Calculating Biological Quantities

CSCI 2897

- Next week ZOOM Recordings
  (Dan @ NIH)

- A1 posted.

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Lecture 3

Lecture 3 Plan

1. **A little notation & vocabulary**

Al **2. What does it mean to "solve" a differential equation?**

3. Checking an analytical solution

Al **4. Creating a numerical solution**

Notation

- “Leibniz” Notation: $\frac{dy}{dt} + y = 2021$

same

- Prime Notation: $y' + y = 2021$

- Dot Notation: $\dot{y} + y = 2021$

"dot" derivatives always mean w.r.t. time.

- Note: $\frac{d^2y}{dt^2} = y'' = \ddot{y}$

Vocab: ODE

- An **ODE** is an ordinary differential equation.
- A **PDE** is a partial differential equation.

- ODEs have ordinary derivatives in them. PDEs have partial derivatives in them.
- Note: partial derivatives come up in Calc 3, but tbh they're not that complicated. Ask me in office hours!

d $\qquad$ ∂ \partial

- Ordinary derivatives look like $\frac{dy}{dx}$ while partial derivates look like $\frac{\partial y}{\partial x}$

Vocab: Order

- The **order** of a differential equation is the highest derivative.
- Examples:
  - $y' + y = \pi$ $\longrightarrow$ first order (y')
  - $\dddot{z} - \ddot{z} = z$ $\longrightarrow$ third order $(\dddot{z})$
  - $\dfrac{d^2y}{dx^2} + 5\left(\dfrac{dy}{dx}\right)^3 = \dfrac{1}{10}$ $\longrightarrow$ second order $\left(\dfrac{d^2y}{dx^2}\right)$

Linearity

- A *n*th order ODE is **linear** if we can write the ODE in this form:

$$a_n(t)\frac{d^ny}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_x(t)\frac{dy}{dt} + a_0(t)y = g(t)$$

derivatives (not non-linear)

functions of t (and only of t) (not of y!)

- Two special cases that come up often are linear first order:

$$a_1(t)y' + a_0(t)y = g(t) \longrightarrow y' + t^2 y = \sin(t)$$

- and linear second order:

$$a_2(t)y'' + a_1(t)y' + a_0(t)y = g(t)$$

$$\frac{1}{t}y'' + 2y = 0$$

Linear

$$\dddot{y} + \sin(t)\dot{y} + y = e^{\pi}$$

- A **nonlinear** ODE is simply one which is not linear.

Ex:

Non Linear

$$y' + y^2 = 0$$

$$\ddot{y} + \sin(\dot{y}) + y = e^{\pi}$$

Practice makes the master!

- **Write down a third order linear ODE.**

	Highest Order?	Linear?
$3y''' + 2y'' + y' = 19$	3	✓
$5y''' + 2ty'' + ty' + 6y = 357$	3	✓
$y''' + y + t^2 = 10$	3	✓

- **Write down a second order non-linear ODE.**

	Highest Order?	Linear?
$y'' + \sin(y) = \pi$	2	no
$\frac{d^2x}{dt^2} + \sin(x) - x^2 + e^{\frac{dx}{dt}} = 1$	2	no
$\ddot{y} + y^2 = 0$	2	no

$$\sqrt{\frac{d^2z}{dt^2}} = z \qquad \ddot{y} + \dot{y}y = 1$$

What does it mean to "solve" an ODE?

- What does it mean to solve $x + 3 = 9$?

claim: $x=5$

plug in: $5+3=9$

$8=9$

↯ contradiction wrong!

Find a value of x, such that, when I plug it into my equation, I get LHS = RHS.

claim: $x=6$

plug in:

$6+3=9$

$9=9$

☺

- Suppose that I give you $\sqrt{z} + z^2 - e^{z-4} = 17$. Is $z = 1$ a solution?

Plug in: $\sqrt{1} + 1^2 - e^{1-4} \stackrel{?}{=} 17$

$1 + 1 - e^{-3} = 17$

$2 - \frac{1}{e^3} \neq 17$ ☹

nope! LHS $\neq$ RHS

- What *is* the solution above? How do we know?

$z=4$

$\sqrt{4} + 4^2 - e^{4-4} \stackrel{?}{=} 17$

$2 + 16 - 1 \stackrel{?}{=} 17$

$17 = 17$

yep! LHS = RHS

ODEs are the same: solving means satisfying

- Example: $\dot{y} = y$. Show that $y = e^t$ is a solution, but that $y = e^{2t}$ is not.

$$\frac{dy}{dt} = y$$

$$e^t$$

I know

$y = e^t$

$\frac{dy}{dt} = e^t$

$e^t = e^t$ $\Rightarrow$ $y = e^t$ solves

LHS = RHS ✓ the ODE $\dot{y} = y$

$y = e^{2t}$

$\frac{dy}{dt} = e^{2t} \cdot 2$

$$\frac{dy}{dt} = y$$

plug in

$2e^{2t} = e^{2t}$

$e^{2t} = 0$

LHS $\neq$ RHS

so $\Rightarrow$ $y = e^{2t}$ does not solve!

