$$\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{bmatrix} = \frac{1}{N} \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-j\omega_0} & e^{-j2\omega_0} & \cdots & e^{-j(N-1)\omega_0} \\ 1 & e^{-j2\omega_0} & e^{-j4\omega_0} & \cdots & e^{-j2(N-1)\omega_0} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & e^{-j(N-1)\omega_0} & e^{-j2(N-1)\omega_0} & \cdots & e^{-j(N-1)(N-1)\omega_0} \end{bmatrix}}_{G} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix}$$

$$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-j\omega_0} & e^{-j2\omega_0} & \cdots & e^{-j(N-1)\omega_0} \\ 1 & e^{-j2\omega_0} & e^{-j4\omega_0} & \cdots & e^{-j2(N-1)\omega_0} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & e^{-j(N-1)\omega_0} & e^{-j2(N-1)\omega_0} & \cdots & e^{-j(N-1)(N-1)\omega_0} \end{bmatrix}}_{H} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{bmatrix}$$

$GH = N\underset{\wedge}{I}$ identity matrix

example: $x[n] = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$

a). Find DTFS coefficients of $x[n]$ using its projection onto the basic signals:

$y_0[n] = e^{(j\frac{\pi}{2}0n)} \ \forall n \Rightarrow y_0[n] = 1$

$y_1[n] = e^{(j\frac{\pi}{2}1n)} \ \forall n \Rightarrow y_1[n] = (e^{j\pi})^n = (-1)^n$

b). show that $GH = MI$

a). $a_0 = \dfrac{\langle x[n], y_0[n] \rangle}{\langle y_0[n], y_0[n] \rangle} = \dfrac{\sum_{n=0}^{1} x[n] y_0^*[n]}{\sum_{n=0}^{1} y_0[n] y_0^*[n]} = \dfrac{1\times1 + 0\times1}{1\times1 + 1\times1} = \dfrac{1}{2}$

$a_1 = \dfrac{\langle x[n], y_1[n] \rangle}{\langle y_1[n], y_1[n] \rangle} = \dfrac{\sum_{n=0}^{1} x[n] y_1^*[n]}{\sum_{n=0}^{1} y_1[n] y_1^*[n]} = \dfrac{1\times1 + 0\times(-1)}{1\times1 + (-1)(-1)} = \dfrac{1}{2}$

$\omega_0 = \frac{2\pi}{N} = \frac{2\pi}{2} = \pi$

$G = \begin{bmatrix} 1 & 1 \\ 1 & e^{-j\pi} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

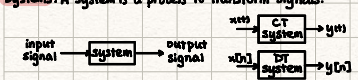
$H = \begin{bmatrix} 1 & 1 \\ 1 & e^{j\pi} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$GH = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \overset{N}{2}I$

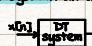
$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \frac{1}{N} G \begin{bmatrix} x[0] \\ x[1] \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

$\begin{bmatrix} x[0] \\ x[1] \end{bmatrix} = H \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$

Systems: A system is a process to transform signals.

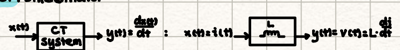


- Hybrid systems: combination of CT and DT systems.
- DT system: an averaging system



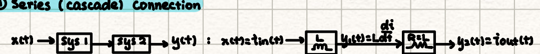
$y[n] = \frac{1}{N} \sum_{k=0}^{N-1} x[n-k]$

- CT: Differentiator

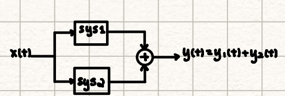


Practical systems are usually realized using an interconnection of multiple sub-systems.

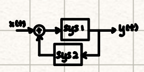
① Series (cascade) connection



② Parallel connection



③ Feedback connection



Basic system properties

① Systems with & without memory: The output of a memoryless system at any given time depends on only the input at the present time.

$u[n] = 2x^2[n] - 3x[n] + 5 \longrightarrow$ memoruless

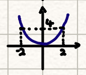
$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$ → with memory

$y[n] = x[n+1]$ → with memory

② Invertibility: A system is invertible if distinct inputs result in distinct outputs

$y(t) = x^3(t)$ → invertible

$y(t) = x^2(t)$ → non-invertible



$x(t) \to \boxed{\text{invertible sys}} \to y(t) = x^3(t)$

For an invertible system, there is an inverse system $y(t) \to \boxed{\text{inverse sys}} \to x(t) = \sqrt[3]{y(t)}$

③ Causality: A system is causal if the output at any given time depends only on the inputs at the present and past times

$y[n] = \sum_{m=n-9}^{n} x[m]$ → causal

$y[n] = \sum_{m=n-4}^{n+5} x[m]$ → Non-causal

$y(t) = x(-t)$ → non-causal

$y[n] = x[n-1]e^{j\omega_0(n+1)}$ → causal

④ Stability: A system is stable if bounded inputs result in bounded outputs. (BIBO stability)

⑤ Time invariance: A system is time-invariant if its characteristics are fixed over time.

