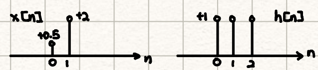
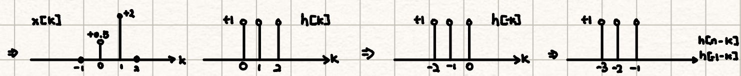
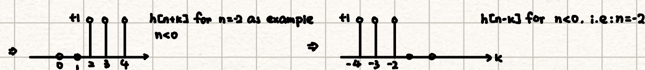
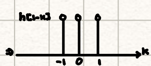
Cont. last week:

$y[n] = x[n] * h[n]$ (convolution sum) $= \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

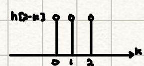








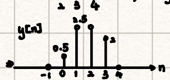
$y[1] = 0.5\times1+1\times2 = 2.5$



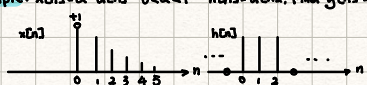
$y[2] = 0.5\times1+2\times1 = 2.5$

$y[3] = 2\times1 = 2$

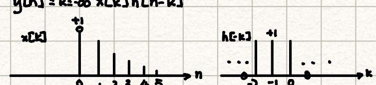
$y[n] = 0$



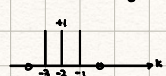
example: $x[n] = \alpha^n u[n]$ $0<\alpha<1$ $h[n] = u[n]$, Find $y[n] = x[n] * h[n]$



$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

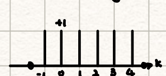


h[n-k] for n<0, e.g: n=-1

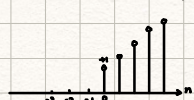


h[n-k], $y[n]=0$

h[n-k] for n>0, e.g: n=4



$y[n] = \sum_{k=0}^{n} \alpha^k (+1) = \frac{1-\alpha^{n+1}}{1-\alpha}$

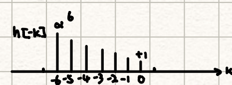


* For a DT LTI system, the input and impulse response are:

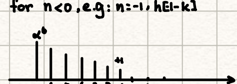
$$x[n] = \begin{cases} 1 & 0 \le n \le 4 \\ 0 & \text{otherwise} \end{cases} \qquad h[n] = \begin{cases} \alpha^n & 0 \le n \le 6 \\ 0 & \text{otherwise} \end{cases}$$

Find $y[n]$

$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

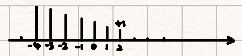


for n<0, e.g: n=-1, h[-1-k]



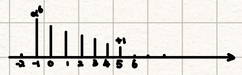
$y[n] = \sum_{k=-\infty}^{\infty} (0) = 0$

for $0 \le n \le 4$, e.g: n=+2 $h[n-k] = h[2-k]$



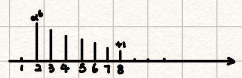
$y[n] = \sum_{k=0}^{n} x[k]h[n-k] = \sum_{k=0}^{n} (+1)\alpha^{n-k} = \alpha^n \sum_{k=0}^{n} (\alpha^{-1})^k = \alpha^n \frac{1-(\alpha^{-1})^{n+1}}{1-\alpha^{-1}} \times \frac{\alpha}{\alpha} = \frac{\alpha^{n+1}-1}{\alpha-1}$

for $5 \le n \le 6$, e.g: $n=5$, $h[n-k]=h[5-k]$



$y[n] = \sum_{k=0}^{4} (+1)(\alpha)^{n-k} = \alpha^n \sum_{k=0}^{4} (\alpha^{-1})^k = \alpha^n \frac{1-(\alpha^{-1})^5}{1-\alpha^{-1}} = \frac{\alpha^{n+1}-\alpha^{n-4}}{\alpha-1}$

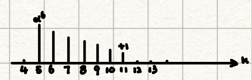
for $7 \le n \le 10$, e.g: $n=8$, $h[n-k]=h[8-k]$



$r=k-n+6 \Rightarrow n-k=6-r$

$y[n] = \sum_{k=n-6}^{4} (+1)(\alpha)^{n-k} = \sum_{r=0}^{10-n} \alpha^{6-r} = \alpha^6 \sum_{r=0}^{10-n} (\alpha^{-1})^r = \alpha^6 \frac{1-(\alpha^{-1})^{11-n}}{1-\alpha^{-1}} = \frac{\alpha^7-\alpha^{n-4}}{\alpha-1}$

for $11 \le n$. e.g: $n=11$, $h[11-k]$

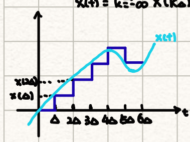


$y[n]=0$

Response of CT LTI systems

Assume arbitary CT signal, $x(t)$.

$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$



$x(t)=t^3$, $\Delta=0.1$

$\delta_{0.1}(t) = \begin{cases} 10 & 0<t<0.1 \\ 0 & \text{otherwise} \end{cases}$

$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(0.1k)\delta_{0.1}(t-0.1k)0.1$

$\delta_{0.1}(t-0.1k) = \delta_{0.1}(0.45-0.1k)$

$\delta_{0.1}$ is not zero for $t=0.45$ if $0<0.45-0.1k<0.1 \Rightarrow -0.45<-0.1k<-0.35$

$\Rightarrow 3.5<k<4.5 \Rightarrow k=0.4$

$t=0.45$ $\hat{x}(0.45) = x(0.1\times4)\times10\times0.1 = x(0.4)$

The smaller Δ is, the more closely $\hat{x}(t)$ resembles $x(t)$

$x(t) = \lim_{\Delta\to0}\hat{x}(t) = \lim_{\Delta\to0}\sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$

$\Rightarrow x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$

$\hat{x}(t) \to \boxed{LTI} \to \hat{y}(t)$

where $\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$, $x(t) = \lim_{\Delta\to0}\hat{x}(t)$

$\hat{y}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)h_\Delta(t-k\Delta)\Delta$

The impulse response $h(t)$ of a CT LTI system is the output of the system when the input is the impulse function $\delta(t)$

$\delta_\Delta(t) = \lim_{\Delta\to0}\delta_\Delta(t) \to h(t) = \lim_{\Delta\to0}h_\Delta(t)$

For $\Delta\to0$:

$x(t) \to \boxed{LTI} \to y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = \underline{x(t)*h(t)}$ convolution integral of $x(t)$ and $h(t)$

$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

example: For a CT LTI system, $h(t)=u(t)$ what is the output for $x(t)=e^{-at}u(t)$, $a\in R^+$

$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$ For $t<0$: $y(t) = \int_{-\infty}^{+\infty} (0)\,d\tau = 0$ $t>0$: $y(t) = \int_0^t e^{-a\tau}u(\tau)(+1)\,d\tau = \int_0^t e^{-a\tau}\,d\tau = \frac{1}{a}(1-e^{-at})$