A note on joint moments

The jk^{th} joint moment of X and Y is defined as $E[X^jY^k] = \begin{cases} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^j y^k f_{XY}(x,y)\,dx\,dy & \text{(continuous)} \\ \sum_i \sum_n x_i^j y_n^k p_{XY}(x_i, y_n) & \text{(discrete)} \end{cases}$

Important note: If $E[XY]=0$, we say that X and Y are orthogonal

Conditional Expectation

The conditional expectation of Y given some outcome $X=x$ is $E[Y|x] = \begin{cases} \sum_{y_k} y_k P_Y(y_k|x) & \text{if X, Y are discrete} \\ \int_{-\infty}^{\infty} y f_Y(y|x)\,dy & \text{if X, Y are continuous} \end{cases}$

Note that $E[Y|x] = g(x)$ since we defining the expectation of Y for some arbitrary $X=x$. What if we take the expectation over all possible value of X?

Law of total expectation

Since $E[Y|x] = g(x)$, and $X=x$, we can define $E[g(x)]$ as: $[f_Y(y|x)f_X(x) = f_{XY}(x,y)]$

$$E[E[Y|x]] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx = \int_{-\infty}^{\infty} E[Y|x] f_X(x)\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (y f_Y(y|x)) f_X(x)\,dy\,dx = \int_{-\infty}^{\infty} y \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = \int_{-\infty}^{\infty} y f_Y(y)\,dy = E[Y]$$

More generally, this holds for some function $h(Y)$, where $E[h(Y)] = E[E[h(y)|x]]$

The k^{th} moment of Y is $E[Y^k] = E[E[Y^k|x]]$

Functions of two RVs

We saw how to calculate the expectation of some $g(X,Y)$. What if we wish to define the CDF and PDF of this function?

Let $Z = g(X,Y)$ and let's say we wish to find its PDF.

$$F_Z(z) = P(Z \leq z) = P(g(X,Y) \leq z) = \iint_{R_z} f_{XY}(x,y)\,dx\,dy$$

where R_z is some 2D region in the (x,y) plane. $R_z = \{x, y : g(x,y) \leq Z\}$. we can find the PDF by differentiating w.r.t. z

example: Let X and Y be two indep. uniform $(0,1)$ RVs, and let $Z = XY$. Find the CDF and PDF of Z.

First note that $R_Z = [0,1]$ since we have the product of x and y

We can express the probability of $Z \leq z$ as follows:

$$F_Z(z) = P(Z \leq z) = P(XY \leq z) = P(X \leq \frac{z}{Y})$$