18 Continuous Functions

Exercise 18.1. Show that the $\varepsilon - \delta$ definition of continuity in $\mathbb{R}$ implies the open set definition

Solution. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy the $\varepsilon - \delta$ definition of continuity, and let $V \subseteq \mathbb{R}$ be open. Fix $x \in f^{-1}(V)$, and choose ε such that the interval $(f(x)-\varepsilon, f(x)+\varepsilon)$ is contained in V. Then there is some $\delta > 0$ such that

$$f((x-\delta, x+\delta)) \subseteq (f(x)-\varepsilon, f(x)+\varepsilon) \subseteq V,$$

which implies that $(x-\delta, x+\delta) \subseteq f^{-1}(V)$. Since x was arbitrary, $f^{-1}(V)$ is open. $\square$

Exercise 18.2. Suppose that $f : X \to Y$ is continuous and $A \subseteq X$. If x is a limit point of A, is $f(x)$ a limit point of $f(A)$?

Solution. No, f could be constant for example. $\square$

Exercise 18.3. Let X and X' denote the same set in the topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively. Let $i : X' \to X$ be the identity function.

(a) Show that i is continuous $\iff$ $\mathcal{T}'$ is finer than $\mathcal{T}$.

(b) Show that i is a homeomorphism $\iff$ $\mathcal{T}' = \mathcal{T}$.

Solution. (a) If i is continuous and U is open in $\mathcal{T}$, then $i^{-1}(U) = U$ is open in $\mathcal{T}'$. Conversely, if $\mathcal{T}'$ is finer than $\mathcal{T}$ and U is open in $\mathcal{T}$, then $i^{-1}(U) = U$ is open in $\mathcal{T}'$.

(b) i and i^{-1} are both identity functions. $\square$

Exercise 18.4. Given $y_0 \in Y$, show that the map $f : X \to X \times Y$ defined by $f(x) = x \times y_0$ is an embedding.

Solution. f is clearly continuous, as it's the product of the identity function with a constant function. Also, it's clear that f is injective with range $X \times \{y_0\}$.

If U is open in X, then $f(U) = U \times \{y_0\}$ is open in $X \times \{y_0\}$, thus f^{-1} is continuous. Therefore, f is an imbedding. $\square$

Exercise 18.5. Show that the subspace (a, b) of $\mathbb{R}$ is homeomorphic with $(0, 1)$, and the subspace $[a, b]$ is homeomorphic with $[0, 1]$.

Solution. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = \frac{x-a}{b-a}$, which is continuous and has a continuous inverse. Then f restricted to the domains (a, b) or $[a, b]$ are both continuous and invertible, so we have the result. $\square$

Exercise 18.6. Find a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous at precisely one point.

Solution. Define f such that

$$f(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

f is only continuous at $x = 0$. $\square$

Exercise 18.7. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous from the right, that is,

$$\lim_{x \to a^+} f(x) = f(a)$$

for each $a \in \mathbb{R}$. Show that f is continuous when considered as a function from $\mathbb{R} \to \mathbb{R}_\ell$.

Solution. Let $[a, b)$ be an open basis set of $\mathbb{R}_\ell$, and let $x_0 \in f^{-1}([a, b))$. Choose some $\varepsilon > 0$ such that $[f(x_0), f(x_0) + \varepsilon) \subseteq [a, b)$. Then there is some $\delta > 0$ such that

$$f((x - \delta, x + \delta)) \subseteq [f(x_0), f(x_0) + \epsilon) \subseteq [a, b).$$

Then $(x - \delta, x + \delta) \in f^{-1}([a, b))$, therefore $f^{-1}([a, b))$ is open and $f : \mathbb{R} \to \mathbb{R}_\ell$ is continuous. $\square$

Exercise 18.8. Let Y be an ordered set in the order topology. Let $f, g : X \to Y$ be continuous.

(a) Show that $\{x \mid f(x) \leq g(x)\}$ is closed in X.

(b) Show that $h(x) = \min(f(x), g(x))$ is continuous.

Solution. (a) Suppose that x is chosen so that $f(x) > g(x)$. We must find a neighborhood of x who's points also satisfy this inequality. Since order topologies are Hausdorff, we can choose two disjoint intervals such that $f(x) \in (a, b)$ and $g(x) \in (c, d)$. Then $f^{-1}(a, b) \cap g^{-1}(c, d)$ is open, contains x, and all points z in this set satisfy $f(z) > g(z)$. Therefore, the complement set $\{x \mid f(x) \leq g(x)\}$ is closed.

(b) We have

$$h(x) = \begin{cases} f(x) & f(x) \leq g(x) \\ g(x) & g(x) \leq f(x) \end{cases}$$

Since the domains in both cases are closed by (a), h is continuous by the pasting lemma. $\square$

Exercise 18.9. Let $\{A_\alpha\}$ be a collection of subsets of X; let $X = \bigcup A_\alpha$. Let $f : X \to Y$; suppose $f|_{A_\alpha}$ is continuous for each α.

(a) If $\{A_\alpha\}$ is finite and each A_α is closed, then f is continuous.

