Induction

Predicate P i.e. P(n): "$n \le 2^n, \forall n \ge 1$"

Three steps:

- Basis
- Hypothesis
- Inductive step

$$[P(1) \cap \forall n\,(\underbrace{P(n)}_{\text{hypothesis}} \Rightarrow \underbrace{P(n+1)}_{\text{inductive step}})] \Rightarrow \forall n P(n)$$

basis (for $P(1)$)

Example 1: Prove $1+2+3+\cdots+n = \frac{n(n+1)}{2}$

Basis: show it holds for n=1: $1 = \frac{1(1+1)}{2}$

Hypothesis: Assume $1+2+3+\cdots n = \frac{n(n+1)}{2}$

Step: Need prove $1+2+\cdots+(n+1) = \frac{(n+1)(n+1+1)}{2}$

$\Rightarrow \frac{n(n+1)}{2} + (n+1)$ = right side $\Rightarrow$ true

Example 2: Prove that the sum of the first n odd positive integers is n^2

Basis: n=1 or "the first integer": $1 = 1^2$

Hypothesis: Assume: $1+3+5+\cdots(2n-1) = n^2$ is true

Step: Need prove it works for the first n+1 odd positive int.

$1+3+5+\cdots(2n+1) = (n+1)^2$

$n^2+2n+1 = (n+1)^2 \Rightarrow$ true

Example 3: Prove that $n < 2^n, \forall n \ge 2$

Basis: It holds n=2 : $2 < 2^2$

Hypothesis: Assume $n < 2^n$ for up to n

Step: Need prove for n+1, we got $n+1 < 2^{n+1} \le 2^n + 2^n = 2^{n+1} \Rightarrow$ true

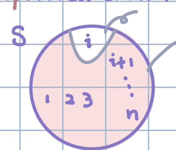
Example 4: Prove that for S its power set 2^S has 2^n subsets

Note: Powerset: $S = \{A, B, C\}$, Powerset S or 2^S, $2^S = \{\emptyset, A, B, C, AB, AC, BC, ABC\} \Rightarrow$ it has 2^3 elements / subsets

Basis: n=1 (obvious)

Hypothesis: Assume it's true for a set with n-elements

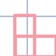
Step: Need show it works for set S where $|S| = n+1$ elements

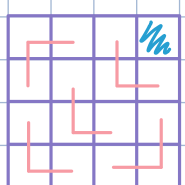


$\to 2^n$ (hypothesis)

$\Rightarrow$ total: $2^n + 2^n$ if we attach i in each element of the remaining powerset

$= 2^{n+1}$

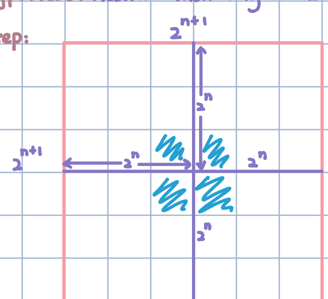
Example 5: Show that every $2^n \times 2^n$ chessboard with any one square blank it can be tiled with  shape tiles



Basis: n=2 as graph

Hypothesis: Assume that any $2^n \times 2^n$ chessboard with any one squre left blank can be tiled by L-shape tiles

Step:



As hypothesis, we can tile all 4 squares in the big squre.

$\Rightarrow$ there's 4 tiles left.

$\Rightarrow$ when we left one square blank, the other 3 can be tiled

$\Rightarrow$ true

Contradiction

Proposition P(n) = T or F, assume $\neg P$ holds $\neg P \Rightarrow \cdots \Rightarrow \cdots \Rightarrow$ False

not

i.e. you cannot grow cherries in February outside in Ontario.

Assume you can grow cherries. $\Rightarrow$ grow $\Rightarrow$ no fruits $\Rightarrow$ true.

Example 1: Show that if (3n+2) = odd, then n = odd

Assume that (3n+2) = odd but n = even. ↖ contradiction if n = even: $3n+2 = 3(2k)+2 = 6k+2 = 2(3k+1) \Rightarrow$ even $\neq$ odd

$\Rightarrow$ if $(3n+2)$ = odd then n = odd

Example 2: if $x^2-5x+4>0$ then $x>0$

Assume toward a contradiction (ATaC) that $x^2-5x+4>0$ but $x \leq 0$

Then $x^2 < 5x-4$ but if $x \leq 0$, then $5x-4<0$ then $x^2<0 \Rightarrow$ contradiction

Example 3: show that $\sqrt{2}$ is irrational

ATaC that $\sqrt{2}$ = rational $\Rightarrow \sqrt{2} = a/b$ where a&b have no common factors

$\Rightarrow 2 = a^2/b^2 \Rightarrow 2b^2 = a^2 \Rightarrow a^2$ = even $\Rightarrow a = 2c$* (we will prove that a^2 = even $\Rightarrow a$ = even)

$\Rightarrow 2^2c^2 = 2b^2 \Rightarrow b^2 = 2c^2$ = even

$\Rightarrow$ this is a contradiction b/c we assume a & b have no common factors but now they are both even (have 2 as common factor)

* proof: ATaC that a^2 = even $\Rightarrow a$ = odd

$a = 2k+1 \Rightarrow a^2 = (2k+1)^2 = 4k^2+1+4k = 4(k^2+k)+1$ = odd $\Rightarrow$ contradiction