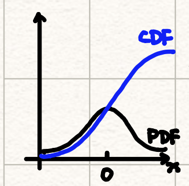
A Note on the PDF / CDF

Consider a standard Gaussian RV:



The CDF is a non-decreasing function of x, meaning its derivative is never negative. therefore:

$$f_X(x) = \frac{d}{dx} F_X(x) \geq 0$$

Independence of two RVs

Random variables X and Y are independent if, for any events where $X \in A$ and $Y \in B$,

$$P(X \in A, Y \in B) = P(X \in A) P(Y \in B)$$

Recall the definition of the joint PMF $P_{XY}(x_j, y_k) = P(X = x_j, Y = y_k)$ where we have the events $\{X = x_j\}$ and $\{Y = y_k\}$.

$$P_{XY}(x_j, y_k) = P(X = x_j, Y = y_k) = P(X = x_j) P(Y = y_k) = P_X(x_j) P_Y(y_k) \quad \text{if X and Y are independent.}$$

In the case of the joint CDF, $F_{XY}(x,y) = P(X \leq x, Y \leq y)$. Define $A = (-\infty, x]$, $B = (-\infty, y]$.

$$F_{XY}(x,y) = P(X \leq x, Y \leq y) = P(X \leq x) P(Y \leq y) = F_X(x) F_Y(y) \quad \text{if X and Y are independent.}$$

We can show that X and Y are independent if the joint CDF is equal to the product of their marginal CDFs.

If we take the partial derivatives of the CDF, we have:

$$f_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y}(F_X(x) F_Y(y)) = F'_X(x) F'_Y(y) = f_X(x) f_Y(y) \quad \text{if X and Y are independent}$$

Example: Determine whether X and Y are independent.

a). $f_{XY}(x,y) = \begin{cases} 2e^{-x-2y} & x, y > 0 \\ 0 & \text{otherwise} \end{cases}$

Find the marginal PDFs:

$$f_X(x) = \int_0^\infty 2e^{-x-2y} dy = \frac{2e^{-x-2y}}{-2} \Big|_0^\infty = e^{-x}$$

$$f_Y(y) = \int_0^\infty 2e^{-x-2y} dx = -2e^{-x-2y} \Big|_0^\infty = 2e^{-2y}$$

Thus, we can see that $f_{XY}(x,y) = f_X(x) f_Y(y) = (e^{-x})(2e^{-2y}) = 2e^{-x-2y}$, for $x, y > 0$

Therefore, X and Y are independent.

b). $f_{XY}(x,y) = \begin{cases} 8xy & \text{if } 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$

Find the marginal PDFs

$$f_X(x) = \int_x^1 8xy\, dy = 4xy^2 \Big|_x^1 = 4x(1-x^2) \Rightarrow f_X(x) = \begin{cases} 4x(1-x^2) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_0^y 8xy\, dx = 4x^2 y \Big|_0^y = 4y^3 \Rightarrow f_Y(y) = \begin{cases} 4y^3 & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Then we can see that $f_{XY}(x,y) \neq f_X(x)f_Y(y)$

Therefore, X and Y are NOT independent