Calculating Biological Quantities

CSCI 2897

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Lecture 4

Last time on CSCI 2897..

1. **How to verify that a function is a solution of an ODE.**
2. **Solving an ODE initial value problem *numerically* by stepping along the solution.**

Lecture 4 Plan

1. **Exponential - Discrete time**
2. **Exponential - Continuous time**
3. **Logistic - Discrete Time**
4. **Logistic - Continuous Time**
5. **Vector fields**
6. **Examples**

Models of population growth

- For any species, at any scale, the number of individuals changes over time in response to:
  - ✓ ✓ resource availability
  - ✓ ✓ competition
  - ∅ ✓ predation
  - ∅ ✓ disease
  - ✓ ✓ weather
  - ✓ ✓ chance events
- Simplest models are called **exponential** and **logistic**.

SARS - COV - 2

W.T. Omicron BA.5

Exponential vs Logistic Growth

- Both models assume that **the environment is constant**. ✓
- Both models assume that there are **no interactions with other species** ✓
    - no competing species, predators, parasites, etc.

- The models differ in their assumptions about available resources:
    - The **exponential growth model** assumes that the amount of resources available to each individual is constant, regardless of population size. (unbounded resources)
    - The **logistic growth model** assumes that fewer resources are available to each individual as the population size increases. (limited resources)

Discrete time exponential growth

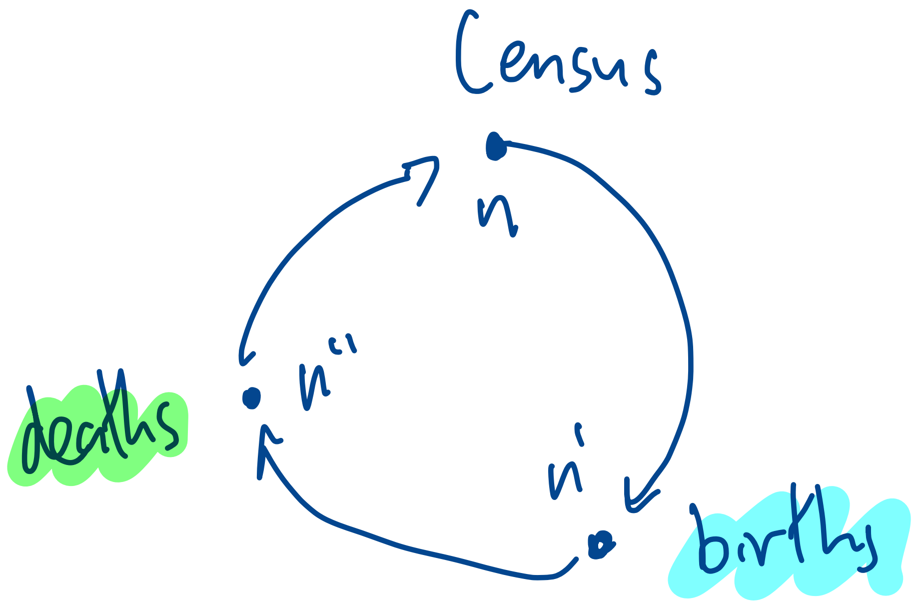
- Let $n(t)$ be the number of individuals at time t.
- Assume that each reproducing parent is replaced by a constant number of individuals R in the next time step.
- This implicitly assumes that all individuals are capable of reproduction, as in a hermaphroditic or asexual species.
    - Can also be applied to species with separate male & female sexes by assuming that the number of offspring is limited by the number of females, and then counting only females.

