26 Compact Spaces

Exercise 26.1. (a) Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the set X; suppose that $\mathcal{T}' \supseteq \mathcal{T}$. What does compactness of X in one of these topologies imply about compactness in the other?

(b) Suppose X is compact Hausdorff under both $\mathcal{T}$ and $\mathcal{T}'$. Show that the topologies are either equal or incomparable.

Solution. (a) Any open cover in $\mathcal{T}$ is also an open cover in $\mathcal{T}'$, so if X is compact in $\mathcal{T}'$, then X is also compact in $\mathcal{T}$.

(b) If $\mathcal{T}' \supseteq \mathcal{T}$ holds, then the identity map $\mathrm{id} : \mathcal{T}' \to \mathcal{T}$ is a continuous bijection, which is a homeomorphism by Theorem 26.6. This means the topologies are equal by the nature of the function id. $\square$

Exercise 26.2. (a) Show that in the finite complement topology on $\mathbb{R}$, every subspace is compact.

(b) If $\mathbb{R}$ has the topology consisting of all sets A such that $\mathbb{R} - A$ is either countable or all of $\mathbb{R}$, is the subspace $[0, 1]$ compact?

Solution. (a) Let $Y \subseteq \mathbb{R}$ and let $\mathcal{A}$ be an open cover of Y. We can assume that $\emptyset \notin \mathcal{A}$ since it has no effect. Now choose some $A \in \mathcal{A}$ arbitrarily, and choose neighborhoods A_α for each $\alpha \in \mathbb{R} - A$. The collection

$$\{A\} \cup \{A_\alpha\}$$

is a finite subcover.

(b) No, for example let $A_n = \mathbb{R} - \{1/k \mid k \geq n\}$ for each $n \in \mathbb{N}$. $\square$

Exercise 26.3. Show that a finite union of compact subspaces of X is compact.

Solution. Let $K_1, \ldots, K_n$ be compact subspaces of X. Let $\mathcal{A}$ be an open cover of $\bigcup K_i$. Then $\mathcal{A}$ has a finite subcover of each individual K_i, so taking the union of these subcovers gives us a finite subcover of $\bigcup K_i$. $\square$

Exercise 26.4. Show that every compact subspace of a metric space is bounded in that metric and is closed. Find a metric space in which not every closed and bounded subspace is compact.

Solution. (1) Let X be a metric space with metric d. Let K be a compact subspace of X. Let

$$\mathcal{A} = \{B_d(x, 1) \mid x \in X\}$$

be an open cover of K. Since K is compact, we can choose finitely many points $x_1, \ldots, x_n$ such that the balls $\{B_d(x_i, 1) \mid 1 \leq i \leq n\}$ cover K. By the triangle inequality, we can find an upper bound for the diameter of K:

$$D \leq \max_{1 \leq i,j \leq n} d(x_i, x_j) + 2.$$

This shows that K is bounded. Also, K is closed by Theorem 26.3 since metric spaces are Hausdorff.

(2) If we instead use the standard bounded metric $\bar{d}$ which induces the standard topology on $\mathbb{R}$, the entire space $\mathbb{R}$ is closed and bounded but not compact. □

Exercise 26.5. Let A and B be disjoint compact subspaces of the Hausdorff space X. Show that there exist disjoint open sets U and V containing A and B, respectively.

Solution. Since B is compact, by Lemma 26.4 we can choose disjoint open sets U_x and V_x for each $x \in A$ such that $x \in U_x$ and $B \subseteq V_x$. The collection $\{U_x\}$ is an open cover of A, so it has a finite subcover $U_{x_1} \cup \cdots \cup U_{x_n}$.

Let $U = \bigcup U_{x_i}$ and $V = \bigcap V_{x_i}$. These are disjoint open sets containing A and B respectively. $\square$

Exercise 26.6. Show that if $f : X \to Y$ is continuous, where X is compact and Y is Hausdorff, then f is a closed map.

Solution. Let K be closed in X. Then K is compact because X is compact. Therefore, $f(K)$ is also compact, so $f(K)$ is closed since Y is Hausdorff. $\square$

Exercise 26.7. Show that if Y is compact, then the projection $\pi_1 : X \times Y \to X$ is a closed map.

Solution. Let C be closed in $X \times Y$. We will show that $X - \pi_1(C)$ is open in X.

Fix some point $x \in X - \pi_1(C)$. Then the line $\pi_1^{-1}(\{x\}) = \{x\} \times Y$ is disjoint from C. By the tube lemma, there is some neighborhood U of x such that $U \times Y$ is disjoint with C. From this we have $U \subseteq X - \pi_1(C)$, which implies that $X - \pi_1(C)$ is open. $\square$

Exercise 26.8. Let $f : X \to Y$; let Y be compact Hausdorff. Then f is continuous if and only if the graph of f,

$$G_f = \{x \times f(x) \mid x \in X\},$$

is closed in $X \times Y$.

