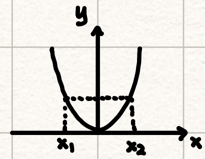
General Transformations



We can extend the previous result to functions with multiple solutions.

In this case, we must sum over all solutions. $f_Y(y) = \frac{f_X(x_1)}{|g'(x_1)|} + \cdots + \frac{f_X(x_n)}{|g'(x_n)|} = \sum_{i=1}^{n} \frac{f_X(x_i)}{|g'(x_i)|}$

Example: Quadratic functions

Let $Y = X^2$ and let X be a standard Gaussian RV. Find the PDF of Y.

$g(x) = y = x^2 \quad g'(x) = 2x$

We have 2 solutions: $x_1 = \sqrt{y}$, $x_2 = -\sqrt{y}$

$$f_Y(y) = \frac{f_X(x_1)}{|g'(x_1)|} + \frac{f_X(x_2)}{|g'(x_2)|} = \frac{f_X(\sqrt{y})}{|2\sqrt{y}|} + \frac{f_X(-\sqrt{y})}{|-2\sqrt{y}|} = \frac{1}{2\sqrt{y}}\left(f_X(\sqrt{y}) + f_X(-\sqrt{y})\right) = \frac{1}{2\sqrt{y}}\left(\frac{1}{\sqrt{2\pi}} e^{-y/2} + \frac{1}{\sqrt{2\pi}} e^{-y/2}\right) = \frac{1}{\sqrt{2\pi y}} e^{-y/2} \quad \text{for } 0 < y < \infty$$

Markov Inequality

Let X be a non-negative RV with mean $\mu = E[X]$

$P(X \geq a) \leq \frac{\mu}{a}$, $a > 0$

Chebyshev Inequality

Let X be a RV with mean μ and variance σ^2

$P(|X - \mu| \geq a) \leq \frac{\sigma^2}{a^2}$, $a > 0$

Example: Manufacturing of resistors

- Some resistors have different values of resistance.
- The avg. resistance is 100 Ohms

If all resistors over 200 ohms will be discarded, what is the max-fraction of resistors to meet such criterion?

Using the Markov inequality, with $\mu = 100$, $a = 200$.

$P(X \geq 200) \leq \frac{100}{200} = 0.5$

The percentage of discarded resistors should not exceed 50% of the total.

Characteristic Functions

The characteristic function (CF) of a random variable x is defined as:

- Continuous RV: $\Phi_X(w) = E[e^{jwx}] = \int_{-\infty}^{\infty} f(x) e^{jwx} dx$
- Discrete RV: $\Phi_X(w) = E[e^{jwx}] = \sum e^{jwx_k} p(x_k)$ where $j = \sqrt{-1}$ and w is a real number

Note that $\Phi_X(w)$ is the Fourier transform of the PDF or PMF of X

Every PDF and its CF form a unique Fourier pair, $\Phi_X(w) \leftrightarrow f_X(x)$

We can recover the PDF of from a CF through the Fourier inversion formula

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_X(w) e^{-jwx}\, dw$$

Notes: By definition: $e^{jw} = \cos(w) + j\sin(w)$

i.e: it is a unit circle with magnitude $|e^{jw}| = 1$

- The magnitude of the CF is maximized at $w=0 \rightarrow |\Phi_X(w)| \leq \Phi_X(0) = 1$