Lecture 8
Segment 2

Matrix Algebra

Matrix Algebra

- Important concepts/topics
  - Matrix addition/subtraction
  - Matrix multiplication/inversion
  - Special types of matrices
    - Correlation matrix
    - Variance/covariance matrix

Matrix Algebra

- A matrix is a rectangular table of known or unknown numbers, e.g.,

$$M = \begin{pmatrix} 1 & 2 \\ 5 & 1 \\ 3 & 4 \\ 4 & 2 \end{pmatrix}$$

Matrix Algebra

- The size, or *order*, of a matrix is given by identifying the number of rows and columns, e.g., the order of matrix M is 4x2

$$M = \begin{pmatrix} 1 & 2 \\ 5 & 1 \\ 3 & 4 \\ 4 & 2 \end{pmatrix}$$

Matrix Algebra

- The *transpose* of a matrix is formed by rewriting its rows as columns

$$M = \begin{pmatrix} 1 & 2 \\ 5 & 1 \\ 3 & 4 \\ 4 & 2 \end{pmatrix} \qquad M^T = \begin{pmatrix} 1 & 5 & 3 & 4 \\ 2 & 1 & 4 & 2 \end{pmatrix}$$

Matrix Algebra

- Two matrices may be added or subtracted only if they are of the same order

$$N = \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 1 & 2 \\ 3 & 1 \end{pmatrix}$$

Matrix Algebra

- Two matrices may be added or subtracted only if they are of the same order

$$M + N = \begin{pmatrix} 1 & 2 \\ 5 & 1 \\ 3 & 4 \\ 4 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 1 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 9 & 6 \\ 4 & 6 \\ 7 & 3 \end{pmatrix}$$

Matrix Algebra

- Two matrices may be multiplied when the number of columns in the first matrix is equal to the number of rows in the second matrix. If so, then we say they are conformable for matrix multiplication.

Matrix Algebra

- Matrix multiplication:

$$R = M^T * N \qquad R_{ij} = \sum (M^T_{ik} * N_{kj})$$

Matrix Algebra

$$R = M^T * N = \begin{pmatrix} 1 & 5 & 3 & 4 \\ 2 & 1 & 4 & 2 \end{pmatrix} * \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 1 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 37 & 38 \\ 18 & 21 \end{pmatrix}$$

Matrix Algebra

- A square matrix has the same number of rows as columns

$$D = \begin{pmatrix} 17 & 14 & 5 \\ 13 & 25 & 7 \\ 18 & 32 & 9 \end{pmatrix}$$

Matrix Algebra

- A square symmetric matrix is such that

  $D = D^T$

$$D = \begin{pmatrix} 17 & 13 & 18 \\ 13 & 25 & 32 \\ 18 & 32 & 9 \end{pmatrix}$$

Matrix Algebra

- Diagonal matrices are square matrices with zeroes in all off-diagonal cells

$$D = \begin{pmatrix} 17 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

Matrix Algebra

- The inverse of a matrix is similar to the reciprocal of a scalar
  - e.g., the inverse of 2 is ½ and their product = 1
- Inverses only exist for square matrices and not necessarily for all square matrices

Matrix Algebra

- An inverse is such that $D * D^{-1} = I$ where I is the identity matrix

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrix Algebra

- The determinant of a matrix is a scalar derived from operations on a square matrix. For example, for a 2x2 matrix A the determinant is denoted as |A| and is obtained as follows:

$$|A| = a_{11}*a_{22} - a_{12}*a_{21}$$

Matrix algebra

- A vector is a matrix with only one row or only one column
  - A row vector is a row of vector elements
  - A column vector is a column of vector elements

$$R = \begin{pmatrix} 4 & 7 & 5 & 3 \end{pmatrix}$$

$$C = \begin{pmatrix} 4 \\ 7 \\ 5 \\ 3 \end{pmatrix}$$

Matrix algebra

- In the next ten slides we will go from a raw data matrix to a correlation matrix!

Raw data matrix

Subjects as rows, variables as columns

$$X_{np} = \begin{pmatrix} 3 & 2 & 3 \\ 3 & 2 & 3 \\ 2 & 4 & 4 \\ 4 & 3 & 4 \\ 4 & 4 & 3 \\ 5 & 4 & 3 \\ 2 & 5 & 4 \\ 3 & 3 & 2 \\ 5 & 3 & 4 \\ 3 & 5 & 4 \end{pmatrix}$$

Row vector of sums (totals)

$$T_{1p} = 1_{1n} * X_{np} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} * \begin{pmatrix} 3 & 2 & 3 \\ 3 & 2 & 3 \\ 2 & 4 & 4 \\ 4 & 3 & 4 \\ 4 & 4 & 3 \\ 5 & 4 & 3 \\ 2 & 5 & 4 \\ 3 & 3 & 2 \\ 5 & 3 & 4 \\ 3 & 5 & 4 \end{pmatrix} = \begin{bmatrix} 34 & 35 & 34 \end{bmatrix}$$

Row vector of means

$$M_{1p} = T_{1p} * N^{-1} = \begin{pmatrix} 34 & 35 & 34 \end{pmatrix} * 10^{-1} = \begin{pmatrix} 3.4 & 3.5 & 3.4 \end{pmatrix}$$