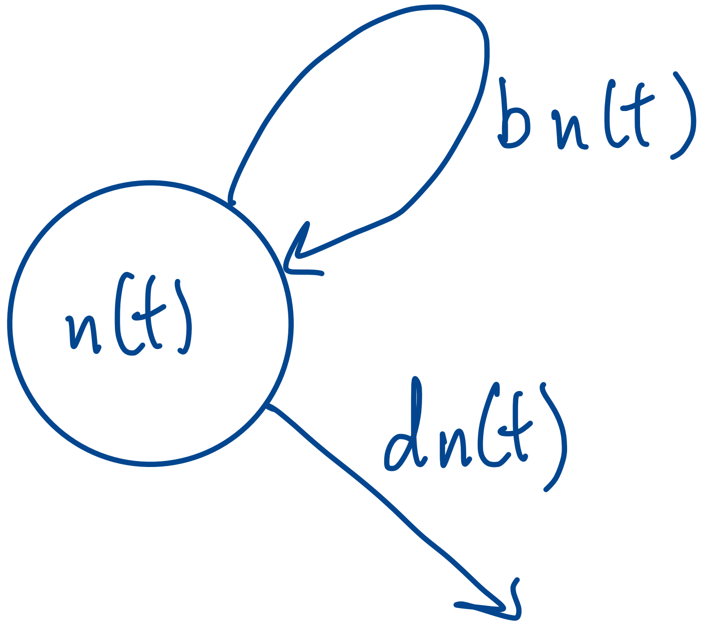
Discrete time exponential growth

- Let $n(t)$ be the number of individuals at time t.
- Assume that each reproducing parent is replaced by a constant number of individuals R in the next time step.
- In this model, we will include just two processes: birth and death.
  - Let b be the number of births per capita per time step
  - Let d be the fraction of the population that dies per time step.

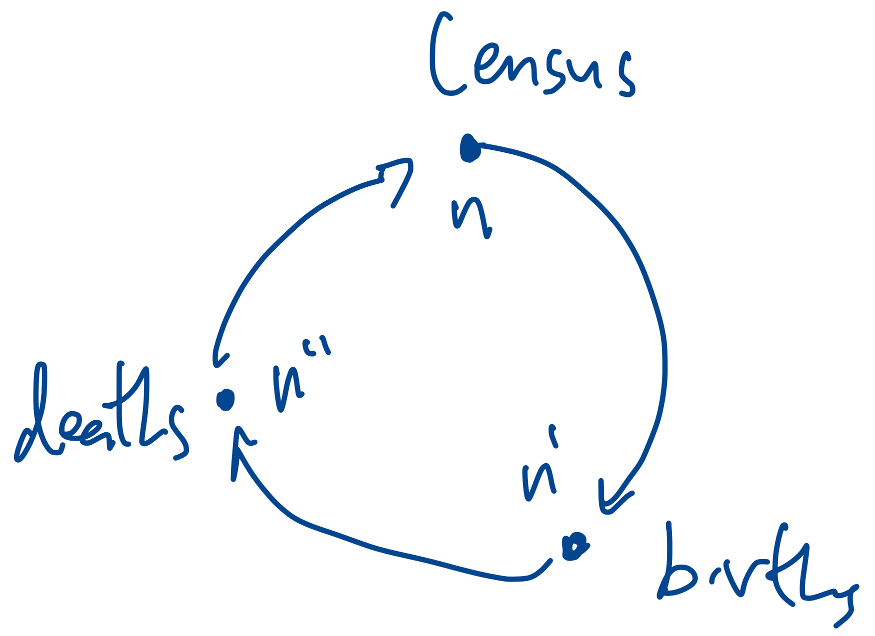
Discrete time exponential growth

- Let $n(t)$ be the number of individuals at time t.
- Let b be the number of births per capita per time step
- Let d be the fraction of the population that dies per time step.
- Let's write down a Life Cycle Diagram and a Flow Diagram for this process.





Discrete time exponential growth



- Let $n(t)$ be the number of individuals at time t.
- Let b be the number of births per capita per time step
- Let d be the fraction of the population that dies per time step.
- Use the life cycle diagram to derive a **recursion** and a **difference equation**.

