Objective: To demonstrate that DT FS coefficients of signal x are the projections of x onto the subspace spanned by complex exponentials

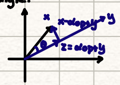
Position, length, and angle in an N-dimensional space ($\mathbb{R}^N$)

① Position: an N-dimentional vector: $x = [x[0], x[1], \cdots x[N-1]]$, $x \in \mathbb{R}^N$, $x[k] \in \mathbb{R}$, $0 \le k \le N-1$

② Length: $\|x\| = \sqrt{\sum_{n=0}^{N-1} (x[n])^2}$

$\|x\| = \sum_{n=0}^{N-1} (x[n])^2$

③ Angle:



$\cos\theta = \frac{\|z\|}{\|x\|}$

$\cos\theta = \|$Projection of x onto the subspace spanned by $y\| / \|x\|$

$\cos\theta = \|\alpha_{opt} y\| / \|x\|$

α_{opt} is the α that minimizes the length of $x - \alpha y$

$d\|x - \alpha_{opt} y\|^2 / d\alpha = \sum_{n=0}^{N-1} -2y[n](x[n] - \alpha_{opt} y[n]) = 0$

$\Rightarrow \alpha_{opt} = \sum_{n=0}^{N-1} x[n]y[n] \Big/ \sum_{n=0}^{N-1} y[n]y[n]$

Inner product in $\mathbb{R}^N$ space

$\langle x, y \rangle = \sum_{n=0}^{N-1} x[n]y[n]$

Properties of inner product in $\mathbb{R}^N$, $x, y \in \mathbb{R}^N$

① $\langle x, y \rangle = \langle y, x \rangle$

② $\langle \gamma x, y \rangle = \langle x, \gamma y \rangle = \gamma \langle x, y \rangle$ $\gamma \in \mathbb{R}$

③ $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

④ $\langle x, x \rangle = \|x\|^2$

$\sum_{n=0}^{N-1} x[n]x[n] = \sum_{n=0}^{N-1} x^2[n]$

$$\alpha_{opt} = \frac{\sum_{n=0}^{N-1} x[n]y[n]}{\sum_{n=0}^{N-1} y[n]y[n]} = \frac{\langle x, y \rangle}{\langle y, y \rangle} = \frac{\langle x, y \rangle}{\|y\|^2}$$

$$\cos\theta = \frac{\|\alpha_{opt} y\|}{\|x\|} = \frac{\left\| \frac{\langle x, y \rangle}{\|y\|^2} y \right\|}{\|x\|} = \frac{1}{\|x\|} \sqrt{\left\langle \frac{\langle x, y \rangle}{\|y\|^2} y, \frac{\langle x, y \rangle}{\|y\|^2} y \right\rangle} = \frac{1}{\|x\|} \sqrt{\frac{\langle x, y \rangle^2}{\|y\|^2 \|y\|^2} \langle y, y \rangle} = \frac{|\langle x, y \rangle|}{\|x\|\|y\|}$$

example: Find the angle between y and $x = \gamma y$, $x \& y \in \mathbb{R}^N$, $\gamma \in \mathbb{R}$

$\alpha_{opt} = \frac{|\langle x, y \rangle|}{\|y\| \cdot \|x\|} = \frac{|\langle \gamma y, y \rangle|}{\|\gamma y\| \|y\|}$

$\Rightarrow \cos\theta = \frac{|\gamma| \|y\|^2}{|\gamma| \|y\| \|y\|} = 1 \Rightarrow \theta = 0$

$\|\gamma y\| = \sqrt{\sum_{n=0}^{N-1} (\gamma y[n])^2} = \sqrt{\gamma^2 \sum_{n=0}^{N-1} (y[n])^2} = |\gamma| \|y\|$

example: Find the angle between y and $x - \alpha_{opt} y$, $x \& y \in \mathbb{R}^N$ $\alpha_{opt} = \frac{\langle x, y \rangle}{\|y\|^2}$

$\cos\theta = \frac{|\langle z, y \rangle|}{\|z\| \|y\|}$, $z = x - \alpha_{opt} y$

$= \frac{|\langle x - \alpha_{opt} y, y \rangle|}{\|z\| \|y\|}$

$= \frac{|\langle x, y \rangle - \alpha_{opt} \langle y, y \rangle|}{\|z\| \|y\|}$

$= \left| \langle x, y \rangle - \frac{\langle x, y \rangle}{\|y\|^2} \|y\|^2 \right| / (\|z\| \|y\|) = 0$

$\Rightarrow \theta = 90°$

Position, Length and Angle in N-dimensional space of complex numbers $\mathbb{C}^N$

① Position: is defined by an N-dimensional vector of complex numbers: $x = [x[0], x[1], \cdots x[N-1]]$ $x[k] \in \mathbb{C}$, $0 \le k \le N-1$

