Pairs of Discrete RVs

↳ $p_{X,Y}(x_j, y_k) = P(\{X=x_j\} \cap \{Y=y_k\}) = P(X=x_j, Y=y_k)$

Intuitively, if we sum over the entire sample space we must have: $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} p_{X,Y}(x_j, y_k) = 1$

What if we want to consider each variable individually?

The marginal PMF describes the behaviour of one of the RVs isolated from the other

$p_X(x_j) = P(X=x_j) = P(X=x_j, Y=\text{anything}) = \sum_{k=1}^{\infty} p_{X,Y}(x_j, y_k)$

	y=0	y=1	y=2
x=0	1/6	1/4	1/8
x=1	1/8	1/6	1/6

Example: Consider 2 RVs, X and Y, with the joint PMF

a) Find $P(X=0, Y\le 1)$ $\quad P(X=0, Y\le 1) = p_{X,Y}(0,0) + p_{X,Y}(0,1) = \frac{1}{6} + \frac{1}{4} = \frac{5}{12}$

b) Find the marginal PMF of X.

Note that the range of X is $\{0,1\}$ and Y is $\{0,1,2\}$

$p_X(0) = p_{X,Y}(0,0) + p_{X,Y}(0,1) + p_{X,Y}(0,2) = \frac{1}{6} + \frac{1}{4} + \frac{1}{8} = \frac{13}{24}$

$p_X(x) = \begin{cases} \frac{13}{24} & x=0 \\ \frac{11}{24} & x=1 \end{cases}$

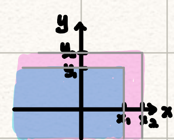
c) Find $P(Y=1 \mid X=0)$

Apply the cond. prob. formula: $P(Y=1 \mid X=0) = \frac{P(X=0, Y=1)}{P(X=0)} = \frac{p_{X,Y}(0,1)}{p_X(0)} = \frac{6}{13}$

Joint CDF

$F_{X,Y}(x,y) = P(\{X\le x\} \cap \{Y \le y\}) = P(X\le x, Y\le y)$

properties:



- If $x_1 \le x_2$ and $y_1 \le y_2$, then $F_{X,Y}(x_1,y_1) \le F_{X,Y}(x_2,y_2)$
- $F_{X,Y}(-\infty, y) = F_{X,Y}(x, -\infty) = 0$
- $F_{X,Y}(\infty,\infty) = 1$
- $F_X(x) = P(X\le x) = P(X \le x, Y \le \text{anything}) = P(X\le x, Y\le \infty) = F_{X,Y}(x, \infty)$

Example: $P(\{a_1 < X \le a_2\} \cap \{b_1 < Y \le b_2\}) = F_{X,Y}(a_2, b_2) - F_{X,Y}(a_2, b_1) - F_{X,Y}(a_1, b_2) + F_{X,Y}(a_1, b_1)$

Example: Let X and Y be two independent Bernoulli RVs with prob. p and q respectively.

Find their Joint PMF and CDF

Note the sample space is $S_{XY} = \{(0,0), (0,1), (1,0), (1,1)\}$

$P_{X,Y}(i,j) = P_X(i)\,P_Y(j)$ for $i,j = 0,1$

Therefore the joint PMF is $P_{X,Y}(0,0) = P_X(0)P_Y(0) = (1-p)(1-q)$

$P_{X,Y}(0,1) = (1-p)q$

$P_{X,Y}(1,0) = p(1-q)$

$P_{X,Y}(1,1) = p \cdot q$

Use the Joint PMF to find the CDF we know $F_{XY}(x,y) = 0$ if $x<0$

$F_{XY}(x,y) = 0$ if $y<0$

$F_{XY}(x,y) = 1$ if $x \geq 1$ and $y \geq 1$

For $0 \leq x < 1$ and $y \geq 1$, $F_{XY}(x,y) = P(X \leq x, Y \leq y) = P(X=0, Y \leq 1) = P(X=0) = 1-p$

Similarly for $0 \leq y < 1$ and $x \geq 1$, $F_{XY}(x,y) = 1-q$

Finally, for $0 \leq x < 1$ and $0 \leq y < 1$, $F_{XY}(x,y) = P(X \leq x, Y \leq y) = P(X=0, Y=0) = (1-p)(1-q)$