Calculating Biological Quantities

CSCI 2897

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Lecture 7

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Last time on CSCI 2897..

1. Haploid models of natural selection

$$\frac{dn}{dt} = r_c n(t)\left(1 - \frac{n(t)}{K}\right) \longrightarrow n(t) = \frac{K}{1 + \left(\frac{K}{n_0} - 1\right)e^{-r_c t}}$$

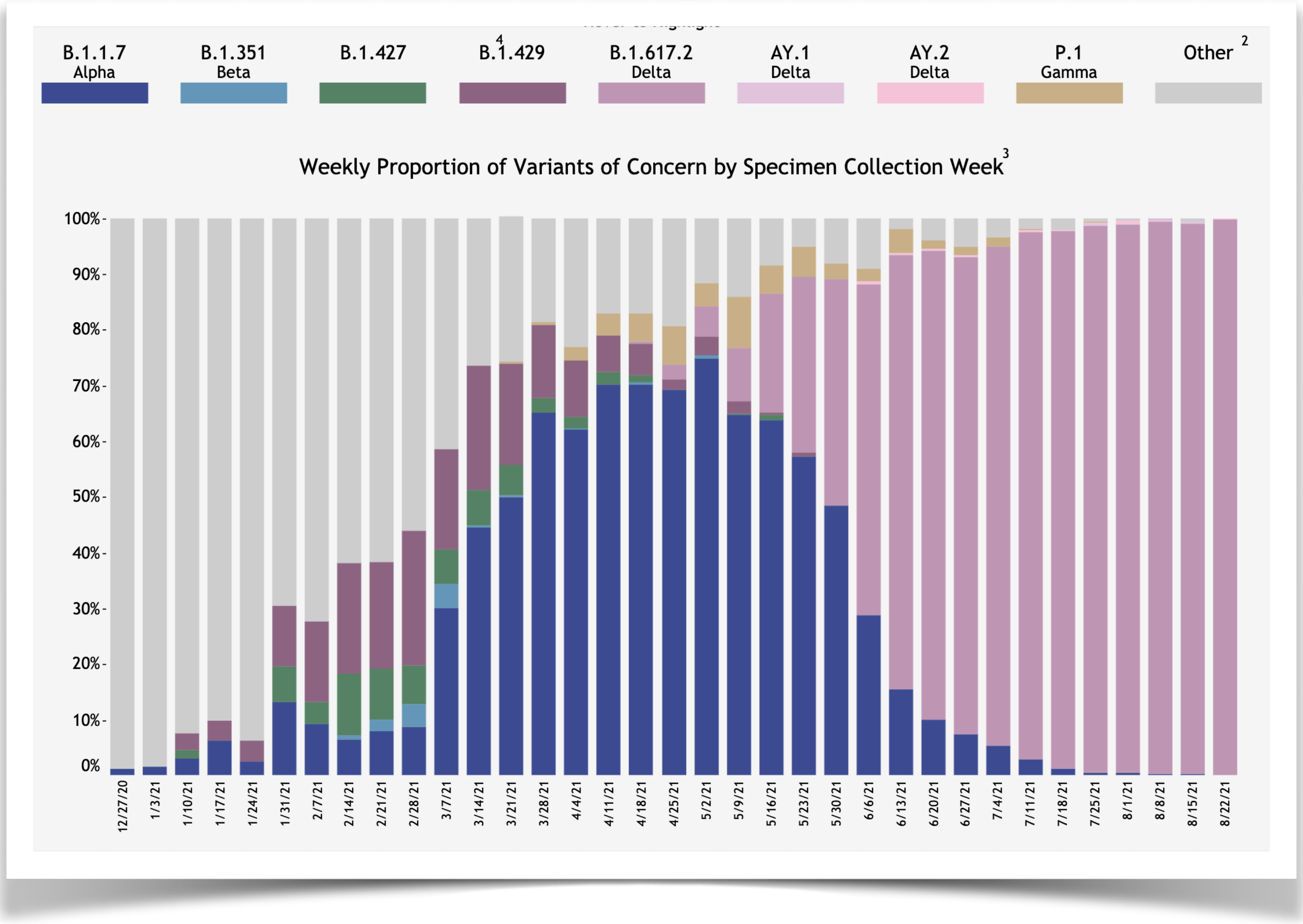
$$\frac{dp}{dt} = s_c p(t)\left(1 - p(t)\right) \longrightarrow p(t) = \frac{1}{1 + \left(\frac{1}{p_0} - 1\right)e^{-s_c t}}$$

$$s_c = (b_A - d_A) - (b_a - d_a) \Big|_{d_A = d_a = 0} \longrightarrow S_c = b_A - b_a$$

S_c pos — A has advantage

neg — a has advantage

$$S_c \Big|_{b_A = b_a = 0} \longrightarrow S_c = d_a - d_A$$



Lecture 7 Plan

1. **Equilibrium solutions**
2. **Explore Logistic & Haploid selection models using Desmos**
3. **Lotka-Volterra Model of Competition**

Equilibrium

A system at **equilibrium** does not change over time. (Plural: **equilibria**.)

