Odd and Even Signals

Odd Signals: $x(t)$ is an odd signal if $x(-t) = -x(t)$

e.g.: $x(t) = \sin(w_0 t) \quad x(-t) = \sin(-w_0 t) = -\sin(w_0 t) = -x(t)$

Odd signals are symmetrical with respect to the origin.

Even Signals: $x(t)$ is an even signal if $x(-t) = x(t)$

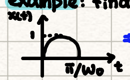
e.g.: $x(t) = \cos(w_0 t) \quad x(-t) = \cos(-w_0 t) = \cos(w_0 t) = x(t)$

Even signals are symmetrical with respect to the vertical axis.

Pratical signals are usually neither odd nor even. But every signal can be written as the sum of an odd signal and an even signal.

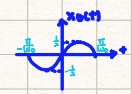
$x(t) = x_o(t) + x_e(t) = Od\{x(t)\} + Ev\{x(t)\}$, where $x_o(t) = \frac{1}{2}(x(t) - x(-t))$, $x_e(t) = \frac{1}{2}(x(t) + x(-t))$

example: find the odd and even parts of $x(t) = \begin{cases} \sin w_0 t & 0 < t < \frac{\pi}{w_0} \\ 0 & \text{otherwise} \end{cases}$

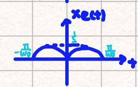


$\Rightarrow x(-t) = \begin{cases} \sin(-w_0 t) & -\frac{\pi}{w_0} < t < 0 \\ 0 & \text{otherwise} \end{cases}$

$\Rightarrow x_o(t) = \frac{1}{2}(x(t) - x(-t)) = \begin{cases} \frac{1}{2}\sin(w_0 t) & -\frac{\pi}{w_0} < t < \frac{\pi}{w_0} \\ 0 & \text{otherwise} \end{cases}$

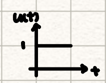


$x_e(t) = \frac{1}{2}(x(t) + x(-t)) = \begin{cases} -\frac{1}{2}\sin(w_0 t) & -\frac{\pi}{w_0} < t < 0 \\ \frac{1}{2}\sin(w_0 t) & 0 < t < \frac{\pi}{w_0} \\ 0 & \text{otherwise} \end{cases}$

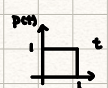


Basic Signals

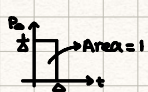
① Unit step function: $u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$



② Pulse: $P(t) = u(t) - u(t-1) = \begin{cases} 1 & 0 < t < 1 \\ 0 & \text{else} \end{cases}$



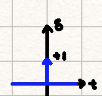
③. Unit Pulse: $P_\Delta(t) = \begin{cases} \frac{1}{\Delta} & 0 < t < \Delta \\ 0 & \text{else} \end{cases}$



④. Unit impulse signal $\delta(t)$

$\delta(t)$ satisfies the following conditions:

1. $\delta(t) = 0$ for $t \neq 0$
2. $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$



$\delta(t)$ closely resembles a lot of practical signals.

$\delta(t)$ is very useful for analyzing signals and systems.

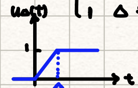
$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)\,dt = x(t_0)$$

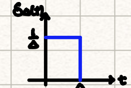
$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

example: $u_\Delta(t) = \begin{cases} 0 & t < 0 \\ \frac{t}{\Delta} & 0 \leq t < \Delta \\ 1 & \Delta \leq t \end{cases} \Rightarrow \delta_\Delta(t) = \frac{du_\Delta(t)}{dt} = \begin{cases} \frac{1}{\Delta} & 0 < t < \Delta \\ 0 & \text{eles} \end{cases}$

$\delta_\Delta(t) = \frac{du_\Delta(t)}{dt} \Rightarrow \delta(t) = \frac{du(t)}{dt}$

$\lim_{\Delta \to 0} u_\Delta(t) = u(t)$

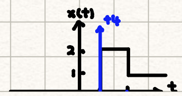




Area: $\Delta \cdot \frac{1}{\Delta} = 1$

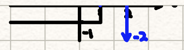
$\Rightarrow \int_{-\infty}^{\infty} \delta_\Delta(t)\,dt = 1 \Rightarrow \delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$

example: What is the derivative of $x(t)$?



$x(t) = -1 + 4u(t-1) - 2u(t-2)$

$\frac{dx(t)}{dt} = 0 + 4\delta(t-1) - 2\delta(t-2)$



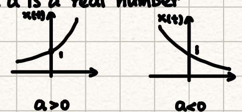
example: Consider $x(t)=\delta(t+2)-\delta(t-2)$ Find the total energy at $y(t)=\int_{-\infty}^{t} x(\tau)\,d\tau$

$$y(t)=\int_{-\infty}^{t} x(\tau)\,d\tau=\begin{cases}0 & t<-2\\ 1 & -2<t<2\\ 0 & t>2\end{cases} \Rightarrow E=\int_{-\infty}^{\infty}|y(t)|^2dt=\int_{-2}^{2}1^2dt=4$$

⑤ complex exponential signal

General format: $x(t)=e^{at}$

1). a is a real number

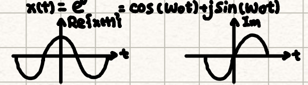


The larger is the magnitude of a, the faster is the decay / growth rate of $x(t)$.

