27 Compact Subspaces of the Real Line

Exercise 27.1. Prove that if X is an ordered set in which every closed interval is compact, then X has the least upper bound property.

Solution. Suppose for a contradiction that S is a nonempty bounded subset of X with no least upper bound. Then the set B of upper bounds of S is nonempty. Let $[a, b]$ be a closed interval containing S. We will show that $[a, b]$ is not compact.

The collection

$$\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 = \{(-\infty, x) \mid x \in S\} \cup \{(x, \infty) \mid x \in B\}$$

is an open cover of X because S has no least upper bound (and no largest element). There is no finite subcover of $[a, b]$ because $\mathcal{A}_1$ and $\mathcal{A}_2$ are disjoint and have no largest interval. $\square$

Exercise 27.2. Let X be a metric space with metric d; let $A \subseteq X$ be nonempty.

(a) Show that $d(x, A) = 0$ if and only if $x \in \bar{A}$.

(b) Show that if A is compact, then $d(x, A) = d(x, a)$ for some $a \in A$.

(c) Define the ϵ-neighborhood of A in X to be the set

$$U(A, \epsilon) = \{x \mid d(x, A) < \epsilon\}$$

Show that $U(A, \epsilon)$ is the union of the open balls $B_d(a, \epsilon)$ for $a \in A$.

(d) Assume A is compact. Let U be an open set containing A. Show that some ϵ-neighborhood of A is contained in U.

(e) Show that the result in (d) need not hold if A is closed but not compact.

Solution. (a) Assume that $d(x, A) = \inf_{a \in A} d(x, a) = 0$. Let $B_d(x, \epsilon)$ be a neighborhood of x. By hypothesis, there is some element $a \in A$ such that $d(x, a) < \epsilon$, so we have $x \in \bar{A}$ since ϵ was arbitrary.

If $d(x, A) > 0$, then the ball $B_d(x, d(x, A))$ does not intersect A, thus $x \notin \bar{A}$.

(b) If A is compact, then the function $a \mapsto d(x, a)$ has a lowest output by the extreme value theorem.

(c) If $x \in U(A, \epsilon)$, then $d(x, A) < \epsilon$, which implies $d(x, a) < \epsilon$ for some $a \in A$. The same argument applies in reverse.

(d) Let $\mathcal{A} = \{B(x, \epsilon_x) \mid x \in A\}$, where ϵ_x is chosen for each $x \in A$ so that the larger ball $B(x, 2\epsilon)$ lies inside U. Let $\mathcal{B}$ be a finite subcover consisting of

$$\{B(x_k, \epsilon_{x_k}) \mid 1 \leq k \leq n\}.$$

Let ϵ be the smallest of the ϵ_k for $1 \leq k \leq n$. For any $a \in B(x_k, \epsilon_{x_k})$, we have

$$B(a, \epsilon) \subseteq B(a, \epsilon_k) \subseteq B(x_k, 2\epsilon_{x_k}) \subseteq U,$$

so the set $B(A, \epsilon)$ is contained in U.

(e) In $\mathbb{R}^2$, let A be the horizontal line $x = 0$, which is closed. Define

$$U = \left\{ x \times y \mid |y| < \frac{1}{1 + x^2} \right\}.$$

This set is open because it's the inverse image of $(0, \infty)$ under the continuous function $x \times y \mapsto \frac{1}{1+x^2} - |y|$. This is a counterexample because $\frac{1}{1+x^2} \to 0$ as x gets large. $\square$

Exercise 27.3. Recall that $\mathbb{R}_K$ denotes $\mathbb{R}$ in the K-topology.

(a) Show that $[0, 1]$ is not compact as a subspace of $\mathbb{R}_K$.

(b) Show that $\mathbb{R}_K$ is connected.

(c) Show that $\mathbb{R}_K$ is not path connected.

Solution. (a) Define an open cover with the sets

$$\{\mathbb{R} - K, (2/3, 2/1), (2/5, 2/3), (2/7, 2/5), \dots\}.$$

(b) Following the hint, note that $(-\infty, 0)$ and $(0, \infty)$ inherit the usual topology on $\mathbb{R}$, meaning they are connected. The set $\{0\}$ is not open in either $[0, \infty)$ or $(-\infty, 0]$, so both of these intervals are connected, and so the entire interval is connected.

(c) Suppose that there were a path $p : [0, 1] \to \mathbb{R}_K$ from 0 to 1. Since $\mathbb{R}_K$ is finer than $\mathbb{R}$, the same function $p : [0, 1] \to \mathbb{R}$ is also continuous, so the intermediate value theorem applies; we have $[0, 1] \subseteq p([0, 1])$. On the other hand, we know that $p([0, 1])$ is compact in K, so since $[0, 1]$ is a closed subset of $p([0, 1])$, it is also compact. This contradicts the result of (a). $\square$

Exercise 27.4. Show that a connected metric space having more than one point is uncountable.

Solution. Let X be a connected metric space with at least two points. Then $X \times X$ is also connected, and so $d(X \times X)$ is connected in $\mathbb{R}$. Since X has at least two points, there is some $b > 0$ such that $[0, b] \in d(X \times X)$. At this point, $X \times X$ is clearly uncountable, and so is X by basic set theory. $\square$

Exercise 27.5. Let X be a compact Hausdorff space; let $\{A_n\}$ be a countable collection of closed sets of X. Show that if each A_n has empty interior in X, then the union $\bigcup A_n$ has empty interior in X.

Solution. Let $x \in \bigcup A_n$; let U be a neighborhood of x. We will show that U is not contained in $\bigcup A_n$.

For each $n \in \mathbb{N}$, choose some $x_n \in A_n$.

Let $U_0 = U$, and inductively for each $n \in \mathbb{N}$, choose a neighborhood $U_n \subseteq U_{n-1}$ such that $\bar{U}_n$ doesn't contain x_n. $\square$