$x[n] \xrightarrow{S} y[n] \Rightarrow x[n-n_0] \xrightarrow{S} y[n-n_0]$

$y(t) = e^{x(t+1)}$ → TI

$y(t) = e^{t x(t+1)}$ → time-variant

⑥ Linearity: If system S is linear and $x_1(t) \xrightarrow{S} y_1(t)$ & $x_2(t) \xrightarrow{S} y_2(t)$

then:

1. $x_1(t) + x_2(t) \xrightarrow{S} y_1(t) + y_2(t)$ Additivity property

2. $a_1 x_1(t) \xrightarrow{S} a_1 y_1(t)$ $a_1 \in \mathbb{C}$ Scaling property

$\Rightarrow a_1 x_1(t) + a_2 x_2(t) \xrightarrow{S} a_1 y_1(t) + a_2 y_2(t)$

$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$: Linear

S: $y(t) = x^2(t)$ Non-linear

$x_1(t) = 2 \to y_1(t) = 4$

$x_2(t) = 3 \to y_2(t) = 9$

$x_1(t) + x_2(t) = 5 \xrightarrow{S} 25 \neq 4+9$

S_2: $y[n] = 5x[n+2]$ Non-linear

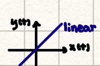
$x_1[n] = 4 \to y_1[n] = 22$

$x_2[n] = 2 \to y_2[n] = 12$

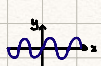
$x_1 + x_2 = 6 \xrightarrow{S_2} 32 \neq 34$

S_3: $x[n] \to \boxed{\text{Linear}} \xrightarrow{z[n]} \oplus \to y_0[n]$ (y_0 added at $\oplus$)

$z[n] = 5x[n]$ Incrementally $y_0 = 2$



S_4: $y(t) = \sin(x(t))$ Non-linear



$x_1(t) = \frac{\pi}{2} \Rightarrow y_1(t) = 1$

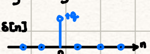
$x_2(t) = \frac{\pi}{2} \Rightarrow y_2(t) = 1$

$x_1(t) + x_2(t) = \pi \xrightarrow{S_4} y(t) = 0 \neq 1+1$

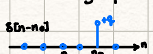
Linear Time-Invariant Systems (LTI systems)

Any DT signal can be represented using impulse signal $\delta[n]$.

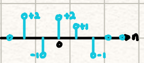
$\delta[n]$



$\delta[n-n_0]$

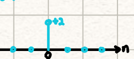


$x[n]$



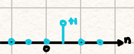
$x[n]\delta[n+1] = x[-1]\delta[n+1]$

$x[n]\delta[n] = x[0]\delta[n]$



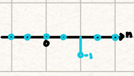
$x[n]\delta[n+2] = x[-2]\delta[n+2]$

$x[n]\delta[n-1] = x[1]\delta[n-1]$



$\Rightarrow x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + x[-1]\delta[n-(-1)] + x[-2]\delta[n-(-2)] = x[n]$

$x[n]\delta[n-2] = x[2]\delta[x-2]$



For an arbitrary $x[n]$: $x[n] = \cdots + x[-2]\delta[n-(-2)] + x[-1]\delta[n-(-1)] + x[0]\delta[n] + x[1]\delta[n-1] + \cdots$

An arbitrary DT signal can be written as the sum of some scaled time-shifted impulse signals: $x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$

LTI system:

$$a_1x_1[n] + a_2x_2[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow a_1y_1[n] + a_2y_2[n]$$

$$a_1x_1[n-n_1] \rightarrow \boxed{\text{LTI}} \rightarrow a_1y_1[n-n_1]$$

$$a_1x_1[n-n_1] + a_2x_2[n-n_2] \longrightarrow \boxed{\text{LTI}} \longrightarrow a_1y_1[n-n_1] + a_2y_2[n-n_2]$$

Impulse response, $h[n]$, of an LTI system is the output of the system when the input is the impulse function.

$$\delta[n] \rightarrow \boxed{\text{LTI}} \rightarrow h[n] \Rightarrow x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \rightarrow \boxed{\text{LTI}} \rightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

The output of an LTI system for an arbitrary input:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \underline{x[n] * h[n]}$$ convolution sum of $x[n]$ and $h[n]$