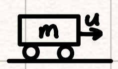
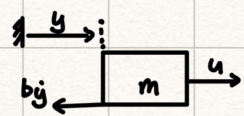
I/O & State Models (Cont'd)

Ex. 2: car model



u = force (input)

Free-Body-Diagram (FBD)



Ass: ①. wheels - negligible inertia

②. friction is viscous, i.e: proportional with velocity $\simeq b\dot{y}$

Newton's Law:

(1). $m\ddot{y} = u - b\dot{y}$ = LTI

Let: y = position; $\dot{y}$ = velocity; b = const.

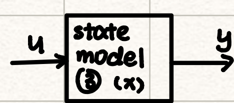
I/O model:

⑤. $m\ddot{y} + b\dot{y} = u \Leftrightarrow 2^{nd}$ order, linear, const. coef. ODE, I.C. $y(0)$, $\dot{y}(0)$

State-model: 2equs. 1^{st} order ODEs, const. coef. ODE

via x = state var. $= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$, $x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} y(0) \\ \dot{y}(0) \end{bmatrix}$

Let: ⑧ $\begin{cases} x_1 = y & (= \text{pos'n}) \\ x_2 = \dot{y} & (= \text{velocity}) \end{cases} \Rightarrow ⑨ \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \ddot{y} \overset{②}{=} -\frac{b}{m}\underset{=x_2}{\dot{y}} + \frac{1}{m}u \end{cases}$



In matrix-vector form:

$$\begin{cases} \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -\frac{b}{m} \end{bmatrix}}_{A\,(2\times2)\text{ metrix}} x + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{B\,(2\times1)\text{ metrix}} u \\ y = \underbrace{[1 \ 0]}_{C\,(1\times2)\text{ metrix}} x \end{cases}$$

dim $x = 2$

General form:

$$\text{LTI} \Leftarrow \begin{cases} \dot{x} = Ax + Bu & (\text{car}) \\ y = Cx + \cancel{D}^{0} u & (\text{RLC circuit}) \end{cases}$$

General: state vector $= \begin{bmatrix} \text{poss'n of masses} \\ \text{velocity of masses} \end{bmatrix}$

For robot link: nonlinear, State model: represented compactly as:

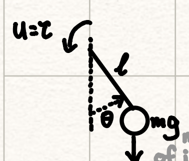
$$\begin{cases} \dot{x} = f(x,u) \\ y = h(x,u) \end{cases}$$

f, h = nonlinear func.

Ex ③: 1 DOF robot arm

u = torque (input)

y = θ = angle (output)



Newton's law for rotational motion:

$$\underset{\text{moment of inertia}}{I}\underset{\text{rotational accel.}}{\ddot{\theta}} = u - \underset{\vec{mg}\times\vec{l}}{mgl\sin\theta} \quad \text{—— ④}$$

⑤ $I\ddot{y} + mgl\sin y = u$

④ & ⑤ → I/O model ⇒ 2^{nd} order ODE, nonlinear, const. coeff   I.C: $\begin{bmatrix} y(0) \\ \dot{y}(0) \end{bmatrix} = \begin{bmatrix} \theta(0) \\ \dot{\theta}(0) \end{bmatrix}$

State-model:

state vector

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot\theta \end{bmatrix} = \begin{bmatrix} y \\ \dot y \end{bmatrix} \in \mathbb{R}^2$

⑥

$u \longrightarrow \boxed{\text{state } (x) \text{ model}} \longrightarrow y$

then: ⑥ $\begin{cases} \dot x_1 = \dot\theta = x_2 \\ \dot x_2 = \ddot\theta = \ddot y \xrightarrow{④ \text{ or } ⑤} -\frac{mgl}{I}\sin \overset{x_1}{\overset{\|}{y}} + \frac{1}{I}\cdot u \quad \text{and } y = x_1 \end{cases}$ $\Rightarrow$ 2 1st order ODE, nonlinear,

From ⑥: $\begin{cases} \dot x = \begin{bmatrix} \dot x_1 \\ \dot x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{mgl}{I}\sin x_1 + \frac{1}{I}u \end{bmatrix} = f(x, u) \\ y = \underset{\text{linear}}{[1 \quad 0]\, x} = h(x, u) \end{cases}$

(To solve) Linearization

Note: const. coeff $\Rightarrow$ time-invariant system

In this course: we will study Linear Time-Invariant System (LTI system)