16 Order / Product / Subspace

Exercise 16.1. Show that if Y is a subspace of X and $A \subseteq Y$, then the subspace topology of A in X is the same as in Y.

Solution. The subspace topology of A in Y is the sets $U \cap A$, where U is open in Y. These are just the sets $(V \cap Y) \cap A = V \cap A$ where V is open in X, which are the same sets in the subspace topology of A in X. □

Exercise 16.2. If $\mathcal{T}, \mathcal{T}'$ are topologies on X and $\mathcal{T}'$ is strictly finer than $\mathcal{T}$, what can be said about the corresponding subspace topologies of $Y \subseteq X$?

Solution. The subspace topology with respect to $\mathcal{T}$ is clearly finer than the one in $\mathcal{T}'$, but not necessarily strictly finer. For example, let Y be the empty set. □

Exercise 16.3. Consider the set $Y = [-1, 1]$ as a subspace of $\mathbb{R}$. Which of the following sets are open in Y? Which are open in $\mathbb{R}$?

- $A = \{x \mid 1/2 < |x| < 1\}$
- $B = \{x \mid 1/2 < |x| \leq 1\}$
- $C = \{x \mid 1/2 \leq |x| < 1\}$
- $D = \{x \mid 1/2 \leq |x| \leq 1\}$
- $E = \{x \mid 0 < |x| < 1 \text{ and } 1/x \notin \mathbb{Z}_+\}$

Solution. A, B, E are open in Y, and A, E are open in $\mathbb{R}$. □

Exercise 16.4. A map $f : X \to Y$ is called an **open map** if $f(U)$ is open in Y for each U open in X. Show that the projection maps π_1 and π_2 are open.

Solution. I will only do π_1. Let U be open in $X \times Y$, and choose $x \in \pi_1(U)$. Then we have $(x, y) \in U$ for some $y \in Y$, so $(x, y) \in (A \times B) \subseteq U$ for some basis set $A \times B$. Then

$$A = \pi_1(A \times B) \subseteq \pi_1(U).$$

Since x was arbitrary, $\pi_1(U)$ is open. □

Exercise 16.5. Let X and X' denote a single set in the topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively; let Y and Y' denote a single set in the topologies $\mathcal{U}$ and $\mathcal{U}'$ respectively. Assume these sets are nonempty. Show that if $\mathcal{T}' \supseteq \mathcal{T}$ and $\mathcal{U}' \supseteq \mathcal{U}$, then the product topology on $X' \times Y'$ is finer than the product topology on $X \times Y$. Does the converse hold?

Solution. This is clearly true because the basis of $X \times Y$ is a subset of the basis of $X' \times Y'$.

For the converse, we will just show $\mathcal{T}' \supseteq \mathcal{T}$. Let U be open in X and choose $x \in U$. Since Y is nonempty, pick an arbitrary $y \in Y$. Then we have $U \times Y$ is open in $X \times Y$, so it's also open in $X' \times Y'$. We can pick a basis set $A \times B \subseteq X' \times Y'$ contained within $U \times Y$ which contains the point (x, y). Therefore, we have $x \in A \subseteq U$, where A is open in X'. Since x was arbitrary, we have shown that $\mathcal{T}' \supseteq \mathcal{T}$ □

Exercise 16.6. Show that the countable collection

$$\{(a, b) \times (c, d) \mid a, b, c, d \in \mathbb{Q}\}$$

is a basis for $\mathbb{R}^2$

Solution. We've shown in a previous exercise that $\{(a, b) \mid a, b \in \mathbb{Q}\}$ is a basis for $\mathbb{R}$, so we can just apply Theorem 15.1 □

Exercise 16.7. Let X be an ordered set. If $Y \subset X$ is convex in X, does it follow that Y is an interval or ray?

Solution. It does not. For example, the set $\{x \mid x^2 < 2\}$ is not expressible as an interval in $\mathbb{Q}$. The key property required for this to hold is completeness. □

Exercise 16.8. If L is a straight line in the plane, describe the topology L inherits as a subspace of $\mathbb{R}_\ell \times \mathbb{R}$ and as a subspace of $\mathbb{R}_\ell \times \mathbb{R}_\ell$.

Solution. The results are summarized below:

	$\mathbb{R}_\ell \times \mathbb{R}$	$\mathbb{R}_\ell \times \mathbb{R}_\ell$
$\uparrow$	$\mathbb{R}$	$\mathbb{R}_\ell$
$\nearrow$	$\mathbb{R}_\ell$	$\mathbb{R}_\ell$
$\rightarrow$	$\mathbb{R}_\ell$	$\mathbb{R}_\ell$
$\searrow$	$\mathbb{R}_\ell$	discrete

□

Exercise 16.9. Show that the dictionary order topology on the set $\mathbb{R} \times \mathbb{R}$ is the same as the product topology $\mathbb{R}_d \times \mathbb{R}$, where $\mathbb{R}_d$ denotes $\mathbb{R}$ in the discrete topology. Compare this topology with the standard topology on $\mathbb{R}^2$.

Solution. Every basis element of $\mathbb{R}_d \times \mathbb{R}$ is also a basis element of $\mathbb{R} \times \mathbb{R}$ in the dictionary ordering. Now let $((a \times b), (c \times d))$ be a basis interval in the dictionary ordering. If $a = c$ and $b < d$, then this is also a basis set of $\mathbb{R}_d \times \mathbb{R}$. Now suppose that $a < c$. Then we have

$$((a \times b), (c \times d)) = (\{a\} \times (a, \infty)) \cup ((a, c) \times \mathbb{R}) \cup (\{c\} \times (-\infty, d)),$$

which is open in $\mathbb{R}_d \times \mathbb{R}$ as the union of 3 open sets. Therefore, the topologies are indeed the same.

By Exercise 16.5, this topology is finer than the standard topology on $\mathbb{R}^2$. □

Exercise 16.10. Let $I = [0, 1]$. Compare the product topology of $I \times I$, the dictionary order topology of $I \times I$, and the topology of $I \times I$ inherited as a subspace of the dictionary order in $\mathbb{R}^2$.

Solution.

- $[0, 1] \times (1/2, 1]$ is open in the product topology, but not the dictionary order (consider the point (0×1)).
- $\{1/2\} \times (0, 1)$ is open in the dictionary order or the space inherited from dictionary order of $\mathbb{R}^2$, but not the product topology.
- From Exercise 16.9, we know the dictionary topology in $\mathbb{R}^2$ is finer than the standard topology. Reducing to $I \times I$ via subspaces, the space inherited from the dictionary topology in $\mathbb{R}^2$ is finer than the product.
- $\{1/2\} \times [0, 1]$ is open in the inherited space, but not the dictionary space.
- To conclude, the only comparison is that the inherited space is strictly finer than the other two.

$\square$