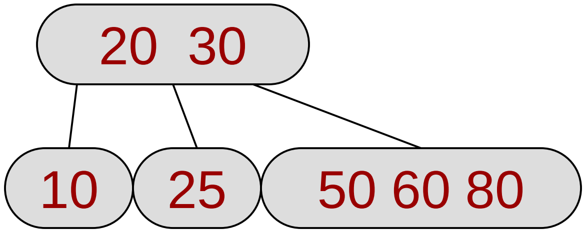
Key: 4-nodes get split as you walk down thus, a leaf will always have room for a value

Add 80



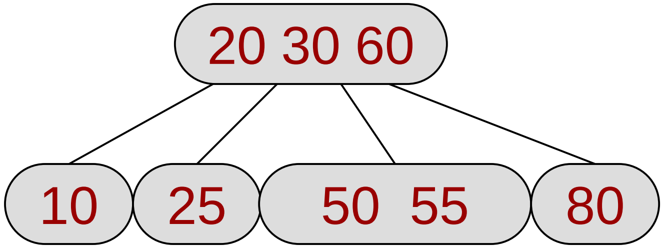
2-3-4 Insertion Exercise 1

Add 55



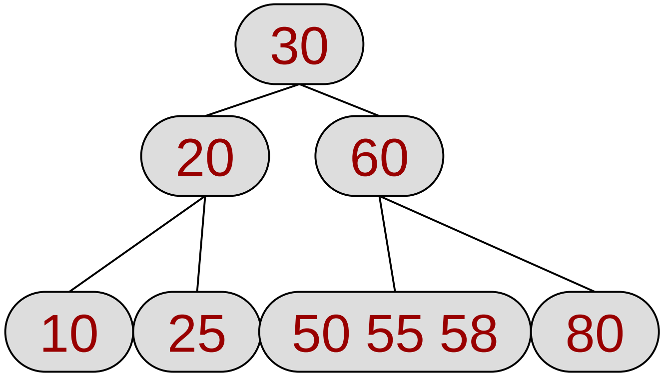
2-3-4 Insertion Exercise 2

Add 58



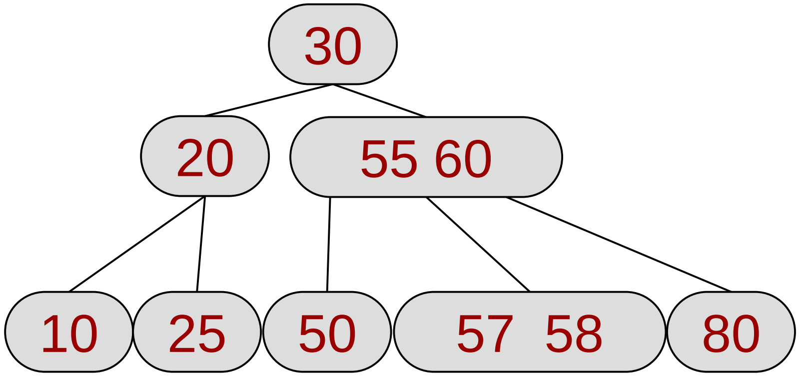
2-3-4 Insertion Exercise 3

Add 57



2-3-4 Insertion Exercise 3

Resulting Tree



B-Trees

- 2-3 and 2-3-4 trees are just instances of a more general data structure known as B-Trees
- Define minimum number of children (degree) for non-leaf nodes, d
  - Non-root nodes must have at least d-1 keys and d children
  - All nodes must have at most 2d-1 keys and 2d children
  - 2-3-4 Tree (d=2)
- Used for disk-based storage and indexing with large value of d to account for large random-access lookup time but fast sequential access time of secondary storage

B Tree Resources

- https://www.cs.usfca.edu/~galles/visualization/BTree.html
- http://ultrastudio.org/en/2-3-4_tree

"Balanced" Binary Search Trees

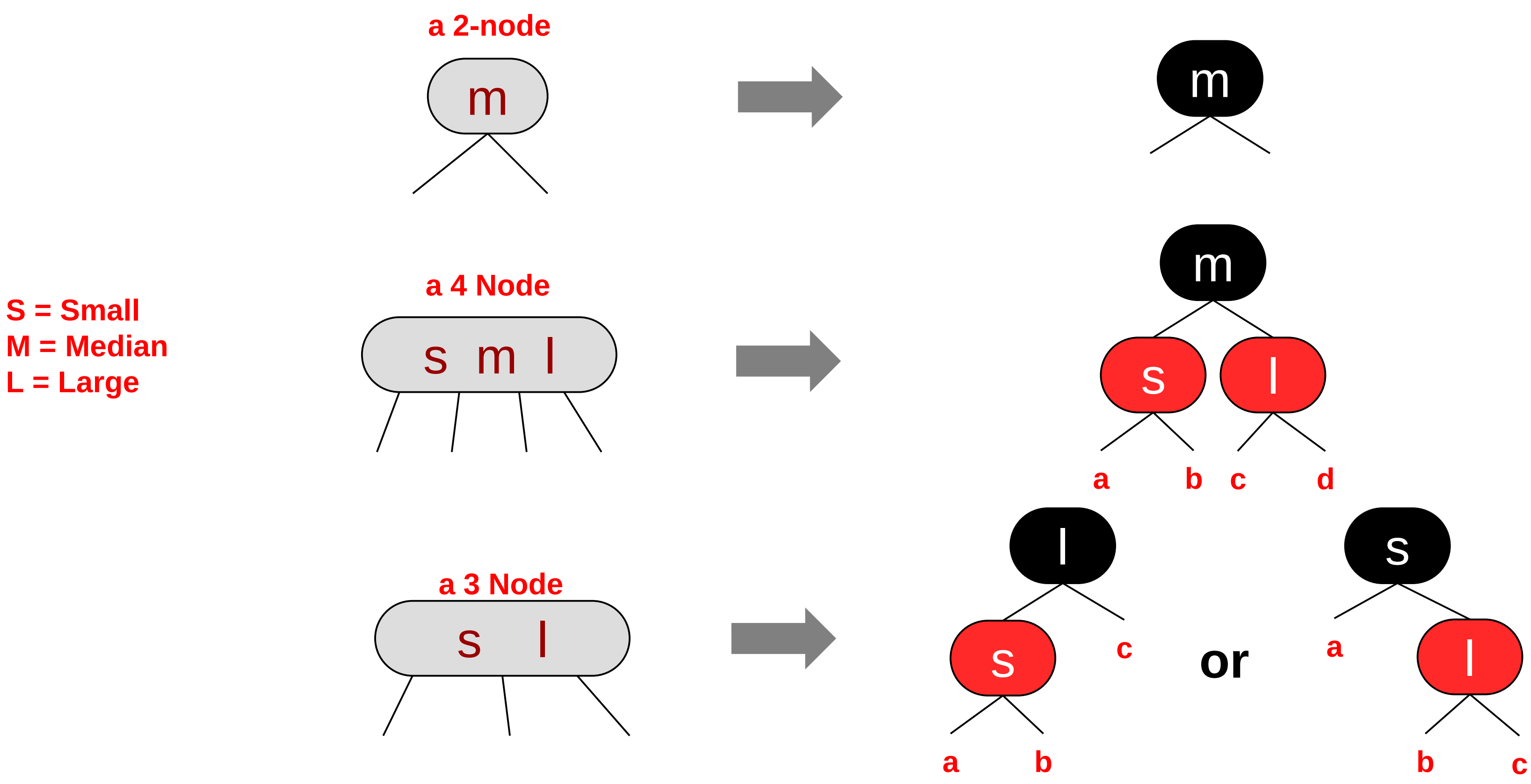
RED BLACK TREES

Red Black Trees

- A red-black tree is a binary search tree
  - Only 2 nodes (no 3- or 4-nodes)
  - Can be built from a 2-3-4 tree directly by converting each 3- and 4- nodes to multiple 2-nodes
- All 2-nodes means no wasted storage overheads
- Yields a "balanced" BST
- "Balanced" means that the height of an RB-Tree is at MOST **twice** the height of a 2-3-4 tree
  - Recall, height of 2-3-4 tree had an upper bound of $\log_2(n)$
  - Thus height or an RB-Tree is bounded by $2*\log_2 n$ which is still $O(\log_2(n))$

