Methodology to prove NPC

Given language L to show $\in NPC$:

a). Prove that a solution can be verified in poly-time

b). Select known $L' \in NPC$

$L' \leq_p L$

b.1. Describe algorithm $f()$, that given instance $x \in L'$ it "translates" it to $f(x)$ s.t. $x \in L' \leq_p f(x) \in L$

b.2. Show $f()$ takes poly time to compute

Example

Formula Satisfiability

A formula Φ with n-Boolean variables and connectives:

$\underset{\text{AND}}{\wedge}, \underset{\text{OR}}{\vee}, \underset{\text{NOT}}{\neg}, \overset{\text{implication}}{\longrightarrow}, \underset{\text{equivalent}}{\longleftrightarrow}, (\ ,\ ) \rightarrow$ parenthesis

Decision Problem

$FSAT = \{\Phi : \text{formula } \Phi \text{ has a satisfying Boolean var assignment st } \Phi = 1\}$

$\Phi = (x_1 \vee \bar{x}_2) \longleftrightarrow (x_3 \rightarrow \bar{x}_4) \vee (\bar{x}_2 \vee \bar{x}_3)$

is there a Boolean assignment to variables $x_1 \ldots x_n$ st $\Phi = 1$

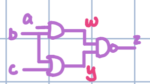
$a \rightarrow b$

a	b	a→b
0	0	1
0	1	1
1	0	0
1	1	1

a). Given a Boolean assignment to the variables $x_1 \cdots x_4$ "plug" values and evaluate Φ to check if $\Phi = 1$ or 0

This takes poly-time

b). Circuit-SAT $\leq_p$ Formula-SAT



that is, given a Boolean circuit we will construct a formula Φ st circuit is satisfiable iff Φ is satisfiable

① name internal lines

②. $\Phi = [(a \wedge b) \leftrightarrow w] \wedge [(b \vee c) \leftrightarrow y] \wedge [(w \wedge y) \leftrightarrow \bar{z}] \wedge z$ (to make $z = 1$)

circuit is satisfiable $(z = 1) \Longleftrightarrow \Phi$ is satisfiable $(\Phi = 1)$

③ $\forall$ gate we introduce a constant number of literals.

Therefore, the construction takes poly-time.

3-CNF-SATISFIABILITY

You are given a formula Φ in Conjuctive Normal Form (CNF)

$\underbrace{(x_1 \vee \bar{x}_2 \vee x_3)}_{\text{clause}} \wedge (\bar{x}_2 \vee \bar{x}_4 \vee x_5) \wedge \cdots$

every clause has 3 literals

$\Phi = (\ ) \wedge (\ ) \wedge \cdots$

SAT solvers

Decision problem

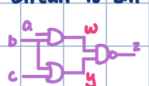
is there a Boolean assignment to variables $x_1 \ldots x_n$ st $\Phi = 1$

ⓐ. Show 3-CNF $\in NP$ similar to formula-SAT

ⓑ. we will prove : circuit-SAT $\leq_p$ 3-CNF-SAT

Given a circuit we will create a 3-CNF Formula Φ st

Circuit is SAT iff Φ is SAT



① name internal lines

② $w \leftrightarrow a \wedge b$. characteristic function:

a	b	w	w↔a∧b
0	0	0	1
0	0	1	0 ✓
0	1	0	1
0	1	1	0 ✓
1	0	0	1
1	0	1	0 ✓
1	1	0	0 ✓
1	1	1	1

find maxterms above

$\Phi_{IN} = (\bar{a} \wedge \bar{b} \wedge w) \vee (\bar{a} \wedge b \wedge w) \vee (a \wedge \bar{b} \wedge w) \vee (a \wedge b \wedge \bar{w})$

complement the above:

$\Phi_{AND} = (a \wedge b \wedge \bar{w}) \vee (a \wedge \bar{b} \wedge \bar{w}) \vee (\bar{a} \wedge b \wedge \bar{w}) \vee (\bar{a} \wedge \bar{b} \wedge w) \Rightarrow$ CNF

$\Phi = \Phi_{AND} \wedge \Phi_{NAND} \wedge \Phi_{OR} \wedge (z = 1)$

introduce pseudo variable k, ℓ

$z=1 \equiv (z \vee k \vee \ell) \wedge (z \vee \bar{k} \vee \ell) \wedge (z \vee k \vee \bar{\ell}) \wedge (z \vee \bar{k} \vee \bar{\ell})$

we claim circuit is satisfiable $\Leftrightarrow \Phi$ has a satisfying assignment

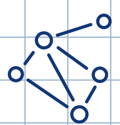
TRUE by construction!

The reduction was poly in time. $\forall$ circuit gate I introduce a constant # of clauses

Clique

Vertex Cover

A subset $V' \subseteq V$ of G that if $(u,v) \in E$ then at least one vertex exists in cover V'



Optimization problem:

Find the minimum vertex cover

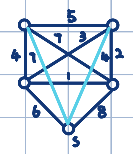
Decision problem:

Does G have a vertex cover of size k?

a). Vertex Cover $\in$ NP

TSP

Given a complete positively weighted G find a HAM CYCLE of minimum weight



Decision problem: is there a TSP of weight k?

ⓐ. TSP $\in NP$

ⓑ. Ham Cycle $\leq_p$ TSP

Given a graph G to find a Ham cycle, add the remaining to create a complete graph.

Then assign weight $w(u,v)$ $\forall$ edge (u,v)

$$w(u,v) = \begin{cases} 0 & \text{if } (u,v) \in G \\ 1 & \text{if } (u,v) \notin G \end{cases}$$

G has a Ham cycle $\Leftrightarrow$ G_{new} has a TSP = 0