13 Topological Spaces / Bases

Exercise 13.1. Let X be a topological space; let A be a subset of X. Suppose that for each $x \in A$ there is an open set U containing x such that $U \subseteq A$. Show that A is open in X.

Proof. For each x, let U_x be an open set containing x such that $U_x \subseteq A$. (Note that the axiom of choice isn't required, as we can let U_x be the largest such open set.) Then I claim that

$$A = \bigcup_{x \in A} U_x.$$

If $x \in A$, then $x \in U_x$ by definition. The reverse inclusion holds since we have $U_x \subseteq A$ for each x. Now A is the union of open sets, and so A is open. $\square$

Exercise 13.2. Consider the nine topologies on the set $X = \{a, b, c\}$ indicated in Example 1 of Section 12. Compare them, that is, for each pair of topologies, determine whether they are comparable, and if so, which is the finer.

Solution. Skipped, as this exercise is trivial. $\square$

Exercise 13.3. Show that $\mathcal{T}_c$, the *countable complement topology*, is a topology on a given set X. Is the collection

$$\mathcal{T}_\infty = \{U \mid X - U \text{ is infinite or empty}\}$$

a topology?

Solution. Clearly $\emptyset$ and X are open in $\mathcal{T}_c$. If (U_α) is a collection of open sets, then

$$X - \bigcup_\alpha U_\alpha = \bigcap_\alpha (X - U_\alpha),$$

which is a subset of some countable $X - U_\alpha$, and is therefore countable. The exception is when $X - U_\alpha = X$ for every α, in which case the intersection is X, which is allowed by our definition.

Let (U_j) be a finite collection of n open sets. Then

$$X - \bigcap_{j=1}^{n} U_j = \bigcup_{j=1}^{n} (X - U_j).$$

A finite union of countable sets is countable, and if $X - U_j = X$ for some j, then the union is X.

Since arbitrary unions and finite intersections of open sets are open, $\mathcal{T}_c$ is a topology.

Consider $\mathcal{T}_\infty$ on the set $\mathbb{N}$. Then $\{2, 4, 6, \dots\}$ and $\{3, 5, 7, \dots\}$ are both open, but their union has complement $\{1\}$ and is not open, so $\mathcal{T}_\infty$ is not a topology. $\square$

Exercise 13.4. (a) If $\{\mathcal{T}_\alpha\}$ is a collection of topologies on X, then $\bigcap T_\alpha$ is a topology. Is $\bigcup T_\alpha$ a topology?

(b) Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on X. Show that there is a smallest topology which contains every $\mathcal{T}_\alpha$ (their least upper bound).

(c) If $X = \{a, b, c\}$, let

$$\mathcal{T}_1 = \{\emptyset, X, \{a\}, \{a, b\}\} \qquad \text{and} \qquad \mathcal{T}_2 = \{\emptyset, X, \{a\}, \{b, c\}\}.$$

Find the intersection and least upper bound topologies

Solution. (a) Every $\mathcal{T}_\alpha$ has both $\emptyset$ and X open, so the intersection has these. Now consider a family of open sets $\{U_\beta\}$ in $\bigcap T_\alpha$. Then they are in each $\mathcal{T}_\alpha$, thus their union is also in the intersection topology. A similar argument applies to finite intersection.

Let X have at least 3 elements and let $\mathcal{T}_1 = \{\emptyset, \{a\}, X\}$ and $\mathcal{T}_2 = \{\emptyset, \{b\}, X\}$. Then $\mathcal{T}_1 \cup \mathcal{T}_2$ is not a topology.

(b) Note that the discrete topology contains *every* topology, so such topologies exist. Then we can let $\mathcal{T}_{\text{sup}}$ be the intersection of all these by part (a).