ODEs are the same: solving means satisfying

- Example: $\dfrac{dy}{dx} = x\sqrt{y}$. Show that $y = \dfrac{1}{16}x^4$ is a solution.

need a 1st deriv.

just x

$$\sqrt{y} = \sqrt{\frac{1}{16}x^4}$$

$$\sqrt{y} = \frac{1}{4}x^2$$

① compute derivatives (to plug in) as needed.

② plug them all in

③ simplify to see if LHS=RHS

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(\frac{1}{16}x^4\right)$$

$$\frac{dy}{dx} = \frac{1}{16}\frac{d}{dx}\left(x^4\right)$$

$$\frac{dy}{dx} = \frac{1}{16}4x^3$$

$$\frac{dy}{dx} = \frac{x^3}{4}$$

$$\frac{x^3}{4} = x\,\frac{1}{4}x^2$$

$$\frac{x^3}{4} = \frac{x^3}{4}$$

LHS=RHS ✓ solves!

ODEs are the same: solving means satisfying

① compute derivatives (to plug in) as needed.

② plug them all in

③ simplify to see if LHS=RHS

- Ex: $y'' - 2y' + y = 0$. For what values of the constant k is $y = kte^t$ a solution?

$y = kte^t$ (product rule)

$y' = k\,1\,e^t + kte^t$

$y' = ke^t(1+t)$

$y'' = ke^t(1+t) + ke^t(1)$

$y'' = ke^t(2+t)$

$ke^t(2+t) - 2ke^t(1+t) + kte^t = 0$

$ke^t\left[(2+t) - 2(1+t) + t\right] = 0$

$ke^t\left[2+t - 2 - 2t + t\right] = 0$

$ke^t\left[\underbrace{2-\cancel{2}}_{0} + \underbrace{t - 2\cancel{t} + t}_{0}\right] = 0$

$ke^t[0] = 0$

What values of k give me RHS = LHS?

All values of k solve!

Some ODEs have *families* of solutions

- Definition: a **family of solutions** is a **set of solutions** that all solve an ODE.
- Typically, a family of solutions will have **arbitrary constants**. The number of constants is typically equal to the order of the ODE.
- Ex: $\dot{y} = y$

$\frac{dy}{dt} = y$

$y(t) = ke^t$ k can be anything!

$y'(t) = ke^t$

$ke^t = ke^t$ true no matter what specific value k is!

- Ex: $\ddot{y} = -y$

$y(t) = A\cos(t) + B\sin(t)$ 2^{nd} order ODE $\rightarrow$ 2 constants.

$y'(t) = -A\sin(t) + B\cos(t)$

LHS $\overset{\checkmark}{=}$ RHS

$y''(t) = -A\cos(t) - B\sin(t)$ Plug in: $-A\cos(t) - B\sin(t) = -(A\cos(t) + B\sin(t))$

Exercise: DIY ODEs

1. Write down a solution to an ODE that has not yet been written down. In other words, write down a function. $y = x^2$
2. Take a couple derivatives and write those down. $y' = 2x \quad y'' = 2$
3. Combine them in an equation to create your own ODE.
4. Then swap with someone else, and **verify** (meaning confirm) the solution.

$$3y - 2y' + 4y'' = 3(x^2) - 2(2x) + 4(2)$$
$$= 3x^2 - 4x + 8$$

$$\boxed{3y - 2y' + 4y'' = 3x^2 - 4x + 8}$$

2nd order linear ODE!

$y = x^2$ solves.

1. ~~t~~take derivs as needed
2. plug in
3. verify LHS = RHS

Challenge: DIY recurrence equations?