2). a is a purely imaginary number: $a=j\omega_0$, $\omega_0\in\mathbb{R}$

Euler's Theorem: $e^{j\alpha}=\cos\alpha+j\sin\alpha$

$x(t)=e^{j\omega_0 t}=\cos(\omega_0 t)+j\sin(\omega_0 t)$

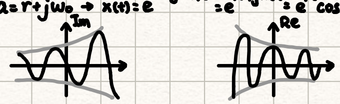


$\cos(\omega_0 t)$ and $j\sin(\omega_0 t)$ are both periodic with period $\frac{2\pi}{|\omega_0|}$. Since $\frac{2\pi/|\omega_0|}{2\pi/|\omega_0|}=1$ is a rational number, $\cos(\omega_0 t)+j\sin(\omega_0 t)$ is also periodic.

The period is $\frac{2\pi}{|\omega_0|}$

3). a is a general complex number.

$a=r+j\omega_0 \rightarrow x(t)=e^{(r+j\omega_0)t}=e^{rt+j\omega_0 t}=e^{rt}\cos(\omega_0 t)+je^{rt}\sin(\omega_0 t)$



Fourier Series Representation of CT periodic signals

The periodic signal $x(t)$ with period T satisfies Dirichlet conditions if

1) $x(t)$ is absolutely integrable over a period: $\int_T |x(t)|\,dt<\infty$
2). $x(t)$ has a finite number of maximum and minimum at every interval Δt
3). $x(t)$ has a finite number of discontinuities at every interval Δt and every discontinuities is finite.

If the periodic signal $x(t)$ with period T satisfies Dirichlet conditions, then it can be expressed as:

$$x(t)=\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}=\sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

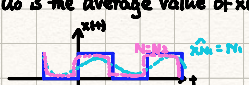
The set $\{a_k\}$, $k\in\mathbb{Z}$ is called **Fourier series coefficients** at $x(t)$ and are obtained from:

$$a_k=\frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt=\frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt$$ **Analysis Relation**

a_0 is obtained from:

$$a_0=\frac{1}{T}\int_T x(t)\,dt$$

a_0 is the average value of $x(t)$ over one period. It is called the **DC component of $x(t)$**.



$$x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t}$$ **Synthesis relation**

$$\hat{x}(t)=\sum_{k=-N}^{N} a_k e^{jk(2\pi/T)t}$$

Example: Find Fourier series coefficients of $x(t)=\cos(2t+\frac{\pi}{4})$

$e^{j\alpha}=\cos\alpha+j\sin\alpha \Rightarrow \cos\alpha=\frac{1}{2}e^{j\alpha}+\frac{1}{2}e^{-j\alpha}$

$e^{-j\alpha}=\cos\alpha-j\sin\alpha$

$\Rightarrow x(t)=\frac{1}{2}e^{j(2t+\frac{\pi}{4})}+\frac{1}{2}e^{-j(2t+\frac{\pi}{4})}$

$=\frac{1}{2}e^{j\frac{\pi}{4}}e^{j2t}+\frac{1}{2}e^{-j\frac{\pi}{4}}e^{-j2t}$

$=\sum_{k=-\infty}^{\infty} a_k e^{jk2t}$

$\Rightarrow a_1=\frac{1}{2}e^{j\frac{\pi}{4}}=\frac{1}{2}(\frac{\sqrt{2}}{2}+j\frac{\sqrt{2}}{2})$ $\quad a_{-1}=\frac{1}{2}e^{-j\frac{\pi}{4}}=\frac{1}{2}(\frac{\sqrt{2}}{2}-j\frac{\sqrt{2}}{2})$

For every other k, $a_k=0$

Using the analysis relation: $a_k=\frac{1}{\pi}\int_0^{\pi}\cos(2t+\frac{\pi}{4})e^{-jk(2\pi/\pi)t}dt$

$w_0 = 2, \ \frac{2\pi}{T} = 2 \Rightarrow T = \pi$

$a_k = \frac{1}{\pi}\int_0^{\pi} \frac{1}{2} e^{j(2t+\frac{\pi}{4})} e^{-jk2t} + \frac{1}{\pi}\int_0^{\pi} \frac{1}{2} e^{-j(2t+\frac{\pi}{4})} e^{-jk2t} \, dt$