

# Notes on The Nonexistence of Certain Statistical Procedures in Nonparametric Problems

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**Objective:** Show that it is impossible to derive a nontrivial upper confidence bound for the mean of non-negative random variables in finite samples without any other restrictions.

## 1 Theorems

Let  $\mathcal{F}$  be a given set of distribution functions  $F, G, \dots$  of a real variable, it must satisfy that:

1. For every  $F \in \mathcal{F}$ ,  $\mu_F = \int_{-\infty}^{\infty} z dF$  exists and is finite.
2. For every real  $m$ , there is an  $F \in \mathcal{F}$  with  $\mu_F = m$ .
3.  $\mathcal{F}$  is **convex**, that is, if  $F, G \in \mathcal{F}$ ,  $\pi$  is a positive fraction, and  $H = \pi F(1 - \pi)G$  then  $H \in \mathcal{F}$ .

Let  $X_1, X_2, \dots \sim F$  denote an infinite sequence of independent RVs, i.e.  $\Pr(X_i \leq z) = F(z)$ . Suppose that a (randomized, sequential) sampling procedure is given, i.e. a set of rules for observing  $X_1, X_2, \dots$  one by one up to a certain stage  $N$  s.t. at each stage the decision whether to continue depends (randomly) on the observed values in hand at that stage. The given procedure is assumed to be closed:

$$P_F(N < \infty) = 1, \tag{1}$$

for each  $F \in \mathcal{F}$ .

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Denote the total outcome of the sampling procedure a random variable,  $V$ , i.e.  $V = (X_1, X_2, \dots, X_N)$ . As in (1), for any event  $A$  defined on the sample space of  $V$ ,  $P_F(A)$  will denote the probability of  $A$  when  $F$  obtains, i.e., when each  $X_i$  is distributed according to  $F$ .

If  $\varphi$  is a real valued function of  $V$ ,  $E_F[\varphi]$  denotes the expected value of  $\varphi$  (if it exists) when  $F$  obtains.

For any real number  $m$ , let  $\mathcal{F}_m$  denote the set of all  $F \in \mathcal{F}$  with  $\mu_F = m$ .

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### 1.1 Theorem I

For each bounded real valued function  $\varphi$  on the sample space of  $V$ ,  $\inf_{F \in \mathcal{F}_m} E_F[\varphi]$  and  $\sup_{F \in \mathcal{F}_m} E_F[\varphi]$  are independent of  $m$ .

In other words, even if  $\mu_F$  is known to equal one of two given values  $m_1$  and  $m_2$ , the sample  $V$  cannot provide effective discrimination between the two hypothetical values.

Let  $H$  be the hypothesis that  $\mu_F = 0$  (i.e.,  $F \in \mathcal{F}_0$ ); i.e. the **null hypothesis**. For any test  $t$ , let  $\beta_F(t)$  denote the probability of rejecting  $H$  in using  $t$  when  $F$  obtains, i.e.  $\beta_F(t)$  denotes the **power function** of  $t$  - the probability that the test correctly rejects the null hypothesis when the alternative hypothesis is true.

1.  $t$  is a **somewhere unbiased** level- $\alpha$  test if  $\beta_F \leq \alpha$  for  $F \in \mathcal{F}_0$  and, for some  $m \neq 0$ ,  $\beta_F \geq \alpha$  for  $F \in \mathcal{F}_m$
2.  $t$  is a **similar** level- $\alpha$  test if  $\beta_F = \alpha$  for each  $F \in \mathcal{F}_0$

Note: level- $\alpha$  test, refers to the probability of rejecting the null hypothesis when the null hypothesis is true, i.e. the false positive rate, a.k.a. type I error.

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Let  $\varphi(V)$  denote the probability prescribed by  $t$  of rejecting  $H$  (the null hypothesis) on observation of  $V$  yields the following corollary.

## 1.2 Corollary 1

If  $t$  is a somewhere unbiased level- $\alpha$  test of  $H$ , or a similar level- $\alpha$  test of  $H$ , then  $\beta_F(t) = \alpha$  for all  $F \in \mathcal{F}$ .

