Example: Calculate the CF of an exponential random variable

$$f_X(x) = \lambda e^{-\lambda x} \qquad \Phi_X(w) = \int_0^\infty \lambda e^{-\lambda x} e^{jwx}\,dx = \int_0^\infty \lambda e^{(jw-\lambda)x}\,dx = \frac{\lambda}{\lambda - jw}$$

Linear Transformation

If $Y = aX + b$, then $\Phi_Y(w) = e^{jwb}\,\Phi_X(aw)$

Calculating moments with the CF

We can use the CF to obtain the moments of X,

$$E[X^n] = \frac{1}{j^n} \cdot \frac{d^n}{dw^n} \Phi_X(w)\Big|_{w=0}$$

Example: Find the mean and variance of an exponential RV.

mean: $\Phi_X'(w) = \frac{\lambda j}{(\lambda - jw)^2}$

$$E[X] = \frac{1}{j}\Phi_X'(0) = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

variance: $\Phi_X''(w) = \frac{-2\lambda}{(\lambda - jw)^3}$

$$E[X^2] = \frac{1}{j^2}\Phi_X''(0) = \frac{-2\lambda}{-\lambda^3} = \frac{2}{\lambda^2}$$

$$VAR[X] = E[X^2] - (E[X])^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Moment Generating Function

The MGF is similar to the CF, where $w = -js$ for some real number s. We then have:

$$M_X(s) = \Phi_X(-js) = E[e^{sX}] = \begin{cases} \int_{-\infty}^{\infty} e^{sx} f(x)\,dx \\ \sum_k e^{sx_k} p_X(x_k) \end{cases}$$

Recall the Taylor expansion: $e^x = 1 + x + \frac{x^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

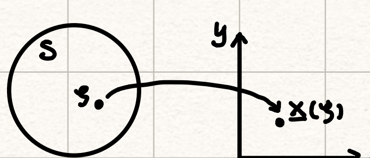
We can write: $e^{sX} = \sum_{k=0}^{\infty} \frac{X^k s^k}{k!} \rightarrow M_X(s) = E[e^{sX}] = \sum_{k=0}^{\infty} E[X^k]\frac{s^k}{k!}$

We can find the k^{th} moment of X by looking at the coefficient of $\frac{s^k}{k!}$ in the series above.

Alternatively, $E[X^n] = \frac{d^n}{ds^n} M_X(s)\Big|_{s=0}$

Multiple Random Variables

In most practical situations, using a single RV is not enough to properly model a system.



Events: We can sometimes denote a 2-dimentional RV with a vector $\underline{X} \in \mathbb{R}^2$, where $\underline{X} = (X, Y)$.

- $A = \{X + Y \geq 10\}$ $\qquad B = \{X^2 + Y^2 \leq 100\}$

Pairs of Discrete RVs

Consider 2 discrete RVs, X and Y. Their joint PMF is

$$P_{X,Y}(x_j, j_k) = P(\{X = x_j\} \cap \{Y = y_k\}) = P(X = x_j, Y = y_k)$$