Example: Mean of a discrete uniform RV

Let X be uniform on $0,1,2\ldots n$. $p_X(x) = \begin{cases} \frac{1}{n+1} & \text{if } x = 0,1,\cdots,n \\ 0 & \text{otherwise} \end{cases}$

$E[X] = 0 \cdot \frac{1}{n+1} + 1 \cdot \frac{1}{n+1} + \cdots + n \cdot \frac{1}{n+1} = \frac{n(n+1)/2}{n+1} = \frac{n}{2}$

Expected value of a function of RVs

Let X be a RV with known PMF $p_X(x)$ and let $Y = g(x)$

By definition, $E[Y] = \sum_y p_Y(y) = \sum_x g(x) p_X(x) = \sum_y \sum_{\{x \mid g(x)=y\}} g(x) p_X(x)$

$= \sum_y y \underbrace{\sum_{\{x \mid g(x)=y\}} p_X(x)}_{P(g(x)=y)} = P(Y=y) = p_Y(y)$

$= \sum_y y p_Y(y)$

Note: In general, $E[g(X)] \neq g(E[X])$

Example Squared value of a dice roll.

$E[g(X)] = E[X^2] = 1^2 \cdot \frac{1}{6} + \cdots + 6^2 \cdot \frac{1}{6} = 15.17$.

$E[X] = 3.5 \Rightarrow g(E[X]) = 3.5^2 = 12.25$

Linearity of expectation

Expectation of a constant $\rightarrow E[a] = a$

$\Rightarrow E[aX + bY + c] = aE[X] + bE[Y] + c$ a.b.c are constants

Variance and standard deviation

The variance and std. deviation are measures of dispersion. It measures the spread of a RV centered around its expected value.

Variance: $\sigma_x^2 = VAR(X) = E[(X - E[X])^2]$