Example: Let X and Y be two independent standard Gaussian RVs. We also have that

$$Z = X, \quad W = \rho X + \sqrt{1-\rho^2}\, Y \quad \text{for some } \rho \in (-1, 1)$$

a). Show that Z and W are jointly Gaussian.

Since X and Y are independent, their joint PDF is

$$f_{XY}(x,y) = f_X(x) f_Y(y) = \frac{1}{2\pi} e^{\left(\frac{-(x^2+y^2)}{2}\right)}$$

For any $aZ + bW$, we have: $aZ + bW = \underbrace{(a+b\rho)}_{\text{constant}} X + \underbrace{b\sqrt{1-\rho^2}}_{\text{constant}}\, Y$

b). Find the joint PDF of Z and W

We have a linear transformation of variables. Therefore, $\begin{bmatrix} Z \\ W \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix}}_{\bar{A}} \begin{bmatrix} X \\ Y \end{bmatrix}$

and we have that $|\bar{A}| = \det(\bar{A}) = \sqrt{1-\rho^2}$

Moreover, $\bar{A}^{-1} = \begin{bmatrix} 1 & 0 \\ \frac{-\rho}{\sqrt{1-\rho^2}} & \frac{1}{\sqrt{1-\rho^2}} \end{bmatrix}$

Finally, we have that $f_{ZW}(z,w) = \frac{f_{XY}(\bar{A}^{-1}\bar{z})}{|\bar{A}|} = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(\frac{-1}{2(1-\ ^2)}(z^2 - 2\rho zw + w^2)\right)$

c). Find the correlation between Z and W

Start by finding the variance of Z and W

$VAR(Z) = VAR(X) = 1$

$VAR(W) = VAR(\rho X + \sqrt{1-\rho^2}\, Y) = \rho^2 VAR(X) + (1-\rho^2) VAR(Y) = 1$

The correlation is defined as $\rho_{ZW} = \frac{COV(Z,W)}{\sigma_Z \sigma_W}$

$$\rho_{ZW} = COV(Z,W) = COV(X, \rho X + \sqrt{1-\rho^2}\, Y) = \rho\, COV(X,X) + \sqrt{1-\rho^2}\, COV(X,Y) = \rho(1) + \sqrt{1-\rho^2}(0) = \rho$$

Sum of RVs

Consider a sum of RVs: $S_n = X_1 + X_2 + \cdots + X_n$. Recall the linearity of the expectation operator:

$$E[S_n] = E[X_1] + E[X_2] + \cdots + E[X_n]$$

and the variance is: $VAR(S_n) = \sum_i \sum_j COV(x_i, x_j) = \sum_i VAR(x_i) + \sum_{i \neq j}\sum COV(x_i, x_j)$

If the variables are independent, then $COV(x_i, x_j) = 0$ when $i \neq j$.

$$VAR(S_n) = \sum_{i=1}^{n} VAR(X_i)$$

PDF of the sum of RVs

To find the PDF of S_n, we can use the characteristic function (CF). Let $Z = X + Y$ where X, Y independent

$$\Phi_Z(\omega) = E[e^{j\omega Z}] = E[e^{j\omega(X+Y)}] = E[e^{j\omega X} e^{j\omega Y}] = E[e^{j\omega X}]E[e^{j\omega Y}] = \Phi_X(\omega)\,\Phi_Y(\omega)$$

When RVs are independent, the CF of their sum is the product of their individual CFs,

$$\Phi_{S_n}(\omega) = \Phi_{X_1}(\omega)\,\Phi_{X_2}(\omega)\cdots\Phi_{X_n}(\omega)$$