Example: We have event $A = \{a \le X \le b\}$. Find the conditional CDF of $X$ given $A$.

$$F_X(x|A) = P(X \le x | A) = P(X \le x | a \le X \le b) = \frac{P(\{X \le x\} \cap \{a \le X \le b\})}{P(A)}$$

If $x < a$, then $F_X(x|A) = 0$.

If $a \le x \le b$, $F_X(x|A) = \frac{P(a \le x \le b)}{P(A)} = \frac{F_X(x) - F_X(a)}{F_X(b) - F_X(a)}$

If $x > b$, $F_X(x|A) = 1$

Therefore, $F_X(x|A) = \begin{cases} 1 & x > b \\ \frac{F_X(x) - F_X(a)}{F_X(b) - F_X(a)} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$

## Total Probability Theorem

Assume we partition the sample space $S$ into disjoint sets $B_1, B_2, \dots, B_n$.

$$F_X(x) = P(X \le x) = \sum_{i=1}^{n} P(X \le x | B_i) P(B_i) = \sum_{i=1}^{n} F_X(x|B_i) P(B_i)$$

And the PDF is $f_X(x) = \frac{d}{dx} F_X(x) = \sum_{i=1}^{n} f_X(x|B_i) P(B_i)$

## Expected value and variance

For a continuous RV, $E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$ and for some function $g(X)$, $E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$

The variance is $VAR[X] = \sigma_X^2 = E[(X - E[X])^2] = \int_{-\infty}^{\infty} (x - E[X])^2 f_X(x)\,dx = E[X^2] - (E[X])^2 \geq 0$

And we still have linearity of the expectation. Let $a$ and $b$ be arbitary constants.

$$E[aX+b] = \int_{-\infty}^{\infty} (ax+b) f_X(x)\,dx = a\int_{-\infty}^{\infty} x f_X(x)\,dx + b\int_{-\infty}^{\infty} f_X(x)\,dx = aE[X]+b$$

In case of the variance, $VAR[aX+b] = E[(aX\cancel{+b} - E[aX\cancel{+b}])^2] = E[(aX - aE[X])^2] = E[a^2(X - E[X])^2] = a^2 VAR[X]$