$$n'(t) = n(t) + b \cdot n(t) = (1+b)n(t)$$

$$n''(t) = n'(t) - d\,n'(t) = (1-d)\,n'(t)$$

$$n(t+1) = n''(t) = (1-d)\,n'(t) = (1-d)(1+b)\,n(t)$$

$$\boxed{n(t+1) = (1-d)(1+b)\,n(t)}$$

recursion

$$\Delta n = n(t+1) - n(t)$$

$$\underbrace{n(t+1) - n(t)}_{\Delta n} = (1-d)(1+b)\,n(t) - n(t)$$

$$\Delta n = [\cancel{1} - d + b - db]\,n(t) - \cancel{n(t)}$$

$$\boxed{\Delta n = [b - d - db]\,n(t)}$$

difference equation

Discrete time exponential growth

- **Recursion**: $n(t+1) = R\, n(t)$

$$n(t+1) = (1+b)(1-d)\, n(t)$$

$$R = (1+b)(1-d)$$

- **Difference**: $\Delta n = (R-1)\, n(t)$

$$\Delta n = [b-d-db]\, n(t)$$

$$R-1 = b-d-db = r$$

- In the biological literature $(R-1)$ is often denoted r. How can we interpret this quantity?

$$\Delta n = r\, n(t)$$

# of new individuals = rate · n(t)

r new individuals per existing individual!

$r = 0 \longrightarrow$ no growth.

$r > 0 \longrightarrow$ growth

$r < 0 \longrightarrow$ decline in population.

$$b-d-db \overset{?}{<} 0$$

$$b-d(1+b) < 0$$

$$b < d(1+b)$$

$$\boxed{\frac{b}{1+b} < d}$$

Discrete time exponential growth

- **Recursion**: $n(t+1) = R\ n(t)$

$n(t+1) = R\,n(t)$

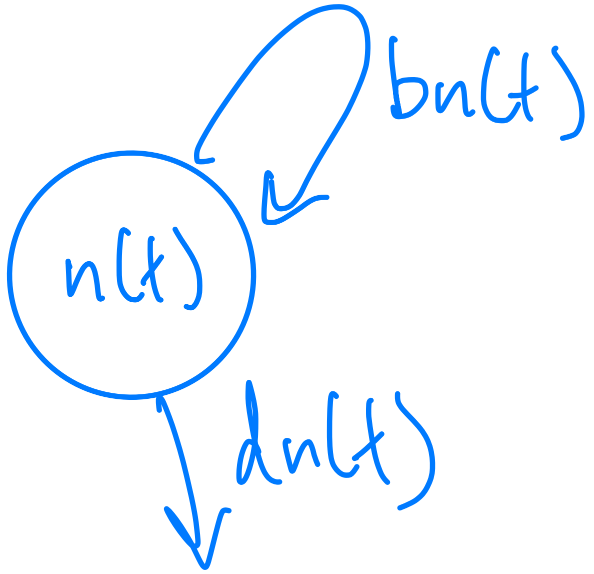
$\Delta n = r\,n(t)$

- **Difference**: $\Delta n = (R-1)\ n(t)$

- In the biological literature $(R-1)$ is often denoted r or r_d .
  - How can we interpret this quantity?
  - r : per-capita change in the number of individuals from one gen. to the next.
  - Sometimes r_d to indicate that this is in $d =$ discrete time.
  - $r_d = R - 1 = (1-d)(1+b) - 1 = b - d - bd$
  - If $R = 1$, then $r_d = 0$, which means no growth—pop. size constant.

Continuous time exponential growth

- What if births and deaths can occur at any time, rather than in specific seasons or time steps?
- Same parameters: per-capita birth rate b and death rate d.
- Using the flow diagram, we can derive the differential equation:



$$\frac{dn}{dt} = bn(t) - dn(t)$$

$$\frac{dn}{dt} = \overbrace{(b-d)}^{r_c}\, n(t)$$

$$\Delta n = \underbrace{(b-d-db)}_{r_d}\, n(t)$$

Continuous time exponential growth

- What if births and deaths can occur at any time, rather than in specific seasons or time steps?
- Same parameters: per-capita birth rate b and death rate d.
- Using the flow diagram, we can derive the differential equation:
  - $\dfrac{dn}{dt} = bn(t) - dn(t) = r_c\, n(t)$
  - where $r_c = b - d$ is called the per-capita growth rate ($c =$ continuous time).

r_c

x^2 x^2

What can we learn from this derivation?

- Notice that r_d became r_c when we took the limit, but
- $r_d = b - d - db$
- $r_c = b - d$
- What the difference, and how can we understand it in terms of modeling?

