Calculating Biological Quantities

CSCI 2897

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Lecture 8

Last time on CSCI 2897..

1. **Equilibrium solutions**

Lecture 8 Plan

1. **Lotka-Volterra Model of Competition**
2. **Consumer-Resource Models**
3. **Reverse engineering an equation — equations to interpretation.**

Lotka-Volterra Competition

Imagine that there are two species, with population sizes $n_1(t)$ and $n_2(t)$.

Let's imagine that each one has the property from Logistic Growth where its growth rate R depends on its population size n, so we have $R_1(n_1)$ and $R_2(n_2)$.

What if one species' growth rate depended on the size of the other population?

Specifically, suppose that species i experiences competition *as if its own species had population* $n_i(t) + \alpha_{ij}\ n_j(t)$. (Here, i could be 1 or 2).

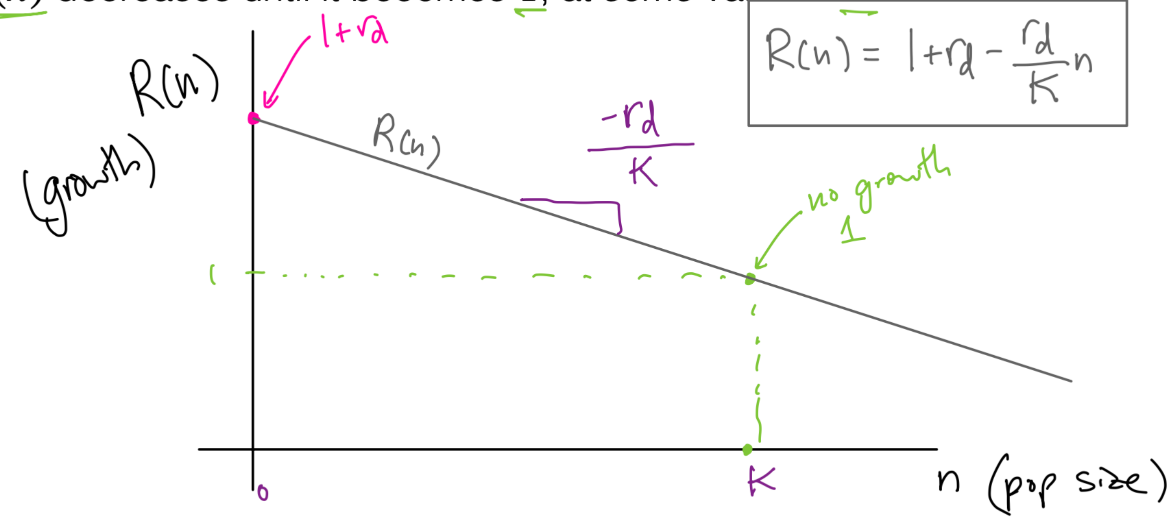
Lotka-Volterra Competition

Remember when we derived the Logistic Growth equation?

Logistic growth in discrete time

- Let's say that when the population size is zero, $R(0) = 1 + r_d$.
  - This is called the **intrinsic rate of growth.**
  - It's what happens when there aren't resource limitations (= prev. model).
- Let's say that $R(n)$ decreases until it becomes 1, at *some* value of n.
- A sketch helps:

$$\boxed{R(n) = 1 + r_d - \frac{r_d}{K} n}$$



Logistic growth in discrete time

- If we write $n(t+1) = R(n)\, n(t)$, we now get
- $n(t+1) = \left(1 + r_d - \frac{r_d}{K} n(t)\right) n(t)$

$$\underbrace{n(t+1)}_{\text{next}} = \underbrace{n(t)}_{\text{prev}} + \underbrace{r_d \left(1 - \frac{n(t)}{K}\right) n(t)}_{\text{change}}$$

We're now going to modify that equation for $R(n)$.

