Problem 1 (10 Marks)

Civil engineers believe that W, the amount of weight (in units of 1000 pounds) that a certain span of a bridge can withstand without structural damage resulting, is normally distributed with mean 400 and standard deviation 40. Suppose that the weight (again, in units of 1000 pounds) of a car is a random variable with mean 3 and standard deviation 0.3.

(a) (5 marks) Using central limit theorem, find the probability that the weight of 100 cars is in the range [360, 450]. (Write your final answer in terms of the Q function.)

Let X_i represent the weight of a car. Let $X = \sum_{i=1}^{100} X_i$

$E[X_i] = 3$. $VAR(X_i) = 0.3^2 = 0.09$

CLT $\rightarrow$ $X \sim N(100 \times 3, 100 \times 0.09)$

$$Z_n = \frac{X - n\mu}{\sigma\sqrt{n}} = \frac{X - 100 \times 3}{0.3\sqrt{100}} = \frac{X-300}{3}$$

$Q(z) = P(Z \leq z)$

$$P(360 \leq X \leq 450) = P\left(\frac{360-300}{3} \leq \frac{X-300}{3} \leq \frac{450-300}{3}\right) = P(20 \leq Z \leq 50) = Q(20) - Q(50)$$

Problem 1 (10 Marks)

Civil engineers believe that W, the amount of weight (in units of 1000 pounds) that a certain span of a bridge can withstand without structural damage resulting, is normally distributed with mean 400 and standard deviation 40. Suppose that the weight (again, in units of 1000 pounds) of a car is a random variable with mean 3 and standard deviation 0.3.

(b) (5 marks) Let n be the number of cars on the bridge. Give an equation for n that will result in probability of structural damage of 0.1. Structural damage occurs when total weight of cars on the bridge is larger than W (defined in statement of question). Hint: $Q(0.1)^{-1} \approx 1.28$. (You don't need to solve the final equation for n).

P1(b) As before, let X_i represent the weight of the car, Let $X = \sum_{i=1}^{100} X_i$

Using CLT, $X \sim N(3n, 0.09n)$

Given $W \sim N(400, 1600)$

For strutual damage, $X - W \geq 0$, $(X-W) \sim N(3n-400, 0.09n+1600)$

$\Rightarrow$ wish to find $P(X-W \geq 0) = 0.01$

$\Rightarrow P(X-W \geq 0) = Q\left(\frac{0-(3n-400)}{\sqrt{0.09n+1600}}\right) = 0.01 \Rightarrow \frac{400-3n}{\sqrt{0.09n+1600}} = Q^{-1}(0.1) = 1.28 \Rightarrow 400-3n = 1.28\sqrt{0.09n+1600}$

Problem 2 (10 Marks)

Student scores on exams given by a certain instructor have mean 74 and standard deviation 14. This instructor is about to give two exams, one to a class of size 25 and the other to a class of size 64.

(a) (3 marks) Using central limit theorem, approximate the probability that the average test score in the class of size 25 exceeds 80. (write your solution in terms of the Q function)

Problem 2 (10 Marks)

Student scores on exams given by a certain instructor have mean 74 and standard deviation 14. This instructor is about to give two exams, one to a class of size 25 and the other to a class of size 64.

(c) (4 marks) Approximate the probability that the absolute difference of the averages of the two classes is greater than 3 points. (write your solution in terms of the Q function)

Define a new variable, $D = M_{64} - M_{25}$

P2(a): looking for the average. i.e. sample mean

if for each student, $X_i \sim N(74, 14^2)$ then $M_n = \frac{1}{n}\sum_{i=1}^{n} X_i$

recall that $E[M_n] = \mu$, $VAR[M_n] = \sigma^2/n$

Using CLT, $M_{25} \sim N(74, 14^2/25)$

$\Rightarrow P(M_{25} > 80) = P\left(\frac{M_{25}-74}{14/5} > \frac{80-74}{14/5}\right) = Q\left(\frac{30}{14}\right)$

What if we need the probability?

Use standard normal table: $Q(\frac{30}{14}) = Q(2.14) = 1 - \Phi(2.14) = 1 - 0.9838 = 0.0162$

P2(c): $D = M_{64} - M_{25}$

$D \sim N(0, 14^2/25 + 14^2/64)$

The desired probability: $P(|D| > 3) = P(D > 3) + P(D < -3)$

Since $E[D] = 0$, $P(D>3) + P(D<-3) = P(D/\sigma_D > 3/\sigma_D) + P(D/\sigma_D < -3/\sigma_D)$ where $\sigma_D = \sqrt{14^2/25 + 14^2/64} \approx 3.3$

$P(|D| > 3) = Q(0.91) + (1 - Q(-0.91)) = 2Q(0.91)$

Problem 3 (10 Marks)

The lifetime X of a device is an exponential random variable with mean of $1/R$. Suppose that due to irregularities in the production process, the parameter R is random and has a uniform distribution in $[1, 11]$.

(b) **(3 marks)** Find the PDF of X.

Find the marginal PDF of X from our joint PDF.

$$f_X(x) = \int_1^{11} \frac{re^{-rx}}{10}\, dr$$

Using integration by parts, we have

$$f_X(x) = \frac{1}{10x^2}\Big[-e^{-rx}(rx+1)\Big]_{r=1}^{r=11}$$

$$= \frac{1}{10x^2}\left[e^{-x}(x+1) - e^{-11x}(11x+1)\right]$$

a): Use cond. prob. $f_{XR}(x,r) = f_X(X|r)\, f_R(r)$

Give some $R=r$. X is expo. RV with para. $r \Rightarrow f_{XR}(x,r) = f_X(X|r)\, f_R(r) = re^{-rx} \cdot \frac{1}{11-1} = \frac{re^{-rx}}{10}$, $x>0$, $1<r<11$

c): Find the mean of X.

take expec. of X using PDF, but easier to use the law of expectation.

$$E[X] = E[E[X|R]] = E[1/R] = \int_1^{11} \frac{1}{r} \frac{1}{10}\, dr = \left[\frac{\ln(r)}{10}\right]_1^{11} = \frac{\ln(11)}{10}$$