Conditional Probability of Two RVs

We have 2 RVs, X and Y, with events A and B, respectively. The probability of $Y \in B$, conditional on $X \in A$ is $P(Y \in B \mid X \in A) = \frac{P(X \in A, Y \in B)}{P(X \in A)}$.

Case 1: X and Y are discrete RVs

The joint PMF: $P_{XY}(x_j, y_k) = P(X = x_j, Y = y_k)$

The conditional PMF of Y given $X = x_j$: $P_Y(y_k \mid x_j) = \frac{P(X = x_j, Y = y_k)}{P(X = x_j)} = \frac{P_{XY}(x_j, y_k)}{P_X(x_j)}$

And if the event $Y \in B$ includes multiple possible outcomes: $P(y \in B \mid X = x_j) = \sum_{y \in B} P_Y(y_k \mid x_j)$

Note the following result: $P_Y(y_k \mid x_j) = \frac{P_{XY}(x_j, y_k)}{P_X(x_j)} \Rightarrow P_{XY}(x_j, y_k) = P_Y(y_k \mid x_j) P_X(x_j)$ where the joint PMF can be expressed as the product of a conditional PMF and a marginal PMF.

If X and Y are independent, then $P_Y(y_k \mid x_j) = P_Y(y_k)$

Case 2: X is discrete and Y is continuous

Case 3: X and Y are continuous

If X is continuous, then $P(X = x) = 0$

$F_Y(y \mid x) = \lim_{h \to 0} F_Y(y \mid x < X \leq x + h) = \lim_{h \to 0} \frac{P(Y \leq y, x < X \leq x + h)}{P(x < X \leq x + h)}$