# Functions of Random Variables

Let $X$ be a continuous RV and $Y=g(X)$ is a function of $X$, then $Y$ itself is a RV. Thus, we should be able to find the CDF and PDF of $Y$.

Example: Let $Y=X^2$

a). Find the CDF and PDF for some arbitary $X$

b). If $X$ is a uniform RV in $[-1,1]$, find the CDF and PDF of $Y$.

a). Note that $y \geq 0$ since all outcomes of $x$ are squared.

$$F_Y(y)=P(Y\leq y)=P(X^2\leq y)=P(-\sqrt{y}\leq X\leq \sqrt{y})$$

$$F_Y(y)=\begin{cases} F_X(\sqrt{y})-F_X(-\sqrt{y}) & y>0 \\ 0 & \text{otherwise}\end{cases}$$

We can find the PDF by differentiating $F_Y$ with respect to $y$.

$$f_Y(y)=\frac{d}{dy}(F_X(\sqrt{y})-F_X(-\sqrt{y}))=\frac{1}{2\sqrt{y}}(f_X(\sqrt{y})+f_X(-\sqrt{y}))$$

(b). Recall the CDF of a uniform RV

$$F_X(x)=\begin{cases} 0 & x<a \\ \frac{x-a}{b-a} & a\leq x\leq b \\ 1 & x>b\end{cases}$$

Replacing the CDF in part (a)

$$F_Y(y)=\begin{cases} 0 & y\leq 0 \\ \frac{\sqrt{y}+1}{1-(-1)}-\frac{-\sqrt{y}+1}{1-(-1)}=\sqrt{y} & 0<y\leq 1 \\ 1 & y>1\end{cases}$$

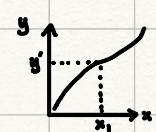
The PDF is $f_Y(y)=F_Y'(y)=\begin{cases} \frac{1}{2\sqrt{y}} & 0<y\leq 1 \\ 0 & \text{otherwise}\end{cases}$

## Method of Transformations

We have found the PDF of $Y=g(X)$ by first finding its CDF and the taking its derivative. However, we can find the PDF directly if

- $g(x)$ is differentiable
- The inverse function $g^{-1}$ is well defined.

Consider a strictly increasing function. ($y=g(x)$ has a single solution)




To find the PDF of $Y$, we differentiate:

$$f_Y(y)=\frac{d}{dy}F_X(x_1)=\frac{dx_1}{dy}F_X'(x_1)=\frac{dx_1}{dy}f_X(x_1)=\frac{f_X(x_1)}{g'(x_1)}$$ note $y=g(x_1)$, $\frac{dx_1}{dy}=\frac{1}{\frac{dy}{dx_1}}=\frac{1}{g'(x_1)}$

Note: if $g(x)$ is strictly decreasing, then $g'(x)$ will be negative, so we need to use $|g'(x)|$ to ensure our prob. is positive.

Example: Affine (linear) functions

Let $Y = aX + b$, and find the PDF of $Y$.

$g(x) = y = ax + b$, $g'(x) = a$ $\Rightarrow$ $x = \frac{y-b}{a}$

$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$