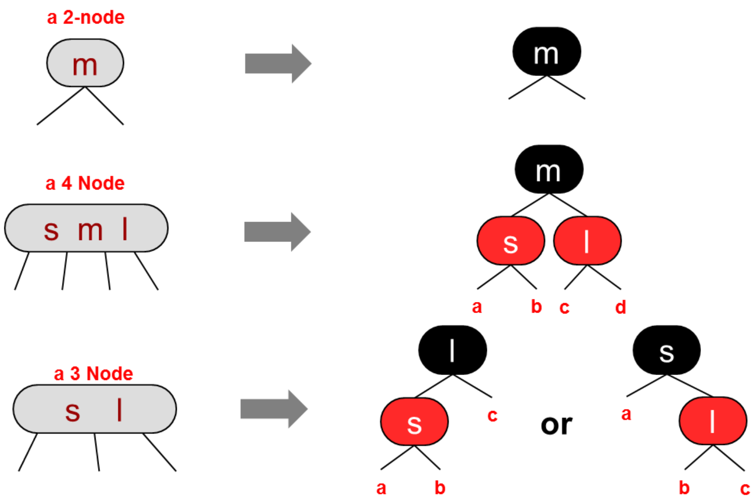
Red Black and 2-3-4 Tree Correspondence

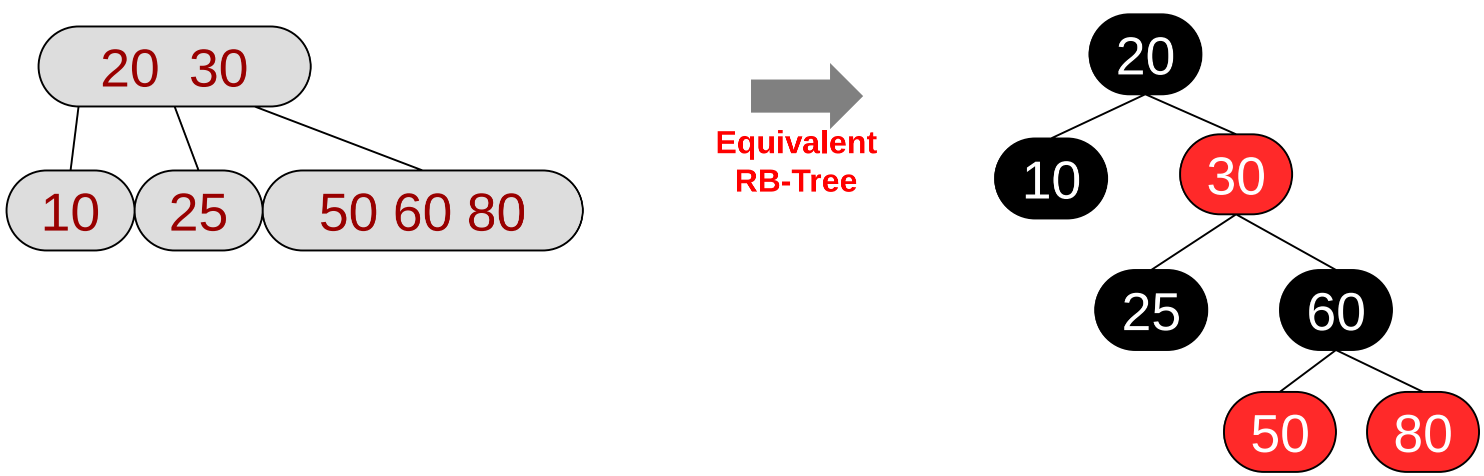
- Every 2-, 3-, and 4-node can be converted to…
  - At least 1 black node and 1 or 2 red children of the black node
  - Red nodes are always ones that would join with their parent to become a 3- or 4-node in a 2-3-4 tree



Red Black Trees

- Below is a 2-3-4 tree and how it can be represented as a directly corresponding RB-Tree
- Notice at most each 2-3-4 node expands to **2** level of red/black nodes
- **Q:** Thus if the height of the 2-3-4 tree was bound by $\log_2 n$, then the height of an RB-tree is bounded by?
- **A:** $2*\log_2 n = O(\log_2 n)$



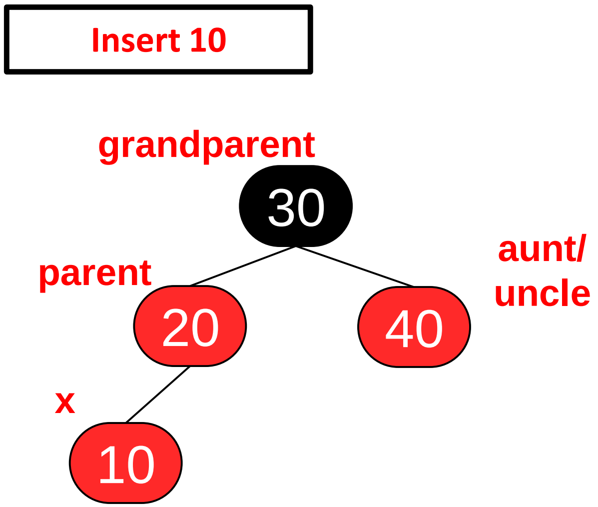


Red-Black Tree Properties

- Valid RB-Trees maintain the invariants that…
- 1. No path from root to leaf has two consecutive red nodes (i.e. a parent and its child cannot both be red)
  - Since red nodes are just the extra values of a 3- or 4-node from 2-3-4 trees you can't have 2 consecutive red nodes
- 2. Every path from leaf to root has the same number of black nodes
  - Recall, 2-3-4 trees are full (same height from leaf to root for all paths)
  - Also remember each 2, 3-, or 4- nodes turns into a black node ***plus*** 0, 1, or 2 red node children
- 3. At the end of an operation the root should always be black
- 4. We can imagine leaf nodes as having 2 non-existent (NULL) black children if it helps

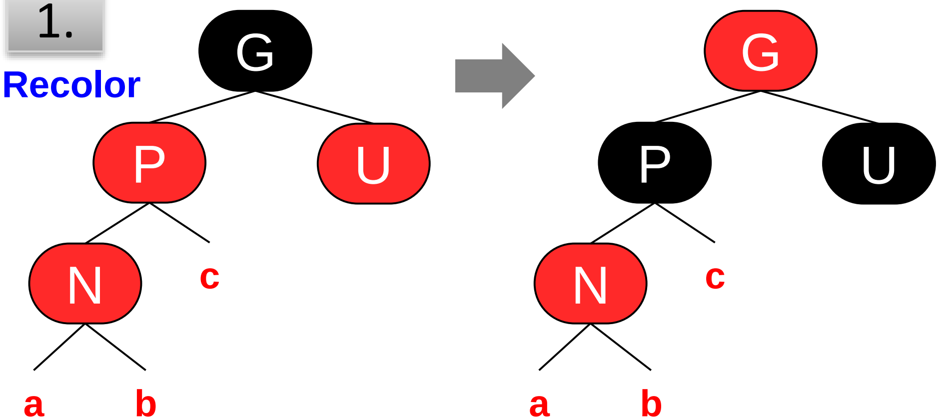
Red-Black Insertion

- Insertion Algorithm:
  - 1. Insert node into normal BST location (at a leaf location) and color it RED
  - 2a. If the node's parent is black (i.e. the leaf used to be a 2-node) then DONE (i.e. you now have what was a 3- or 4-node)
  - 2b. Else perform fixTree transformations then repeat step 2 on the parent or grandparent (whoever is red)
- fixTree involves either
  - recoloring or
  - 1 or 2 rotations and recoloring
- Which case of fixTree you perform depends on the color of the new node's "aunt/uncle"

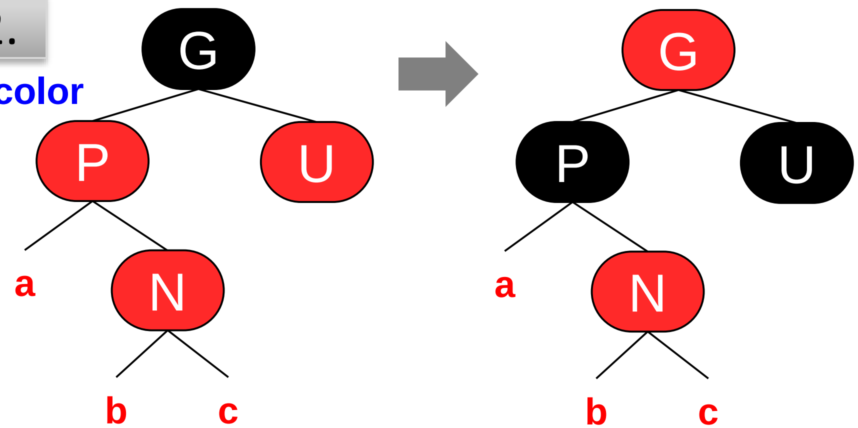


fixTree Cases

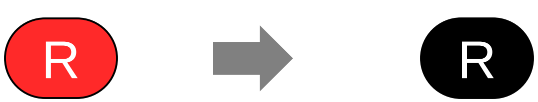
1.
Recolor

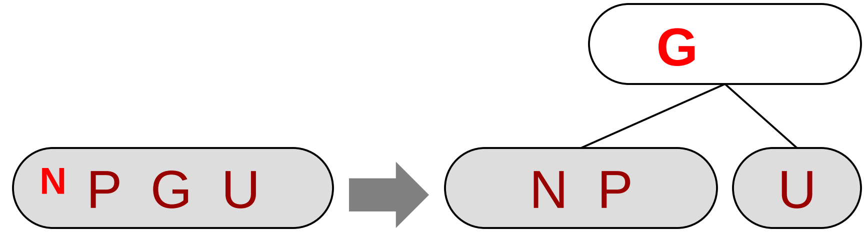


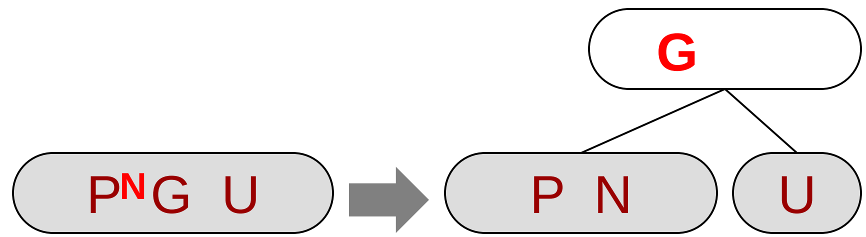
2.
Recolor



3.
Recolor Root





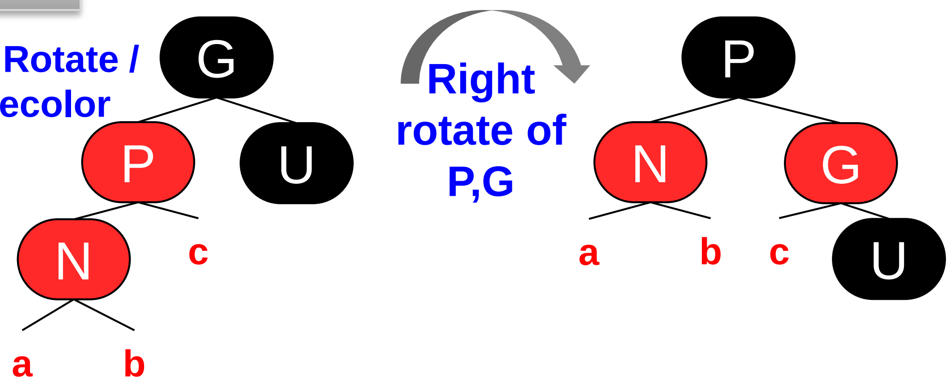


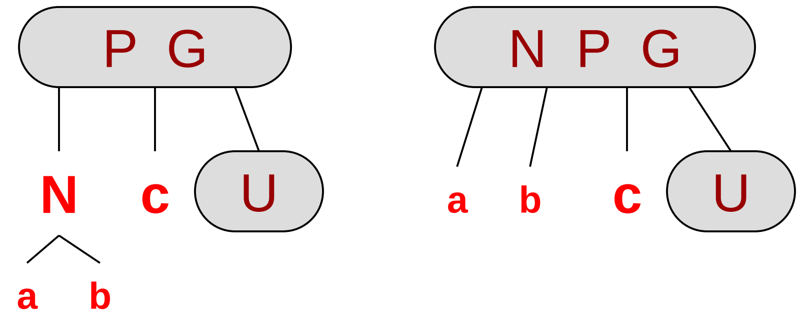
Note: For insertion/removal algorithm we consider non-existent leaf nodes as black nodes

fixTree Cases

4.

