Example: In the prisoner's dilemma, the guard's answer provided no information regarding the probability of A's execution.

- $P(E_A) = \frac{1}{3}$
- $P(E_A | G_B) = \frac{1}{3}$

The events {A executed} and {Guard says B} are independent.

Independent ≠ Mutually exclusive

If A and B are mutually exclusive, and have non zero probability:

- $P(A \cap B) = P(\phi) = 0$
- $P(A)P(B) > 0$
- They cannot be independent: $P(A \cap B) = P(A)P(B)$

Conditional independence

Two events are conditionally independent given an event C with $P(C) > 0$ if $P(A \cap B | C) = P(A|C)P(B|C)$

Assume A and B are conditionally independent. Recall conditional prob.: $P(A|B) = \frac{P(A \cap B)}{P(B)}$

By conditioning on C, we have:

$$P(A | B, C) = \frac{P(A \cap B | C)}{P(B|C)} = \frac{P(A|C)P(B|C)}{P(B|C)} = \frac{P(A|C)P(B|C)}{P(B|C)} = P(A|C)$$

Note:
- Events that are indep. are not necessarily conditionally indep.
- Events that are conditionally indep. are not necessarily indep.

Example: We have a regular coin and a fake 2 headed coin. We choose one coin at random and toss it twice

- A: 1^{st} toss is H.
- B: 2^{nd} toss is H.
- C: Regular coin was selected

Are A and B independent? Note that A and B are conditionally indep. given C

⇒ Find $P(A)$, $P(B)$, $P(A \cap B)$

$$P(A) = P(A|C)P(C) + P(A|C^c)P(C^c)$$

$$= \frac{1}{2} \times \frac{1}{2} + 1 \times \frac{1}{2} = \frac{3}{4}$$

$$P(B) = P(B|C)P(C) + P(B|C^c)P(C^c)$$

$$= \frac{1}{2}\times\frac{1}{2} + 1\times\frac{1}{2} = \frac{3}{4}$$

$\Rightarrow P(A)P(B) = \frac{9}{16}$

$\Rightarrow P(A\cap B) = P(A\cap B|C)P(C) + P(A\cap B|C^c)P(C^c)$

$$= P(A|C)P(B|C)P(C) + P(A|C^c)P(B|C^c)P(C^c)$$

$$= \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} + 1\times1\times\frac{1}{2} = \frac{5}{8} \neq \frac{9}{16}$$

$\Rightarrow P(A\cap B) \neq P(A)P(B)$

$\Rightarrow$ A and B are not indep.