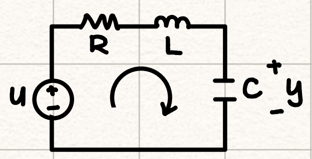
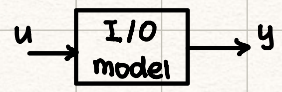
I/O and State Models

Ex1: Voltage control of RLC circuit



$\Rightarrow$



u = source voltage (input) $\rightarrow$ variable used to control (offset) system

y = voltage across C (output) $\rightarrow$ variable to measure and/or control

$$\left.\begin{aligned} &\text{KVL: } Ri + L\cdot\frac{di}{dt} + y - u = 0 \quad ① \\ &\text{KCL: } i = C\frac{dy}{dt} \quad ② \end{aligned}\right\}\ t \text{ is time}$$

Eliminate i to get I/O model between u & y

Take $\frac{d}{dt}$ in ② and sub-in ①: $LC\frac{d^2y}{dt^2} + RC\frac{dy}{dt} + y = u$ ③ $\rightarrow$ I/O model for RLC circuit

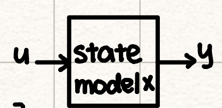
2nd order ODE, linear, const. coeff.

To solve it, we need 2 I.C. $y(0)$ & $\dot{y}(0)$, where $\dot{y} = \frac{dy}{dt}$

$\rightarrow$ Get a State (space) Model by introducing auxilary variables to covert from

a 2nd order ODE $\rightarrow$ a set of 2 1st order ODE

Let ⊛ $\begin{cases} x_1 = y \text{ (= voltage across C) generalizable to any RLC circuit} \\ x_2 = i \text{ (= current through L)} = C\frac{dy}{dt} \end{cases}$



and $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$, state vector $X(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} y(0) \\ C\cdot\dot{y}(0) \end{bmatrix}$

$$\frac{dx_1}{dt} \overset{⊛}{=} \frac{dy}{dt} \overset{KCL}{=} \frac{1}{C}\cdot i \overset{⊛}{=} \frac{1}{C}x_2$$

$$\frac{dx_2}{dt} = \frac{di}{dt} \overset{KCL}{=} C\cdot\frac{d^2y}{dt^2} \overset{③}{=} -\frac{R}{L}\underbrace{\boxed{C\frac{dy}{dt}}}_{x_2} - \frac{1}{L}\underbrace{\boxed{y}}_{x_1} + \frac{1}{L}u$$

Overall State-model $\quad X(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$

$$④ \begin{cases} \dot{x}_1 = \frac{1}{C}x_2 \\ \dot{x}_2 = -\frac{1}{L}x_1 - \frac{R}{L}x_2 + \frac{1}{L}u \\ y = x_1 \end{cases} \Rightarrow \begin{aligned} \dot{x} &= \overset{A}{\boxed{\begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}}}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \overset{B}{\boxed{\begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}}}\cdot u \\ y &= \underset{C}{\boxed{[1 \quad 0]}}\, x \end{aligned}$$

⑤ $\begin{cases} \dot{x} = Ax + Bu & \text{where } D = 0 \\ y = Cx + Du & \text{general state model (matrix form)} \end{cases}$

E.g 3 capacitors, 2 inductors $\Rightarrow$ 5 state variables, 5 1st ODE, $x \in \mathbb{R}^5$, A is 5×5