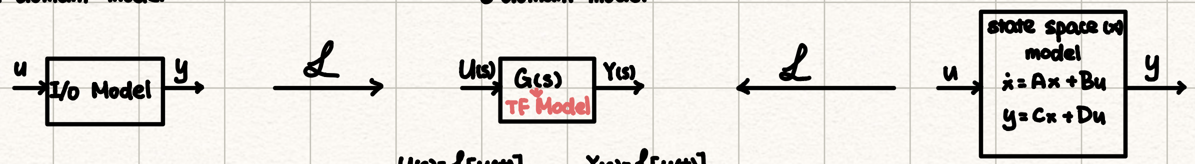
Transfer Function Model

LTI, time domain model — s-domain model



$U(s) = \mathcal{L}[u(t)]$ $\quad$ $Y(s) = \mathcal{L}[y(t)]$

Given LTI, I/O model

① $\underset{\frac{d^n y}{dt^n}}{y^{(n)}} + a_{n-1}y^{(n-1)} + \cdots + a_1\dot{y} + a_0 y = b_m u^{(m)} + \cdots + b_0 u$, where n = order of the model

$\left.\begin{matrix} a_i, i = 0, 1, \cdots n \\ b_j, j = 0, 1, \cdots m \end{matrix}\right\}$ are const. real coeff

0 initial condition

Ass: 0 I.C.: $y(0) = \dot{y}(0) = \cdots = y^{(n-1)}(0) = 0$, $u(0) = \dot{u}(0) = \cdots = u^{(m-1)}(0) = 0$

Take LT on both side of ①, use linearity prop. of LT and derivative prop. & 0 I.C.

$$\underbrace{s^n Y(s)}_{\mathcal{L}[y^{(n)}]} + a_{n-1}\underbrace{s^{n-1}Y(s)}_{\mathcal{L}[y^{(n-1)}]} + \cdots + a_1 sY(s) + a_0 Y(s) = b_m \underbrace{s^m U(s)}_{\mathcal{L}[u^{(m)}]} + \cdots + b_1 \underbrace{sU(s)}_{\mathcal{L}[\dot{u}]} + b_0 U(s)$$

Then isolate $Y(s)$, $U(s)$:

$$[s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0]\, Y(s) = [b_m s^m + \cdots + b_1 s + b_0]\, U(s)$$

$$\Rightarrow \frac{Y(s)}{U(s)} = \underbrace{\frac{b_m s^m + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}}_{\text{T.F model of ①}} \triangleq G(s) = \frac{N(s)}{D(s)}$$

, $N(s), D(s)$ = polynominals in s, $\deg(D(s)) = n \geq \deg(N(s)) = m \Rightarrow$ proper TF

real-rational func. of order n

$$\Rightarrow ② \quad Y(s) = G(s) \overset{\text{multiplication}}{\cdot} U(s) \xrightarrow{\mathcal{L}^{-1}} y(t) = g(t) \overset{\text{convolution}}{*} u(t), \text{ where } g(t) = \mathcal{L}^{-1}[G(s)] = \text{impulse response}$$

Given LTI, state space model

③ $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$; $x \in \mathbb{R}^n$ (state vector), $u \in \mathbb{R}$, $y \in \mathbb{R}$, A.B.C.D = const. matrices of appropriate dimension

Ass: 0 I.C. $x(0) = 0$

Want to obtain a TF model: i.e. $Y(s) = G(s) \cdot U(s)$. What's $G(s)$?

Use the same method:

Take L.T in ③ on both sides: $Y(s) \overset{\text{by linearity of LT}}{=} CX(s) + DU(s)$ $\quad X(s) = \mathcal{L}[x(t)] = \begin{bmatrix} \mathcal{L}[x_1(t)] \\ \vdots \\ \mathcal{L}[x_n(t)] \end{bmatrix} \in \mathbb{R}^n$

deriv. prop. of L.T.

0 I.C.

$s \cdot \mathcal{L}[x(t)] = A\mathcal{L}[x(t)] + BU(s)$

④ $\Rightarrow \begin{cases} sX(s) = AX(s) + BU(s) \\ Y(s) = C\underset{\uparrow}{X(s)} + DU(s) \end{cases}$

intermidiary variable $\Rightarrow$ eliminate $X(s)$ by substitution from 1st equ.

$\Rightarrow (sI_n - A)X(s) = BU(s) \Rightarrow X(s) = (sI_n - A)^{-1} \cdot B \cdot U(s)$

$(n \times n)$

$$\text{Sub in the } 2^{nd} \text{ equ: } Y(s) = [\underbrace{\underbrace{\overset{\text{row vec.}}{C}(sI_n - A)^{-1} \cdot \overset{\substack{(n\times 1)\\ \text{col. vec.}}}{B}}_{\text{scalar}} + \overset{\text{scalar}}{D}}_{G(s)}] \underset{\text{multiplication}}{\cdot} U(s) \quad (5)$$

T.F model of the state space model (5)