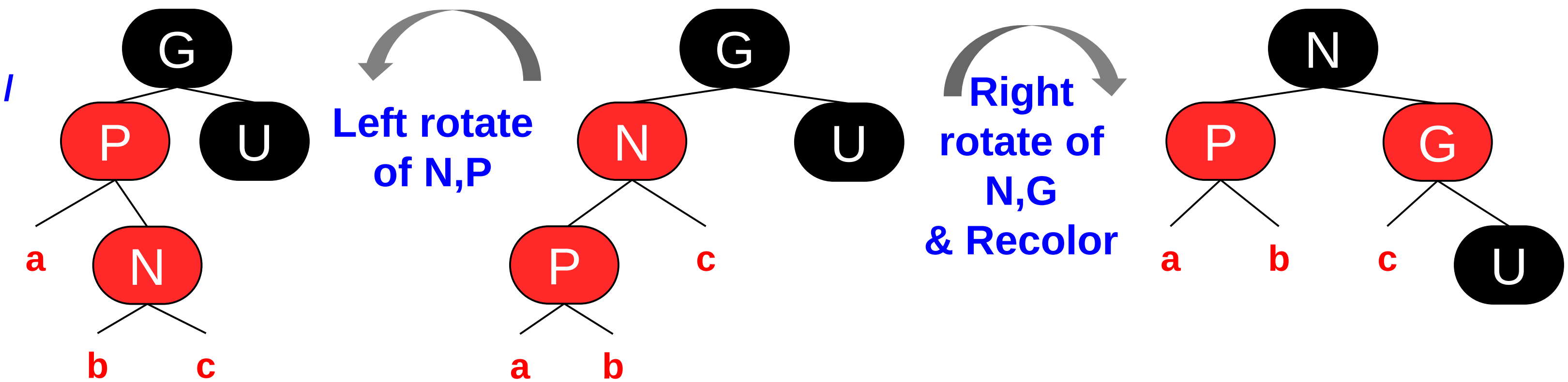
1 Rotate / Recolor

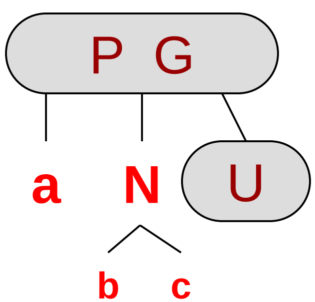


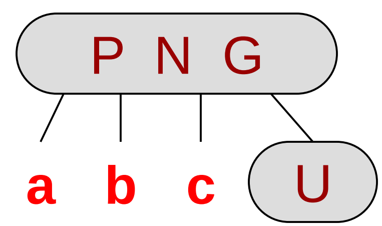


5.

2 Rotates / Recolor



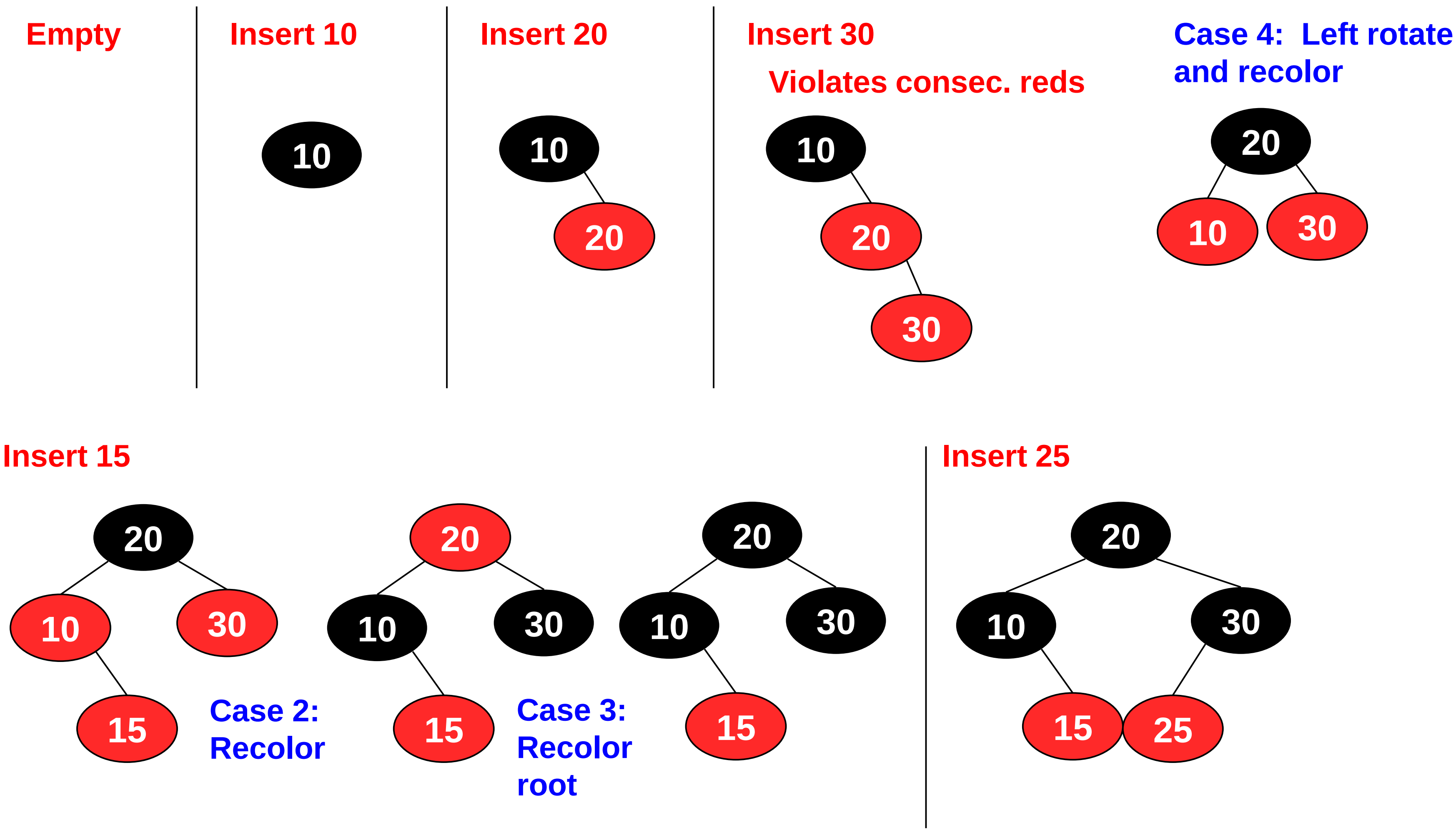




www.cse.ohio-state.edu/~gurari/course/cis680Ch11.html

Insertion

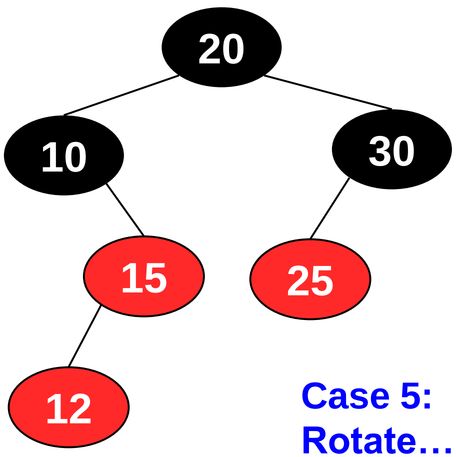
- Insert 10, 20, 30, 15, 25, 12, 5, 3, 8