Matrix of means

$$M_{np} = 1_{n1} * M_{1p} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} * \begin{pmatrix} 3.4 & 3.5 & 3.4 \end{pmatrix} = \begin{pmatrix} 3.4 & 3.5 & 3.4 \\ 3.4 & 3.5 & 3.4 \\ 3.4 & 3.5 & 3.4 \\ 3.4 & 3.5 & 3.4 \\ 3.4 & 3.5 & 3.4 \\ 3.4 & 3.5 & 3.4 \\ 3.4 & 3.5 & 3.4 \\ 3.4 & 3.5 & 3.4 \\ 3.4 & 3.5 & 3.4 \\ 3.4 & 3.5 & 3.4 \end{pmatrix}$$

Matrix of deviation scores

$$D_{np} = X_{np} - M_{np} = \begin{pmatrix} 3 & 2 & 3 \\ 3 & 2 & 3 \\ 2 & 4 & 4 \\ 4 & 3 & 4 \\ 4 & 4 & 3 \\ 5 & 4 & 3 \\ 2 & 5 & 4 \\ 3 & 3 & 2 \\ 5 & 3 & 4 \\ 3 & 5 & 4 \end{pmatrix} - \begin{pmatrix} 3.4 & 3.5 & 3.4 \\ 3.4 & 3.5 & 3.4 \\ 3.4 & 3.5 & 3.4 \\ 3.4 & 3.5 & 3.4 \\ 3.4 & 3.5 & 3.4 \\ 3.4 & 3.5 & 3.4 \\ 3.4 & 3.5 & 3.4 \\ 3.4 & 3.5 & 3.4 \\ 3.4 & 3.5 & 3.4 \\ 3.4 & 3.5 & 3.4 \end{pmatrix} = \begin{pmatrix} -.4 & -1.5 & -.4 \\ -.4 & -1.5 & -.4 \\ -1.4 & .5 & .6 \\ .6 & -.5 & .6 \\ .6 & .5 & -.4 \\ 1.6 & .5 & -.4 \\ -1.4 & 1.5 & .6 \\ -.4 & -.5 & -1.4 \\ 1.6 & -.5 & .6 \\ -.4 & 1.5 & .6 \end{pmatrix}$$

Sums of squares and Cross-products matrix

$$S_{xx} = D_{pn}^{T} * D_{np} =$$

$$\begin{pmatrix} -.4 & -.4 & -1.4 & .6 & .6 & 1.6 & -1.4 & -.4 & 1.6 & -.4 \\ -1.5 & -1.5 & .5 & -.5 & .5 & .5 & 1.5 & -.5 & -.5 & 1.5 \\ -.4 & -.4 & .6 & .6 & -.4 & -.4 & .6 & -1.4 & .6 & .6 \end{pmatrix} * \begin{pmatrix} -.4 & -1.5 & -.4 \\ -.4 & -1.5 & -.4 \\ -1.4 & .5 & .6 \\ .6 & -.5 & .6 \\ .6 & .5 & -.4 \\ 1.6 & .5 & -.4 \\ -1.4 & 1.5 & .6 \\ -.4 & -.5 & -1.4 \\ 1.6 & -.5 & .6 \\ -.4 & 1.5 & .6 \end{pmatrix}$$

$$= \begin{pmatrix} 10.4 & -2.0 & -.6 \\ -2.0 & 10.5 & 3.0 \\ -.6 & 3.0 & 4.4 \end{pmatrix}$$

Variance-covariance matrix

$$C_{xx} = S_{xx} * N^{-1} = \begin{pmatrix} 10.4 & -2.0 & -.6 \\ -2.0 & 10.5 & 3.0 \\ -.6 & 3.0 & 4.4 \end{pmatrix} * 10^{-1} = \begin{pmatrix} 1.04 & -.20 & -.06 \\ -.20 & 1.05 & .30 \\ -.06 & .30 & .44 \end{pmatrix}$$

Diagonal matrix of standard deviations

$$S_{xx} = (\mathrm{Diag}(C_{xx}))^{1/2} = \begin{pmatrix} 1.02 & 0 & 0 \\ 0 & 1.02 & 0 \\ 0 & 0 & .66 \end{pmatrix}$$

Correlation matrix

$R_{xx} = S_{xx}^{-1} * C_{xx} * S_{xx}^{-1} =$

$$\begin{pmatrix} 1.02^{-1} & 0 & 0 \\ 0 & 1.02^{-1} & 0 \\ 0 & 0 & .66^{-1} \end{pmatrix} * \begin{pmatrix} 1.04 & -.20 & -.06 \\ -.20 & 1.05 & .30 \\ -.06 & .30 & .44 \end{pmatrix} * \begin{pmatrix} 1.02^{-1} & 0 & 0 \\ 0 & 1.02^{-1} & 0 \\ 0 & 0 & .66^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} 1.00 & -.19 & -.09 \\ -.19 & 1.00 & .44 \\ -.09 & .44 & 1.00 \end{pmatrix}$$

Matrix Algebra

- Important concepts/topics
  - Matrix addition/subtraction
  - Matrix multiplication/inversion
  - Special types of matrices
    - Correlation matrix
    - Variance/covariance matrix