The corollary illustrates that all tests of the value of  $\mu$ , assuming that it exists, are unsuccessful.

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Let  $I$  denotes a confidence set for  $\mu_F$ , i.e.  $I$  is a (randomized) function of  $V$ .

For any real  $m$ , let  $C[m]$  denote the event that  $I$

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## 1.3 Corollary II

If  $P_F(C[\mu_F]) \geq 1 - \alpha$  for all  $F \in \mathcal{F}$ , then  $P_F(C[m]) \geq 1 - \alpha$  for all  $m$  and all  $F \in \mathcal{F}$ .

*Proof.* For each  $m$ , let  $p_m(V) = P(C[m] | V) = P_F(C[m])$ ,  $0 \leq p \leq 1$ . Consider a fixed  $m$ , since  $\mu_F = m$ , for all  $F \in \mathcal{F}_m$ , we have that  $P_F(C[\mu_F]) = P_F(C[m]) = E_F[p_m] \geq 1 - \alpha$  for  $F \in \mathcal{F}_m$ . By Theorem I, for each bounded real valued function  $\varphi$  on the sample space of  $V$ ,  $\inf_{F \in \mathcal{F}_m} E_F[\varphi]$  and  $\sup_{F \in \mathcal{F}_m} E_F[\varphi]$  are independent of  $m$ . Hence,  $P_F(C[m]) = E_F[p_m] \geq 1 - \alpha$  for all  $F \in \mathcal{F}$ . Proof is complete for arbitrary  $m$ .  $\square$

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## 1.4 Corollary III

Suppose that there exists at least one  $F \in \mathcal{F}$  such that  $P_F(I \text{ is a set bounded from below}) = 1$ . Then  $\inf_{F \in \mathcal{F}} \{P_F(C[\mu_F])\} = 0$ .

*Proof.* For each  $n = 1, 2, \dots$

Let  $B_n$  denote the event that  $I \in [-n, \infty)$  - i.e.,  $I$  is bounded from below;

Let  $q_n(V)$  denote the probability of  $B_n$  given  $V$ ;  $0 \leq q_n \leq q_{n+1} \leq 1$ .

Let  $\bar{B}_n$  denote the complement of  $B_n$ , i.e. the event that  $I \in (-\infty, -n]$ ;

Let  $1 - q_n(V)$  denote the probability of  $\bar{B}_n$  given  $V$ .

Let  $F$  be a distribution in  $\mathcal{F}$  s.t.  $I$  is bounded from below w.p. 1 when  $F$  obtains. By the Monotone Convergence Theorem, we can swap the expectation and the limit:  $E_F[\lim_n q_n] = \lim_n E_F[q_n] = \lim_n P_F(B_n) = P_F(I \text{ is bounded from below}) = 1 \implies \lim_n q_n(V) = 1$  except on a set of points  $V$  of  $P_F$ -measure zero.

For any  $m < -n$ ,  $p_m(V) = \Pr(m \in I | V) \leq \Pr(\bar{B}_n | V) = 1 - q_n(V)$ , thus, except on a  $P_F$ -null set,

$$\lim_{m \rightarrow -\infty} p_m(V) = 0. \quad (2)$$

Since  $P_F(C[m]) = E_F[p_m]$  for all  $m$ , it follows from (2), by Lebesgue's theorem for boundedly convergent sequences (Bounded Convergence Theorem - corollary of Dominated Convergence Theorem), that

$$\lim_{m \rightarrow -\infty} P_F(C[m]) = \lim_{m \rightarrow -\infty} E_F[p_m] = E_F \left[ \lim_{m \rightarrow -\infty} p_m \right] = 0. \quad (3)$$

By Corollary 2:  $\inf_{G \in \mathcal{F}} \{P_G(C[\mu_G])\} = \inf_{G \in \mathcal{F}} \inf_m \{P_G(C[m])\}$ . It follows from (3) that the common value of these infima is zero.  $\square$

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Consider the problem of constructing a suitable point estimator for  $\mu_F$ . Let  $M$  be an estimator, that is, a real valued (randomized) function of  $V$ . Suppose that when  $F$  obtains, the expected loss in using  $M$  is  $E_P[L(M - \mu_P)] = r_P(M)$ , where  $L(m)$  is bounded from below and  $\lim_{m \rightarrow \infty} L(m) = \infty$  or  $\lim_{m \rightarrow -\infty} L(m) = \infty$  (e.g.,  $L(m) = |m|$ ,  $L(m) = m^2$ ).