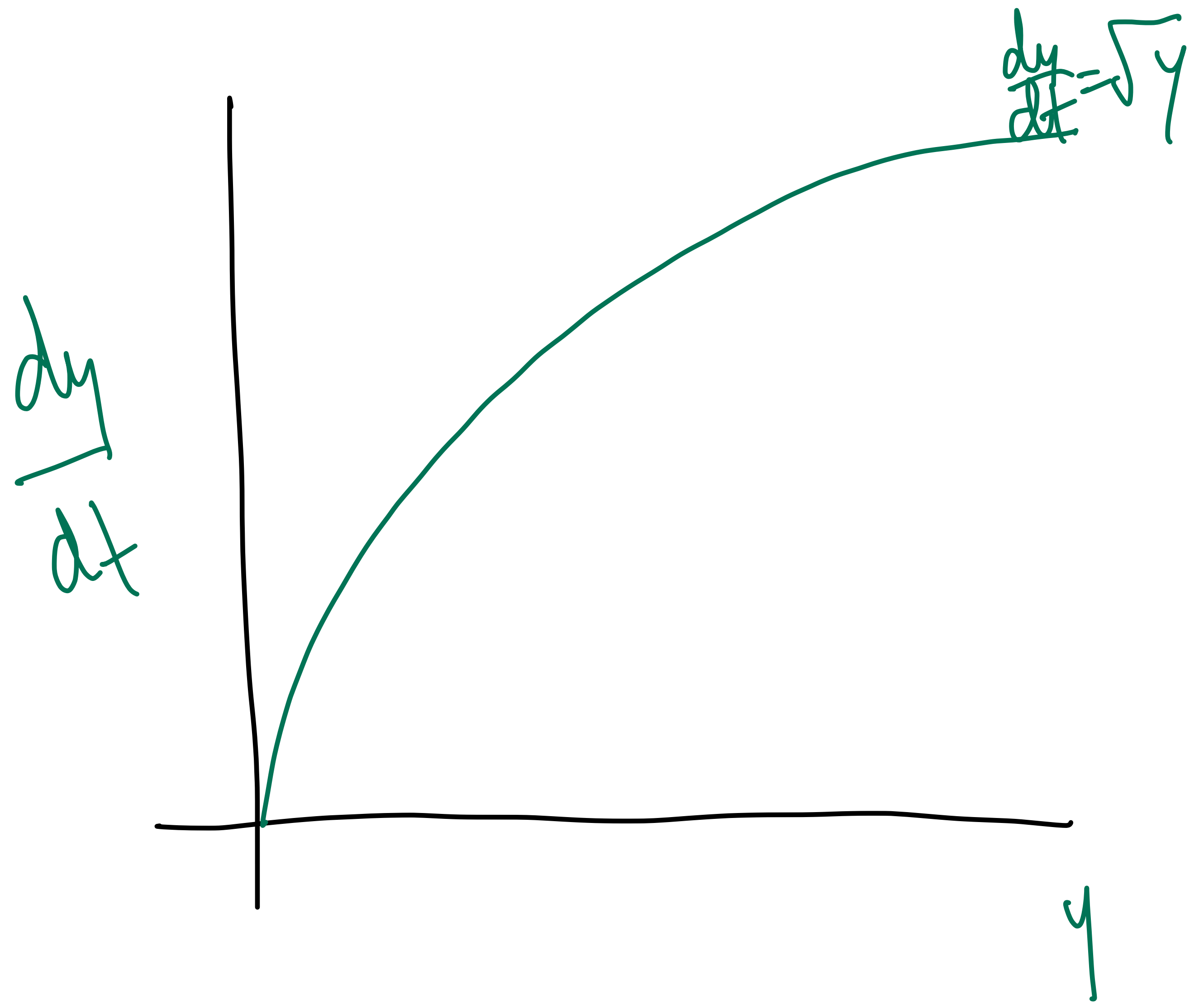
- On the last slide, we made up our own ODEs and solutions. Can you puzzle out how to do the same kind of thing, but with a **recurrence** equation?
- Recall that a recurrence equation looks like: $n(t+1) =$ some function of $n(t)$

Numerical Solutions to *initial value* problems

- Remember this? Can we write down a recipe for *approximately* solving this?

$$\frac{dy(t)}{dt} = \sqrt{y}, \quad y(0) = 1$$

what is y, when t=0 ?



(turn by turn directions / GPS)

Numerical Solutions to *initial value problems*

- Goal of numerical solution: generate a set of points $\left(t_n, y(t_n)\right)$ that approximate the analytical solution.

t values

y at each of those t values

subscript n keeps track of recursion... the steps in our GPS

- Why might we want to do this?
  - analytical solution $(y = f(t))$ too difficult!
  - analytical solution impossible.
  - simulate.

- There are many ways to *numerically solve differential equations*, but here is one, referred to as **Euler's Method.**

To solve $y' = f(t, y)$, with $y(t_0) = y_0$ use the formulas

$$y_{n+1} = y_n + \Delta t \cdot f(t_n, y_n)$$ ← recursion

$$t_{n+1} = t_n + \Delta t$$

$$\underset{\text{next } y}{y_{n+1}} = \underset{\text{prev } y}{y_n} + \underbrace{\Delta t \cdot f(t_n, y_n)}_{\text{run} \cdot \text{slope}} = y_n + \Delta y$$

rise $= \Delta y$

$$\underset{\text{next } t}{t_{n+1}} = \underset{\text{prev } t}{t_n} + \underset{\text{run.}}{\Delta t}$$