State Models & Linearization

General form of a NL state model

① $\begin{cases} \dot{x} = f(x,u) \\ y = h(x,u) \end{cases}$ $\quad x \in \mathbb{R}^n,\ x = \begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix}$ = state vector $\quad \left.\begin{matrix} u \in \mathbb{R} = \text{input (scalar)} \\ y \in \mathbb{R} = \text{output (scalar)} \end{matrix}\right\}$ SISO system (Single input Single output)

f, h non-linear function $\rightarrow f(x,u) = f(x_1, \cdots x_n, u) = \begin{bmatrix} f_1(x_1 \cdots x_n, u) \\ \vdots \\ f_n(x_1, \cdots x_n, u) \end{bmatrix}$

$h(x,u) = h(x_1, \cdots x_n, u)$

General form of a LTI state model

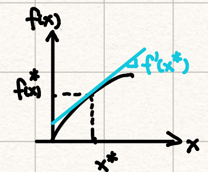
② $\begin{cases} \dot{x} = Ax + Bu \\ y = Dx + Cu \end{cases}$ $\quad x \in \mathbb{R}^n, u \in \mathbb{R}, y \in \mathbb{R}$ SISO LTI system

$A = (n \times n)$ matrix $\quad B = (n \times 1) \quad C = (1 \times n) \quad D = (1 \times 1)$ (scalar)

Q: How do we get form ① to a model as in ② (since this our focus in this course)? ⇒ Linearization

Idea: Use linearization of func. on the RHS of ①

Recap: Let $f: \mathbb{R} \rightarrow \mathbb{R}$



linearization of f around (at) x^* is

$y = f(x^*) + f'(x^*)(x - x^*)$ (+ H.O.T) Higher Order Term ↗ ignored

Let $f: \mathbb{R}^m \rightarrow \mathbb{R}^n$, similar rel:

for linearization around (at) x^*:

$y = f(x^*) + \underbrace{\boxed{\left.\frac{\partial f}{\partial x}\right|_{x=x^*}}}_{(n \times m)} \cdot (x - x^*)$ (Jacobian matrix) $\quad$ where $\left.\frac{\partial f}{\partial x}\right|_{x=x^*} = \begin{bmatrix} \cdots \boxed{\left.\frac{\partial f_i}{\partial x_j}\right|_{x^*}} \cdots \end{bmatrix}$ (m×m matrix)

$f = \begin{bmatrix} f_1 \\ f_i \\ f_n \end{bmatrix}$ $\quad$ partial derivative of f_i w.r.t x_j (with respect to)

We will use this method but first we have to pick linearization point called the operating point.

We will pick (x^*, u^*) op point as a equilibria condition for sys ①. i.e.

s.t $\quad f(x^*, u^*) = 0$ ③

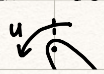
Note: Eq points condition (x^*, u^*) are s.t if set $u = u^* =$ const. in time, $x(0) = x^*$ (I.C), then the solution of ① is

$x(t) = x^*, \forall t \geq 0$ const. in time

⇓

take $\dot{x}(t) = 0 = f(x^*, u^*)$

Ex: 1 DOF robot link $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$



$\begin{cases} \dot{x}_1 = x_2 \end{cases}$ $\quad ① \triangleq f(x,u) = \begin{bmatrix} f_1(x_1, x_2, u) \\ f_2(x_1, x_2, u) \end{bmatrix}$

$$\begin{cases} \dot{x}_2 = -\frac{mgl}{I}\sin x_1 + \frac{1}{I}u \end{cases} \triangleq h(x,u)$$
$$y = x_1$$

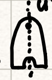
Want to obtain a linearized model around (at) (x^*, u^*) eq. pt.

First step is to find eq. pts. (x^*, u^*)

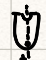
$$f(x^*, u^*) = \phi \Leftrightarrow \begin{cases} x_2 = 0 \\ -\frac{mgl}{I}\sin x_1 + \frac{1}{I}u = 0 \end{cases}$$

$$\text{set } u^* = \text{const.} \Rightarrow \begin{cases} x_2^* = 0 \\ \sin x_1^* = \frac{u^*}{mgl} \end{cases}$$

$u^* = 0$

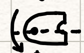
set $u^* = 0 \Rightarrow (x_1^*, x_2^*) = (0, 0) \rightarrow$  (DOWN) $\theta = 0$

$\sin x_1^* = 0$

$(x_1^*, x_2^*) = (\pi, 0) \rightarrow$  (UP) $\theta = 0$

$u^* = 0$

$\sin x_1^* = 1$

set $u^* = mgl \rightarrow (x_1^*, x_2^*) = (\frac{\pi}{2}, 0) \rightarrow$  (HORIZ) $\theta = \frac{\pi}{2}$

$u^* = mgl$