# Notes on Quantile Risk Control: A Flexible Framework for Bounding the Probability of High-Loss Predictions

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**Abstract:** Presents bounds on quantiles of the loss distribution of a predictor.

## 1 Introduction

**Goal:** Produce predictive algorithms that are *rigorous*, i.e. produce bounds that can be trusted with high confidence, and *flexible*, i.e. applicable to an array of loss-related risk measures and adaptive to the difficulty of the instance.

**Contribution:** Produce lower confidence bounds on the cumulative distribution function (CDF) of a predictor's loss distribution, which can be converted into an upper bound on the quantile function. Provide bounds for risk measures that can be expressed as weighted integrals of the quantile function, a.k.a. quantile-based risk measures (QBRM): expected loss, VaR, CVaR, and VaR-interval.

**Definition 1.1** (Expected Loss)**.** Mean loss over the test distribution.

**Definition 1.2** (Value at Risk (VaR))**.** The $\beta$-VaR measures the maximum loss incurred on a specific quantile, after excluding a $1-\beta$ proportion of high-loss outliers, i.e. maximum loss incurred with probability $\beta$.

**Definition 1.3** (Conditional Value at Risk (CVaR))**.** The $\beta$-CVaR measures the mean loss for the worst $1-\beta$ proportion of the population.

**Definition 1.4** (VaR-interval)**.** Optimizing an uncertain loss quantile that belongs to a known range, i.e. for a range of $\beta$ values.

### 1.1 Setting and Notation

Assume a black-box predictor $h : \mathcal{Z} \to \hat{\mathcal{Y}}$ that maps from an input space $\mathcal{Z}$ to a space of predictions $\hat{\mathcal{Y}}$.

Assume a loss function $\ell : \hat{\mathcal{Y}} \times \mathcal{Y} \to \mathbb{R}$ that quantifies the quality of a prediction $\hat{Y}$ with respect to the target output $Y$.

Let $(Z, Y)$ be drawn from an unknown data distribution $\mathcal{D}$ over $\mathcal{Z} \times \mathcal{Y}$ and define the random variable $X \triangleq \ell(h(Z), Y)$ to be the loss induced by $h$ on $\mathcal{D}$.

Recall that the CDF of a random variable $X$ is defined as $F(x) \triangleq P(X \leq x)$; i.e. $F$ is the CDF of the loss RV $X$

**Goal:** Produce rigorous upper bounds on the risk $R(F)$ for a class of risk measures $R \in \mathcal{R}$, given a set of validation loss samples $X_{1:n}$.

### 1.2 Quantile-based Risk Measures

Recall that the quantile function is defined as $F^{-1}(p) \triangleq \inf\{x : F(x) \geq p\}$.

**Definition 1.5** (Quantile-based Risk Measure)**.** Let $\psi(p)$ be a weighting function such that $\psi(p) \geq 0$ and $\int_0^1 \psi(p)dp = 1$. The quantile-based risk measure defined by $\psi$ is

$$R_\psi(F) \triangleq \int_0^1 \psi(p)F^{-1}(p)dp.$$

## 2 Quantile Risk Control

Quantile Risk Control (QRC) achieves rigorous control of quantile-based risk measures (QBRM), by inverting a one-sided goodness-of-fit test statistic to produce a lower confidence bound on the loss CDF. This can subsequently be used to form a family of upper confidence bounds that hold for any QBRM.

More formally, specify a confidence level $\delta \in (0,1)$ and let $X_{(1)} \leq \ldots \leq X_{(n)}$ denote the order statistics of the validation loss samples. QRC consists of the following high-level steps:

1. Choose a one-sided test statistic of the form $S \triangleq \min_{1\leq i\leq n} s_i\left(F\left(X_{(i)}\right)\right)$, where $F$ is the (unknown) CDF of $X_1, \ldots, X_n$.
2. Compute the critical value $s_\delta$ such that $P\left(S \geq s_\delta\right) \geq 1-\delta$.
3. Construct a CDF lower confidence bound $\hat{F}_n$ defined by coordinates $\left(X_{(1)}, b_1\right), \ldots, \left(X_{(n)}, b_n\right)$ where $b_i \triangleq s_i^{-1}\left(s_\delta\right)$.
4. For any desired QBRM defined by weighting function $\psi$, report $R_\psi\left(\hat{F}_n\right)$ as the upper confi dence bound on $R_\psi(F)$.

It can be shown that $P\left(R_\psi(F) \leq R_\psi\left(\hat{F}_n\right)\right) \geq 1-\delta$ for any QBRM weighting function $\psi$.

QRC can also be used to bound the risk of multiple predictors simultaneously by setting the critical value to $\delta' \triangleq \delta/m$, where $m$ is the number of predictors.

### 2.1 CDF Lower Bounds are Risk Upper Bounds

### 2.2 Inverting Goodness-of-fit Statistics to Construct CDF Lower Bounds

### 2.3 Conservative CDF Completion

### 2.4 Interpreting Standard Tests of Uniformity as Minimum-type Statistics $S$

### 2.5 Bounding Multiple Predictors Simultaenously

### 2.6 Novel Truncated Berk-Jonest Statistics