(c) The intersection is $\{\emptyset, X, \{A\}\}$, and the least upper bound is

$$\{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$$

□

Exercise 13.5. Show that if $\mathcal{A}$ is a basis for a topology on X, then the topology generated by $\mathcal{A}$ is the intersection of the topologies containing $\mathcal{A}$. Prove the same if $\mathcal{A}$ is a subbasis

Solution. Suppose $\mathcal{A}$ is a basis of $\mathcal{T}$. Let $\mathcal{T}'$ be any topology containing all of $\mathcal{A}$. Then $\mathcal{T}'$ contains every possible union of $\mathcal{A}$, which is just $\mathcal{T}$.

Now suppose $\mathcal{A}$ is instead a subbasis generating $\mathcal{T}$. Again let $\mathcal{T}'$ be a topology containing all of $\mathcal{A}$. Then $\mathcal{T}'$ contains every combination of unions and finite intersections of sets in $\mathcal{A}$, so $\mathcal{T} \subseteq \mathcal{T}'$. □

Exercise 13.6. Show that the topologies of $\mathcal{R}_\ell$ and $\mathbb{R}_K$ are incomparable.

Solution. No basis element of $\mathbb{R}_K$ lies inside $[-1, 1)$ and contains -1.

No basis element of $\mathbb{R}_\ell$ lies inside $(-1, 1) - K$ and contains 0 by the Archimedean principle. □

Exercise 13.7. Consider the following topologies on $\mathbb{R}$:

$\mathcal{T}_1$ = the standard topology

$\mathcal{T}_2 = \mathbb{R}_K$

$\mathcal{T}_3$ = the finite complement topology

$\mathcal{T}_4$ = the upper limit topology

$\mathcal{T}_5 =$ the topology having all sets $(-\infty, a)$ as a basis.

Now compare them all

Solution. ($\mathcal{T}_3 \subset \mathcal{T}_1 \subset \mathcal{T}_2, \mathcal{T}_4$) We've seen that $\mathcal{T}_1 \subsetneq \mathcal{T}_2, \mathcal{T}_4$ in the textbook. Let U be open in $\mathcal{T}_3$. If U is empty, we're done. Otherwise, choose $x \in U$. Since there are only finitely many gaps in U, we can find the minimum distance ε from x to a gap. Then $(x-\varepsilon, x+\varepsilon)$ is an open subset of U in $\mathcal{T}_1$ and contains x. Since x was arbitrary, U is open in $\mathcal{T}_1$.

($\mathcal{T}_3 \perp \mathcal{T}_5$) We have $\mathbb{R} - \{0\}$ is open in $\mathcal{T}_3$ but not $\mathcal{T}_5$. Also, $(-infty, 0)$ is open in $\mathcal{T}_5$, but not $\mathcal{T}_3$.

($\mathcal{T}_5 \subset \mathcal{T}_1$) Every basis element $(-\infty, a)$ of $\mathcal{T}_5$ is also open in $\mathcal{T}_1$. On the other hand, $(0, 1)$ is open in $\mathcal{T}_1$, but not $\mathcal{T}_5$.

($\mathcal{T}_2 \perp \mathcal{T}_4$) This argument is similar to Exercise 6.

These comparisons are sufficient. □

Exercise 13.8. (a) Show that the countable collection

$$\mathcal{B} = \{(a, b) \mid a < b \text{ and } a, b \in \mathbb{Q}\}$$

is a basis which generates the standard topology on $\mathbb{R}$.

(b) Show that the collection

$$\mathcal{B} = \{[a, b) \mid a < b \text{ and } a, b \in \mathbb{Q}\}$$

doesn't generate the lower limit topology on $\mathbb{R}$.

Solution. (a) Clearly every element of $\mathcal{B}$ is open in the standard topology. Now let (x, y) be a basis set of the standard topology, and choose z in the interval. By density of $\mathbb{Q}$, there are rational numbers a and b such that $x < a < z < b < y$, so that z is in the basis set $(a, b) \in \mathcal{B}$.

(b) If a is irrational, then $[a, \infty)$ is open in $\mathbb{R}_\ell$, but no element of $\mathcal{B}$ contained inside $[a, \infty)$ has a in it. In fact, $\mathbb{R}_\ell$ has no countable basis. □