If event A can happen in m-ways, and event B can happen in n-ways.

Rule of product There are $(n \times m)$-ways that A & B can happen

Rule of Sum There are $(m+n)$-ways that A or B can happen

Example: You have 5 Latin, 7 Greek, 10 French books. In how many ways one can select two different books?

$5\times 7 + 5\times 10 + 7\times 10 = 155$ ways

In how many ways one can select any two books?

22×21 ways

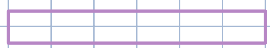
Permutations

Combinations

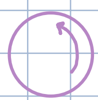
$P(n,r)$ = the # ways to arrange r out of n distinct objects where the order is important.

$P(n,r) = n\cdot(n-1)\cdot(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$

Example: In how many ways n-people can sit to form a ring?



sitting on a table $P(n,n)$



$\frac{P(n,n)}{n} = (n-1)!$

If not all objects are distinct, but q_1 of 1st kind, q_2 of 2nd kind, q_3 of 3rd kind, $\cdots$ q_t of tth kind

# of ways to permute objects: $\frac{n!}{q_1!q_2!\cdots q_t!}$

Example: five dashes & eight dots can be arranged in $\frac{13!}{5!8!}$ ways

Example: Show that $(k!)!$ is divisible by $(k!)^{(k-1)!}$

$\frac{(k!)!}{(k!)^{(k-1)!}}$ = some integer

Use a combinatorial argument: Consider $k!$ objects as follows:

k of 1st kind

k of 2nd kind

$\vdots$

k of $(k-1)!$ kind

total of $k!$ objects

In how many ways can we permute them?

$\frac{k!}{\underbrace{k!k!\cdots k!}_{(k-1)!}} = \frac{k!}{(k!)^{(k-1)!}}$ Q.E.D.

Example: Among 10 billion #s between $1 \sim 10{,}000{,}000{,}000$ how many contain the digit "1"?

Look at range: $0 \sim 9{,}999{,}999{,}999$, how many numbers do not contain 1?

"1" can go into any location. ⇒ In 9^{10} ways

⇒ $9^{10}-1$ ways do not contain "1" from the original range, which means that $10^{10} - (9^{10}-1)$ contain "1"

Combinations

$C(n,r)$ = # of combinations of r out of n-objects where order does not matter.

$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!} = C(n,n-r) = \binom{n}{r}$ "n choose r"

Example: How many diagonals a decagon has?

There are 10 corners for a decagon.

$C(10,2)$ to connect 10 points

10 actual decagon sides.

⇒ $C(10,2) - 10 = 35$

Example: In how many ways can 3 numbers be selected from $1, 2, \cdots 300$ so that their unique sum is divisible by 3.

There are 100 #s that leave remainder 1 — Group ①

" " " " " 2 — Group ②

" " " " " 0 — Group ③

3 from ① : $C(100,3)$

3 from ② : $C(100,3)$

3 from ③ : $C(100,3)$

1 from each group: $C(100,1)^3 = 100^3$

⇒ total = 1,485,100

Example: 11 scientists work on a secret project. They lock the doc into a cabinet s.t. the cabinet can open iff at least 6 scientists are present with their keys.

a). What's the smallest # of locks needed?

b). What's the smallest # of keys each scientist needs to have?

a). For every group of 5 scientists, 1 lock they cannot open

For every 2 groups of 5 scientists, at least 1 lock should be different

(or you will have 6 or more scientists that cannot open).

$\Rightarrow \forall$ group of 5, $\exists$ 1 lock

$\Rightarrow$ Result: $C(11,5) = 462$ locks

b). Everytime an individual scientist goes to a group of 5, he needs to have key open the lock those five cannot.

$C(10,5) = 252$ keys