19 The Product Space

Exercise 19.1. Suppose the topology on each space X_α is given a basis $\mathcal{B}_\alpha$. The collection of all sets

$$\prod_{\alpha \in J} B_\alpha,$$

where $B_\alpha \in \mathcal{B}_\alpha$ for each α, is a basis for the box topology.

The collection of sets of the same form, where $B_\alpha \in \mathcal{B}_\alpha$ for finitely many α and $B_\alpha = X_\alpha$ for the remaining indices, is a basis of the product topology.

Solution. Let $\prod U_\alpha$ be a basis element of the box topology. Then for each $(x_\alpha)_{\alpha \in J} \in \prod U_\alpha$, there are basis elements $B_\alpha \in \mathcal{B}_\alpha$ such that $x_\alpha \in B_\alpha \subseteq U_\alpha$ for each α. Thus,

$$(x_\alpha)_{\alpha \in J} \in \prod B_\alpha \subseteq \prod U_\alpha,$$

which shows that these products of basis elements form a basis for the box topology.

A similar argument applies with the product topology. □

Exercise 19.2. Let A_α be a subspace of X_α for each α. Then $\prod A_\alpha$ is a subspace of $\prod X_\alpha$ in either the box or product topologies.

Solution. Suppose $\prod U_\alpha$ is a basis element of $\prod A_\alpha$ in the box topology. We can write

$$\prod U_\alpha = \prod (A_\alpha \cap V_\alpha) = \prod A_\alpha \cap \prod V_\alpha,$$

where each V_α is open in X_α. This is just $\prod A_\alpha$ intersected with some basis element of $\prod X_\alpha$, so the subspace relationship holds. A similar argument applies with the product topology. □

Exercise 19.3. If each X_α is a Hausdorff space, then $\prod X_\alpha$ is Hausdorff in both the product and box topologies.

Solution. Since the box topology is finer than the product topology, it suffices to only consider the product topology.

Suppose $(x_\alpha)_{\alpha \in J}$ and $(y_\alpha)_{\alpha \in J}$ are distinct points in $\prod X_\alpha$. Then there is some $\kappa \in J$ such that $x_\kappa \neq y_\kappa$. Since X_κ is Hausdorff, we can choose nonintersecting neighborhoods U and V contain x_κ and y_κ respectively. Then

$$(x_\alpha)_{\alpha \in J} \in \pi_\kappa^{-1}(U) \qquad \text{and} \qquad (y_\alpha)_{\alpha \in J} \in \pi_\kappa^{-1}(V),$$

and the inverse images are disjoint neighborhoods, thus the product space is Hausdorff. □

Exercise 19.4. Show that $(X_1 \times \cdots \times X_{n-1}) \times X_n$ is homeomorphic to $X_1 \times \cdots \times X_n$.

Solution. Let $f((x_1, \dots, x_{n-1}), x_n) = (x_1, \dots, x_n)$ be our candidate for a homeomorphism. If $U = U_1 \times \cdots \times U_n$ is open in the codomain, then

$$f^{-1}(U) = (U_1 \times \cdots \times U_{n-1}) \times U_n,$$

which is open in the domain. Conversely, if $U = (U_1 \times \cdots \times U_{n-1}) \times U_n$ is open in the domain, then

$$f(U) = U_1 \times \cdots \times U_n$$

is open in the codomain. Therefore, f is a homeomorphism. $\square$

Exercise 19.5. One of the implications in Theorem 19.6 holds for the box topology. Which is it?

Solution. Firstly, it's easy to check that projections are continuous in the box topology. Therefore, if $f : A \to \prod X_\alpha$ is continuous, then each composition $\pi_\alpha \circ f$ is continuous. $\square$

Exercise 19.6. Let $x_1, x_2, \dots$ be a sequence of points in the product space $\prod X_\alpha$. Show that this sequence converges to the point x if and only if the sequence $\pi_\alpha(x_1), \pi_\alpha(x_2), \dots$ converges to $\pi_\alpha(x)$ for each α. Is this fact true if one uses the box topology?

Solution. Suppose that $x_1, x_2, \dots$ converges to x, and fix an index β. Let $V \subseteq X_\alpha$ be a neighborhood of $\pi_\alpha(x)$. Then since π_α is continuous, we have $\pi_\alpha^{-1}(V)$ is a neighborhood of x, and thus contains almost every x_n. This means that $\pi_\alpha(x_n) \in V$ for almost all n, so that $\pi_\alpha(x_1), \pi_\alpha(x_2), \dots$ converges to $\pi_\alpha(x)$. This part works fine in the box topology.

Conversely, suppose that $\pi_\alpha(x_1), \pi_\alpha(x_2), \dots$ converges to $\pi_\alpha(x)$ for each α. Let $\prod U_\alpha$ be a basis element of $\prod X_\alpha$ containing x. In the **product** topology, we have $U_\alpha = X_\alpha$ for all but finitely α. Now the set of n such that $x_n \notin U$ is

$$\bigcup \{n \mid \pi_\alpha(x_n) \notin U_\alpha\},$$

which is the finite union of finite sets, and thus finite. So in the product topology, $(x_n) \to x$.