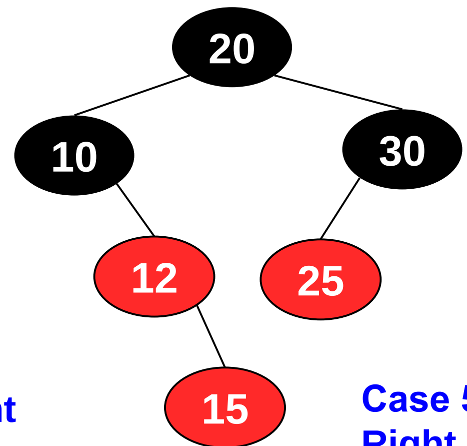
Insertion

- Insert 10, 20, 30, 15, 25, 12, 5, 3, 8

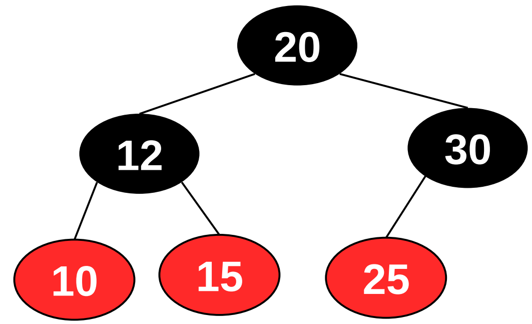
Insert 12



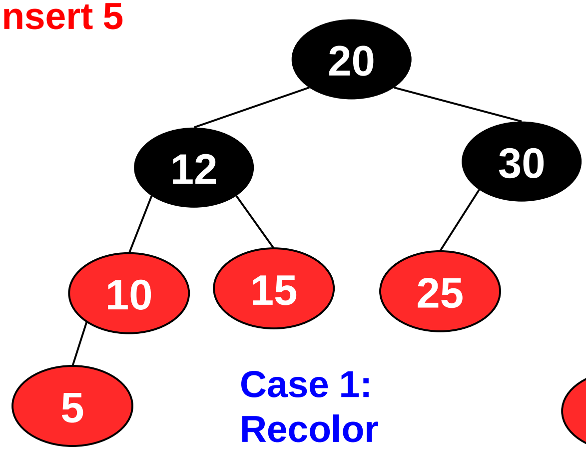
Case 5: Right Rotate…



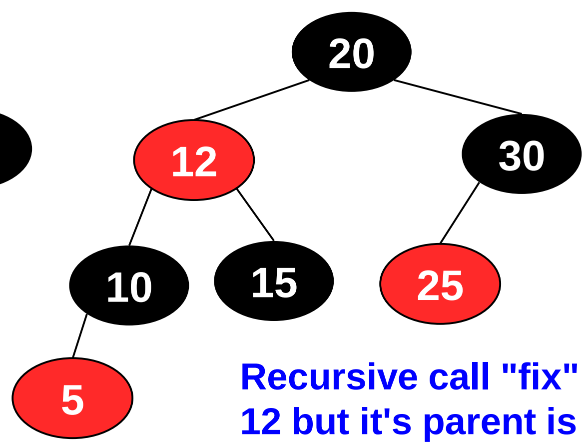
Case 5: … Right Rotate and recolor



Insert 5



Case 1: Recolor

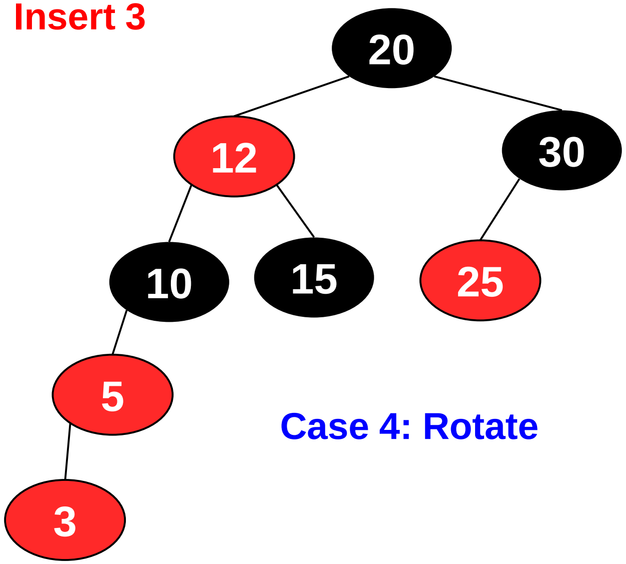


Recursive call "fix" on 12 but it's parent is black so we're done

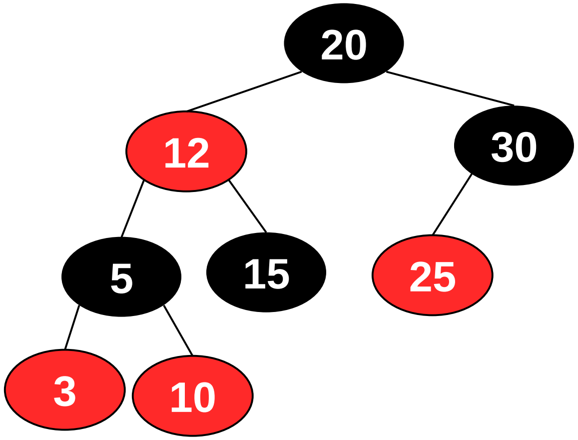
Insertion

- Insert 10, 20, 30, 15, 25, 12, 5, 3, 8

Insert 3

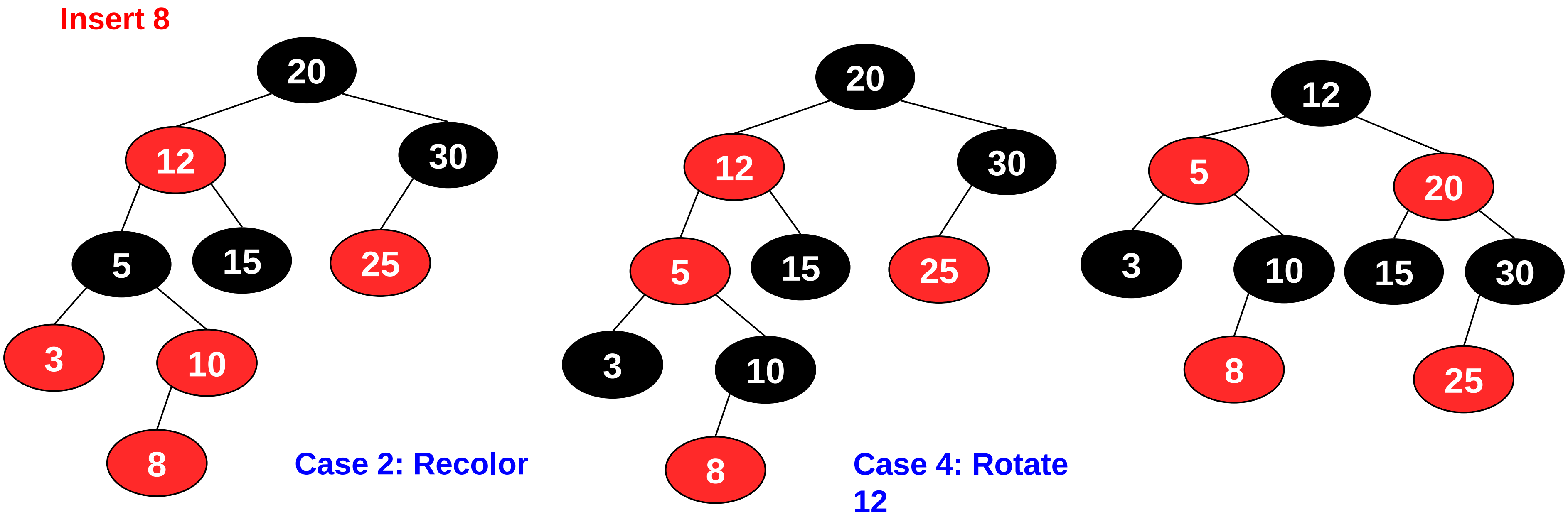


Case 4: Rotate



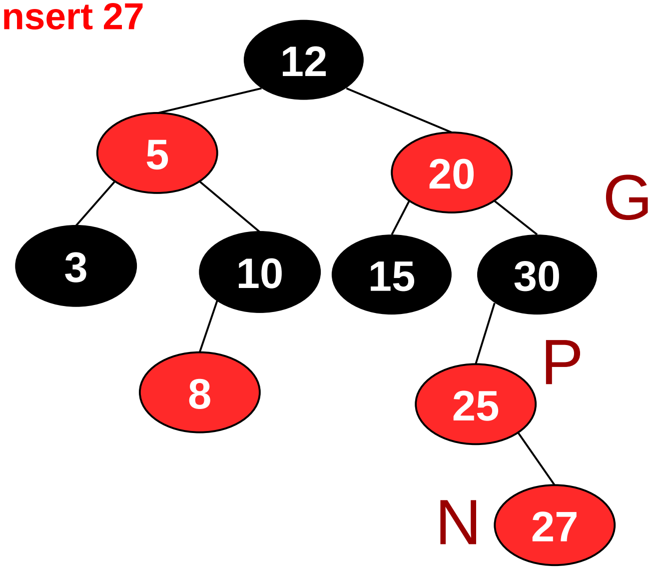
Insertion

- Insert 10, 20, 30, 15, 25, 12, 5, 3, 8