$-db$

per-timestep per-capita birth and death rates.

As we make timestep smaller, b gets smaller
d gets smaller

$\Downarrow$

b·d gets really small

Δt gets very small
bd gets miniscule
$r_d \approx r_c$

Alternative:

In cont. time, you simply can't be born and die in the same dt.
$\Longrightarrow$ no bd term in r_c

Aside:

- Don't worry! We'll solve this equation (and the next one) numerically and analytically in the next two classes!
- Here is a picture of my dog to tide you over:

King II?

ok

Logistic growth in discrete time

- Many factors slow pop. growth, including declining resource availability, increase predation, higher incidence of disease, and so on.
- Logistic model **describes theses processes indirectly** by assuming that the population replacement number R declines with increasing population size.
- We therefore write it as $R(n)$.
- Let's say that when the population size is zero, $R(0) = 1 + r_d$.

same as exp. growth when $n=0$.

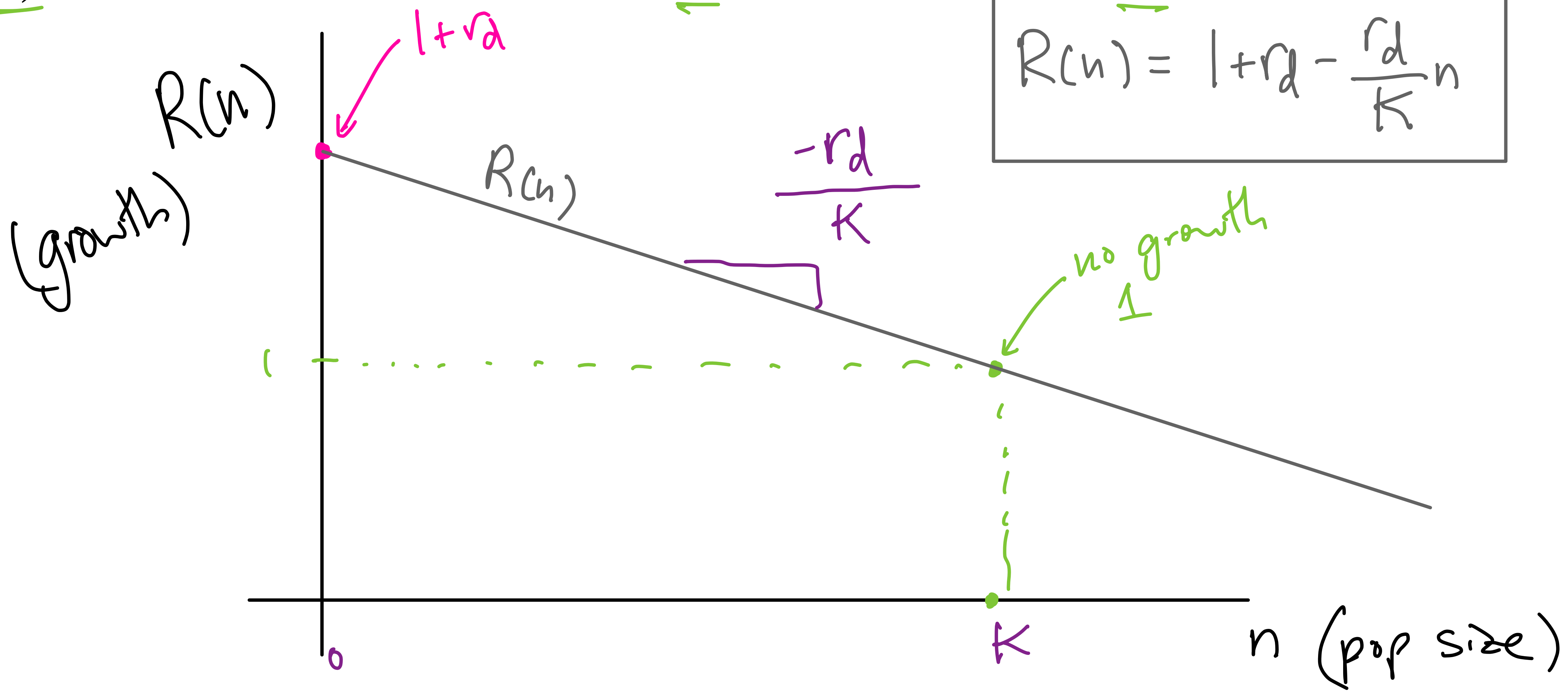
    - This is called the **intrinsic rate of growth.**
    - It's what happens when there aren't resource limitations (= prev. model).
- Let's say that $R(n)$ decreases until it becomes 1, at *some* value of n.