In the box topology, the union may not be finite, so this argument doesn't work. A counterexample is the set $X = \mathbb{R}^\omega$ in the box topology, and $x_n = (1/n, 1/n, \dots)$. We have $(\pi_\alpha(x_n)) \to 0$ for each α, but the open set $\prod(-1/n, 1/n)$ actually contains no x_n! $\square$

Exercise 19.7. Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ who's sequences are those with only finitely many nonzero terms. What is the closure of $\mathbb{R}^\infty$ in the product and box topologies?

Solution. I claim that $\mathbb{R}^\infty$ is closed in the box topology. If a sequence x_n has infinitely many nonzero terms, we can construct open sets around these terms in $\mathbb{R}$ which all exclude 0. Any sequence within these open sets must then have infinitely many nonzero terms, and thus is not in $\mathbb{R}^\infty$.

On the other hand, I claim that the closure of $\mathbb{R}^\infty$ is $\mathbb{R}^\omega$ in the product topology. Then if x_n is a sequence, when we construct open sets around each term, almost all of them must be all of $\mathbb{R}$ and contain 0. We can therefore create a new sequence y_n which is zero almost everywhere and intersects the neighborhood of x_n. $\square$

Exercise 19.8. Given sequences $(a_1, a_2, \dots)$ and $(b_1, b_2, \dots)$ of real numbers with $a_i > 0$ for all i, define $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ by the equation

$$h(x_1, x_2, \dots) = (a_1 x_1 + b_1, a_2 x_2 + b_2, \dots).$$

Show that h is a homeomorphism in the product topology. What about the box topology?

Solution. Let $h_i(x) = a_i x + b_i$ for each i. In the product topology, we have h is continuous if and only if each $\pi_i \circ h$ is continuous. But we have

$$\pi_i \circ h = h_i \circ \pi_i,$$

and both h_i and π_i are continuous, therefore h is continuous in the product topology.

In the box topology, let $U = U_1 \times U_2 \times \dots$ be a basis element. Then we have

$$h^{-1}(U) = h_1^{-1}(U_1) \times h_2^{-1}(U_2) \times \dots,$$

which is also open, so h is continuous in the box topology.

Similarly, h^{-1} has the same form and is continuous in the product and box topologies, thus h is a homeomorphism in both topologies. In fact, we only need each h_i to be a homeomorphism for this to work. $\square$

Exercise 19.9. Show that the choice axiom is equivalent to the statement that for each family $\{A_\alpha\}_{\alpha \in J}$ of nonempty sets, with $J \neq \emptyset$, the cartesian product

$$\prod_{\alpha \in J} A_\alpha$$

is nonempty.

Solution. Suppose the axiom of choice is true, and let $\{A_\alpha\}$ be a collection of sets. Then the collection $\{A'_\alpha\} = \{\{\alpha\} \times A_\alpha\}_{\alpha \in J}$ is pairwise disjoint, so by the axiom of choice, choose C containing one element of each A'_α. Then C is actually a function $J \to \bigcup A_\alpha$ such that $C(\alpha) \in A_\alpha$ for each α. In other words, C is an element of the product.

Conversely, suppose that products are nonempty, and let $\{A_\alpha\}_{\alpha \in J}$ be a collection of disjoint sets. Choose some $f \in \prod_{\alpha \in J} A_\alpha$. Define C by the equation

$$C = \{f(\alpha) \mid \alpha \in J\}.$$

This set satisfies the requirements for the axiom of choice. $\square$

Exercise 19.10. Let A be a set; let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of spaces; and let $\{f_\alpha\}_{\alpha \in J}$ be an indexed family of functions $f_\alpha : A \to X_\alpha$.

(a) Show that there is a unique coarsest topology $\mathcal{T}$ on A relative to which each of the functions f_α is continuous.

(b) Redundant

(c) Show that a map $g : Y \to A$ is continuous relative to $\mathcal{T}$ if and only if each map $f_\alpha \circ g$ is continuous.

(d) Let $f : A \to \prod X_\alpha$ be defined by the equation

$$f(a) = (f_\alpha(a))_{\alpha \in J};$$

let Z denote the subspace $f(A)$ of the product space X_α. Show that the image under f of each set in $\mathcal{T}$ is open in Z.

Solution. (a,b) Define $\mathcal{T}$ by the subbasis

$$\{f_\alpha^{-1}(U) \mid \alpha \in J \text{ and } U \text{ open in } X_\alpha\}.$$

(c) Suppose that $g : Y \to A$ and each map $f_\alpha \circ g$ is continuous. Let $f_\alpha^{-1}(U)$ be a subbasis element of $\mathcal{T}$. Then

$$g^{-1}(f_\alpha^{-1}(U)) = (f_\alpha \circ g)^{-1}(U),$$

which is also open, thus g is continuous.

(d) Let U be open in A, and let $\mathbf{x} \in f(U)$, so that we can choose $a \in U$ where $\mathbf{x} = f(a)$. Using the basis of $\mathcal{T}$ derived from the subbasis, there is a basis set W such that

$$a \in W = \bigcap_{k=1}^{n} f_{\alpha_k}^{-1}(V_k) \subseteq U$$

for some indices $\alpha_1, \dots, \alpha_n$ and some open sets $V_1, \dots, V_n$. Then we have

$$\mathbf{x} \in \bigcap (Z \cap V_k) = \bigcap f(f_{\alpha_k}^{-1}(V_k)) \subseteq f(W) \subseteq f(U),$$

which shows that $f(U)$ is open because the finite intersection $\bigcap(Z \cap V_k)$ is open and contained in $f(U)$. $\square$
