Recall the total prob. theorem, partition S into $B_1, B_2, \dots B_n$ disjoint sets.

$$E[X] = \sum_{i=1}^{n} E[X|B_i]P(B_i) \quad \text{and} \quad E[g(X)] = \sum_{i=1}^{n} E[g(x)|B_i]P(B_i)$$

Example:

Consider a game where we roll a dice and you get the value of your roll squared if the roll is even, and you get zero otherwise. Your friend rolls the dice and tells you the outcome is a prime number.

(a). What is the conditional exp. value?

Let X be the outcome of the roll.

First, find P(B), where $B = \{2, 3, 5\}$: $P(B) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$

The conditional PMF is

$$P_x(x|B) = \frac{P(\{X=x\} \cap B)}{P(B)} = \begin{cases} \frac{1}{3} & x = 2, 3, 5 \\ 0 & x = 1, 4, 6 \end{cases}$$

Define $g(x) = \begin{cases} x^2 & \text{if even} \\ 0 & \text{otherwise} \end{cases}$

Solve for the conditional exp. value of g(x):

$$E[g(x)|B] = \sum_x g(x) P_x(x|B) = 0 \cdot 0 + 2^2 \cdot (\tfrac{1}{3}) + 0 \cdot \tfrac{1}{3} + 4^2 \cdot 0 + 0 \cdot \tfrac{1}{3} + 6^2 \cdot 0 = \tfrac{4}{3}$$

(b). What is the unconditional exp. value?

Define our 2 disjoint sets B and B^c $P_x(x|B^c) = \begin{cases} \frac{1}{3} & x = 1, 4, 6 \\ 0 & \text{otherwise} \end{cases}$

Exp. value of g(x)

$$E[g(x)] = E[g(x)|B]P(B) + E[g(x)|B^c]P(B^c) = P(B)\sum_x g(x) P_x(x|B) + P(B^c)\sum_x g(x) P_x(x|B^c)$$

$$= \tfrac{1}{2} \times \tfrac{4}{3} + \tfrac{1}{2} \times (0 \cdot \tfrac{1}{3} + 2^2 \cdot 0 + 0 \cdot 0 + 4^2 \cdot \tfrac{1}{3} + 0 \cdot 0 + 6^2 \cdot \tfrac{1}{3})$$

$$= \tfrac{1}{2} \times \tfrac{4}{3} + \tfrac{1}{2} \times \tfrac{52}{3} = \tfrac{28}{3} = 9.33$$

Poisson RVs

A Poisson distribution describes the probability of a given number of events occurring in a fixed interval of time or space. The outcome of the RV represents the number of occurrences

Poisso PMF: $P_x(k) = \frac{e^{-\alpha} \alpha^k}{k!}$, $k = 0, 1, \cdots$, α is the average number of occurrences in a specified interval of time or space