Fourier Series Representation of Discrete time periodic signals

If $x[n]$ is a periodic signal with period N, then:

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n} \quad \text{Synthesis ①}$$

$\{a_k\}$ is called Fourier Series coefficient of $x[n]$ and is obtained from:

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n] e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=\langle N\rangle} x[n] e^{-jk(2\pi/N)n} \quad \text{Analysis ②}$$

Two ways to express ①

$$x[n] = a_0 e^{j0\omega_0 n} + \cancel{a_1 e^{j1\omega_0 n}} + \cdots + \cancel{a_{N-1} e^{j(N-1)\omega_0 n}}$$

$$x[n] = \cancel{a_1 e^{j1\omega_0 n}} + \cancel{a_2 e^{j2\omega_0 n}} + \cdots + a_N e^{jN\omega_0 n}$$

$$\Rightarrow a_0 \cancel{e^{j0\omega_0 n}}^{1} = a_N e^{jN\omega_0 n}, \text{ where } e^{jN\omega_0 n} = e^{jN(2\pi/N)n} = e^{j2\pi n} = \cancel{\cos(2\pi n)}^{1} + j\cancel{\sin(2\pi n)}^{0} = 1$$

$$\Rightarrow \quad a_0 = a_N$$

For a periodic DT signal with periodic N, the FS coefficients are also periodic with period N

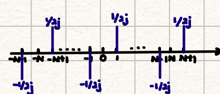
$$a_k = a_{k+N}$$

example:

Find the FS coefficients of $x[n] = \sin(\omega_0 n)$, where $\frac{2\pi}{\omega_0}$ is a rational number.

Assume $\frac{2\pi}{\omega_0} = N$, $N \in \mathbb{Z}$.

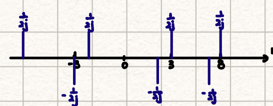
$$x[n] = \sin(\omega_0 n) = \frac{1}{2j} e^{j(2\pi/N)n} - \frac{1}{2j} e^{-j(2\pi/N)n}$$



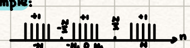
as. $x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n} \Rightarrow a_1 = \frac{1}{2j}, \ a_{-1} = -\frac{1}{2j}$

Assume $\frac{2\pi}{\omega_0} = \frac{N}{M}$, $M, N \in \mathbb{Z}$

$$x[n] = \sin(\omega_0 n) = \sin\left(\frac{2\pi M}{N} n\right) = \frac{1}{2j} e^{jM(2\pi/N)n} - \frac{1}{2j} e^{-jM(2\pi/N)n} \Rightarrow \begin{cases} a_{+M} = \frac{1}{2j} & k = +M \\ a_{-M} = -\frac{1}{2j} & k = -M \end{cases}$$



example:



Find FS coefficients

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$$

$$a_k = \frac{1}{N}\sum_{n=-N/2}^{N/2} x[n] e^{-jk(2\pi/N)n} = \frac{1}{N}\sum_{n=-N_1}^{N_1} 1\cdot e^{-jk(2\pi/N)n} \qquad \sum_{P=0}^{H} \alpha^P = \begin{cases} \frac{1-\alpha^{H+1}}{1-\alpha} & \alpha \neq 1 \\ H+1 & \alpha = 1 \end{cases}$$

Let $m = n + N_1$, $n = m - N_1$, $a_k = \frac{1}{N}\sum_{m=0}^{2N_1} e^{-jk(2\pi/N)(m-N_1)}$

$$= \frac{1}{N} e^{jk(2\pi/N)N_1} \sum_{m=0}^{2N_1} e^{-jk(2\pi/N)m}$$

For $k = 0, \pm N, \pm 2N \cdots$, $e^{-jk(2\pi/N)} = 1 \Rightarrow a_k = \frac{1}{N}(1)^{N_1} \sum_{m=0}^{2N_1} (1)^m$

$$= \frac{1}{N}(2N_1+1) = \frac{2N_1+1}{N}$$

For k otherwise: $a_k = \frac{1}{N} e^{jk(2\pi/N)N_1} \cdot \frac{1 - e^{-jk(2\pi/N)(2N_1+1)}}{1 - e^{-jk(2\pi/N)}}$

