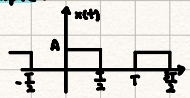
Fourier Series Representation of periodic signals

example:



$a_k = ?$

$$\begin{aligned} a_k &= \frac{1}{T}\int_0^T x(t) e^{-jk\omega_0 t} dt \\ &= \frac{1}{T}\int_0^{\frac{T}{2}} A e^{-jk\omega_0 t} dt \\ &= \frac{A}{T}\cdot\frac{1}{-jk\omega_0}\left(e^{-jk\omega_0 t}\right)\Big|_0^{\frac{T}{2}} \quad , \frac{2\pi}{T} = \omega_0 \\ &= \frac{A}{-jTk(2\pi/T)}\left[e^{-jk(2\pi/T)T/2} - 1\right] \\ &= \frac{jA}{2\pi k}\left[e^{-jk\pi} - 1\right] \end{aligned}$$

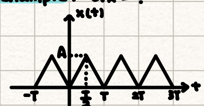
$$\begin{aligned} a_0 &= \frac{1}{T}\int_0^T x(t)\, dt \\ &= \frac{1}{T}\int_0^{T/2} A\, dt \\ &= \frac{A}{T} t \Big|_0^{T/2} \\ &= \frac{A}{2} \end{aligned}$$

$$e^{-jk\pi} = \cos(k\pi) - j\sin(k\pi) \Rightarrow e^{-jk\pi} = \begin{cases} 1 & k \text{ is even} \\ -1 & k \text{ is odd} \end{cases}$$

$$\Rightarrow a_k = \begin{cases} 0 & k \text{ is even} \\ -jA/(\pi k) & k \text{ is odd} \end{cases}$$

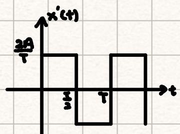
$$x(t) = \frac{A}{2} + \sum_{m=-\infty}^{\infty} \frac{-jA}{\pi(2m+1)} e^{j(2m+1)(2\pi/T)t}$$

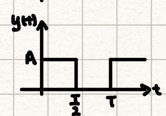
example: $a_k = ?$



$$\begin{aligned} a_k &= \frac{1}{T}\int_0^T x(t) e^{-jk(2\pi/T)t} dt \\ &= \frac{1}{T}\left(\int_0^{T/2} \frac{2At}{T} e^{-jk(2\pi/T)t} dt + \int_{T/2}^{T}\left(\frac{-2At}{T} + 2A\right) e^{-jk(2\pi/T)t} dt\right) \end{aligned}$$

example: $a_k = ?$





$$y(t) = \frac{A}{2} + \sum_{m=-\infty}^{\infty} \frac{-jA}{\pi(2m+1)} e^{j(2m+1)(2\pi/T)t}$$

$$x'(t) = y(t)\Big|_{A:\frac{4A}{T}} - \frac{2A}{T} = \frac{4A}{T2} + \sum \frac{-j(4A/T)}{\pi(2m+1)} e^{-j(2m+1)(2\pi/T)} - \frac{2A}{T}$$

$$\begin{aligned} x(t) &= \int x'(t)\, dt \\ &= \int \sum_{m=-\infty}^{\infty} \frac{j(4A/T)}{\pi(2m+1)} e^{j(2m+1)(2\pi/T)t} dt \\ &= \sum_{m=-\infty}^{\infty} \frac{j^2 4A/T}{\pi(2m+1)} \cdot \frac{1}{j(2m+1)(2\pi/T)} e^{j(2m+1)(2\pi/T)t} + C \end{aligned}$$

$$\Rightarrow a_k = \begin{cases} 0 & k \text{ is even} \\ \frac{2A}{\pi^2 k^2} & k \text{ is odd} \end{cases} \quad , C = a_0 = \frac{AT}{2}$$

Trigonometric FS representation of CT periodic signals

If $x(t)$ is a periodic signal with period T, and it satisfies Dirichlet conditions, then $x(t)$ can be written as:

$$x(t) = a_0 + \sum_{k=1}^{\infty}\left(a_k\cos(k\omega_0 t) + b_k\sin(k\omega_0 t)\right)$$

$\{a_k\}$ and $\{b_k\}$ are called trigonometric Fourier Series coefficient of $x(t)$ and can be obtained from:

$$a_k = \frac{2}{T}\int_T x(t)\cos(k\omega_0 t)\, dt \qquad a_0 = \frac{1}{T}\int_T x(t)\, dt$$

$$b_k = \frac{2}{T}\int_T x(t)\sin(k\omega_0 t)\, dt$$

Properties of FS representation of CT periodic signals

If $\{a_k\}$ is the FS coeff. of $x(t)$, the relation between $x(t)$ & $\{a_k\}$ is denoted by: $x(t) \overset{FS}{\longleftrightarrow} a_k$

① Linearity

$x(t) \overset{FS}{\longleftrightarrow} a_k$, $y(t) \overset{FS}{\longleftrightarrow} b_k$, if $Ax(t) + By(t)$ is periodic, then $Ax(t) + By(t) \overset{FS}{\longleftrightarrow} Aa_k + Bb_k$.

