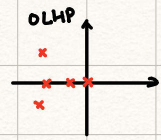
Final Value Theorem (FVT) (prelude to feedback)

THM: Suppose $Y(s)$ is a rational, proper, complex fcn. If $y(\infty)=\lim_{t\to\infty} y(t)$ exists finite, then it is given by

final value of $y(t)$ (steady-state value)

$$y(\infty)=\lim_{s\to 0} sY(s)$$

FVT: If $Y(s)$ has only poles in OLHP, and at most one pole in $s=0$, then $y(\infty)=\lim_{s\to 0} sY(s)$



The hypothesis of FVT can be checked by using Routh criteria

Ex: $Y(s)=\frac{1}{s} \Leftrightarrow y(t)=1\!\!1(t) \Rightarrow y(\infty)=1$ ✓

$$\lim_{s\to 0} sY(s)=\lim_{s\to 0} s\cdot\frac{1}{s}=1$$

$Y(s)=\frac{1}{s+1} \Leftrightarrow y(t)=e^{-t}1\!\!1(t) \Rightarrow y(\infty)=0$

$$\lim_{s\to 0} sY(s)=\lim_{s\to 0} s\cdot\frac{s}{s+1}=0$$

$Y(s)=\frac{1}{s^2+1} \Leftrightarrow y(t)=\sin t\, 1\!\!1(t) \Rightarrow y(\infty)$ does not exists.

can't apply FVT (Note: $\lim_{s\to 0} sY(s)=\lim_{s\to 0}\frac{s}{s^2+1}=0$ WRONG!)

$Y(s)=\frac{1}{s^2} \Leftrightarrow y(t)=t1\!\!1(t). \Rightarrow y(\infty)=\infty$ "blows up"

can't apply FVT

$G(s)=\frac{s+3}{s^3+2s^2+5s+1}$ $\quad U(s)=\frac{1}{s}$ $\quad u(t)=1\!\!1(t)$ $\quad \underset{u(t)}{u}\rightarrow \boxed{G} \rightarrow \underset{y(\infty)=?}{Y}$

Find $y(\infty)$ by using FVT & Routh Critiria

$$Y(s)=\underbrace{\frac{s+3}{s^3+2s^2+5s+1}}_{G}\cdot\underbrace{\frac{1}{s}}_{u} \Rightarrow Y(s) \text{ has one pole in } s=0$$

these poles would be in OLHP

Apply Routh Cri:

s^3	1	5	0
s^2	2	1	0
s	$\frac{9}{2}$		
s^0	1		

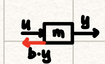
$b_1=-\frac{1}{2}\begin{vmatrix}1 & 5\\ 2 & 1\end{vmatrix}=\frac{9}{2}$

$\rightarrow$ No sign variation $\Rightarrow$ all routes are in OLHP

$\Rightarrow$ can use FVT: $y(\infty)=\lim_{s\to 0} s\cdot Y(s)=3$

Intro to Feedback Control

Consider cruise control problem:



y = velocity $\quad m$ = car mass $\quad b$ = friction coeff $\quad u$ = force

Want to find u s.t. y approaches and maintains a desired velocity y_r ← refer.

model: x = position $\quad u=\dot{x}$

$m\ddot{x} = u - b\dot{x}$ $\quad$ $m\dot{y} = u - by$

$Y(s) = \dfrac{U(s)}{sm+b}$

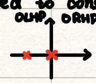
$G(s) = \dfrac{\frac{1}{m}}{s+\frac{b}{m}} = \dfrac{a}{s+b}$, a, b = perameters depend on model & actual plant

Goal: $y(\infty) = y_r$; want: $y(t) \xrightarrow{t\to\infty} y_r$ $\quad$ or: for $e(t) = y_r - y(t)$, want $e(\infty) = 0$

tracking error

Try use FVT to find $u(t)$, assumed to constant: $u(t) = u_0 \cdot \mathbb{1}(t)$, find u_0

Then $Y(s) = \overset{G}{\dfrac{a}{s+b}} \cdot \overset{U}{\dfrac{u_0}{s}}$



can apply FVT: $y(\infty) = \lim_{s\to 0} sY(s) = \dfrac{a \cdot u_0}{b}$

Want $y(\infty) = y_r$

Find $u_0 = \frac{b}{a} y_r$ and $u(t) = \frac{b}{a} y_r \cdot \mathbb{1}(t)$ $\Rightarrow$ open loop control (no feedback)

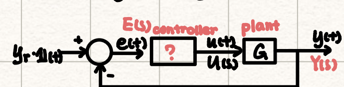
Note: O.L control needs: ① precise knowledge of parameters a, b (mass friction coeff)

② is not robust to imperfect param. knowledge / variation

③ not robust to possible disturbances affecting system (e.g. need higher force if car moves on a slope)

④. cannot affect speed of tracking/convergence to y_r

Using Feedback: $e(t) = y_r \cdot \mathbb{1}(t) - y(t)$



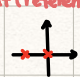
find ? block, s.t. $y(t)$ for the CLOSED-LOOP system satisfies

$y(t) \xrightarrow{t\to\infty} y_r$ or $e(t) \xrightarrow{t\to\infty} 0$

Simplest: try $? = \overset{\text{constant}}{K}$ (proportional control) Find $Y(s)$ and $E(s)$

$\Rightarrow Y(s) = \dfrac{G(s)\cdot k}{1+G(s)k} \cdot \boxed{\dfrac{y_r}{s}}$ input (reference) $\quad$ $[G(s) = \dfrac{a}{s+b}]$

$= \dfrac{ak}{s+b+ak} \cdot \dfrac{y_r}{s}$ $\rightarrow$



$\Rightarrow$ use FVT: $y(\infty) = \lim_{s\to 0} s \cdot Y(s) = \dfrac{ak}{b+ak} \cdot y_r$, $k > 0$.

Note: as $k \gg$ $\Rightarrow$ $y(\infty) \simeq y_r$ (high gain proportional controller)