For a discrete time model, at equilibrium, it must be true that:

$n(t+1) = n(t)$

$\Delta n = 0$

no change

$\Delta S = 0$

$\Delta I = 0$

$\Delta R = 0$

For a continuous time model, at equilibrium, it must be true that:

$$\frac{dn}{dt} = 0$$

(no change)

$$\frac{dS}{dt} = 0 \qquad \frac{dI}{dt} = 0 \qquad \frac{dR}{dt} = 0$$

Sometimes we call an equilibrium a **steady state**.

Equilibrium

A system at **equilibrium** does not change over time. (Plural: **equilibria**.)

What is the equilibrium / what are the equilibria for our haploid frequency equation?

$$\frac{dp}{dt} = s_c p(t)\big(1 - p(t)\big)$$

Want: $\frac{dp}{dt} \overset{\text{set}}{=} 0$ ①

solve for variables (not our parameters) ②

① $\frac{dp}{dt} = 0 \implies s_c\, p(t)\big(1-p(t)\big) = 0$

② a $s_c \neq 0 \implies p(t)\big(1-p(t)\big) = 0$

$x(1-x) = 0$ $\quad x = 0$, $x = 1$

$p(t) = 0$ or $p(t) = 1$

b $s_c = 0$ $\quad 0 \cdot p(t)\big(1-p(t)\big) = 0$

$0 = 0$

$p(t)$ can be anything!

Note: we're always solving for equilibrium values of the *variables*, not the *parameters*.

Equilibrium

If system is at equilibrium, it will stay there indefinitely — unless "bumped".

A system at **equilibrium** does not change over time. (Plural: **equilibria**.)

What is the equilibrium / what are the equilibria for our haploid frequency equation?

$$\frac{dp}{dt} = s_c p(t)\big(1 - p(t)\big)$$

$s_c \neq 0$ $p(t) = 0$ or $p(t) = 1$

When $s_c \neq 0$, either a goes to 100% ($p = 0$)

or A goes to 100% ($p = 1$)

$s_c = 0$ $p(t)$ can be anything!

no advantage in fitness, the system stays at whatever p it starts at.

Note: we're always solving for equilibrium values of the *variables*, not the *parameters*.

Stability

An equilibrium is **locally stable** if a system near that equilibrium approaches it. This property is called **locally attracting**.

Suppose @ equilibrium

→ jiggle/bump system. ⟶ goes back to equil.

An equilibrium is **globally stable** if a system approaches that equilibrium *regardless* of its initial position.

An equilibrium is **unstable** if a system near the equilibrium moves away from it. This property is called **repelling**.

Suppose @ equilibrium

→ jiggle/bump system. ⟶ ~~goes~~ does not go back to equil.

Stability

Are the equilibria for our haploid allele frequency equation stable or unstable?

$$\frac{dp}{dt} = s_c p(t)\big(1 - p(t)\big)$$

$p_{equil} = 0$ Let $p = p_{equil} + \varepsilon$

$p = \varepsilon$ "tiny!"

$$\frac{dp}{dt} = \underbrace{\overset{+}{s_c} \cdot \overset{+}{\varepsilon} \, (1 - \overset{+}{\varepsilon})}_{+ \text{ or } - \text{ ?}}$$

