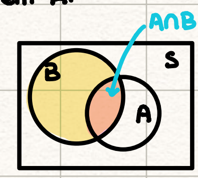
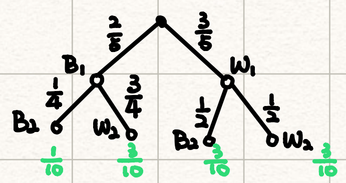
Conditional Probability

Sometimes 2 events are related, and the occurrence of event B alters the likelihood of observing event A.



- Denoted as $P(A \mid B)$
- If B is known to have occurred, then A can only occur if $A \cap B$ occurs.
- Assume $P(B) > 0$, $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$
- This also implies $P(A \cap B) = P(A \mid B)P(B) = P(B \mid A)P(A)$
- Example: We have 2 black and 3 white balls. We select 2 balls without replacement.



$P(B_1 \mid B_2) = \frac{1}{4}$

Probability of drawing white after drawing a black: $P(W_2 \mid B_1) = \frac{3}{4}$

What's the probability of observing both black?

$P(B_1 \cap B_2) = P(B_1)P(B_2 \mid B_1) = \frac{2}{5} \times \frac{1}{4} = \frac{1}{10}$

Total probability theorem

Assume we partition $S = B_1 \cup B_2 \cdots \cup B_n$,

↳ B_i's are disjoint

↳ An event A can be partitioned as $A = A \cap S = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n)$

Events $(A \cap B_i)$ are also disjoint.

By axiom 3, $P(A) = P(A \cap B_1) + P(A \cap B_2) + \cdots + P(A \cap B_n)$

↳ $P(A) = P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + \cdots + P(A \mid B_n)P(B_n)$

Example: As before, we have 2 black and 3 white balls, select 2 w/o replacement.

↳ What is the probability the scond ball is white?

$P(W_2) = P(W_2 \mid B_1)P(B_1) + P(W_2 \mid W_1)P(W_1)$

$= \frac{3}{4} \times \frac{2}{5} + \frac{1}{2} \times \frac{3}{5} = \frac{3}{5}$

Bayes' Rule (Theorem)

Recall the implications of conditional probability: $P(A \cap B) = P(A \mid B)P(B) = P(B \mid A)P(A)$

This allows us to derive Bayes' rule: $P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$

- We have proposition A and evidence B
- $P(A)$ is the "a prior probability" or "prior"
- $P(A|B)$ is the "posteriori probability" or "posterior"
- $\frac{P(B|A)}{P(B)}$ is the support B provides for A