Recall:

Let us partition the sample space into two mutually exclusive events, A and A^c

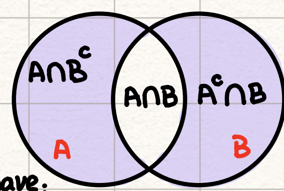
Corollary 1: Since $S = A \cup A^c$ and $A \cap A^c = \phi$, by axiom 2 and 3 we must have: $P(S) = P(A^c) + P(A) = 1 \Rightarrow P(A^c) = 1 - P(A)$

Corollary 2: By axiom 1, we have that $P(B) \geq 0$. Thus, from corollary 1, we have that:

$$P(A) = 1 - P(A^c) \leq 1 \Rightarrow 0 \leq P(A) \leq 1$$

Corollary 3: The empty set has probability zero. $P(\phi) = 1 - P(S) = 0$

Corollary 4: Union of 2 events $P(A \cup B)$ that are not necessarily mutually exclusive.



Decompose $A \cup B$, A and B as unions of disjoint events. By axiom 3, we have:

$$P(A \cup B) = P(A \cap B^c) + P(B \cap A^c) + P(A \cap B)$$

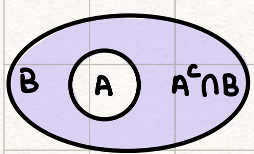
$$P(A) = P(A \cap B^c) + P(A \cap B)$$

$$P(B) = P(B \cap A^c) + P(A \cap B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Corollary 5: Since $P(A \cap B) \geq 0$, from corollary 4, we must have that $P(A \cup B) \leq P(A) + P(B)$

Corollary 6: Let's say $A \subset B$. By axiom 1: $P(A^c \cap B) \geq 0$



↳ $P(A) \leq P(A) + P(A^c \cap B) = P(B)$

↳ $P(A) \leq P(B)$

Discrete sample spaces

Recall our 3 coins example:

- $S = \{HHH, HHT, \cdots T\}$
- If S consists of n equally likely outcomes, then, for some event A, $P(A) = \frac{\text{\# of elements in } A}{n}$
- Consider the event A "at least 2 tails in a row"

↳ S has $n = 8$ posibile outcomes

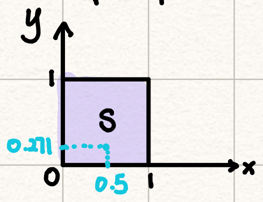
↳ $A = \{HTT, TTH, TTT\}$

↳ $P(A) = \frac{3}{8}$

Continuous sample spaces

Consider an experiment where we choose two numbers between 0 and 1 at random: $S = \{(x, y) \in \mathbb{R} : 0 \leq x \leq 1, 0 \leq y \leq 1\}$

The sample space would look like this:

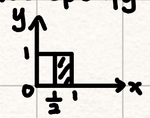


The probability that we take on specific values, $A = \{(0.5, 0.271)\}$, is zero, $P(A) = 0$

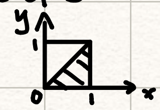
This is because our sample space is uncountably infinite

In such cases, we specify events as sub intervals of S

Event $x \geq \frac{1}{2}$



event $x \geq y$



Computing probability by counting

Consider a finite sample space where all outcomes are equally likely $P(A) = \dfrac{\#\text{ of elements in } A}{\text{total }\#\text{ of elements}}$

Sampling with ordering and with replacement

We have 4 coloured balls in an urn: R, G, B, Y. After selecting a ball, we replace it with an identical one.

If we select 3 balls, what sequence will we observe? $\underbrace{\text{RRR}}_{\text{tuple}}$, ...RRY, ..., RYR, ...YYY

In this experiement, there are $4^3 = 64$ possible outcomes. If we have n subjects and choose k, we would have n^k k-tuples

Note: If our number of choices is different between stages (e.g. k urns with n_k balls per urn), then the number of ordered k-tuples is $\#\text{ of k-tuples} = n_1 \cdot n_2 \cdots n_k$