Q.2. $\boxed{\text{13 marks}}$ An information source generates 1 with a probability of p and a 0 with a probability of $1-p$. Consider triplets (groups of three) of bits wherein each bit is generated independently.

2(a) $\boxed{\text{5 marks}}$ Let X be the number of ones in each triplet. Determine S_X and find pmf of X.

$S_X = \{0, 1, 2, 3\}$

$P(X=0) = \binom{3}{0} p^0 (1-p)^3 = (1-p)^3$

$P(X=1) = \binom{3}{1} p^1 (1-p)^2 = 3p(1-p)^2$

$P(X=2) = 3p^2(1-p)$

$P(X=3) = p^3$

2(b) $\boxed{\text{5 marks}}$ Let Y be the decimal value equivalent to each triplet of bits and assume $p = 0.5$. Find the **expected value** of Y **given** that the **right-most bit is equal to one**.
The mapping from binary to decimal values is given in the following table.

binary	000	001	010	011	100	101	110	111
decimal	0	1	2	3	4	5	6	7

Answer: Since $p = 0.5$, $P[Y = y] = \frac{1}{8}$ for any $y \in S_Y$. The conditional pmf is then given by

$$P[Y \text{is odd}|b_0 = 1] = \frac{1/8}{1/2} = \frac{1}{4}$$
$$P[Y \text{is even}|b_0 = 1] = 0$$

The conditional expected value of Y given $b_0 = 1$ is

$$E[Y|b_0 = 1] = \sum_{y \in S_Y} yP[Y = y|b_0 = 1] = \frac{1}{4}(1 + 3 + 5 + 7) = 4$$

2(c) $\boxed{\text{3 marks}}$ Consider the following events: $A = \{Y \geq 4\}$ and $B = \{Y \text{ is even}\}$. Are A and B independent? Justify your answer.

Answer:

$$P[A] = P[Y = 4] + P[Y = 5] + P[Y = 6] + P[Y = 7] = \frac{1}{2}$$
$$P[B] = P[Y = 0] + P[Y = 2] + P[Y = 4] + P[Y = 6] = \frac{1}{2}$$
$$P[A \cap B] = P[Y \geq 4 \cap Y \text{is even}] = P[Y = 4] + P[Y = 6] = \frac{1}{4} = P[A]P[B]$$

The events A and B are therefore independent.

3. In a communication system, the information bits are sent in packets of 1000 bits each. During the transmission of a packet, each bit is corrupted independently with probability p. The system hA n error correction mechanism, such that if the number of corrupted bits in a packet is 3 or fewer, the packet can be correctly decoded by the receiver.

a) Write an expression for the probability that a packet can be correctly decoded

$$P[\text{correct}] = \sum_{k=0}^{3} \binom{1000}{k} p^k (1-p)^{1000-k}$$