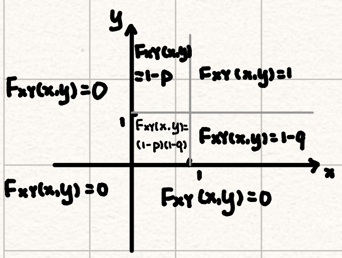
Joint CDF



$$F_X(x)=\begin{cases} \cdots & x=\cdots \\ \cdots & x=\cdots \\ \cdots & x=\cdots \\ \cdots & x=\cdots \end{cases}$$

Joint PDF

The joint PDF of two RVs is $f_{XY}(x,y)=\dfrac{\partial^2 F_{XY}(x,y)}{\partial x\,\partial y}$

If the pair $\bar{X}=(X,Y)\in B$, then the corresponding probability is $P(\bar{X}\in B)=\iint_B f_{XY}(x,y)\,dx\,dy$

And we can recover the CDF by finding the probability over the event $\{X\le x\}\cap\{Y\le y\}$

$$F_{XY}(x,y)=P(X\le x, Y\le y)=\int_{-\infty}^{y}\int_{-\infty}^{x} f_{XY}(x',y')\,dx'\,dy' \text{ (not derivative) } =\int_{-\infty}^{y}\int_{-\infty}^{x} f_{XY}(\xi,\gamma)\,d\xi\,d\gamma$$

The joint PDF gives us more flexibility than the CDF to define the probability of some event. Recall our example $P(a_1<X\le a_2, b_1<Y\le b_2)=F_{XY}(a_2,b_2)-F_{XY}(a_1,b_2)-F_{XY}(a_2,b_1)+F_{XY}(a_1,b_1)$

Using the joint PDF, we have: $P(a_1<X\le a_2, b_1<Y\le b_2)=\int_{b_1}^{b_2}\int_{a_1}^{a_2} f_{XY}(x,y)\,dx\,dy$

If we take the limit over all possible outcomes, $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy=1$.

Example: Let X and Y be 2 jointly continuous RVs with joint PDF $f_{XY}(x,y)=\begin{cases} x+cy^2 & 0\le x\le 1,\ 0\le y\le 1 \\ 0 & \text{otherwise}\end{cases}$

a) Find the constant c

we must have that $1=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy=\int_0^1\int_0^1 x+cy^2\,dx\,dy=\int_0^1 \frac{x^2}{2}+cxy^2\Big|_0^1\,dy=\int_0^1 \frac{1}{2}+cy^2\,dy=\frac{1}{2}+\frac{c}{3}=1 \Rightarrow c=\frac{3}{2}$

b) Find $P(0\le X\le \frac{1}{2}, 0\le Y\le \frac{1}{2})$

$P(0\le X\le \frac{1}{2}, 0\le Y\le \frac{1}{2})=\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} x+\frac{3}{2}y^2\,dx\,dy=\frac{3}{32}$

Marginal PDF

We can find the marginal PDF of X by integrating over all possible outcomes of Y.

$$f_X(x)=\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

We can also find the marginal PDF by taking the derivative of the marginal CDF.

$$f_X(x)=\frac{d}{dx}F_X(x)=\frac{\partial}{\partial x}F_{XY}(x,\infty)=\frac{\partial}{\partial x}\int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{XY}(\xi,y)\,dy\,d\xi=\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$