② Time shift

$x(t) \overset{FS}{\longleftrightarrow} a_k$, then $x(t-t_0) \overset{FS}{\longleftrightarrow} e^{-jk\omega_0 t_0} a_k$, $\omega_0 = \frac{2\pi}{T}$

proof: $a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt \qquad b_k = \frac{1}{T}\int_T x(t-t_0) e^{-jk\omega_0 t} dt \qquad \tau = t - t_0 \Rightarrow t = \tau + t_0 \Rightarrow dt = d\tau$

$$\begin{aligned} \Rightarrow b_k &= \frac{1}{T}\int_T x(\tau) e^{-jk\omega_0(\tau+t_0)} d\tau \\ &= \frac{1}{T}\int_T x(\tau) e^{-jk\omega_0\tau} e^{-jk\omega_0 t_0} d\tau \\ &= \frac{1}{T} e^{-jk\omega_0 t_0} \underbrace{\int_T x(\tau) e^{-jk\omega_0\tau} d\tau}_{a_k} \\ &= e^{-j\omega_0 k t_0} a_k \end{aligned}$$

③ Time reversal

$x(t) \overset{FS}{\longleftrightarrow} a_k$, then $x(-t) \overset{FS}{\longleftrightarrow} a_{-k}$

proof: $x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

$x(-t)=\sum_{k=-\infty}^{\infty} a_k e^{-jk\omega_0 t}$

$m=-k,\ k=-m$

$x(-t)=\sum_{m=\infty}^{-\infty} a_{-m} e^{jm\omega_0 t}=\sum_{k=-\infty}^{\infty} a_{-k} e^{jk\omega_0 t}$

④ Time scaling

$x(t)\overset{FS}{\longleftrightarrow} a_k$, then $x(\alpha t)\overset{FS}{\longleftrightarrow} a_k$

proof: $x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ $\Rightarrow$ same FS coeff.

$x(\alpha t)=\sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\omega_0)t}$

⑤ Conjugation

$x(t)\overset{FS}{\longleftrightarrow} a_k$, then $x^*(t)\overset{FS}{\longleftrightarrow} a^*_{-k}$

example:

If $x(t)\overset{FS}{\longleftrightarrow} a_k$, find the FS coeff. of:

a) $x(t-t_0)+x(t+t_0)$

b). $Ev\{x(t)\}$: Even part. $x_e(t)$

c). $Re\{x(t)\}$

d). $x(3t-1)$

a). $x(t-t_0)+x(t+t_0)\overset{FS}{\longleftrightarrow} b_k$

$b_k=e^{-jk\omega_0 t_0}a_k+e^{jk\omega_0 t_0}a_k$

$=a_k(e^{-jk\omega_0 t_0}+e^{jk\omega_0 t_0})$

$=2a_k\cos(k\omega_0 t_0)$

b). $Ev\{x(t)\}\overset{FS}{\longleftrightarrow} b_k$

$Ev\{x(t)\}=\frac{x(t)+x(-t)}{2}$

$b_k=\frac{a_k+a_{-k}}{3}$

c). $Re\{x(t)\}\overset{FS}{\longleftrightarrow} b_k$

$\frac{x(t)+x^*(t)}{2}=Re\{x(t)\}$

$b_k=\frac{a_k+a^*_{-k}}{2}$

d). $x(3t-1)\overset{FS}{\longleftrightarrow} b_k$

If period of $x(t)$ is T, then the period of $x(3t)$ is $\frac{T}{3}$

$x(3t-1)\overset{FS}{\longleftrightarrow} e^{-jk(2\pi/\frac{T}{3})}a_k=e^{-jk6\pi/T}a_k$

If $x(t)$ is **a real signal**: $\left.\begin{array}{l} x(t)\overset{FS}{\longleftrightarrow} a_k \\ x(-t)\overset{FS}{\longleftrightarrow} a^*_{-k}\end{array}\right\} a_k=a^*_{-k}$

$\Rightarrow Re\{a_k\}+jIm\{a_k\}=Re\{-a_k\}-jIm\{a_{-k}\}\Rightarrow Re\{a_k\}=Re\{a_{-k}\}\Rightarrow Re\{a_k\}$ is an even signal.

$Im\{a_k\}=-Im\{a_{-k}\}\Rightarrow Im\{a_k\}$ is an odd signal.

Exercise: $|a_k|$ is an even signal. $\measuredangle a_k$ is an odd signal.

