## Case 3: X and Y are continuous

If X is continuous, then $P(X=x)=0$

$$F_Y(y|x)=\lim_{h\to 0} F_Y(y|x<X\le x+h)=\lim_{h\to 0}\frac{P(Y\le y,\ x<X\le x+h)}{P(x<X\le x+h)}$$

$$=\lim_{h\to 0}\left[\int_{-\infty}^{y}\int_{x}^{x+h} f_{XY}(\varepsilon,\tau)\,d\varepsilon\, d\tau \Big/ \int_{x}^{x+h} f_X(\varepsilon)\,d\varepsilon\right]\approx \lim_{h\to 0}\frac{\left(\int_{-\infty}^{y} f_{XY}(x,\tau)\,d\tau\right)\cdot h}{f_X(x)\cdot h}=\frac{\int_{-\infty}^{y} f_{XY}(x,\tau)\,d\tau}{f_X(x)}$$

$$f_Y(y|x)=\frac{d}{dy}F_Y(y|x)=\frac{f_{XY}(x,y)}{f_X(x)}$$

Note that, if X and Y are independent, then $f_Y(y|x)=\frac{f_X(x)f_Y(y)}{f_X(x)}=f_Y(y)$

### Total Probability

$P(Y\in y_k|X=x_j)=P_Y(y_k|x_j)$

We saw that $P(Y\in B|X=x_j)=\sum_{y_k\in B} p_Y(y_k|x_j)$. What if we want to find $P(Y\in B)$?

$$P(Y\in B)=\sum_{x_j}\sum_{y_k\in B} p_{XY}(x_j,y_k)=\sum_{x_j}\sum_{y_k\in B} p_Y(y_k|x_j)p_X(x_j)=\sum_{x_j} p_X(x_j)\sum_{y_k\in B}p_Y(y_k|x_j)=\sum_{x_j}P(Y\in B|X=x_j)p_X(x_j)$$

If X and Y are continuous, then $P(Y\in B|X=x)=\int_B f_Y(y|x)\,dy$

If we integrate over all possible outcomes of X, then we have

$$P(Y\in B)=\int_{-\infty}^{\infty}P(Y\in B|X=x)f_X(x)\,dx$$

### Bayes Rule

We have determined that $f_Y(y|x)=\frac{f_{XY}(x,y)}{f_X(x)}$, therefore, $f_{XY}(x,y)=f_Y(y|x)f_X(x)=f_X(x|y)f_Y(y)$

We can state Bayes rule as following:

$$f_Y(y|x)=\frac{f_X(x|y)f_Y(y)}{f_X(x)}$$

**example:** Let X and Y be two RVs with joint PDF $f_{XY}(x,y)=\begin{cases} x^2/4+y^2/4+xy/6 & 0\le x\le 1,\ 0\le y\le 2 \\ 0 & \text{otherwise}\end{cases}$ Given that $Y=y$ and $0\le y\le 2$

a) Find the conditional PDF of X

$$f_X(x|y)=\frac{f_{XY}(x,y)}{f_Y(y)}$$

Find the marginal PDF of Y: $f_Y(y)=\int_0^1 x^2/4+y^2/4+xy/6\,dx=\frac{3y^2+y+1}{12}$

Therefore, $f_X(x|y)=\begin{cases}\frac{3x^2+3y^2+2xy}{3y^2+y+1} & 0\le x\le 1 \\ 0 & \text{otherwise}\end{cases}$

b). Find $P(X<\frac{1}{2}|Y=y)$

From part a), we have $P(X<\frac{1}{2}|Y=y)=\int_0^{\frac{1}{2}}\frac{3x^2+3y^2+2xy}{3y^2+y+1}\,dx=\frac{1}{3y^2+y+1}\left[x^3+3xy^2+x^2y\right]_0^{\frac{1}{2}}=\frac{\frac{1}{8}+\frac{3}{2}y^2+\frac{y}{4}}{3y^2+y+1}$

By our own design, $P(X<\frac{1}{2}|Y=y)$ depends on y.