(b) Find an example where the collection $\{A_\alpha\}$ is countable and each A_α is closed, but f is not continuous.

(c) $\{A_\alpha\}$ is said to be locally finite if each $x \in X$ has a neighborhood that intersects only finitely many A_α. Show that if $\{A_\alpha\}$ is locally finite and each A_α is closed, then f is continuous.

Solution. (a) This is true by induction and the pasting lemma.

(b) Define $A_0 = \{0\}$ and $A_n = [1/(n+1), 1/n]$ for each $n \in \mathbb{N}$. The union is $[0,1]$. Define $f : [0,1] \to \mathbb{R}$ so that $f(x)$ is 1 if $x = 0$, and 0 otherwise. Then f is not continuous, but it is continuous when restricted to each A_n.

The key here was to choose the A_n so that some of the closed sets have a nonclosed union, and then have another set include the limit point.

(c) We will show that the arbitrary union of a locally finite collection of closed sets $\{A_\alpha\}$ is closed. Suppose that $x \notin \bigcup A_\alpha$. By local finiteness, there is some neighborhood U containing x which intersects only finitely many A_α; call these intersections $A_1, \ldots, A_n$. The union of these finitely many closed sets is closed, so there is a neighborhood V of x which intersects none of them. Now $U \cap V$ is a neighborhood of x disjoint from $\bigcup A_\alpha$, thus the union is closed.

Following the proof of the pasting lemma, for each closed set C we have

$$f^{-1}(C) = \bigcup (f|_{A_\alpha})^{-1}(C),$$

which is the union of locally finite closed sets, and thus closed. □

Exercise 18.10. Let $f : A \to B$ and $g : C \to D$ be continuous functions. Let us define a map $f \times g : A \times C \to B \times D$ by the equation

$$(f \times g)(a \times c) = (f(a) \times g(c)).$$

Show that $f \times g$ is continuous.

Solution. Let $U \times V \subseteq B \times D$ be open. Then we have

$$\begin{aligned}(f \times g)^{-1}(U \times V) &= \{a \times c \mid f(a) \in U \text{ and } g(c) \in V\} \\ &= f^{-1}(U) \times g^{-1}(V),\end{aligned}$$

which is open because f and g are continuous. □

Exercise 18.11. Let $F : X \times Y \to Z$. We say that F is continuous in each variable separately if, for each $y_0 \in Y$, the map $h : X \to Z$ defined by $h(x) = F(x, y_0)$ is continuous, and similarly with the other variable. Show that if F is continuous, then it's continuous in each variable separately.

Solution. Fix $y_0 \in Y$, and define $h(x) = F(x, y_0)$. Let U be an open set in Z. Then

$$h^{-1}(U) = \{x \in X \mid F(x, y_0) \in U\}$$

Let $x \in h^{-1}(U)$, so that $F(x, y_0) \in U$. Then we have $(x, y_0) \in F^{-1}(U)$. Since F is continuous, let $A \times B$ be a neighborhood of (x, y_0) contained in $F^{-1}(U)$. Then

$$\begin{aligned}h(A) = F(A, y_0) &\subseteq F(A \times B) \subseteq U \\ A &\subseteq h^{-1}(U),\end{aligned}$$

so $h^{-1}(U)$ is open and h is continuous. □

Exercise 18.12. Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by the equation

$$F(x, y) = \begin{cases} xy/(x^2 + y^2) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Show that f is continuous in each variable, but not continuous.

Solution. Fix $y_0 \in \mathbb{R}$. We have $F(x, 0) = 0$ for all x, so let's assume $y_0 \neq 0$. Then

$$h(x) = F(x, y_0) = \frac{xy_0}{x^2 + y_0^2},$$

which is continuous by real analysis.

Now consider the function $h(x) = F(x, x)$. We have $h(0) = 0$, but $h(x) = 1/2$ for any nonzero x, so h is not continuous at $x = 0$. Since $\delta : x \mapsto (x, x)$ is continuous and $h = F \circ \delta$, F cannot be continuous. □

Exercise 18.13. Let $A \subseteq X$; let $f : A \to Y$ be continuous; let Y be Hausdorff. Show that if f may be extended to a continuous function $g : \overline{A} \to Y$, then g is uniquely determined by f.

Solution. Suppose there existed two continuous extensions $g_1, g_2 : \overline{A} \to Y$, and let x be any point in $\overline{A} - A$. Assume for a contradiction that $g_1(x) \neq g_2(x)$. Then because Y is Hausdorff, we can choose disjoint open sets $U, V \subseteq Y$ such that $g_1(x) \in U$ and $g_2(x) \in V$. We have

$$x \in g_1^{-1}(U) \cap g_2^{-1}(V),$$

so this intersection is open and nonempty, thus it intersects A. Choosing a so that

$$a \in A \cap g_1^{-1}(U) \cap g_2^{-1}(V),$$

we have $g_1(a) = g_2(a) = f(a) \in U \cap V$, contradicting that U and V are disjoint. Therefore, $g_1 = g_2$. □
