*Supplementary Exercises: Topological Groups

A topological group G is a group that is also a topological space satisfying the T_1 axiom, such that the group operation $G \times G \to G$ and the inverse $G \to G$ are continuous.

Exercise 1. Let H denote a group that is also a topological space satisfying the T_1 axiom. Show that H is a topological group if and only if the map $x \times y \mapsto x \cdot y^{-1}$ is continuous.

Solution. Suppose the map $f(x \times y) = x \cdot y^{-1}$ is continuous. Then f is continuous with respect to each variable, so the map $g(y) = f(1, y) = y^{-1}$ is also continuous and is the inverse operation. Then since $x \times y \mapsto x \times y^{-1}$ is continuous, we have

$$p(x \times y) = f(x \times y^{-1}) = x \cdot y$$

is continuous. $\square$

Exercise 2. Show that $(\mathbb{Z}, +)$, $(\mathbb{R}, +)$, $(\mathbb{R}_+, \cdot)$, $(S^1, \cdot)$, and $\mathrm{GL}(n)$ are topological groups.

Solution. Subgroups of topological groups are topological groups, so we can skip $(\mathbb{Z}, +)$. We know $\mathbb{R}$ satisfies the T_1 axiom and that addition and negation are continuous, so $(\mathbb{R}, +)$ is a topological group.

The space $\mathbb{C} \setminus \{0\}$ satisfies the T_1 axiom and both multiplication and inverses are continuous, so $(\mathbb{C}^\times, \cdot)$ is a topological group, and so are it's subspaces $(\mathbb{R}_+, \cdot)$ and $(S^1, \cdot)$.

Multiplication of matrices seems complicated, but every component is just a combination of addition and multiplication in $\mathbb{R}$, so the entire multiplication is continuous. The same goes for inverses. $\square$

Exercise 3. Let H be a subspace and subgroup of G. Show that both H and $\bar{H}$ are topological groups.

Solution. Subspaces preserve the T_1 axiom and continuity, so H is a topological group.

All we need to show is that $\bar{H}$ is closed under the operation of G, in other words $f(\bar{H} \times \bar{H}) \subseteq \bar{H}$. Indeed, we have

$$f(\bar{H} \times \bar{H}) = f(\overline{H \times H}) \subseteq \overline{f(H \times H)} \subseteq \bar{H}$$

by continuity of f and Theorem 18.1. $\square$