Sampling with ordering and without replacement

Consider a similar scenario: 4 coloured balls, except we no longer replace them. The number of outcomes in the first draw is 4, decreasing by 1 after each draw. If we pick 3 balls, there are $4 \cdot 3 \cdot 2 = 24$ possible outcomes.

More generally, the number of ordered k-tuples $= n(n-1)(n-2)\cdots(n-k+1)$

Permutations

Let $k=n$. The number of n-tuples when sampling without replacement is also called the number of permutations

$$n(n-1)(n-2)\cdots(2)(1) = n! \quad \text{n factorial}$$

If we go back to the k-tuple scenario, where $k \leq n$, we can write the k-permutation as

$$n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!} = P_k^n$$

Example: Birthday Problem

What is the probability, $P(A)$, that at least 2 people in the room share the same birthday?

Idea: Use the complement, i.e, what is the probability that nobody shares the same birthday?

$$P(A^c) = \frac{365}{365} \times \frac{364}{365} \times \cdots \times \frac{365-k+1}{365} = \left(\frac{1}{365}\right)^k \frac{365!}{(365-k)!}$$

$$P(A) = 1 - P(A^c)$$

If we have $k=23$, then: $P(A) = 1 - 0.4927 = 50.7\%$

For $k=100$, $P(A) = 99.99997\%$

Sampling without ordering or replacement

Consider k objects. There are $k!$ possible ways of ordering them. However if order does not matter, then we have a single combination of size k.

Now consider n objects, where $n > k$. From before, the k-permutation is $n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$

Let C_k^n denote the combinations of size k from a set of size n. We can order this combination in $k!$ possible ways, i.e,

$$C_k^n k! = \frac{n!}{(n-k)!} \rightarrow C_k^n = \frac{n!}{k!(n-k)!} = \binom{n}{k} \quad \text{Binomial coeff.}$$

Note: Choosing k objects out of n is the same as choosing $n-k$. $\binom{n}{k} = \binom{n}{n-k}$