# 23 Connected Spaces

**Exercise 23.1.** Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on $X$. If $\mathcal{T}' \supseteq \mathcal{T}$, what does connectedness in one topology imply about connectedness in the other?

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*Solution.* The statement of $X$ being connected is a universal quantifier over open sets, which means that if $X$ is connected in $\mathcal{T}'$ and $\mathcal{T}' \supseteq \mathcal{T}$, then $X$ is also connected in $\mathcal{T}$.

This applies for any statement which is a universal quantifier like this. If it were an existential quantifier, then the converse would hold instead. If it were a mix, then we can't use this method. $\square$

**Exercise 23.2.** Let $A_n$ be a sequence of connected subspaces of $X$ such that $A_n \cap A_{n+1}$ is nonempty for each $n$. Show that $\bigcup A_n$ is connected.

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*Solution.* Let $B_n = \bigcup_{k=1}^{n} A_k$. Each individual $B_n$ is connected by an inductive argument, so we have

$$\bigcup A_n = \bigcup B_n$$

is connected since all of the $B_n$ have some $a \in A_1$ in common. $\square$

**Exercise 23.3.** Let $\{A_\alpha\}$ be a collection of connected subspaces of $X$; let $A$ be a connected subspace of $X$. Show that if $A \cap A_\alpha \neq \emptyset$ for all $\alpha$, then $A \cup \bigcup A_\alpha$ is connected.

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*Solution.* Suppose $A \cup \bigcup A_\alpha$ is separated as $U \cup V$. Assume WLOG that $A \subseteq U$. Then each $A_\alpha$ intersects $U$, so each $A_\alpha$ is contained in $U$. Therefore, $A \cup \bigcup A_\alpha \subseteq U$, contradicting the assumption that $V$ is empty. $\square$

**Exercise 23.4.** Show that if $X$ is an infinite set, it is connected in the finite complement topology.

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*Solution.* Suppose that $X = U \cup V$ is a separation of $X$. Then

$$U \cap V = X - (X - U) \cup (X - V)$$

is infinite, so clearly $U$ and $V$ aren't disjoint. $\square$

**Exercise 23.5.** A space is **totally disconnected** if it's only connected subspaces are one-point sets. Show that if $X$ has the discrete topology, then $X$ is totally disconnected. Does the converse hold?

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*Solution.* Suppose $X$ has the discrete topology; let $A \subseteq X$ have two points $a, b \in A$. Then the partition $A = \{a\} \cup (A - \{a\})$ is a separation of $A$. Therefore, $X$ is totally disconnected.

Example 4 shows that $\mathbb{Q}$ is totally disconnected, so the converse doesn't hold. $\square$

**Exercise 23.6.** Let $A \subseteq X$. Show that if $C$ is a connected subspace of $X$ that intersects both $A$ and $X - A$, then $C$ intersects $\mathrm{Bd}(A)$.

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*Solution.* If $C$ didn't intersect $\mathrm{Bd}(A)$, we could write

$$C = (C \cap \mathrm{Int}(A)) \cup (C \cap \mathrm{Int}(X - A)),$$

which is a separation of $C$, contradicting that $C$ is connected. $\square$

**Exercise 23.7.** Is the space $\mathbb{R}_\ell$ connected?

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*Solution.* No, it is totally disconnected in fact. We have $\mathbb{R} = (-\infty, x) \cup [x, \infty)$ for all $x \in \mathbb{R}$. $\square$

**Exercise 23.8.** Determine whether or not $\mathbb{R}^\omega$ is connected in the uniform topology.

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*Solution.* We can actually just use the same proof as with the box topology, separating $\mathbb{R}^\omega$ into bounded and unbounded sequences. $\square$

**Exercise 23.9.** Let $A$ be a proper subset of $X$ and let $B$ be a proper subset of $Y$. If $X \times Y$ is connected, show that

$$(X \times Y) - (A \times B)$$

is connected.

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*Solution.* It helps to visualize this. Let $C(x \times y) = (X \times \{y\}) \cup (\{x\} \times Y)$ be the "plus shape" centered at $x \times y$. Each of these individually is connected. Furthermore, we have

$$(X \times Y) - (A \times B) = \bigcup_{x \in X - A} \bigcup_{y \in Y - B} C(x \times y),$$

and each union has at least one common point, so the entire thing is connected. $\square$

**Exercise 23.10.** Let $\{X_\alpha\}_{\alpha\in J}$ be an indexed family of connected spaces; let $X = \prod X_\alpha$. Let $a = (a_\alpha)$ be a fixed point.

(a) Given any finite subset $K \subseteq J$, let $X_K$ denote the subspace of $X$ consisting of all points $x = (x_\alpha)$ such that $x_\alpha = a_\alpha$ for each $\alpha \notin K$. Show that $X_K$ is connected.

(b) Show that the union $Y$ of all $X_K$ is connected.

(c) Show that $X = \overline{Y}$, so that $X$ is connected.

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*Solution.* (a) $X_K$ is homeomorphic to a finite product, thus it is connected.

(b) All $X_K$ have the common point $a$, so $Y = \bigcup X_K$ is connected.

(c) Let $b = (b_\alpha)$ be any point, and let $\prod U_\alpha$ be a neighborhood of $(b_\alpha)$, where $U_\alpha = X_\alpha$ for all but finitely many $\alpha$. Let $K$ denote the finitely many indices where $U_\alpha \neq X_\alpha$. Then the sequence

$$c_\alpha = \begin{cases} b_\alpha & \alpha \in K \\ a_\alpha & \alpha \notin K \end{cases}$$

is in both $U$ and $X_K$, thus $b$ is in the closure of $Y$. Since $b$ was arbitrary, we have $\bar{Y} = X$, which is connected because $Y$ is connected. $\square$

**Exercise 23.11.** Let $p : X \to Y$ be a quotient map. Show that if each set $p^{-1}(\{y\})$ is connected, and if $Y$ is connected, then $X$ is connected.

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*Solution.* Suppose $X = U \cup V$ is a separation. Then each $p^{-1}(\{y\})$ must lie within either $U$ or $V$, thus $U$ and $V$ are saturated. But this implies $p(U)$ and $p(V)$ form a separation of $Y$, which contradicts that $Y$ is connected. $\square$

**Exercise 23.12.** Let $Y \subseteq X$; let $X$ and $Y$ be connected. Show that if $A$ and $B$ form a separation of $X - Y$, then $Y \cup A$ and $Y \cup B$ are connected.

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*Solution.* It REALLY helps to draw this.

By the symmetry of the problem, we only need to show that $Y \cup A$ is connected.

Suppose that $Y \cup A$ was separated as $U \cup V$, where WLOG we assume that $Y \subseteq U$. Note that this implies $V \subseteq A$. We will show that $V$ is open and closed in $X$.

Since $V \subseteq A$, we have

$$V \text{ open in } Y \cup A \implies V \text{ open in } A \implies V \text{ open in } X.$$

A similar argument applies to show that $V$ is closed in $X$, thus $V$ and $X - V$ form a separation of $X$. $\square$