We can exploit this linear transformation to express the PDF $f_{ZW}(z,w)$ as follows:

$$f_{XY}(x,y)\,dx\,dy \approx f_{ZW}(z,w)\,dP$$

$$f_{ZW}(z,w) = \frac{f_{XY}(x,y)}{|dP/dxdy|} \quad \text{where } \left|\frac{dP}{dxdy}\right| = \frac{|ae-bc|(dxdy)}{dxdy} = |\bar{A}| = \det(\bar{A})$$

where $|A|$ is the determinant of $\bar{A}$. Therefore if we know the joint PDF of X and Y, we can express the joint PDF of Z, W as: $f_{ZW}(z,w) = \frac{f_{XY}(\bar{A}^{-1}\bar{z})}{|A|}$

Example: Let X and Y be two independent standard Gaussian RVs and let $Z=2X-Y, W=-X+Y$ Find $f_{ZW}(z,w)$

Since X and Y are independent, we have $f_{XY}(x,y) = f_X(x)f_Y(y) = \frac{1}{2\pi}e^{-\frac{(x^2+y^2)}{2}}$

By inspection, we can see that $\bar{A} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \Rightarrow |\bar{A}| = ae-bc = 1$

$\bar{A}^{-1} = \frac{1}{ae-bc}\begin{bmatrix} e & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \Rightarrow \bar{A}^{-1}\bar{z} = \begin{bmatrix} z+w \\ z+2w \end{bmatrix}$

$$f_{ZW}(z,w) = \frac{f_{XY}(\bar{A}^{-1}\bar{z})}{|\bar{A}|} = \frac{1}{2\pi}\exp\left(\frac{-((z+w)^2+(z+2w)^2)}{2}\right)$$

Two Jointly Gaussian RVs

Useful note: In genearal, the variance of the sum of two RVs is $VAR(aX+bY) = a^2VAR(X) + b^2VAR(Y) + 2abCOV(X,Y)$

Two RVs, X and Y, are said to be jointly Gaussian (normal) if their sum, $Z=aX+bY$, has a Gaussian distribution.

If $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$ then $Z \sim N(a\mu_X + b\mu_Y, a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\sigma_{XY})$ where $\sigma_{XY} = COV(X,Y) = \rho\sigma_X\sigma_Y$

If X, Y are independent, then $\rho=0$, $Z \sim N(a\mu_X + b\mu_Y, a^2\sigma_X^2 + b^2\sigma_Y^2)$

For the jointly Gaussian X and Y, their joint PDF is:

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{XY}^2}}\exp\left\{\frac{-1}{2(1-\rho_{XY}^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho_{XY}\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)\right]\right\}$$

In addition, their marginal PDFs are also Gaussian $\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \frac{1}{\sqrt{2\pi}\sigma_X}e^{\frac{-(x-\mu_X)^2}{2\sigma_X^2}}$ $\quad \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \frac{1}{\sqrt{2\pi}\sigma_Y}e^{\frac{-(y-\mu_Y)^2}{2\sigma_Y^2}}$