This means that for some large fixed value of n, we have a high probability of being close to the true mean

If we wish to reach to more powerful conclusion, we can look at the strong law of large numbers:

$$P(\lim_{n\to\infty} M_n = \mu) = 1$$

where we have a more dynamic approach. In this statement, we say that whenever we add another observation to our sequence, we converge towards μ and stay there.

Central limit theorem

Let $X_1, X_2, \dots X_n$ be iid RVs with mean μ and variance σ^2. Their sum is also a RV:

$$S_n = X_1 + X_2 + \cdots X_n$$

Now we consider the zero-mean, unit variance RV defined as:

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}} = \frac{\sqrt{n}\cdot M_n - \mu}{\sigma}$$

then $\lim_{n\to\infty} P(Z_n \leq z) = \frac{1}{2\pi}\int_{-\infty}^{z} e^{-x^2/2}\,dx$

In other words, the random variable Z_n conveges in distribution to the standard normal RV as n goes to ∞.

Proof of the CLT

We can express Z_n as $Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}} = \frac{1}{\sigma\sqrt{n}}\sum_{k=1}^{n}(X_k - \mu)$

The characteristic function of Z_n is:

$$\Phi_{Z_n}(w) = E[e^{jwZ_n}] = E[\exp(\frac{jw}{\sigma\sqrt{n}}\sum_{k=1}^{n}(X_k-\mu))] = \prod_{k=1}^{n} E[e^{jw(X_k-\mu)/\sigma\sqrt{n}}]$$ since the RVs are indep.

$$= (E[e^{jw(X-\mu)/\sigma\sqrt{n}}])^n$$ since the RVs are iid.

Next, we take the Taylor expansion of $E[e^{jw(X-\mu)/\sigma\sqrt{n}}]$:

$$E[1 + \frac{jw}{\sigma\sqrt{n}}(X-\mu) + \frac{(jw)^2}{2!\,\sigma^2 n}(X-\mu) + R(w)]$$

$$= 1 + \frac{jw}{\sigma\sqrt{n}}\overset{0}{E[(X-\mu)]} + \frac{(jw)^2}{2!\,\sigma^2 n}\overset{\sigma^2}{E[(X-\mu)^2]} + E[R(w)]$$

$$= 1 - \frac{w^2}{2n} + \underline{E[R(w)]}$$ goes to 0 as $n \to \infty$ (faster than $\frac{w^2}{2n}$ so ignore it)

Going back to the CF of Z_n,

$$\Phi_{Z_n}(w) \approx (1 - \frac{w^2}{2n})^n$$

$$\Rightarrow \lim_{n\to\infty}\Phi_{Z_n}(w) = \lim_{n\to\infty}(1 - \frac{w^2}{2n})^n = e^{-w^2/2}$$

Recall that $e^{-\omega^2/2}$ is the CF of a standard Gaussian RV. Therefore, as n gets larger, we have that the CDF of Z_n is:

$$\lim_{n\to\infty} F_{Z_n}(z) = \frac{1}{2\pi}\int_{-\infty}^{z} e^{\frac{-x^2}{2}}\,dx$$

Example: Consider the sum of n iid Bernoulli RVs, $X_i \sim \text{Bernoulli}(P)$. We previously saw that this sum can be represented by a Binomial RV. $S_n \sim \text{Binomial}(n, p)$. As n increases, this Binomial RV can be reasonably approximated by a Gaussian (normal) distribution: $S_n \sim N(np, np(1-p))$.