General Case: From TF to state-space model

Given: $G(s) = \dfrac{b_m s^m + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$, $m<n$ strictly proper TF, case $m=n$ just @ the end.

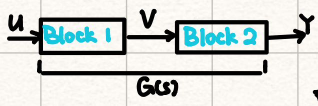
we can write a state-space model (a state-space linearization of G):

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \quad \text{where: } A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & & & \cdots & 1 \\ -a_0 & -a_1 & \cdots & & -a_{n-1} \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \; C = [b_0 \; b_1 \cdots b_m \; \underbrace{0 \cdots 0}_{\text{last } n-m-1 \text{ zeros (bc } m<n)}], \; D = 0$$

Note: A entries ⇠ coeff of denominator ("flipped" signs)

C entries ⇠ coeff of numerator

$$G(s) = \underbrace{\frac{1}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}}_{\text{Block 1}} \underbrace{(b_m s^m + \cdots + b_0)}_{\text{Block 2}}$$



$Y(s) = G(s) \cdot U(s)$

$$\begin{cases} V(s) = \text{Block 1} \cdot U(s) \\ Y(s) = \text{Block 2} \cdot V(s) \end{cases}$$

Look @ Block 1: find the s.s. model:

$$V(s) = \frac{U(s)}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \Leftrightarrow s^n V(s) + a_{n-1}s^{n-1}V(s) + \cdots + a_0 V(s) = U(s)$$

$\downarrow \mathcal{L}^{-1}$ 0.I.C $\qquad \downarrow \mathcal{L}^{-1}$

$$V^{(n)} + a_{n-1}V^{(n-1)} + \cdots + a_1 \dot{V} + a_0 V = u \quad n^{th} \text{ order ODE}$$

$\rightarrow$ Let: $\begin{cases} x_1 = V \\ x_2 = \dot{V} \\ \vdots \\ x_n = V^{(n-1)} \end{cases}$ $x \in \mathbb{R}^n$ $\Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = V^{(n)} = -a_{n-1}x_n - \cdots - a_1 x_2 - a_0 x_1 + u \end{cases}$

state variables

$$\Rightarrow \begin{cases} \dot{x} = Ax + Bu \\ V = x_1 \end{cases}$$

For Block 2:

$$Y(s) = (b_m s^m + \cdots + b_1 s + b_0) V(s)$$

$\downarrow \mathcal{L}^{-1}$, 0 I.C

$$y = b_m V^{(m)} + \cdots + b_1 \dot{V} + b_0 V \overset{\text{use New state variables}}{=} b_m x_{m+1} + \cdots + b_1 x_2 + b_0 x_1 = \overbrace{[b_0 \; b_1 \cdots b_m \; 0 \cdots 0]}^{C} x + \overbrace{[0]}^{D} u$$

What if $m=n$? Long (-tedious) division

$$\frac{b_n s^n + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} = d + \text{strictly proper TF}$$

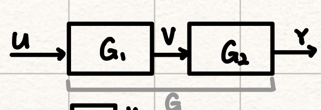
$Y(s) = (d + \text{stri. proper TF}) \cdot U(s)$

$\downarrow \mathcal{L}^{-1}$ $\quad D = d + 0$

$y = d \cdot u + [\quad]$

TFs and Block-Diagram Manipulation

Series (Cascade) Connection

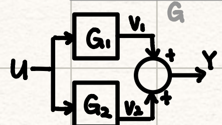


$Y = G_2 \cdot V$

$V = G_1 \cdot u$

$\Rightarrow G = G_1 \cdot G_2$

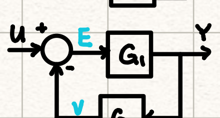
Parallel Connection



$Y = V_1 + V_2$

$\Rightarrow G = G_1 + G_2$

Feedback Connection

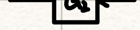


$Y = G_1 \cdot E$

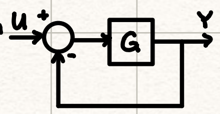
$E = u - V$

$V = G_2 Y$

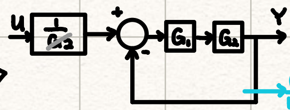
$\Rightarrow Y = \dfrac{G_1}{1 + G_1 G_2} u \Rightarrow G = \dfrac{G_1}{1 + G_1 G_2}$



Unit Feedback Connection



$\Longrightarrow$



$$\frac{G_1\cancel{G_2}}{1+G_1G_2} \Rightarrow \frac{G_1}{1+G_1G_2}$$