① $\begin{cases} \dot{x} = f(x,u) \\ y = h(x,u) \end{cases} \xrightarrow{\text{linearization}} \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$ LTI model

at point s.t. $\dot{x} = 0,\ u = u^* \Leftrightarrow f(x^*, u^*) = 0$

Linearize RHS ① at (x^*, u^*)

② $\begin{cases} \dot{x} = \underbrace{f(x^*, u^*)}_{\text{0 at } (x^*,u^*)} + \dfrac{\partial f}{\partial x}\Big|_{(x^*,u^*)} \cdot \overbrace{(x - x^*)}^{\delta x} + \dfrac{\partial f}{\partial u}\Big|_{(x^*,u^*)} \overbrace{(u - u^*)}^{\delta u} + \underbrace{H.O.T}_{\text{ignored}} \\ y = h(x^*, u^*) + \dfrac{\partial h}{\partial x}\Big|_{(x^*,u^*)} \cdot \overbrace{(x - x^*)}^{\delta x} + \dfrac{\partial h}{\partial u}\Big|_{(x^*,u^*)} \overbrace{(u - u^*)}^{\delta u} \end{cases}$

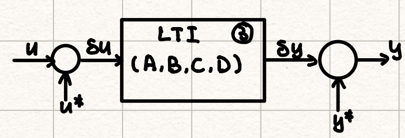
Let $\begin{cases} \delta x(t) = x(t) - x^* \\ \delta u(t) = u(t) - u^* \end{cases} \longrightarrow$ deviation from (x^*, u^*): $(\delta x, \delta u)$

$\delta y = y - y^* = y - h(x^*, u^*)$ change of variable

Then ③ $\begin{cases} \delta \dot{x} = \dot{x} - \underbrace{\dot{x}^*}_{\text{0 at } (x^*)} \overset{②}{=} \overbrace{\boxed{\dfrac{\partial f}{\partial x}\Big|_{(x^*,u^*)}}}^{A} \delta x + \overbrace{\boxed{\dfrac{\partial f}{\partial u}\Big|_{(x^*,u^*)}}}^{B} \delta u \\ \delta y = \underbrace{\boxed{\dfrac{\partial h}{\partial x}\Big|_{(x^*,u^*)}}}_{C} \delta x + \underbrace{\boxed{\dfrac{\partial h}{\partial u}\Big|_{(x^*,u^*)}}}_{D} \delta u \end{cases}$

↳ LTI model for ① around (x^*, u^*)

$\begin{cases} \delta \dot{x} = A\delta x + B\delta u \\ \delta y = C\delta x + B\delta u \end{cases}$



$u = u^* + \delta u,\quad x = x^* + \delta x,\quad y = y^* + \delta y$

Note: A.B.C.D depedent on (x^*, u^*);
δx (stable)
δu (input) depend on (x^*, u^*).

Ex. 1 DOF robot arm $\quad x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$

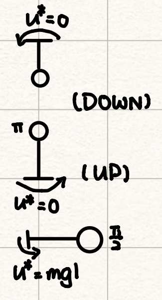
④ $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{mgL}{I} \cdot \sin x_1 + \dfrac{1}{I} u \\ y = x_1 \end{cases}$

Equalibira condition:

$u^* = 0 \Rightarrow (x^*, u^*) = \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, 0\right)$

$(x^*, u^*) = \left(\begin{bmatrix} \pi \\ 0 \end{bmatrix}, 0\right)$

$u^* = mgl \Rightarrow (x^*, u^*) = \left(\begin{bmatrix} \frac{\pi}{2} \\ 0 \end{bmatrix}, mgl\right)$



$f(x,u) = \begin{bmatrix} x_2 \\ -\dfrac{mgl}{I} \sin x_1 + \dfrac{1}{I} u \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2, u) \\ f_2(x_1, x_2, u) \end{bmatrix}$

$h(x,u) = x_1$

Then $A = \dfrac{\partial f}{\partial x} = \begin{array}{c} \begin{array}{cc} \frac{\partial f_i}{\partial x_1} & \frac{\partial f_i}{\partial x_2} \end{array} \\ \begin{bmatrix} 0 & 1 \\ -\frac{mgl}{I}\cos x_2 & 0 \end{bmatrix} \end{array} \Leftrightarrow \dfrac{\partial f_i}{\partial x_j} \quad B = \dfrac{\partial f}{\partial u} = \begin{array}{c} \frac{\partial f_i}{\partial u} \\ \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix} \end{array} \quad C = \dfrac{\partial h}{\partial x} = \begin{array}{c} \begin{array}{cc} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} \end{array} \\ \begin{bmatrix} 1 & 0 \end{bmatrix} \end{array} \quad D = \dfrac{\partial h}{\partial u} = 0$

$\frac{\partial f_2}{\partial x_1}$ $\frac{\partial f_2}{\partial x_2}$ $\frac{\partial f_2}{\partial u}$

Now at equalibria point

① $(x^*, u^*) = \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, 0\right) \Rightarrow A = \begin{bmatrix} 0 & 1 \\ -\frac{mgl}{I} & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \end{bmatrix},\ D = 0$

$$\Rightarrow \begin{cases} \delta\dot{x} = A\delta x + B\delta u \\ \delta y = C\delta x + D\delta u \end{cases}, \text{ where } \begin{matrix} \delta x = x - x^* = x \\ \delta u = u - u^* = u,\ \delta y = y \end{matrix}$$

②. $(x^*, u^*) = \left(\begin{bmatrix} \pi \\ 0 \end{bmatrix}, 0\right) \Rightarrow A = \begin{bmatrix} 0 & 1 \\ +\frac{mgl}{I} & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \end{bmatrix},\ D = 0$

different model at equalibria ②

$$\Rightarrow \begin{cases} \delta\dot{x} = A\delta x + B\delta u \\ \delta y = C\delta x + D\delta u \end{cases}, \text{ where } \begin{matrix} \delta x = x - x^* = x - \begin{bmatrix} \pi \\ 0 \end{bmatrix} \\ \delta u = u - u^* = u,\ dy = y \end{matrix}$$

③. $(x^*, u^*) = \left(\begin{bmatrix} \frac{\pi}{2} \\ 0 \end{bmatrix}, mgl\right) \Rightarrow A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \end{bmatrix},\ D = 0$

$$\Rightarrow \begin{cases} \delta\dot{x} = A\delta x + B\delta u \\ \delta y = C\delta x + D\delta u \end{cases}, \text{ where } \begin{cases} \delta x = x - x^* = x - \begin{bmatrix} \frac{\pi}{2} \\ 0 \end{bmatrix} \\ \delta u = u - u^* = u - mgl \\ \delta y = x_1 - \frac{\pi}{2} \end{cases}$$