Solution. ($\Longrightarrow$) Suppose f is continuous. Let $x \times y$ be a point such that $f(x) \neq y$. Since Y is Hausdorff, we can choose disjoint open sets U_1 and U_2 containing $f(x)$ and y respectively. Also, the set $f^{-1}(U_1)$ is open, so

$$f^{-1}(U_1) \times U_2$$

is a neighborhood of $x \times y$ which doesn't intersect G_f. We didn't use the assumption that Y is compact in this direction.

($\Longleftarrow$) Suppose that G_f is closed. Let C be a closed set in Y. Then the intersection $G_f \cap (X \times C)$ is closed, so by Exercise 26.7, the projection

$$\pi_1(G_f \cap (X \times C)) = f^{-1}(C)$$

is closed. Therefore, f is continuous. □

Exercise 26.9. Generalize the tube lemma as follows:

Let A and B be subspaces of X and Y, respectively; let N be an open set in $X \times Y$ containing $A \times B$. If A and B are compact, then there exist open sets U and V in X and Y, respectively, such that

$$A \times B \subseteq U \times V \subseteq N.$$

Solution. The tube lemma is this in the special case where A consists of a single point in X and $B = Y$.

We have $A \times B$ is compact; let $\mathcal{A}$ be a finite open cover of $A \times B$ consisting of open sets $A_i \times B_i \subseteq N$ for $1 \leq i \leq n$. For each $b \in B$, let J_b be the set of indices i such that $b \in B_i$. Let U_b be the union of A_i for $i \in J_b$, and let V_b be the intersection of all B_i for $i \in J_b$.

By construction, $U_b \times V_b$ contains the line $A \times \{b\}$, so the collection

$$\{U_b \times V_b \mid b \in B\}$$

is an open cover of $A \times B$. By a similar method, we can create an open set contained in N which contains $A \times B$. $\square$

Exercise 26.10. (a) Prove the following partial converse to the uniform limit theorem:

Let $f_n : X \to \mathbb{R}$ be a sequence of continuous functions, with $f_n(x) \to f(x)$ for each $x \in X$. If f is continuous, and if the sequence f_n is monotone increasing, and if X is compact, then the convergence is uniform.

(b) Give examples to show that both hypotheses are necessary.

Solution. (a) Fix $\epsilon > 0$. For each $x \in X$, choose an integer N_x such that

$$f(x) - f_{N_x}(x) < \epsilon.$$

Since $f - f_{N_x}$ is continuous, we can choose a neighborhood U_x of x such that

$$f(a) - f_{N_x}(a) < \epsilon$$

for all $a \in U_x$. Since X is compact, we can choose finitely many $x_1, \ldots, x_n \in X$ such that $U_{x_1} \cup \cdots \cup U_{x_n} = X$. Let $N = \max\{N_{x_1}, \ldots, N_{x_n}\}$, and we have

$$f(x) - f_N(x) < \epsilon$$

for all $x \in X$. ☐

Exercise 26.11. Let X be a compact Hausdorff space. Let $\mathcal{A}$ be a collection of closed connected subsets of X that is simply ordered by proper inclusion. Then

$$Y = \bigcap_{A \in \mathcal{A}} A$$

is connected.

Solution. Suppose $C \cup D$ is a separation of Y. Then C and D are both compact since they are closed in X. We can create disjoint open sets U and V containing C and D respectively, by Exercise 26.5.

Consider the set $A - (U \cup V)$ for each $A \in \mathcal{A}$. Each of these sets is closed in X, and also nonempty since A is connected. The intersection

$$\bigcap_{A \in \mathcal{A}} (A - (U \cup V))$$

is then nonempty since X is compact. This contradicts the fact that $Y \subseteq U \cup V$. $\square$

Exercise 26.12. Let $p : X \to Y$ be a closed continuous surjective map such that $p^{-1}(\{y\})$ is compact for each $y \in Y$. (Such a map is called a **perfect map**) Show that if Y is compact, then X is compact.

Solution. Let $\mathcal{A}$ be an open cover of X. For each $y \in Y$, consider the collection of $A \in \mathcal{A}$ which intersect $p^{-1}(\{y\})$. This has a finite subcover of $p^{-1}(\{y\})$, which we will call $\mathcal{A}_y$.

Let $V_y = \bigcup \mathcal{A}_y$. Since p is a closed map, the set $U_y = Y - p(X - V_y)$ is open and contains y for each $y \in Y$. We have

$$p^{-1}(U_y) = X - p^{-1}(p(X - V_y)) \subseteq V_y.$$

Since Y is compact, we can choose finitely many $y_1, \ldots, y_n$ such that the sets U_{y_k} cover Y. Then their inverse images under p cover X, thus the sets V_{y_k} cover X. Since each V_{y_k} is just a finite union of sets in $\mathcal{A}$, we have our finite subcover of $\mathcal{A}$.

We didn't use the assumption that p is continuous or surjective. It just needs to be a closed map. $\square$

Exercise 26.13. Let G be a topological group.

(a) Let A and B be subspaces of G. If A is closed and B is compact, show $A \cdot B$ is closed.

Solution. $\square$