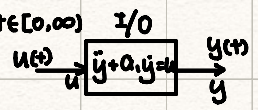
Mathmatical Models

Time Domain $t \in [0, \infty)$

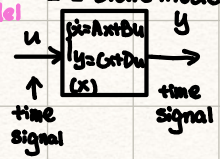
I/O



Ex. RLC

Linearization → LTI state model

NL state model



Ex. 1 DOF robot arm

Transfer function (S-Domain) $\mathcal{L}$ (Laplace transform) $s \in \mathbb{C}$

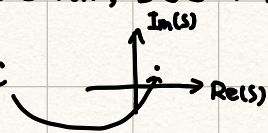
$U(s) = \mathcal{L}[u(t)]$

Laplace transform review

Let $f: [0, +\infty) \to \mathbb{R}; f(t)$ be a signal. The laplace transf. of $f(t)$, denoted by $F(s)$ is defined as:

$$F(s) = \mathcal{L}[f(t)] = \int_0^{+\infty} f(t) \cdot e^{-st} \cdot dt, \quad s \in \mathbb{C}, \quad F(s) \in \mathbb{C}$$

$t \in \mathbb{R}^+ \quad f(t) \xrightarrow{\mathcal{L}} F(s) \quad ; s \in \mathbb{C}$



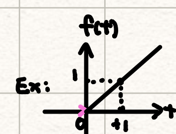
Note: L.T. of $f(t)$ exists ($\exists$)

if $f(t)$ is piecewise continuous of exponential order

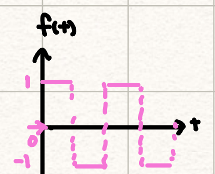
$\exists\, k, \rho, T$ s.t. $|f(t)| \le k e^{\rho t}; \quad t \le T$

Ex: $f(t) = 1\!\!1(t)$

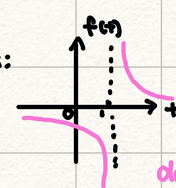
Ex:



Ex:



Ex:



$f(t) = \frac{1}{t-1}$ does NOT have L.T.

Typical L.T. pairs

$f(t)$	$F(s)$
$1\!\!1(t)$	$\frac{1}{s}$
$t\, 1\!\!1(t)$	$\frac{1}{s^2}$
$t^k\, 1\!\!1(t)$	$\frac{k!}{s^{k+1}}$
$e^{at}\, 1\!\!1(t)$	$\frac{1}{s-a}$
$t^k e^{at}\, 1\!\!1(t)$	$\frac{k!}{(s-a)^{k+1}}$
$\sin(at)\, 1\!\!1(t)$	$\frac{a}{s^2+a^2}$
$\cos(at)\, 1\!\!1(t)$	$\frac{s}{s^2+a^2}$

Properties of L.T.:

1). Linearity: $\mathcal{L}[c_1 f(t) + c_2 g(t)] = c_1 \mathcal{L}[f(t)] + c_2 \mathcal{L}[g(t)]$, $c_1, c_2 \in \mathbb{R}$ const.

2). Time-differentiation: If $\dot{f}(t)$ or $f'(t)$ or $\frac{d}{dt} f(t)$ is Laplace transferable, $\mathcal{L}[\dot{f}(t)] = s \cdot \mathcal{L}[f(t)] - f(0) = s \cdot F(s) - f(0)$, where $\mathcal{L}[f(t)] = F(s)$

multiplication

$\frac{d^n f}{dt^n}$ multiplication $F(s)$

By induction: $\mathcal{L}[f^{(n)}(t)] = s^n \cdot \mathcal{L}[f(t)] - s^{n-1}\cdot f(0) \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$

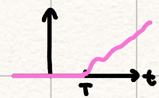
3). Integration: $\mathcal{L}[\int_0^t f(\tau)d\tau] = \frac{1}{s} F(s)$

4). Convolution: Let: $(f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau$

multiplication of L.T.

Then $\mathcal{L}[(f*g)(t)] = \mathcal{L}[f(t)]\cdot\mathcal{L}[g(t)] = F(s)\cdot G(s)$

5). Time delay: $T>0$. $\mathcal{L}[f(t-T)\cdot \mathbb{1}(t-T)] = e^{-Ts}\cdot F(s)$



6). Multiplication by t: $\mathcal{L}[t\cdot f(t)] = -\frac{dF(s)}{ds}$

Shift in s: $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$

Ex: $y(t) = (t\cdot e^{-2t} + t\sin t)\mathbb{1}(t)$

$Y(s) = \mathcal{L}[y(t)]$

properties
pairs

$= \mathcal{L}[te^{-2t}\mathbb{1}(t)] + \mathcal{L}[t\sin(t)\mathbb{1}(t)]$

$= \frac{1}{(s+2)^2} + \begin{cases} \text{Met 1 (property 6)}: -\frac{d}{ds}\left[\frac{1}{s^2+1}\right] \\ \text{Met 2}: \sin t = e^{jt} + e^{-jt} \end{cases}$