If $x(t)$ is **a real and even signal**:

$a_k=a^*_{-k}$ ①

$\left.\begin{array}{l} x(t)\overset{FS}{\longleftrightarrow} a_k \\ x(-t)\overset{FS}{\longleftrightarrow} a_{-k}\end{array}\right\} a_k=a_{-k}$ ②

① & ② $\Rightarrow a_{-k}=a^*_{-k}\Rightarrow a_k=a^*_k$

$Re\{a_k\}+jIm\{a_k\}=Re\{a_k\}-jIm\{a_k\}\Rightarrow Im\{a_k\}=-Im\{a_k\}\Rightarrow 2Im\{a_k\}=0\Rightarrow Im\{a_k\}=0$

If $x(t)$ is **a real and odd signal:**

$a_k=a^*_{-k}$ ①

$\left.\begin{array}{l} x(t)\overset{FS}{\longleftrightarrow} a_k \\ x(-t)\overset{FS}{\longleftrightarrow} -a_{-k}\end{array}\right\} a_k=-a_{-k}$ ③

① & ③ $\Rightarrow a^*_{-k}=-a_{-k}\Rightarrow a^*_k=-a_k$

$Re\{a_k\}-jIm\{a_k\}=-Re\{a_k\}-jIm\{a_k\}$

$Re\{a_k\}=-Re\{a_k\}\Rightarrow 2Re\{a_k\}=0\Rightarrow Re\{a_k\}=0$

$\Rightarrow \{a_k\}$ is purely imaginary for every k.

If $x(t)$ is **a real signal**, the FS coeff. of $x_e(t)$ ($Ev\{x(t)\}$) is also a real signal

$a_k=a^*_{-k}$ ①

$x_e(t)=\frac{x(t)+x(-t)}{2}\overset{FS}{\longleftrightarrow}\frac{a_k+a_{-k}}{2}$ $\Rightarrow x_e(t)\overset{FS}{\longleftrightarrow}\frac{a_k+a^*_k}{2}=Re\{a_k\}$

If $x(t)$ is **a real signal**, the FS coeff. of the odd part of $x(t)$, $x_o(t)$ ($Od\{x(t)\}$) is a purely imaginary

$a_k = a_{-k}^*$ ①

$x_o(t) = \frac{x(t)-x(-t)}{2} \xleftrightarrow{FS} \frac{a_k - a_{-k}}{2}$ $\Rightarrow x_o(t) \xleftrightarrow{FS} \frac{a_k - a_k^*}{2} = \frac{j2Im\{x(t)\}}{2} = jIm\{a_k\}$

⑥ Parseval's Theorem:

For periodic $x(t)$ with period T and FS coefficient $\{a_k\}$: $\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$

Total Power: $P_{tot} = \lim_{\tau\to\infty} \frac{1}{2\tau}\int_{-\tau}^{+\tau} |x(t)|^2 dt$ $\Rightarrow$ For periodic $x(t)$ with period T: $P_{tot} = \frac{1}{T}\int_T |x(t)|^2 dt$

$y_k(t) = a_k e^{jk(2\pi/T_k)t}$ $\Rightarrow P_{tot}.y_k = \frac{1}{T_k}\int_{T_k} |a_k e^{jk(2\pi/T_k)t}|^2 dt$ $\Rightarrow x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t}$

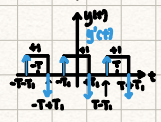
$= \frac{1}{T_k}\int_0^{T_k} |a_k|^2 dt$

$= |a_k|^2$

The total power of $x(t)$ is equal to the sum of the "total powers of all of the complex exponential signals that are presented in the synthesis relation of $x(t)$

example: Use the FS coeff. of $x(t)$ to find the FS coeff. of $y(t)$

impulse train



$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t}$

$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk(2\pi/T)t} dt$

$= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jk(2\pi/T)t} dt$

$= \frac{1}{T}\int_{-\infty}^{\infty} \delta(t) e^{-jk(2\pi/T)t} dt$

$= \frac{1}{T} e^{-jk(2\pi/T)0} \int_{-\infty}^{\infty} \delta(t) dt$

$= \frac{1}{T} \times 1 \times 1 = \frac{1}{T}$

$y'(t) = x(t+T_1) - x(t-T_1) \xleftrightarrow{FS} e^{jk(2\pi/T)T_1} \frac{1}{T} - e^{-jk(2\pi/T)T_1} \frac{1}{T}$

$b_k = \frac{1}{T}(e^{jk(2\pi/T)T_1} - e^{-jk(2\pi/T)T_1})$

$= \frac{1}{T} 2j\sin(k(2\pi/T)T_1)$

$\Rightarrow y'(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk(2\pi/T)t}$

$y(t) = \int y'(t) dt = \sum_{k=-\infty}^{\infty} b_k \cdot \frac{1}{jk(2\pi/T)} e^{jk(2\pi/T)t} + C$

$\Rightarrow C_k = \frac{2j\sin(k(2\pi/T)T_1)}{T} \cdot \frac{1}{jk(2\pi/T)} = \frac{\sin(k(2\pi/T)T_1)}{k\pi}$

$C_0 = \frac{1}{T}\int_T y(t) dt = \frac{2T_1}{T}$