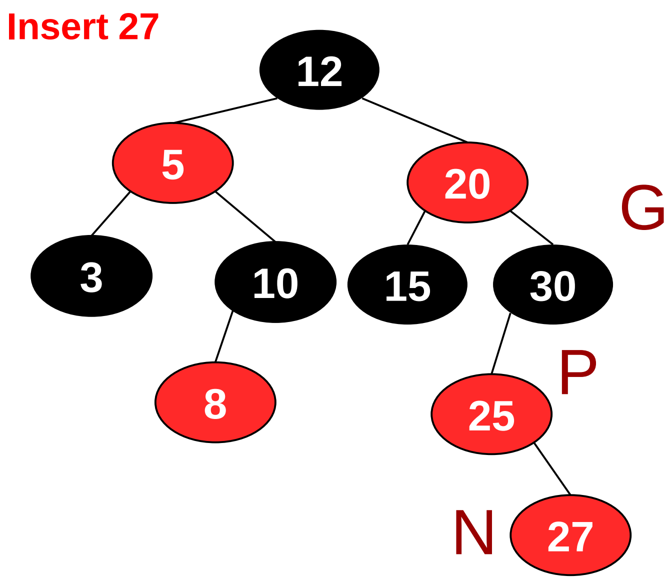


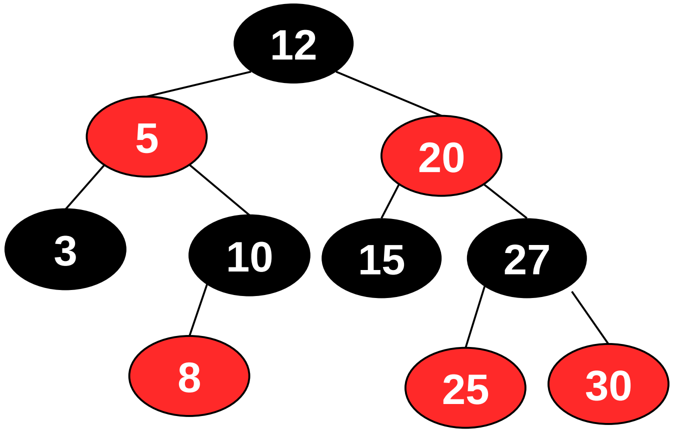
Insertion Exercise 1

Insert 27



Insertion Exercise 1



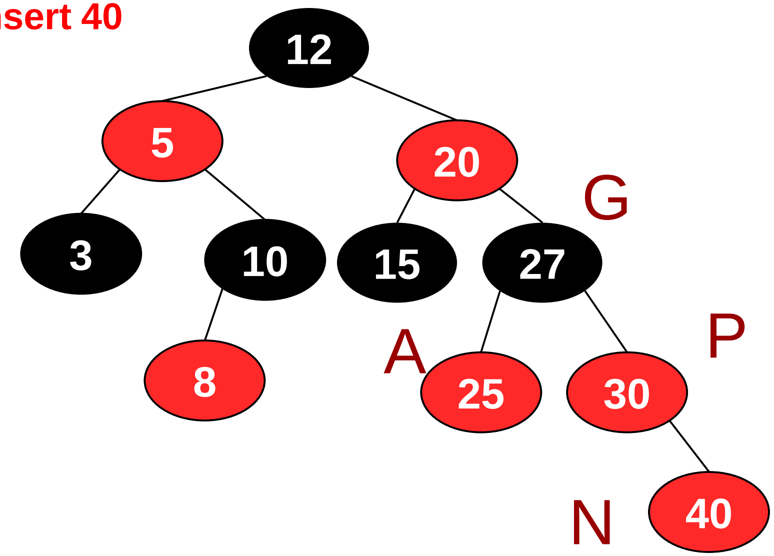


This is case 5.

1. Left rotate around P
2. Right rotate around N
3. Recolor

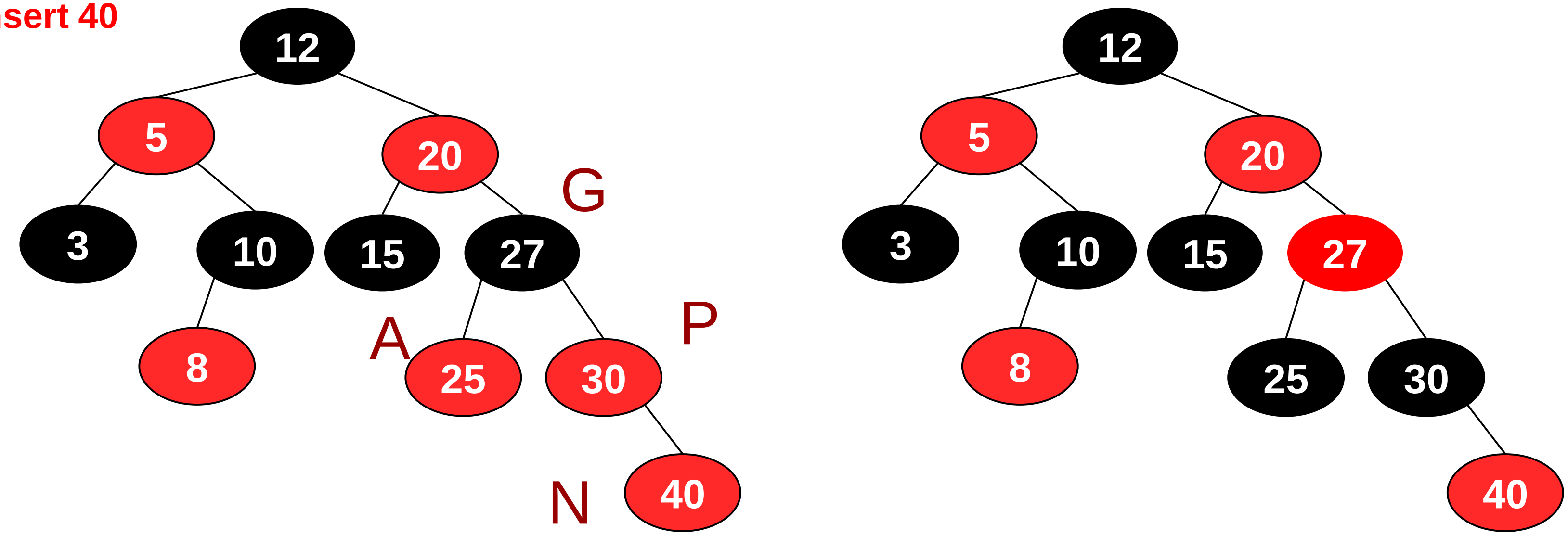
Insertion Exercise 2

Insert 40



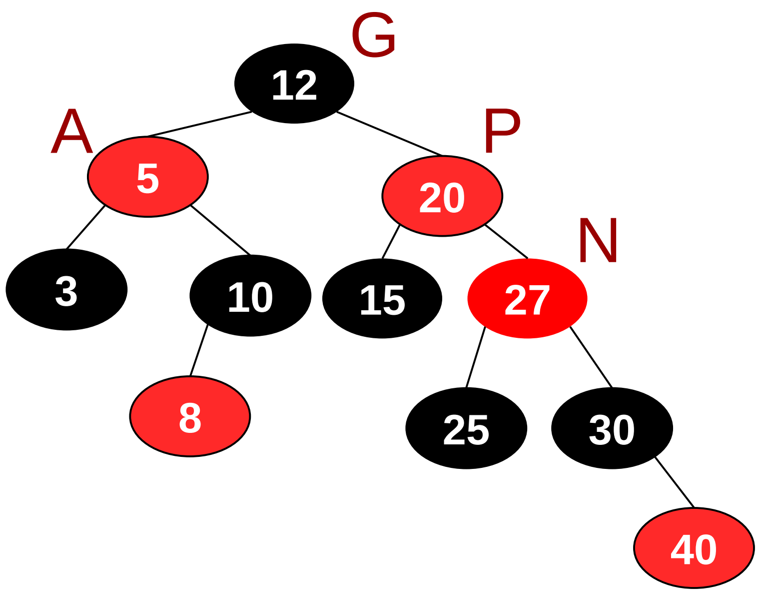
Insertion Exercise 2

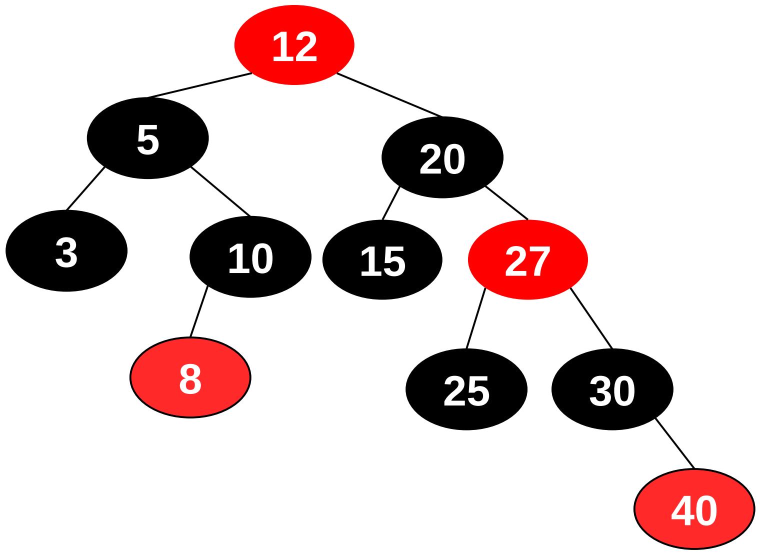
Insert 40



Aunt and Parent are the same color. So recolor aunt, parent, and grandparent.

Insertion Exercise 2

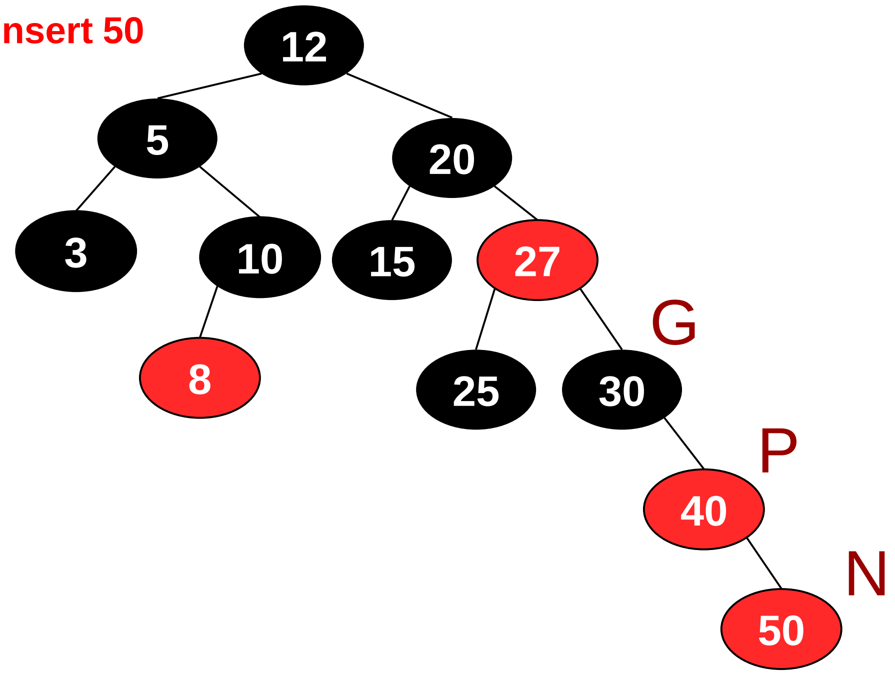




Aunt and Parent are the same color. So recolor aunt, parent, and grandparent.