replacement (no growth)

Logistic growth in discrete time

- Let's say that when the population size is zero, $R(0) = 1 + r_d$.
    - This is called the **intrinsic rate of growth.**
    - It's what happens when there aren't resource limitations (= prev. model).
- Let's say that $R(n)$ decreases until it becomes 1, at *some* value of n.
- A sketch helps:

$$\boxed{R(n) = 1 + r_d - \frac{r_d}{K} n}$$



Logistic growth in discrete time

- If we write $n(t+1) = \underbrace{R(n)}\, \underbrace{n(t)}$, we now get
- $n(t+1) = \left(1 + r_d - \frac{r_d}{K} n(t) \right) n(t)$

$$\underbrace{n(t+1)}_{\text{next}} = \underbrace{n(t)}_{\text{prev}} + \underbrace{r_d \left(1 - \frac{n(t)}{K} \right) n(t)}_{\text{change}}$$

Logistic growth in discrete time

- If we write $n(t+1) = R(n)\ n(t)$, we now get

- $n(t+1) = n(t) + r_d\ n(t)\left(1 - \frac{n(t)}{K}\right)$ recursion

- $\Delta n = r_d\ n(t)\left(1 - \frac{n(t)}{K}\right)$ difference

Logistic growth in *continuous time*

- If we also assume that r is a function of n, and that it declines from $r(0) = r_c$ to $r(K) = 0$, then we can also get the ODE:

- $$\frac{dn}{dt} = r_c\, n(t)\left(1 - \frac{n(t)}{K}\right)$$ ✓

discrete time

$\updownarrow$

recursion / difference Eq.

continuous time

$\updownarrow$

differential equations.

QUICK QUIZ, HOT SHOT

- $$\frac{dn}{dt} = r_c\, n(t)\left(1 - \frac{n(t)}{K}\right)$$

- Order? highest derivative $\frac{dn}{dt}$ → $\boxed{1^{st}}$
- Linear or nonlinear?
- $\boxed{\text{ODE}}$ or PDE?

nonlinear

$$\frac{dn}{dt} = r_c\left(n(t) - \frac{n^2(t)}{K}\right)$$

$$\frac{dn}{dt}$$

$$\frac{\partial n}{\partial t}$$

Understanding an ODE with a vector field

"wind"

