Asymptotics

Performance or complexity: - Time
- Space

5, 3, 7, 16 (n-numbers) → IN → i.e: sorting → OUT → 3, 5, 7, 16 (sorted)

Algorithm
Computer
Toring Machine

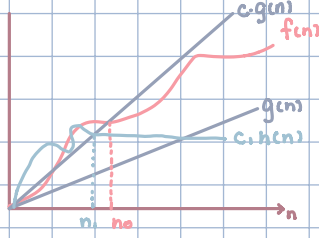
Alg 1: $n^4 + 3 \cdot n$ (in terms of n) $\rightarrow O(n^4)$

Alg 2: $700n^3 + 10^6 n^2 + 10^6 n$ $\rightarrow O(n^3)$

$\Rightarrow$ Alg 2 is actually much faster

Big-Oh

We say that $f(n) = O(g(n))$, iff $O(g(n)) = \{f(n)$: $\exists$ exist a positive constant C and n_0, such that $0 \le f(n) \le C \cdot g(n)$ $\forall n \ge n_0\}$



after n_0, $c \cdot g(n) \ge f(n)$ $\Rightarrow$ Asymptotically this is an upper bound

example:

$13n + 7 \in O(n)$

$13n + 7 \le 14n \Rightarrow n_0 = 7$

$\frac{1}{2}n^2 - 3n = O(n^2)$

We need to prove: $\frac{1}{2}n^2 - 3n \le C \cdot n^2$

$\Rightarrow \frac{1}{2} - \frac{3}{n} \le C$, true for $C = \frac{1}{2}$ (or larger) $\forall n \ge 1 = n_0$

$n! = 1 \cdot 2 \cdot 3 \cdots n$

$n! \le \underbrace{n \cdot n \cdots n}_{n \text{ times}} = n^n$

$\Rightarrow n! = O(n^n)$

What about $\log n!$

$n! = O(n^n) \Leftrightarrow \log n! = O(n \log n)$

$f(n) = O(g(n))$

$f(n) = \Omega(h(n))$

Big-Omega

We say $f(n) = \Omega(h(n))$ iff $\Omega(h(n)) = \{f(n)$: $\exists$ positive constants C_1 & n_1, s.t. $0 \le C_1 h(n) \le f(n)$ $\forall n \ge n_1\}$

$\Rightarrow$ A "lower" bound for $f(n)$

example:

$f(n) = 1 + 2 + 3 + \cdots + n = \Omega(?)$

$f(n) = 1 + 2 + \cdots + n \ge (\frac{n}{2}) + (\frac{n}{2}+1) + (\frac{n}{2}+2) + \cdots + n \ge \underbrace{(\frac{n}{2}) + (\frac{n}{2}) + \cdots + (\frac{n}{2})}_{\frac{n}{2} \text{ times}} \ge \frac{n}{2} \cdot \frac{n}{2} = \frac{n^2}{4}$

$\Rightarrow$ for $C_1 = \frac{1}{4}$, $n_1 = 1$, $f(n) = \Omega(n^2)$

$f(n) = \frac{1}{2}n^2 - 3n = O(n^2) = \Omega(?)$

Assume it's $\Omega(n^2)$, then $C_1 n^2 \le \frac{1}{2}n^2 - 3n \Leftrightarrow C_1 \le \frac{1}{2} - \frac{3}{n}$

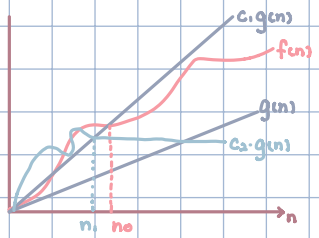
Allow $n \ge 7$ then: $C_1 \le \frac{1}{2} - \frac{3}{7} \Leftrightarrow C_1 \le \frac{7}{14} - \frac{6}{14} = \frac{1}{14}$

$\Rightarrow$ for $C_1 \le \frac{1}{14}$, $f(n) = \Omega(n^2)$ and $= O(n^2)$ $\Big\} = \Theta(n^2)$

Big-theta

We say that $f(n) = \Theta(g(n))$, $\Theta(g(n)) = \{f(n)$: $\exists$ positive constants C_1, C_2 and n_0 s.t. $0 \le C_1 g(n) \le f(n) \le C_2 g(n)$ $\forall n \ge n_0\}$ $\Rightarrow$ An asymptotic "tight bound"

Desirable but not always possible



$f(n) = O(g(n))$
$f(n) = \Omega(g(n))$
$\Big\} \Rightarrow f(n) = \Theta(n)$

example:

Arithmetic series: $1 + 2 + \cdots + n = \Omega(n^2)$

$= \frac{n(n+1)}{2} = \frac{1}{2}n^2 + \frac{1}{2}n = O(n^2) \Rightarrow 1 + 2 + \cdots + n = \Theta(n^2)$

Prove that: $\sum_{i=1}^{n} i^k = \Theta(n^{k+1})$

Need show:

$O(n^{k+1})$

$\sum_{i=1}^{n} i_k \le \sum_{i=1}^{n} n^k = n \cdot n^k = n^{k+1}$ QED

$\Omega(n^{k+1})$ $f(n) = \sum_{i=1}^{n} i^k$

$\Rightarrow 2f(n) = \sum_{i=1}^{n} i^k + \sum_{i=1}^{n} (n-i+1)^k$

$1^k + 2^k + \cdots n^k$ $\quad$ $n^k + (n-1)^k + \cdots 1^k$

$= \sum_{i=1}^{n} i^k + (n-i+1)^k$ every pair adding is always bigger than $(\frac{n}{2})^k$

$\geq \sum_{k=\frac{n}{2}}^{n} \left(\frac{n}{2}\right)^k = \frac{n^{n+1}}{2^{n+1}} = \Omega(n^{n+1})$

Asymptotics properties

- Transitivity: $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n)) \Rightarrow f(n) = \Theta(h(n))$
- Symmetry: $f(n) = \Theta(g(n))$ iff $g(n) = \Theta(f(n))$
- Transpose: $f(n) = O(g(n))$ iff $g(n) = \Omega(f(n))$
- Cookbook: $n^a \in O(n^b)$ iff $a \leq b$

  $\log_a n \in O(\log_b n)$ $\forall a, b$

  $c^n \in O(d^n)$ iff $c \leq d$

  if $f(n) \in O(f'(n))$ & $g(n) \in O(g'(n))$, then $f(n) \cdot g(n) = O(f'(n) \cdot g'(n))$

  (not derivative, just another function) $f(n) + g(n) = O(\max \in f'(n), g'(n))$

- Best case: min time of an algorithm → Ω
- Worst case: max .. .. → O
- average case
- expected case → Randomized Algorithm
- amortized case

Difference:
- Randomized Algorithm: At certain point, it "flips a coin" and depends on the output (0 or 1), it does different steps. i.e: Monte Carlo, Las Vegas, Quick Sort
- Probabilistic Analysis: Based on probability, we make assumptions depends on the input