Calculating Biological Quantities

CSCI 2897

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Lecture 3

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Lecture 3 Plan

1. **A little notation & vocabulary**
2. **What does it mean to "solve" a differential equation?**
3. **Checking an analytical solution**
4. **Creating a numerical solution**

Notation

- "Leibniz" Notation: $\frac{dy}{dt} + y = 2021$

- Prime Notation: $y' + y = 2021$

- Dot Notation: $\dot{y} + y = 2021$

- Note: $\frac{d^2y}{dt^2} = y'' = \ddot{y}$

Vocab: ODE

- An **ODE** is an ordinary differential equation.
- A **PDE** is a partial differential equation.

- ODEs have ordinary derivatives in them. PDEs have partial derivatives in them.
- Note: partial derivatives come up in Calc 3, but tbh they're not that complicated. Ask me in office hours!

- Ordinary derivatives look like $\dfrac{dy}{dx}$ while partial derivates look like $\dfrac{\partial y}{\partial x}$

Vocab: Order

- The **order** of a differential equation is the highest derivative.
- Examples:
  - $y' + y = \pi$
  - $\dddot{z} - \ddot{z} = z$
  - $\dfrac{d^2y}{dx^2} + 5\left(\dfrac{dy}{dx}\right)^3$

Linearity

- A *n*th order ODE is **linear** if we can write the ODE in this form:

$$a_n(t)\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_x(t)\frac{dy}{dt} + a_0(t)y = g(t)$$

- Two special cases that come up often are linear first order:

$$a_1(t)y' + a_0(t)y = g(t)$$

- and linear second order:

$$a_2(t)y'' + a_1(t)y' + a_0(t)y = g(t)$$

- A **nonlinear** ODE is simply one which is not linear.

Practice makes the master!

- **Write down a third order linear ODE.**

- **Write down a second order non-linear ODE.**

What does it mean to "solve" an ODE?

- What does it mean to solve $x + 3 = 9$?

- Suppose that I give you $\sqrt{z} + z^2 - e^{z-4} = 17$. Is $z = 1$ a solution?

- What *is* the solution above? How do we know?

ODEs are the same: solving means satisfying

- Example: $\dot{y} = y$. Show that $y = e^t$ is a solution, but that $y = e^{2t}$ is not.

ODEs are the same: solving means satisfying

- Example: $\dfrac{dy}{dx} = x\sqrt{y}$. Show that $y = \dfrac{1}{16}x^4$ is a solution.

ODEs are the same: solving means satisfying

- Ex: $y'' - 2y' + y = 0$. For what values of the constant k is $y = kte^t$ a solution?

Some ODEs have *families* of solutions

- Definition: a **family of solutions** is a **set of solutions** that all solve an ODE.
- Typically, a family of solutions will have **arbitrary constants**. The number of constants is typically equal to the order of the ODE.
- Ex: $\dot{y} = y$
- Ex: $\ddot{y} = -y$

Exercise: DIY ODEs

1. Write down a solution to an ODE that has not yet been written down. In other words, write down a function.
2. Take a couple derivatives and write those down.
3. Combine them in an equation to create your own ODE.
4. Then swap with someone else, and **verify** (meaning confirm) the solution.

Challenge: DIY recurrence equations?

- On the last slide, we made up our own ODEs and solutions. Can you puzzle out how to do the same kind of thing, but with a **recurrence** equation?
- Recall that a recurrence equation looks like: $n(t+1) =$ some function of $n(t)$

Numerical Solutions to *initial value problems*

- Remember this? Can we write down a recipe for *approximately* solving this?

$$\frac{dy(t)}{dt} = \sqrt{y}, \quad y(0) = 1$$

Numerical Solutions to *initial value problems*

- Goal of numerical solution: generate a set of points $\left(t_n, y(t_n)\right)$ that approximate the analytical solution.
- Why might we want to do this?
- There are many ways to *numerically solve differential equations*, but here is one, referred to as **Euler's Method.**

To solve $y' = f(t, y)$**, with** $y(t_0) = y_0$ **use the formulas**

$$y_{n+1} = y_n + \Delta t \cdot f(t_n, y_n)$$

$$t_{n+1} = t_n + \Delta t$$