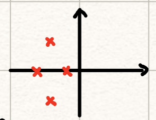
Internal Asy Stability v.s. BIBO Stability

$$\underset{\downarrow}{\text{Internal Asy Stability}} \quad \text{v.s.} \quad \underset{\downarrow}{\text{BIBO Stability}}$$
$$A \qquad\qquad\qquad G(s)$$

$\text{eig}(A) \supseteq \text{poles of } G$



Theorem 3: If all eig(A) are in OLHP then for any B, C, D. the TF $G(s) = C(sI_n - A)^{-1}B + D$ has all poles in OLHP

i.e: int. asy stability $\xRightarrow{\text{implies}}$ BIBO stability

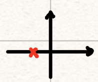
Note: No matter what values for B, C, D, i.e. no matter how input affects system: $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$ and no matter which variable we measure as output, the TF will be BIBO stable

Note: The reverse is NOT true, i.e. If sys is BIBO stable, this does NOT mean that it is int. asy stable. (b/c there could be cancellations between numerator & denominator of $G(s)$)

Ex: $\begin{cases} \dot{x} = \overset{A}{\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}} x + \overset{B}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} u \\ y = \overset{C}{\begin{bmatrix} 0 & 1 \end{bmatrix}} x \end{cases}$ Find TF.

$$G(s) = \underset{C}{\begin{bmatrix} 0 & 1 \end{bmatrix}} \underset{(sI_2-A)^{-1}}{\begin{bmatrix} s-1 & -1 \\ 0 & s+1 \end{bmatrix}^{-1}} \underset{B}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \underbrace{\frac{1}{(s-1)(s+1)}}_{\det(sI_2-A)} \begin{bmatrix} s+1 & 1 \\ 0 & s-1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



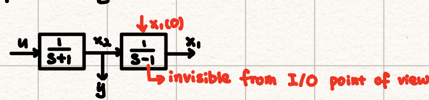
$$= \frac{1}{(s-1)(s+1)} \cdot (s-1) = \frac{1}{s+1} \Rightarrow \text{pole at } p = -1\text{, sys is BIBO stable}$$

Q: is it int asy stable? Check if $\text{eig}(A) \in$ OLHP.

$$\det(\lambda I_2 - A) = 0 \quad \begin{vmatrix} \lambda-1 & -1 \\ 0 & \lambda+1 \end{vmatrix} = 0 \quad (\lambda-1)(\lambda+1) = 0 \Rightarrow \begin{cases} \lambda_1 = 1 \rightarrow \text{ORHP} \\ \lambda_2 = -1 \end{cases} \Rightarrow \text{sys NOT asy. stable (unstable)}$$

Due to that pole/zero cancellation, there is a part of sys. that is NOT "visible" from input to output.

$$\begin{cases} \dot{x}_1 = x_1 + x_2 \\ \dot{x}_2 = -x_2 + u \\ y = x_2 \end{cases} \overset{LT}{\Longleftrightarrow} \begin{cases} (s-1)x_1 = x_2 \Rightarrow x_1 = \frac{x_2}{s-1} \\ (s+1)x_2 = u \Rightarrow y = \frac{u}{s+1} = x_2 \\ y = x_2 \end{cases}$$



The Routh Criteria

an algebraic test to check whether the roots of some polynomial are in OLHP (without explicitly finding them)

Given: $p(s) = s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0$

"1" normalized

$$\begin{array}{lllllll}
s^n & 1 & a_{n-2} & a_{n-4} & \cdots & \cdots & \text{"even" coeff} \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots & \cdots & \text{"odd" coeff} \\
s^{n-2} & b_1 & b_2 & b_3 & \cdots & \cdots & \\
s^{n-3} & c_1 & c_2 & c_3 & \cdots & \cdots & \text{where } b_i, c_i \text{ etc. built as following} \\
\vdots & & & & & & \\
s & & & & & & \\
s^0 & & & & & &
\end{array}$$

$$b_1 = -\frac{1}{a_{n-1}} \begin{vmatrix} 1 & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix} \; ; \quad b_2 = -\frac{1}{a_{n-1}} \begin{vmatrix} 1 & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix} \quad b_3 = -\frac{1}{a_{n-1}} \begin{vmatrix} 1 & a_{n-6} \\ a_{n-1} & a_{n-7} \end{vmatrix}$$

$$c_1 = -\frac{1}{b_1} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix} \; ; \quad c_2 = -\frac{1}{b_2} \begin{vmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{vmatrix} \quad c_3 = -\frac{1}{b_1} \begin{vmatrix} a_{n-1} & a_{n-7} \\ b_1 & b_4 \end{vmatrix}$$

Based on the Routh table: The roots of $p(s)$ are all in OLHP if and only if the first column of the Routh Table/array has NO SIGN variations (in particular no zero value) The # of sign variation is equal to the # of roots in ORHP

Ex: $n=2: s^2 + a_1 s + a_0 = 0 \rightarrow$ quadratic

$$\begin{array}{c|cc} s^2 & 1 & a_0 \\ s & a_1 & 0 \\ s^0 & b_1 & b_2 \end{array} \qquad b_1 = -\frac{1}{a_1}\begin{vmatrix} 1 & a_0 \\ a_1 & 0 \end{vmatrix} = \frac{a_1 a_0}{a_1} = a_0 \Leftrightarrow \begin{cases} a_0 > 0 \\ a_1 > 0 \end{cases}$$

Ex: $p(s) = s^5 + s^4 - s^3 - 2s^2 + s - 2$

$$\begin{array}{c|ccc} s^5 & 1 & -1 & 1 \\ s^4 & 1 & -2 & -2 \\ s^3 & b_1=1 & b_2=3 & 0 \\ s^2 & c_1=-5 & -2 & 0 \\ s & d_1=\frac{13}{5} & 0 & 0 \\ 1 & -2 & 0 & 0 \end{array} \qquad b_1 = -\frac{1}{1}\begin{vmatrix} 1 & -1 \\ 1 & -2 \end{vmatrix} = 1 \quad b_2 = -\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = 3$$

Not all roots are in OLHP. it's not stable and there are 3 ORHP roots