Let  $\rho(F)$  be a real valued functional on  $\mathcal{F}$ .

**Definition 1.1** (Uncontrollable).  $\rho$  is uncontrollable (from above) if there exists no real valued (randomized) function of  $V$ , say  $S$ , such that  $\inf_{F \in \mathcal{F}} \{P_F(\rho(F) < S)\} > 0$ .

The following corollary shows there is no estimator  $M$  s.t. the expected loss  $r_F(M)$  is bounded in  $F$ .

## 1.5 Corollary IV

*For any estimator  $M$ ,  $r_P(M)$  is uncontrollable.*

*Proof.* w.l.o.g. assume that  $\lim_{m \rightarrow \infty} L(m) = \infty$  and replace  $L(m)$  by  $L(m) - \inf_a L(a)$ .

Further assume w.l.o.g. that  $L$  is non-negative, with  $\inf_m L(m) = 0$ .

Consider a fixed estimator  $M$ , and  $L_F = L(M - \mu_F)$ .

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**Chebyshev's Inequality** Let  $X$  (integrable) be a random variable with finite non-zero variance  $\sigma^2$  (and thus finite expected value  $\mu$ ). Then for any real number  $k > 0$ ,

$$\Pr(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

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**Markov's Inequality** If  $X$  is a nonnegative random variable and  $a > 0$ , then the probability that  $X$  is at least  $a$  is at most the expectation of  $X$  divided by  $a$ :

$$P(X \geq a) \leq \frac{E(X)}{a}$$

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Since  $L_F \geq 0$ , by Chebyshev's and/or Markov's Inequality, considering the cases  $r_F = 0$ ,  $0 < r_F < \infty$ , and  $r_F = \infty$  separately that  $P_F(L_F \leq \alpha r_F) \geq 1 - \frac{1}{\alpha}$  for all  $\alpha > 0$  and all  $F$ . Since

$$P(L_F \geq \alpha r_F) \leq \frac{E(L_F)}{\alpha r_F} = \frac{1}{\alpha} \implies P(L_F \leq \alpha r_F) \geq 1 - \frac{1}{\alpha}$$

Suppose, contrary to Corollary 4, that there exists a random variable  $S$  with distribution determined by  $V$ , and a positive constant  $\beta$ , such that  $P_F(r_F < S) \geq \beta$  for all  $F \in \mathcal{F}$ . We can assume w.l.o.g. that  $S$  is always

positive. Choose and fix an  $\alpha > 0$  such that  $\beta - \frac{1}{\alpha} > 0$ . Let  $Y = \sup\{m : L(m) \leq \alpha S\}$  and define  $I$  to be the random interval  $[M - Y, \infty)$ . Then  $P_F(I \text{ is bounded from below}) = 1$  for each  $F$ . Also, for each  $F \in \mathcal{F}$ ,

$$\begin{aligned}
P_F(C[\mu_F]) &= P_F(M - \mu_F \leq Y) \\
&\geq P_F(L_F \leq \alpha S) \\
&\geq P_F(L_F \leq \alpha S, r_F < S) \\
&\geq P_F(L_F \leq \alpha r_F, r_F < S) \\
&\geq P_F(L_F \leq \alpha r_F) + P_F(r_F < S) - 1 \\
&\geq 1 - \frac{1}{\alpha} + \beta - 1 \\
&> 0.
\end{aligned}$$

This contradiction to Corollary 3 establishes Corollary 4. The proof shows that if  $M$  is an estimator such that  $r_P(M)$  is controllable, then  $\mu_P$  is controllable, contrary to Corollary 3, which can be used to show the uncontrollability of certain parameters, such as the variance of  $F$ , the difference between the mean and median values of  $F$ , and the supremum of the points of increase of  $F$ .  $\square$