- $$\frac{dn}{dt} = r_c \, n(t)$$

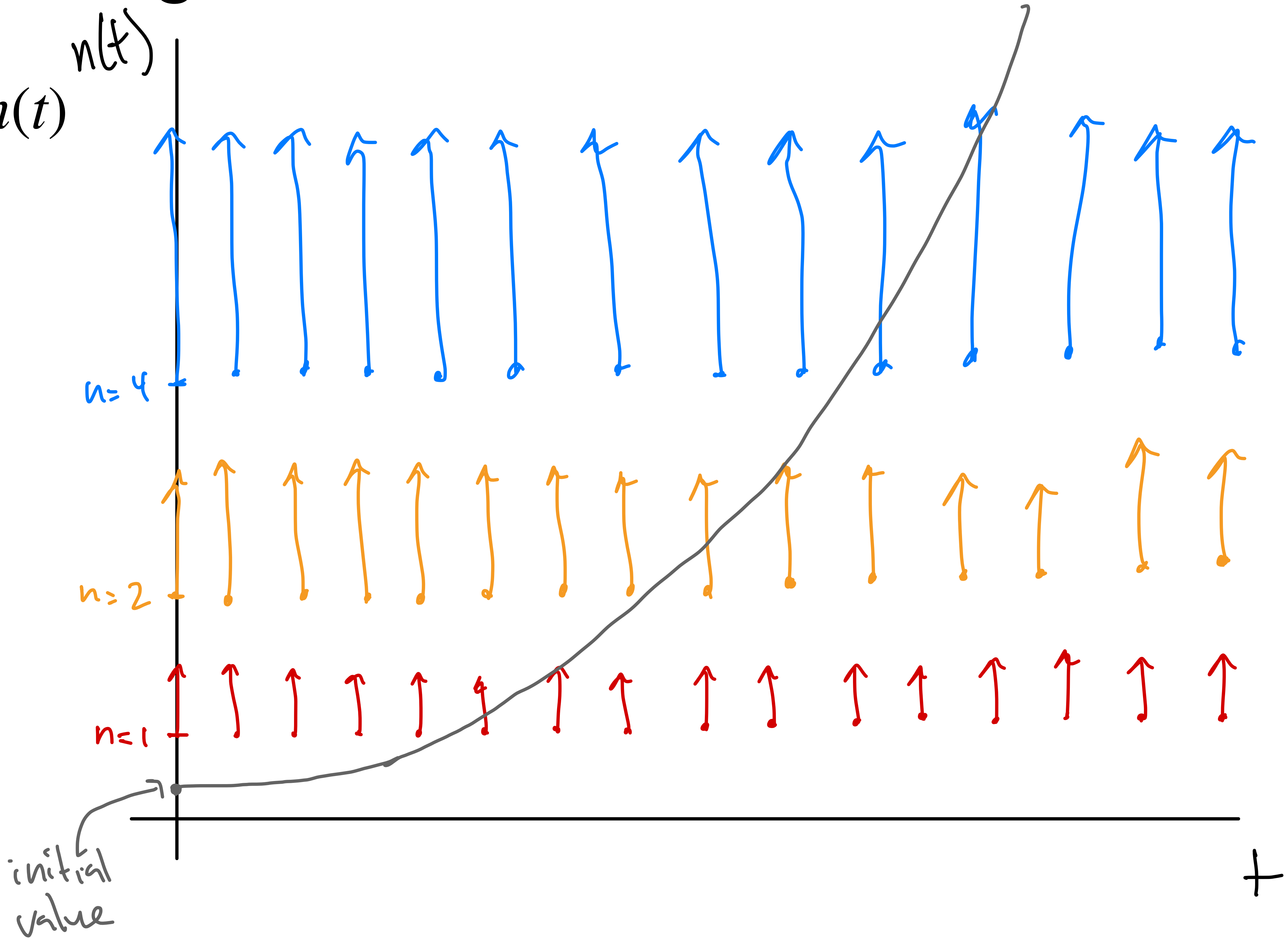
$r_c > 0$
(growth)

$r_c = 1$

$\frac{dn}{dt} = 1 \cdot 4 = 4$

$\frac{dn}{dt} = 1 \cdot 2 = 2$

$\frac{dn}{dt} = 1 \cdot 1 = 1$



bigger arrow = stronger wind = steeper slope

vectors (arrows) express direction and magnitude of growth/change

exponential growth.

