Exp. Value of a function of two RVs

Let's say we have a function of two RVs. $g(X,Y)$. Its expected value is

$$E[g(X,Y)] = \begin{cases} \sum_j \sum_k g(x_j, y_k) P_{XY}(x_j, y_k) & \text{Discrete} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{XY}(x,y)\, dx\, dy & \text{Continuous} \end{cases}$$

Example: X and Y are continuous RVs with joint PDF $f_{XY}(x,y) = \begin{cases} x+y & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$ Find $E[XY^2]$

$$E[XY^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy^2 f_{XY}(x,y)\,dx\,dy = \int_0^1\int_0^1 xy^2(x+y)\,dx\,dy = \int_0^1\int_0^1 x^2y^2 + xy^3\,dx\,dy = \int_0^1 \left[\frac{x^3y^2}{3} + \frac{x^2y^3}{2}\right]_0^1 dy = \int_0^1 \frac{y^2}{3} + \frac{y^3}{2}\,dy = \frac{17}{72}$$

Linearity of Expectation

$E[aX+b] = aE[X]+b$

The expectation of a sum of RVs is $E[X+Y] = E[X] + E[Y]$

$$E[X+Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y) f_{XY}(x,y)\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{XY}(x,y)\,dx\,dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{XY}(x,y)\,dx\,dy = \int_{-\infty}^{\infty} x f_X(x)\,dx + \int_{-\infty}^{\infty} y f_Y(y)\,dy = E[X] + E[Y]$$

Expectation and Independence

If we have two independent continuous RVs, we saw that $f_{XY}(x,y) = f_X(x) f_Y(y)$

Let $Z = XY$: $E[Z] = E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{XY}(x,y)\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x f_X(x))(y f_Y(y))\,dx\,dy = \left\{\int_{-\infty}^{\infty} x f_X(x)\,dx\right\}\left\{\int_{-\infty}^{\infty} y f_Y(y)\,dy\right\} = E[X]\cdot E[Y]$

This same property carries over to some functions of RVs.

Let $g(X,Y) = g_1(X) g_2(Y)$. $E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(x) g_2(y) f_X(x) f_Y(y)\,dx\,dy = E[g_1(X)]\cdot E[g_2(Y)]$

Covariance

The covariance gives us some information about how X and Y are statistically related.

$$COV(X,Y) = E[(X-E[X])(Y-E[Y])] = E[XY - XE[Y] - YE[X] + E[X]E[Y]] = E[XY] - 2E[X]E[Y] + E[X]E[Y] = E[XY] - E[X]E[Y]$$

If $E[X]=0$ and/or $E[Y]=0$, then $COV(X,Y) = E[XY]$

The correlation coefficient is a normalized measure of the covariance

$\rho_{XY} = \frac{COV(X,Y)}{\sigma_X \sigma_Y}$ $(-1 \le \rho \le 1)$

std. deviation

Uncorrelated RVs

If X and Y are uncorrelated: $COV(X,Y) = 0 \rightarrow E[XY] = E[X]E[Y]$, $\rho_{XY} = 0$

- If X and Y are independent, then they are uncorrelated.
- However, if X and Y are correlated, they may or may not be independent.

[illegible]