$$= \frac{1}{N} e^{jk(2\pi/N)N_1} \frac{e^{-jk(2\pi/N)(N_1+\frac{1}{2})}}{e^{-jk(\pi/N)}} \times \frac{e^{jk(2\pi/N)(N_1+\frac{1}{2})} - e^{-jk(2\pi/N)(N_1+\frac{1}{2})}}{e^{jk\pi/N} - e^{-jk\pi/N}}$$

$$= \frac{1}{N} \frac{\sin(2\pi k/N)(N_1+\frac{1}{2})}{\sin(\pi k/N)}$$

Properties of DT FS coefficients

$x[n] \overset{FS}{\longleftrightarrow} a_k$, Period: N

$y[n] \overset{FS}{\longleftrightarrow} b_k$, Period: N

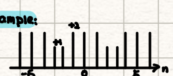
① Linearity

$Ax[n] + By[n] \overset{FS}{\longleftrightarrow} Aa_k + Bb_k$, Period: N

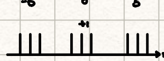
② Time shifting

$x[n-n_0] \overset{FS}{\longleftrightarrow} e^{-jk(2\pi/N)n_0} a_k$

example:



$x[n] = x_1[n] + x_2[n]$



$x_1[n]$

$x_2[n]$

$x[n] \overset{FS}{\longleftrightarrow} a_k \quad x_1[n] \overset{FS}{\longleftrightarrow} b_k \quad x_2[n] \overset{FS}{\longleftrightarrow} c_k$

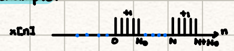
$$a_k = b_k + c_k$$

$$b_k = \begin{cases} \frac{\sin(3\pi k/5)}{5\sin(\pi k/5)} & k \neq 0, \pm 5, \pm 10 \cdots \\ \frac{3}{5} & k = 0, \pm 5, \pm 10 \cdots \end{cases}$$

$$c_k = \begin{cases} 1 & k = 0, \pm 5, \pm 10 \cdots \\ 0 & k \neq 0, \pm 5, \pm 10 \cdots \end{cases}$$

$$\Rightarrow a_k = \begin{cases} \frac{\sin(3\pi k/5)}{5\sin(\pi k/5)} & k \neq 0, \pm 5 \cdots \\ \frac{8}{5} & k = 0, \pm 5 \cdots \end{cases}$$

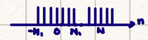
example:



Find FS coefficients of x[n] for:

a). $N_0=4$, $N=7$

b). $N_0=3$, $N=6$

a). 

$x[n] \xleftrightarrow{FS} a_k$

$$a_k = \begin{cases} \frac{1}{N}\cdot\frac{\sin((2\pi k/N)(N_1+\frac{1}{2}))}{\sin(\pi k/N)} & k\neq 0, \pm N, \\ \frac{2N_1+1}{N} & k=0, \pm N, \pm 2N, \cdots \end{cases}$$

Total number of non-zero samples in one period: $N_0+1 = 2N_1+1 \Rightarrow N_1 = \frac{N_0}{2}$

For $N_0=4$, $N=7$

$$a_k = \begin{cases} \frac{1}{7}\cdot\frac{\sin(5\pi k/7)}{\sin(\pi k/7)} & k\neq 0, \pm 7, \pm 14, \cdots \\ \frac{5}{7} & k=0, \pm 7, \pm 14 \cdots \end{cases}$$

$b_k = e^{-jk(2\pi/7)2} a_k \qquad y[n] \xleftrightarrow{FS} b_k$

b). $b_k = \frac{1}{6}\sum_{n=0}^{3}(e^{-jk(2\pi/6)})^n$

For $k\neq 0, \pm 6, \cdots$ $\quad b_k = \frac{1}{6}\frac{1-e^{-jk(2\pi/6)4}}{1-e^{-jk(2\pi/6)}} = \frac{1}{6}\cdot\frac{e^{-jk(2\pi/3)}}{e^{-jk(\pi/6)}}\cdot\frac{e^{jk(2\pi/6)2}-e^{-jk(2\pi/6)2}}{e^{jk(\pi/6)}-e^{-jk(\pi-6)}} = \frac{1}{6}e^{-jk(\pi/2)}\frac{\sin(2\pi k/3)}{\sin(k\pi/6)}$