② Length: $\|x\|^2 = \sum_{n=0}^{N-1} \|x[n]\|^2 = \sum_{n=0}^{N-1} x[n] x^*[n]$

③ Angle: $\cos\theta = \frac{\|\alpha_{opt} y\|}{\|x\|}$, θ: Angle between x, y, $x \& y \in \mathbb{C}^N$

$\alpha_{opt} = \frac{\langle x, y \rangle}{\|y\|^2}$

$\Rightarrow \cos\theta = \frac{|\langle x, y \rangle|}{\|x\| \|y\|}$

Properties of inner product

① $\langle x, y \rangle = \langle y, x \rangle^*$

$\langle y, x \rangle^* = \left(\sum_{n=0}^{N-1} x^*[n] y[n] \right)^* = \sum_{n=0}^{N-1} x[n] y^*[n] = \langle x, y \rangle$

② $\langle \gamma x, y \rangle = \gamma \langle x, y \rangle$, $\gamma \in \mathbb{C}$

$\langle x, \gamma y \rangle = \gamma^* \langle x, y \rangle$

③ $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

$\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$

④ $\|x\|^2 = \langle x, x \rangle$

$\|x\|^2 = \sum_{n=0}^{N-1} |x[n]|^2 = \sum_{n=0}^{N-1} x[n] x^*[n] = \langle x, x \rangle$

example: Find the angle between y and $x-\alpha_{opt}y$ where x & $y \in \mathbb{C}^N$ $\alpha_{opt}=\frac{\langle x,y\rangle}{\|y\|^2}$

$$\cos\theta=\frac{|\langle y, x-\alpha_{opt}y\rangle|}{\|y\|\,\|x-\alpha_{opt}y\|}$$

$$\begin{aligned}\langle y, x-\alpha_{opt}y\rangle &= \langle y,x\rangle-\langle y,\alpha_{opt}y\rangle\\ &=\langle y,x\rangle-\alpha^*_{opt}\langle y,y\rangle\\ &=\langle y,x\rangle-\left(\frac{\langle x,y\rangle}{\|y\|^2}\right)^*\|y\|^2\\ &=\langle y,x\rangle-\frac{\langle x,y\rangle^*}{\|y\|^2}\|y\|^2\\ &=\langle y,x\rangle-\langle y,x\rangle=0\end{aligned}$$

If $\langle x,y\rangle=0$, then x and y are orthogonal and the projection of x onto the subspace spanned by y is zero.

Projecting onto subspaces

To find a better approximation of a vector. we can find the projection of that vector onto a subspace that is obtained by multiple vectors

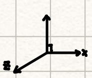
subspace $S=\{z\in\mathbb{C}^N,\ z=\sum_{k=0}^{K-1}\alpha_k y_k,\ \alpha_k\in\mathbb{C}\ \ y_k\in\mathbb{C}^N\}$ $K\leq N$

All the vectors z in sub space S can be expressed as a linear combination of vector y1, … yk-1.

If it is not possible to express any of the vectors yk in the y1, … yk-1 set as a combination of the remaining vectors in this set, then the set is a linearly independent set.

For a linearly independent set each pair of vectors, ym & yn, are orthogonal. $0\leq m,n\leq K-1$

i.e: $\langle y_m, y_n\rangle=\begin{cases}0 & m\neq n\\ \|y_m\|^2 & m=n\end{cases}$



For a linearly indep. set $y_1,\cdots y_{K-1}$ the projection of x onto the vector y_k is:

$$\alpha_{opt,k}=\frac{\langle x,y_k\rangle}{\|y_k\|^2}$$

DT Fourier Series coefficient of the signal x is the projection of x onto the subspace spanned by

$$\phi_k[n]=e^{jk(\frac{2\pi}{N})n}\qquad n=0,1,\cdots,N-1$$

$$\langle\phi_p,\phi_q\rangle=\sum_{k=0}^{N-1}e^{jp(\frac{2\pi}{N})k}\,e^{-jq(\frac{2\pi}{N})k}=\begin{cases}1 & p=q\\ 0 & p\neq q\end{cases}$$

$$a_k=\frac{\langle x, e^{jk2\pi n/N}\rangle}{\|e^{jk2\pi n/N}\|^2}=\frac{1}{N}\sum_{n=0}^{N-1}x[n]e^{-jk2\pi n/N}$$

Matrix Representation of DT FS coefficients.

$$a_k=\frac{1}{N}\sum_{n=0}^{N-1}x[n]e^{-jk\omega_0 n}\qquad \omega_0=\frac{2\pi}{N}$$

$$a_0=\frac{1}{N}(x[0]e^{-j\omega_0 0\cdot 0}+x[1]e^{-j\omega_0 0\cdot 1}+\cdots x[N-1]e^{-j\omega_0 0(N-1)})$$

$$a_1=\frac{1}{N}(x[0]e^{-j\omega_0\cdot 1\cdot 0}+x[1]e^{-j\omega_0\cdot 1\cdot 1}+\cdots x[N-1]e^{-j\omega_0\cdot 1\cdot(N-1)})$$

$$\vdots$$

$$a_{N-1}=\cdots$$