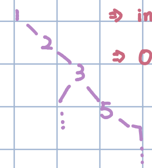
Red - Black Trees

Solve the imbalance issue of BSTs (AVL trees, B trees ⋯ also solve the issue)

example: insert 1, 2, 3, 5, 7 ⋯



⇒ imbalance!

⇒ $O(h)$ ⇒ goal: $O(n)$

Definition: is a BST and each node has a color black / red

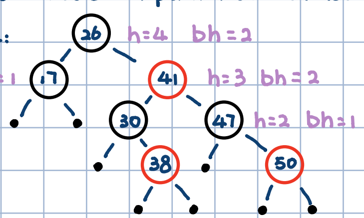
Convention: 1. all leaves are NULL and has the color black

2. root's parent is NULL and black

Red-black Properties

1. Every node is either red / black
2. The root is black
3. Every leaf is black
4. if node is red then both children are black (no two red nodes in row!!)
5. For each node, all path from that node to descendant leaves have same number of black nodes

Example:



Definitions about heights

1. Height of node is # edges in longest path to leaf
2. Black height of node x : bh(x) is # black nodes on path from x to a leaf (not counting x)

Balance proof

Claim 1: any node with height h has black-height $\geq \frac{h}{2}$

Proof: by property 4, $\leq h/2$ nodes on path from node to leaf are red

⇒ $\geq \frac{h}{2}$ are black

Claim 2: subtree rooted at any node x contains at least $(2^{bh(x)}-1)$ internal nodes

Proof by induction on height of x

Basis: height of $x=0$ ⇒ x is leaf ⇒ $bh(x)=0$

The subtree at x has 0 internal nodes

$2^{bh(x)}-1=0$

Step: let height of x be h, black-height $bh(x)=b$

Any child of x has height $h-1$ and black-height: b if child is red

$b-1$ if child is black

By hypothesis: each child has $\geq 2^{bh(x)}-1$ internal nodes.

Thus, subtree rooted at x has $\geq 2(2^{bh(x)-1}-1)+1 = 2^{bh(x)}-1$ internal nodes

Lemma: a red-black tree with n internal nodes has height $\leq 2\log(n+1)$

Proof: Let h and b be the height and black-height of root

By claim 1 & 2, $n \geq 2^{b}-1 \geq 2^{h/2}-1$

Add 1 and take log: $\log(n+1) \geq \frac{h}{2}$

⇒ $h \leq 2\log(n+1)$

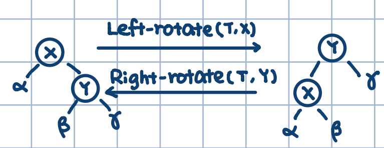
Operations

Min, Max, Successor, Predecessor, Search are same as BST, all have time: $O(h)$ ⇒ $O(\log n)$

Insert & Delete (may violate properties)

⇒ Rotation: helps fix violations

- tree-restruction
- Inorder walk & BST property are preserved.



- Time : $O(1)$

Insert : Search and insert and color node red.

i.e INSERT (T, z)

search z, insert z, and color it red

Property check: