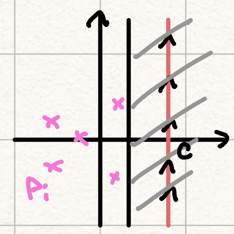
Inverse Laplace Transform (Review)

$$\text{time domain } f(t) \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} \text{s domain } F(s) = \int_0^{\infty} f(t)e^{-st}dt$$

Given $F(s)$. $f(t)=\mathcal{L}^{-1}\{F(s)\}=\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}ds$

where c is longer than all real-parts of poles of $F(s)$



We'll use the residue them:

$$f(t)=\sum \text{Res}[F(s)e^{st}, s=p_i]\cdot 1\!\!1(t) \quad \text{where } p_i = \text{pole of } F(s)$$

1. Let p_i = pole (simple), $\text{Res}[F(s)\cdot e^{st}, s=p_i] = F(s)\cdot e^{st}\cdot(s-p_i)\big|_{s=p_i}$
2. If p_i = double pole, $\text{Res}[F(s)e^{st}, s=p_i] = \frac{d}{ds}[F(s)e^{st}\cdot(s-p_i)^2]\big|_{s=p_i}$
3. If p_i = pole of multiplicity r, $\text{Res}[F(s)e^{st}, s=p_i] = \frac{1}{(r-1)!}\cdot\frac{d^{r-1}}{ds^{r-1}}[F(s)e^{st}\cdot(s-p_i)^r]\big|_{s=p_i}$

Ex: $F(s)=\frac{3}{(s+3)(s+9)^2}$, $f(t)=?$

Met 1: PFE $\rightsquigarrow \frac{A}{s+3}+\frac{Bs-C}{(s-9)^2}+\frac{D}{s+9} \xrightarrow{\text{look up LT table}} A\cdot e^{-3t}\cdots$ (find coeff A.B.C.D)

Met 2: Residue them: $f(t)=\text{Res}[F(s)\cdot e^{st}, s=-3]+\text{Res}[F(s)e^{st}, s=-9]$

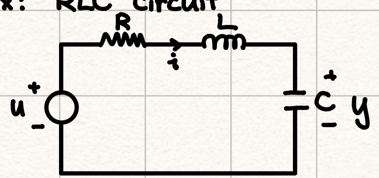
$$=\left[\frac{3e^{st}}{(s+9)^2}\Big|_{s=-3} + \frac{d}{ds}\left[\frac{3e^{st}}{s+3}\right]\Big|_{s=-9}\right]\cdot 1\!\!1(t)$$

$$=\frac{e^{-3t}}{12}\cdot 1\!\!1(t) + 3\cdot\frac{te^{st}(s+3)-e^{st}}{(s+3)^2}\Big|_{s=-9}\cdot 1\!\!1(t)$$

$$=\frac{e^{-3t}}{12}\cdot 1\!\!1(t) + \frac{1}{12}(-te^{-9t}\cdot 6 - e^{-9t})\cdot 1\!\!1(t)$$

Solution of ODEs using LT

Ex: RLC circuit



$u \rightarrow$ [I/O Model] $\rightarrow y$

I/O model: $LC\ddot{y}+RC\dot{y}+y=u$ —— (*)

Let $LC=\frac{1}{3}$, $RC=\frac{4}{3}$: $\frac{1}{3}\ddot{y}+\frac{4}{3}\dot{y}+y=u$

Ass: $y(0)=2$, $\dot{y}(0)=1$, $u(t)=1\!\!1(t)$

Use LT to move the ODE eq. to s-domain. From (*), using LT properties, take LT:

$$Y(s)=\mathcal{L}[y(t)], \quad U(s)=\mathcal{L}[u(t)]$$

→ by **linearity**: $\frac{1}{3}\mathcal{L}[\ddot{y}]+\frac{4}{3}\mathcal{L}[\dot{y}]+Y(s)=U(s)=\frac{1}{s}$

$\mathcal{L}[\ddot{y}] = s^2 Y(s) - s y(0) - \dot{y}(0) = s^2 Y(s) - 2s - 1$

$\mathcal{L}[\dot{y}] = s \cdot Y(s) - y(0) = sY(s) - 2$

$\Rightarrow$ our unknown is $Y(s)$ $\Rightarrow$ $Y(s) \cdot \left[\frac{1}{3}s^2 + \frac{4}{3}s + 1\right] = \overset{U(s)}{\overset{\parallel}{\frac{1}{s}}} + \overbrace{\frac{2s}{3} + \frac{1}{3} + \frac{8}{3}}^{\text{due to I.C.}}$

$\Rightarrow Y(s) = \frac{3}{\underset{\downarrow\, U(s)}{s}(s^2+4s+3)} + \frac{(2s+9)}{(s^2+4s+3)}$ (ratios of poly in s)

$\Rightarrow y(t) = \mathcal{L}^{-1}[Y(s)]$ $\Rightarrow$ use residue: $s^2+4s+3 = (s+3)(s+1)$, simple pole

$$\Rightarrow y(t) = \underbrace{\frac{3e^{st}}{s^2+4s+3}\Big|_{s=0}}_{\text{Res @ } s=0} + \underbrace{\frac{3e^{st}}{s(s+3)}\Big|_{s=-1}}_{@\ s=-1} + \underbrace{\frac{3e^{st}}{s(s+1)}\Big|_{s=-3}}_{@\ s=-3} + \frac{(2s+9)e^{st}}{s+3}\Big|_{s=-1} + \frac{(2s+9)e^{st}}{s+1}\Big|_{s=-3}$$

$$= \underbrace{\left(1 - \frac{3}{2}e^{-1} + \frac{1}{2}e^{-3t}\right)\mathbb{1}(t)}_{\text{due to } U(s)} \underset{\substack{\downarrow \\ \text{superposition}}}{+} \underbrace{\left(\frac{7}{2}e^{-t} - \frac{3}{2}e^{-3t}\right)\mathbb{1}(t)}_{\text{due to I.C.}}$$