Understanding an ODE with a *vector field*

K = carrying capacity

- $$\frac{dn}{dt} = r_c\, n(t)\left(1 - \frac{n(t)}{K}\right)$$

$r_c = 1$ $\quad$ $K = 10$

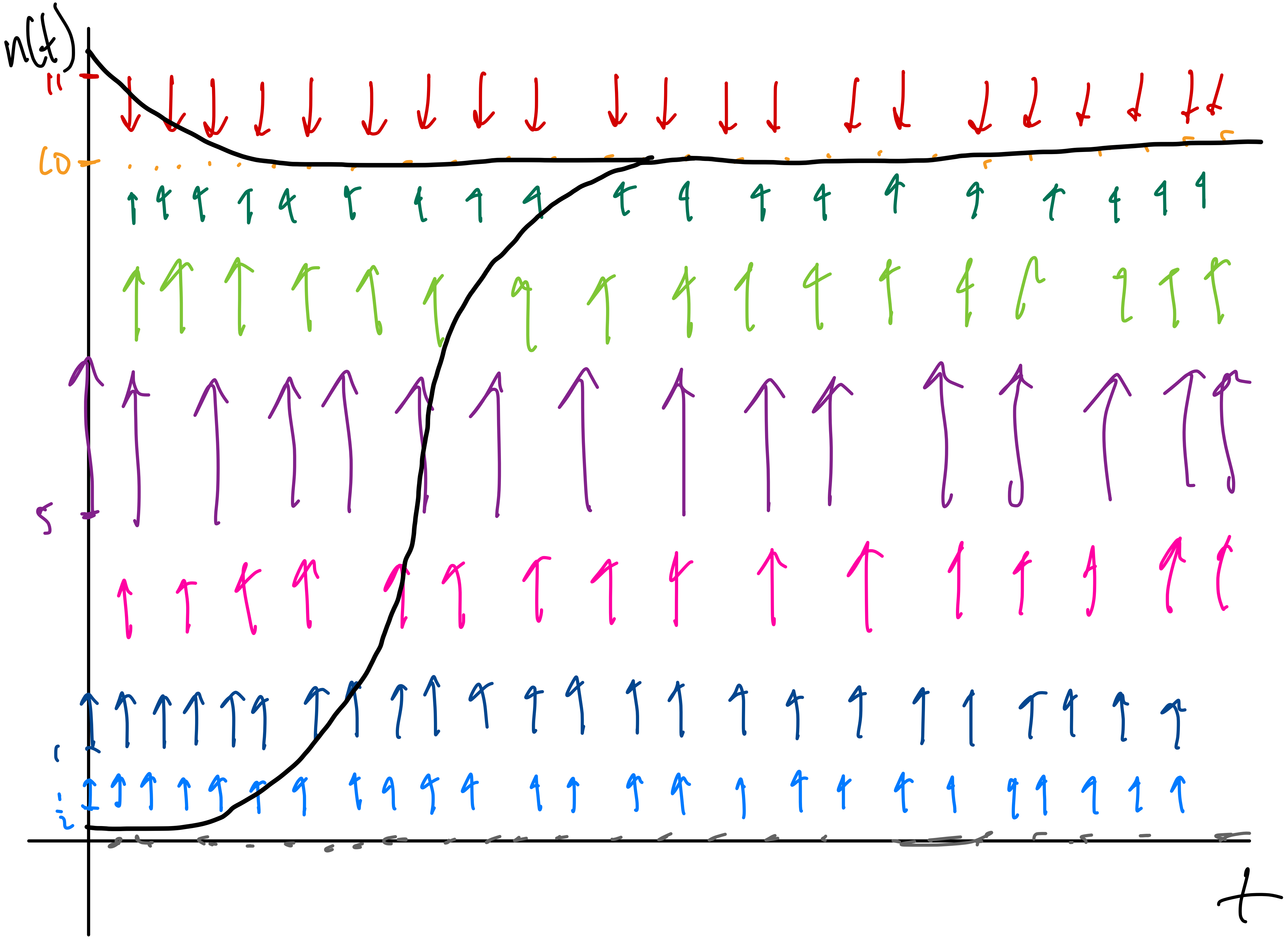
$n = 11$: $\frac{dn}{dt} = 11\left(1 - \frac{11}{10}\right) = -\frac{11}{10}$

$n = 10$: $\frac{dn}{dt} = 10\left(1 - \frac{10}{10}\right) = 0$

$n = 5$: $\frac{dn}{dt} = 5\left(1 - \frac{5}{10}\right) = \frac{5}{2}$

$n = 1$: $\frac{dn}{dt} = 1\left(1 - \frac{1}{10}\right) = \frac{9}{10}$

$n = \frac{1}{2}$: $\frac{dn}{dt} = \frac{1}{2}\left(1 - \frac{1}{20}\right) = \frac{1}{2}\left(\frac{19}{20}\right) = \frac{19}{40}$



Examples of logistic growth

- Mable & Otto (2001) — cultivated both haploid & diploid *S. cerevisiae* (yeast) in two separate flasks.
- Diploid yeast cells are *bigger* and thus take up more resources.