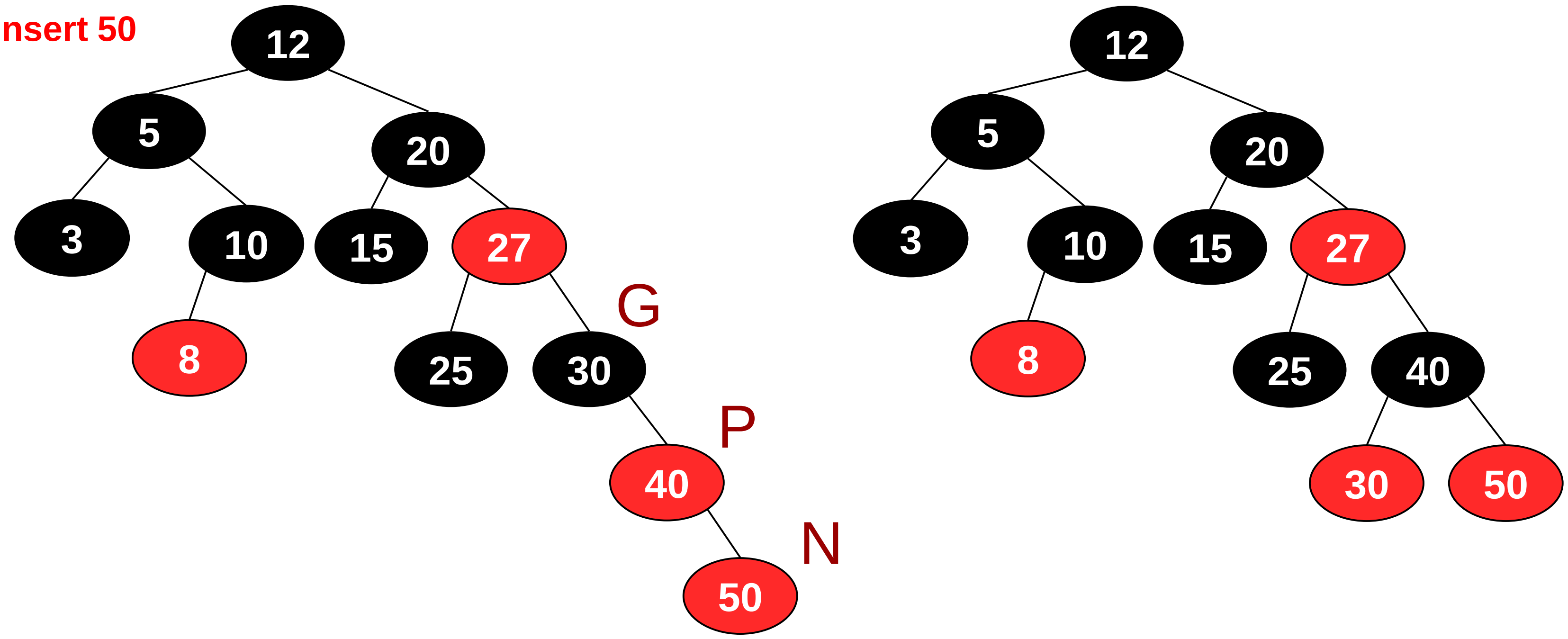
Insertion Exercise 3

Insert 50



Insertion Exercise 3

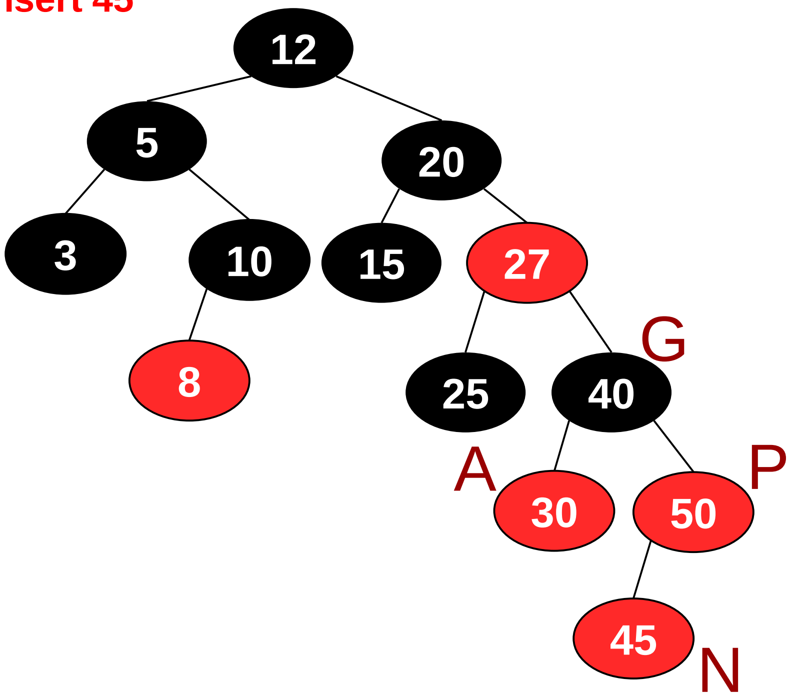
Insert 50



Remember, empty nodes are black.
Do a left rotation around P and recolor.