Lotka-Volterra Competition

Let $R_i = (1 + r_i) + \left(\dfrac{-r_i}{K_i}\right)\Big(n_i(t) + \alpha_{ij}n_j(t)\Big)$

Let's plug in this reproductive factor into *each* of our two update equations:

$$n_1(t+1) =$$

$$n_2(t+1) =$$

Lotka-Volterra Competition

We can write similar equations in continuous time:

$$\frac{dn_1}{dt} =$$

$$\frac{dn_2}{dt} =$$

Lotka-Volterra Competition

Quick check: if the species don't interact, then:

which implies that…

$$\frac{dn_1}{dt} = r_1 n_1(t)\left(1 - \frac{n_1(t) + \alpha_{12}\; n_2(t)}{K_1}\right) =$$

$$\frac{dn_2}{dt} = r_2 n_2(t)\left(1 - \frac{n_2(t) + \alpha_{21}\; n_1(t)}{K_2}\right) =$$

Interpretation:

Also note: this model is *symmetric* in that relabeling $1 \leftrightarrow 2$ produces the same equations.

Lotka-Volterra…Competition?

$$\frac{dn_1}{dt} = r_1 n_1(t)\left(1 - \frac{n_1(t) + \alpha_{12}\, n_2(t)}{K_1}\right)$$

$$\frac{dn_2}{dt} = r_2 n_2(t)\left(1 - \frac{n_2(t) + \alpha_{21}\, n_1(t)}{K_2}\right)$$

What if α_{12} is negative? How does an increase in n_2 affect $\frac{dn_1}{dt}$?

Lotka-Volterra…Competition?

α_{12}	α_{21}	Relationship

$$\frac{dn_1}{dt} = r_1 n_1(t)\left(1 - \frac{n_1(t)+\alpha_{12}\, n_2(t)}{K_1}\right)$$

$$\frac{dn_2}{dt} = r_2 n_2(t)\left(1 - \frac{n_2(t)+\alpha_{21}\, n_1(t)}{K_2}\right)$$

Let's code up the Lotka-Volterra model to explore!

$$\frac{dn_1}{dt} = r_1 n_1(t)\left(1 - \frac{n_1(t) + \alpha_{12}\, n_2(t)}{K_1}\right)$$

$$\frac{dn_2}{dt} = r_2 n_2(t)\left(1 - \frac{n_2(t) + \alpha_{21}\, n_1(t)}{K_2}\right)$$

with initial conditions

$$n_1(0) = a$$

$$n_2(0) = b$$

Consumer-Resource Models

So far we've been thinking about resources as constant.

- **light** striking a patch of land
- **nutrients** in a river flowing past a location

But in many situations, the resources *get depleted* as they are consumed.

- Bears eat salmon—and decrease the salmon population as a consequence!

We can account for these phenomena using a *consumer-resource* model.

Consumer-Resource Models: General Structure

Here is the general form of a consumer-resource model. What do you see?

$$\frac{dn_1}{dt} = f(n_1) - g(n_1, n_2)$$

$$\frac{dn_2}{dt} = \epsilon g(n_1, n_2) - h(n_2)$$

Construct a flow diagram:

Consumer-Resource Models: General Structure

$$\frac{dn_1}{dt} = f(n_1) - g(n_1, n_2)$$

$$\frac{dn_2}{dt} = \epsilon g(n_1, n_2) - h(n_2)$$

$f(n_1)$: rate of change of the resource via means other than consumption $(n_2 = 0)$.

$g(n_1, n_2)$: rate of consumption of the resource by the consumer.

ϵ : the conversion factor by which resource units $\rightarrow$ consumer units.

$h(n_2)$: rate at which the number of consumers changes without resources $(n_1 = 0)$.

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TABLE 3.3

Consumer-resource models. Examples of functions that can be used in the consumer-resource model (3.16), where n_1 refers to the level of resources (e.g., number of prey) and n_2 refers to the level of consumers (e.g., number of predators).