For $k=0, \pm 6, \pm 12\cdots$ $\quad b_k = \frac{4}{6}$

③ Frequency shifting

$e^{+j(2\pi/N)Mn} x[n] \longleftrightarrow a_{k-M}$

④ Conjugation

$x^*[n] \longleftrightarrow a^*_{-k}$

⑤ Time reversal: $x[-n] \xleftrightarrow{FS} a_k$

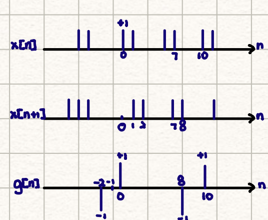
All of the relations for the FS coefficients of real, even and odd CT signals, apply to DT signals.

⑥ First difference:

$x[n]-x[n-1] \xleftrightarrow{FS} (1-e^{-jk(2\pi/N)})a_k$

example: $x[n] = \begin{cases} 1 & 0\leq n\leq 7 \\ 0 & 8\leq n\leq 9 \end{cases}$, $N=10$ $\qquad g[n] = x[n]-x[n-1]$

Using FS coeff. of g[n] find the FS of x[n]



$g[n] \xleftrightarrow{FS} b_k$

$b_k = \frac{1}{N}\sum_{n=\langle N\rangle} g[n]e^{-jk(2\pi/N)n}$

$= \frac{1}{10}(1\times e^{0} - 1\times e^{-jk(2\pi/10)8})$

$= \frac{1}{10}(1-e^{-jk(2\pi/10)8})$

$g[n] = x[n]-x[n-1]$

$\Rightarrow b_k = a_k - e^{-jk(2\pi/N)}a_k$

$a_k = \frac{b_k}{1-e^{-jk(2\pi/10)}}$

$= \frac{1}{10}\cdot\frac{1-e^{-jk(2\pi/10)8}}{1-e^{-jk(2\pi/10)}}$

example: $x[n] \xleftrightarrow{fs} a_k$

1) x[n] is real and even

2). The period of x[n] is 10

3). $a_{11}=5$

4). $\frac{1}{10}\sum_{n=0}^{9}|x[n]|^2 = 50$

Find the x[n] max

FS coefficients are real and even

$a_{11}=5 \Rightarrow a_1 = 5 = a_{-1}$

$\frac{1}{10}\sum_{n=0}^{9}|x[n]|^2 = \frac{1}{10}\sum_{n=-1}^{8}|x[n]|^2$

Parseval:

$\frac{1}{10}\sum_{n=-1}^{8}|x[n]|^2 = \sum_{k=-1}^{8}|a_k|^2 = 50$

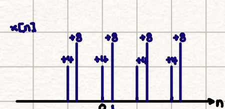
$|a_{-1}|^2+|a_0|^2+|a_1|^2+\sum_{k=2}^{8}|a_k|^2 = 50$

$\Rightarrow a_0 = a_2 = a_3 = a_4 = \cdots a_8 = 0$

$x[n] = \sum_{k=-1}^{8} a_k e^{jk(2\pi/10)n} = 5e^{-j(\pi/5)n} + 5e^{j(\pi/5)n} = 10\cos(\frac{\pi n}{5})$

$\Rightarrow \max\{x[n]\} = 10$

example: Find the FS coefficient of $x[n] = \sum_{m=-\infty}^{\infty}(4\delta[n-4m]+8\delta[n-4m-1])$



$a_k = \frac{1}{N}\sum_{n=0}^{3} x[n]e^{-jk(2\pi/N)n}$

$= \frac{1}{4}(4+8e^{-jk(2\pi/4)})$

$a_0 = \frac{1}{4}(4+8) = 3$

$a_1 = 1+2e^{-j(\pi/2)}$

$a_2 = 1+2e^{-j(\pi/2)2}$

$a_3 = 1+2e^{-j(\pi/2)3}$