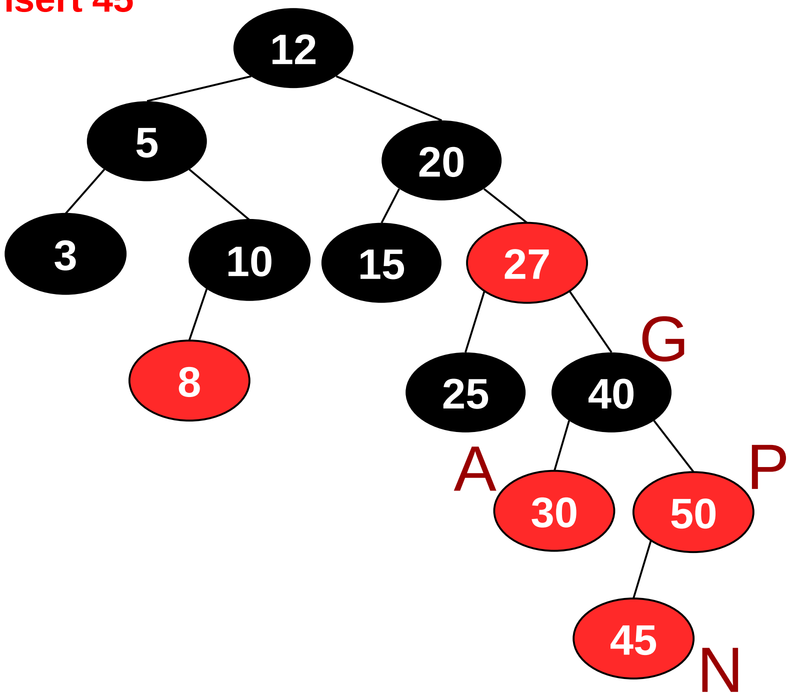
Insertion Exercise 4

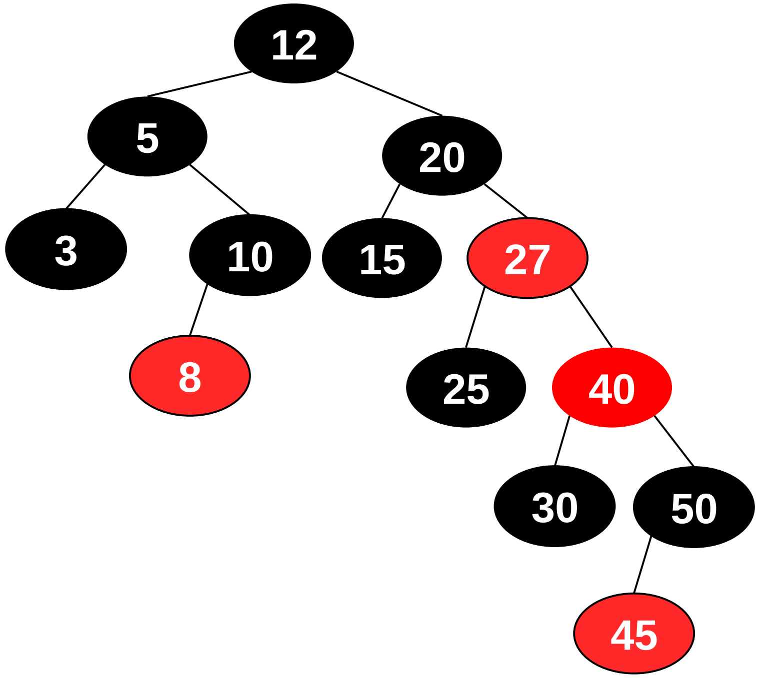
Insert 45



Insertion Exercise 4

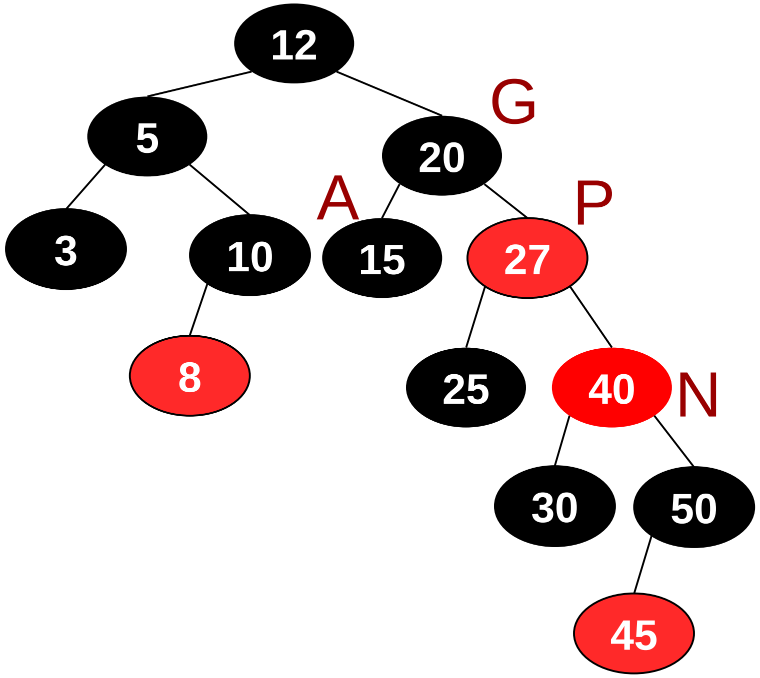
Insert 45



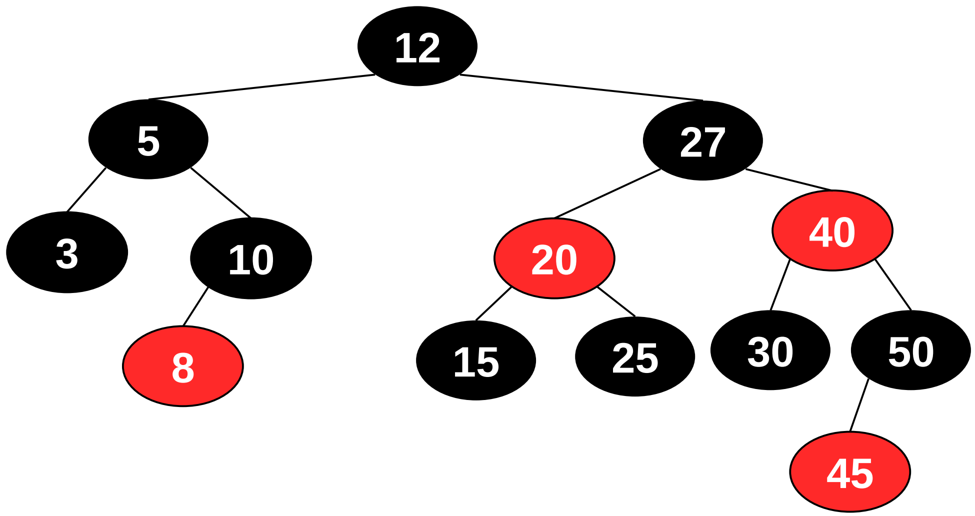


Aunt and Parent are the same color.
Just recolor.

Insertion Exercise 4

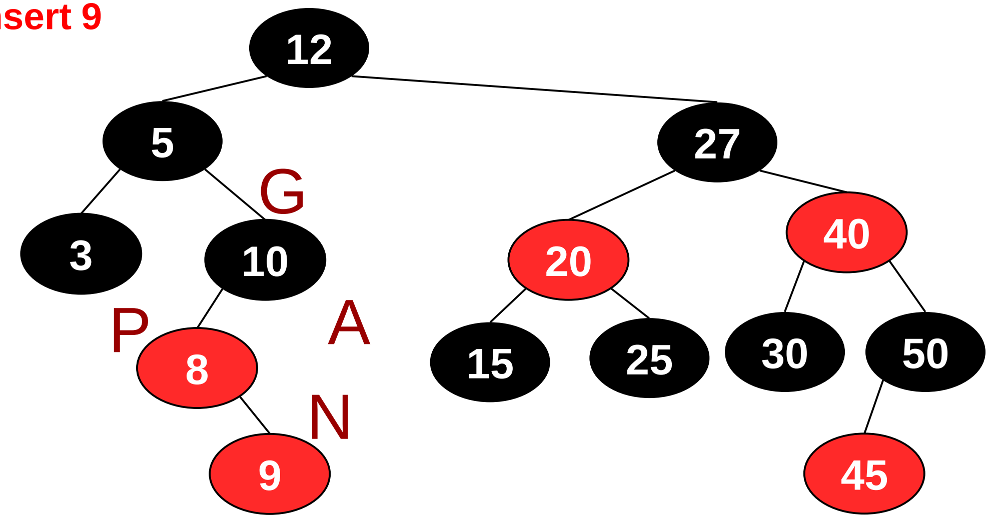


Final Result

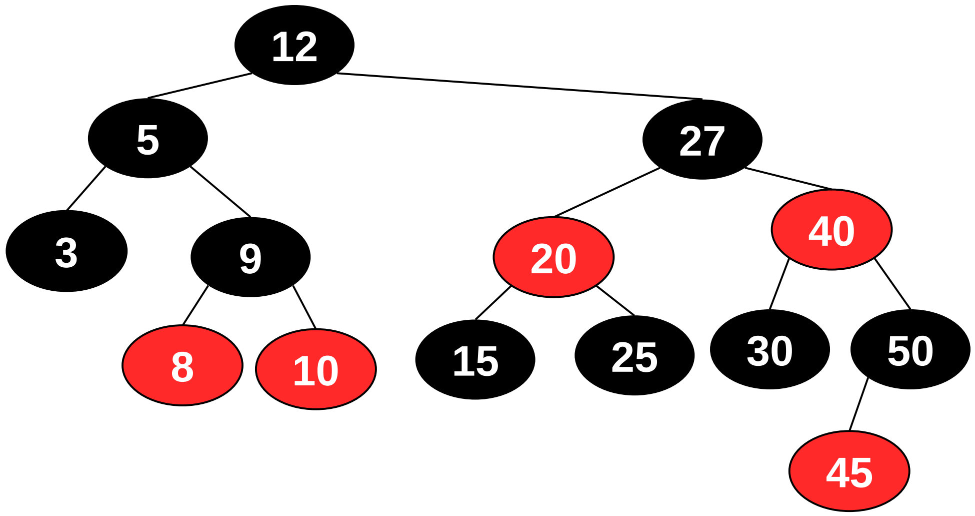


Insertion Exercise 5

Insert 9



Insertion Exercise 5



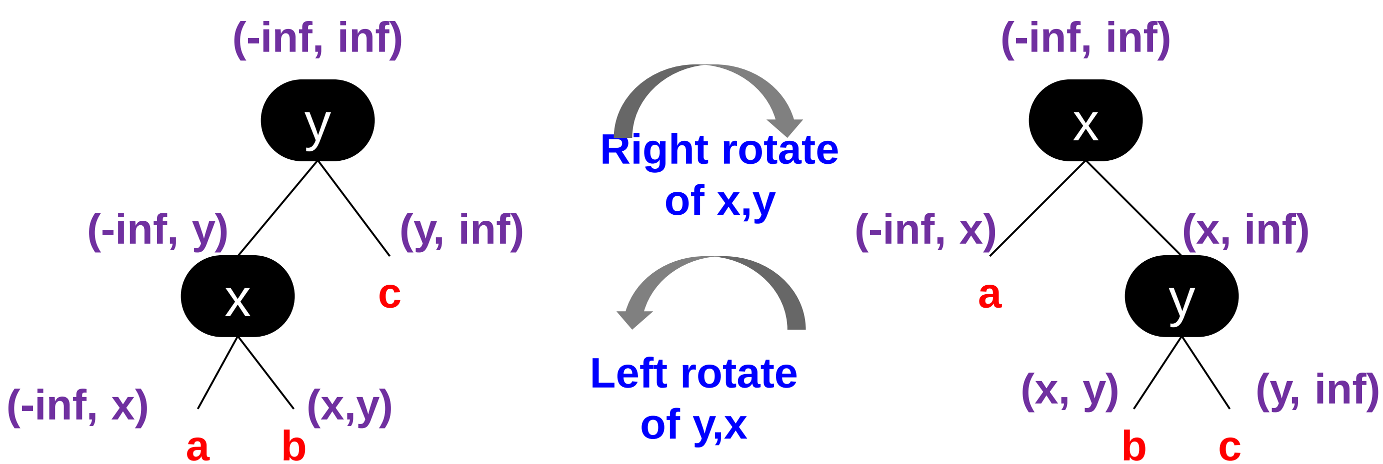
RB-Tree Visualization & Links

- https://www.cs.usfca.edu/~galles/visualization/RedBlack.html

RB TREE IMPLEMENTATION

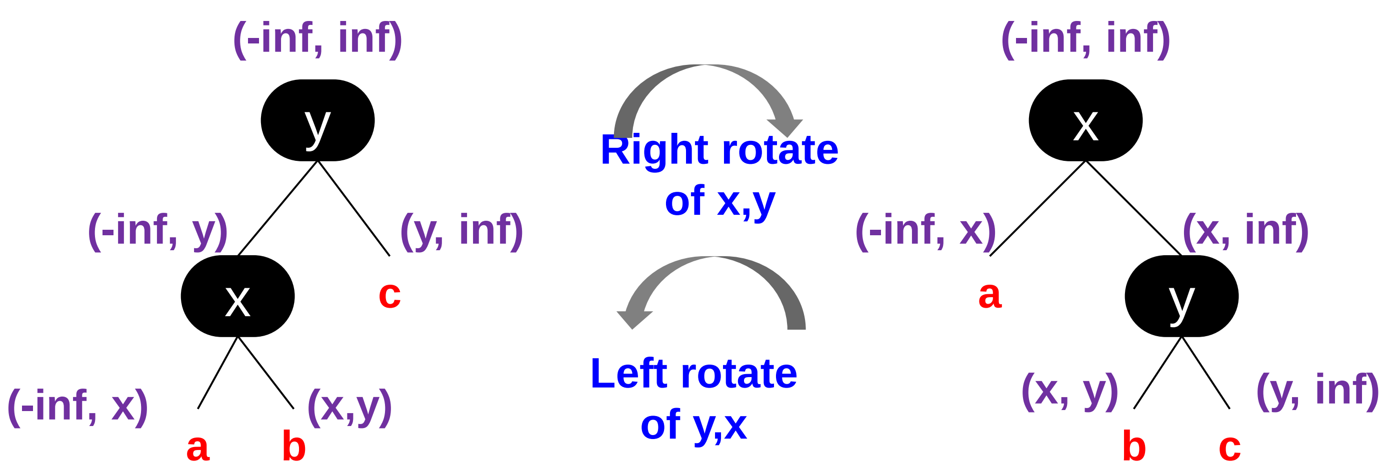
Hints

- Implement private methods:
  - findMyUncle()
  - AmIaRightChild()
  - AmIaLeftChild()
  - RightRotate
  - LeftRotate
    - Need to change x's parent, y's parent, b's parent, x's right, y's left, x's parent's left or right, and maybe root