positive! $s_c > 0$ unstable $p = 0$

negative! $s_c < 0$ stable $p = 0$

$p_{equil} = 1$ Let $p = p_{equil} - \varepsilon$

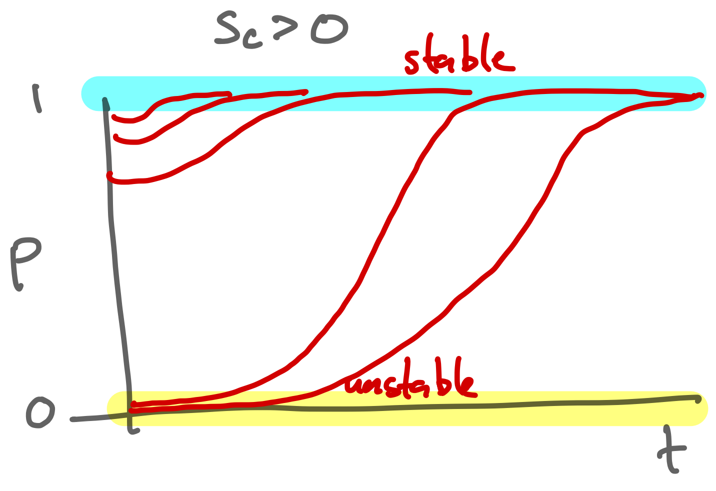
$p = 1 - \varepsilon$

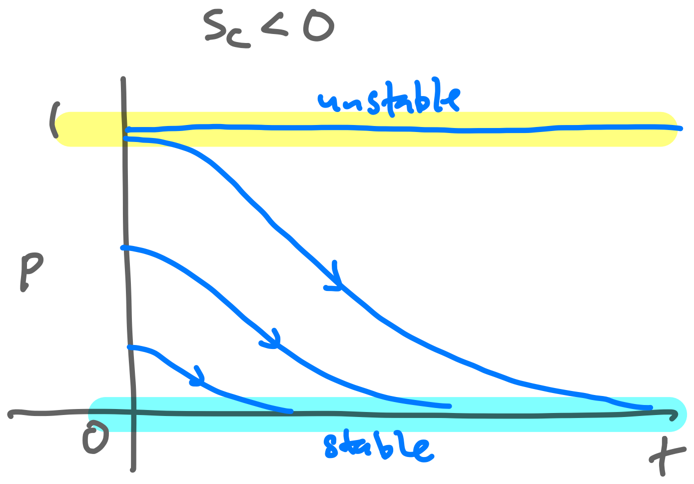
$$\frac{dp}{dt} = s_c \cdot (1-\varepsilon)\big(1 - (1-\varepsilon)\big)$$

$$= \overset{+}{s_c} \, (\overset{+}{1-\varepsilon})(\overset{+}{\varepsilon})$$

positive! $s_c > 0$ stable $p = 1$

negative! $s_c < 0$ unstable $p = 1$





Bonus

Identify the equilibrium/a of the logistic growth equation, and characterize stability.

$$\dot{n} = r\,n\left(1 - \frac{n}{K}\right)$$

$\dot{n} = 0$

$0 = rn\left(1 - \frac{n}{K}\right)$

$r \neq 0$

$0 = n\left(1 - \frac{n}{K}\right)$

either $\boxed{n = 0}$

$1 - \frac{n}{K} = 0$

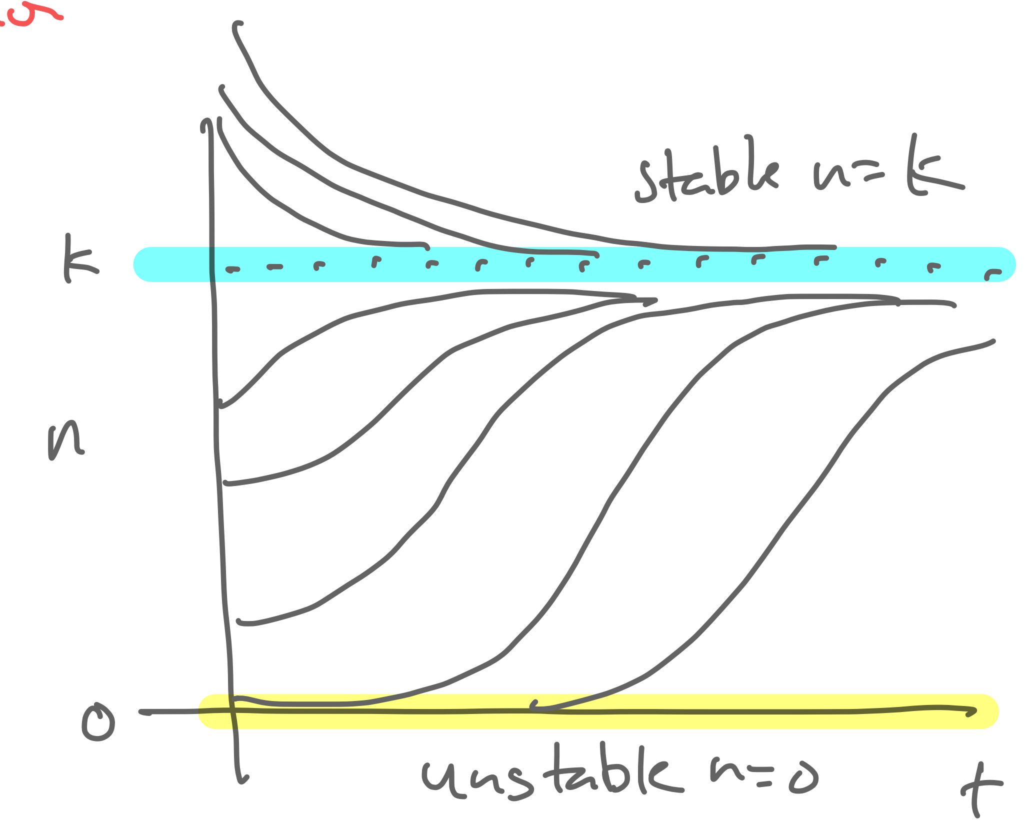
$1 = \frac{n}{K}$

$\boxed{n = K}$

haploid selection

$p = 0$

$p = 1$



https://www.desmos.com/calculator

1. Plot the haploid selection model for $0 \leq t \leq 20$ and $0 \leq p \leq 1$.
2. Include all 4 parameters b_a, b_A, d_a, d_A
3. Learn how to animate.
4. Learn how to show multiple solutions at once.
5. Place a slope-field point on the plot.