Function	Description
$f(n_1) = \theta$	Inflow of resources at a constant rate
$f(n_1) = -\psi$	Outflow of resources at a constant rate
$f(n_1) = r\, n_1$	Constant per capita growth of resource species
$f(n_1) = r n_1\left(1 - \frac{n_1}{K}\right)$	Per capita growth of resource species declines linearly with resource level (logistic)
$f(n_1) = r\, n_1\, e^{-\alpha n_1}$	Per capita growth of resource species declines exponentially with resource level
$g(n_1, n_2) = a\, c\, n_1\, n_2$	Linear (type I) rate of resource consumption
$g(n_1,n_2) = \frac{a\, c\, n_1}{b + n_1} n_2$	Saturating (type II) rate of resource consumption
$g(n_1,n_2) = \frac{a\, c\, n_1^k}{b + n_1^k} n_2$	Generalized (type III) rate of resource consumption
$h(n_2) = \delta\, n_2$	Constant per capita death rate of consumer
$h(n_2) = (\delta\, n_2)\, n_2$	Per capita death rate of consumer increases linearly with consumer population size

Otto & Day, Table 3.3

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Consumer-Resource Models: Const. Immigration

$$\frac{dn_1}{dt} = \theta - a\ c\ n_1\ n_2$$

$$\frac{dn_2}{dt} = \epsilon\ a\ c\ n_1\ n_2 - \delta\ n_2$$

Consumer-Resource Models: Const. Immigration

$$\frac{dn_1}{dt} = \theta - a\ c\ n_1\ n_2$$

$$\frac{dn_2}{dt} = \epsilon\ a\ c\ n_1\ n_2 - \delta\ n_2$$

Example: a nutrient flows into a lake at a constant rate.
A population of algae uses that nutrient to grow.

Consumer-Resource Models: Const. Immigration

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Example: a nutrient flows into a lake at a constant rate. A population of algae uses that nutrient to grow.

Extension: How could we explore a situation where the nutrient no longer flows into the lake? What other scenarios might we explore?

Consumer-Resource Models: Const. Immigration

$$\frac{dn_1}{dt} = \theta - a\ c\ n_1\ n_2$$

What is an equilibrium solution to these equations?

$$\frac{dn_2}{dt} = \epsilon\ a\ c\ n_1\ n_2 - \delta\ n_2$$

Consumer-Resource Models: General Structure

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Consumer-Resource Models: Predator-Prey

$$\frac{dn_1}{dt} = r\ n_1 - a\ c\ n_1\ n_2$$

What happens to n_1 if there is no n_2?

$$\frac{dn_2}{dt} = \epsilon\ a\ c\ n_1\ n_2 - \delta\ n_2$$

Consumer-Resource Models: Predator-Prey

$$\frac{dn_1}{dt} = r\ n_1 - a\ c\ n_1\ n_2$$

$$\frac{dn_2}{dt} = \epsilon\ a\ c\ n_1\ n_2 - \delta\ n_2$$

What happens to n_2 if there is no n_1?

Consumer-Resource Models: Predator-Prey

$$\frac{dn_1}{dt} = r\ n_1 - a\ c\ n_1\ n_2$$

What is an equilibrium solution to these equations?

$$\frac{dn_2}{dt} = \epsilon\ a\ c\ n_1\ n_2 - \delta\ n_2$$

Working Backward: Equations to Interpretation

By now, we're pretty good at taking a system and working forward, from behavior into a diagram and then into equations.

Sometimes, it's useful to be able to work in reverse, by taking an equation, and then interpreting the equation in terms of biology.

For example, here's the logistic equation with constant hunting or harvesting.

$$\frac{dn}{dt} = rn(t)\left(1 - \frac{n(t)}{K}\right) - \theta$$

Working Backward: Equations to Interpretation

However, all 3 of these equations could be described as "logistic growth with constant hunting or harvesting." So what's the difference in *meaning*?

1. $$\frac{dn}{dt} = rn(t)\left(1 - \frac{n(t)}{K}\right) - \theta$$

2. $$\frac{dn}{dt} = rn(t)\left(1 - \frac{n(t)}{K}\right) - Hn(t)$$

3. $$\frac{dn}{dt} = rn(t)\left(1 - \frac{n(t)}{K}\right) - H(n(t) - P)$$

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Working Backward: Equations to Interpretation

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2. $$\frac{dn}{dt} = rn(t)\left(1 - \frac{n(t)}{K}\right) - Hn(t)$$

Working Backward: Equations to Interpretation

However, all 3 of these equations could be described as "logistic growth with constant hunting or harvesting." So what's the difference in *meaning*?

3. $$\frac{dn}{dt} = rn(t)\left(1 - \frac{n(t)}{K}\right) - H(n(t) - P)$$