Hints

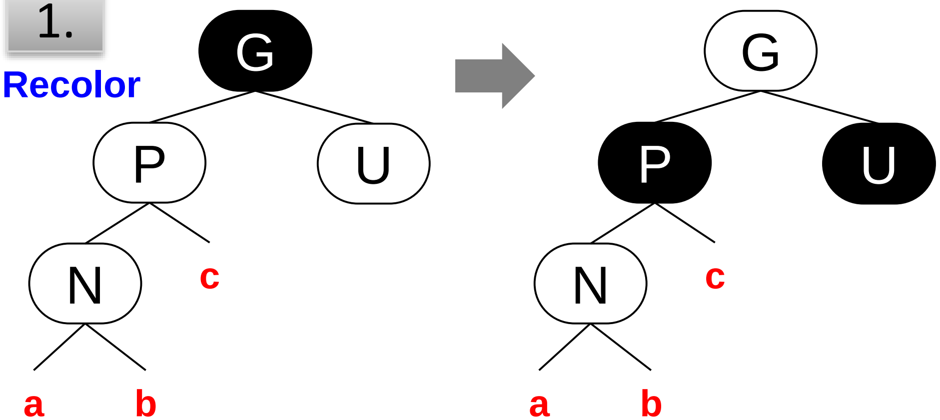
- You have to fix the tree after insertion if...
- Watch out for traversing NULL pointers
  - node->parent->parent
  - However, if you need to fix the tree your grandparent...
- Cases break down on uncle's color
  - If an uncle doesn't exist (i.e. is NULL), he is (color?)...



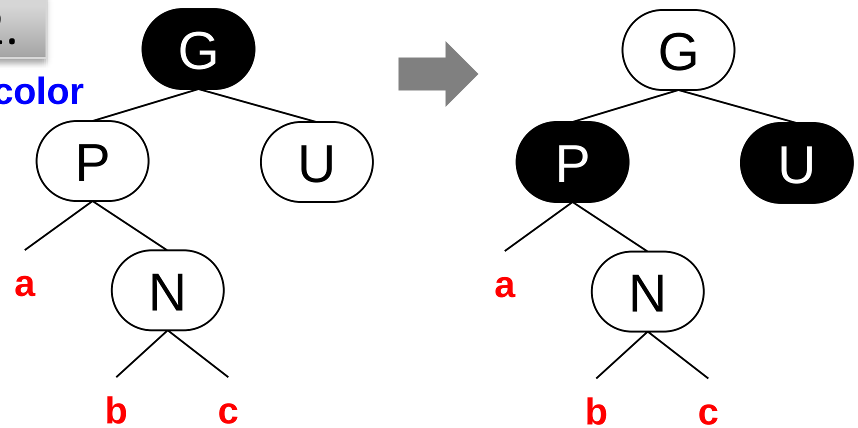
FOR PRINT

fixTree Cases

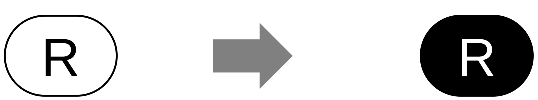
1\.
Recolor

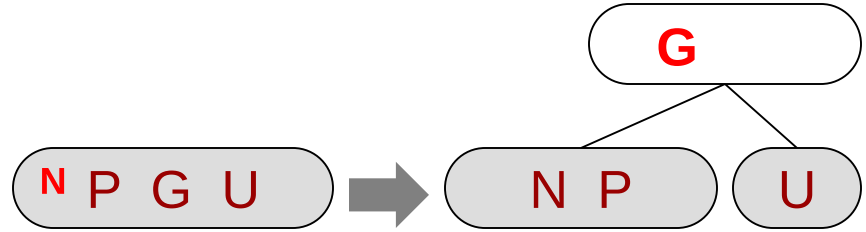


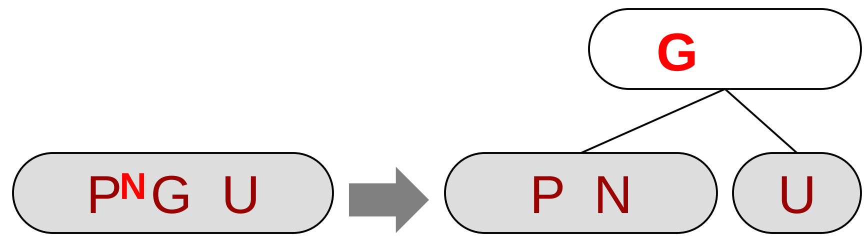
2\.
Recolor



3\.
Recolor Root





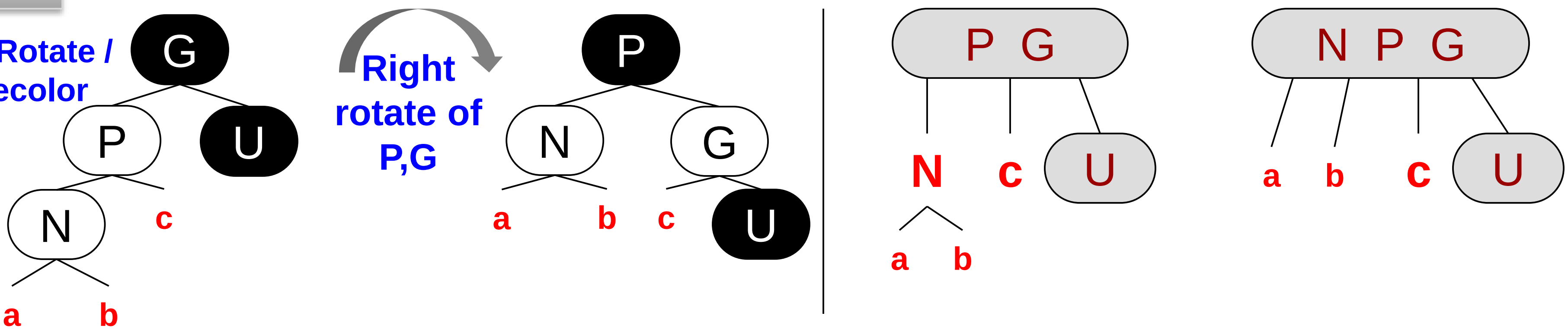


Note: For insertion/removal algorithm we consider non-existent leaf nodes as black nodes

fixTree Cases

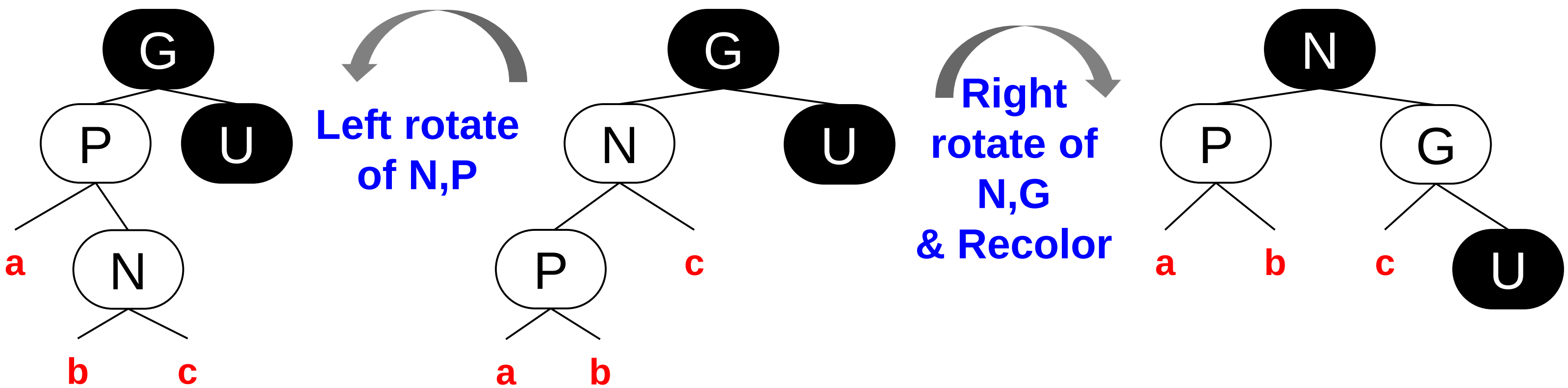
4.

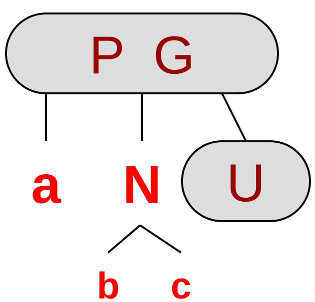
1 Rotate / Recolor

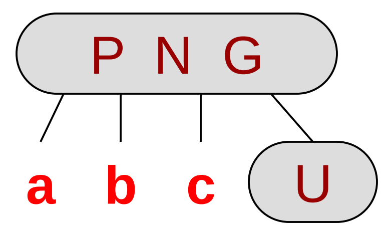


5.

2 Rotates / Recolor







www.cse.ohio-state.edu/~gurari/course/cis680Ch11.html