21 Metric Spaces (Continued)

Exercise 21.1. Let $A \subseteq X$. If d is a metric for the topology of X, show that $d|A \times A$ is a metric for the subspace topology of A.

Solution. Let $d' = d|A \times A$. We have

$$B_{d'}(x, \epsilon) = \{a \in A \mid d(a, x) < \epsilon\} = A \cap B_d(x, \epsilon).$$

$\square$

Exercise 21.2. Let X and Y be metric spaces with metrics d_X and d_Y, respectively. Let $f : X \to Y$ have the property that for every pair $x_1, x_2 \in X$,

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2).$$

Show that f is an imbedding. It is called an isometric imbedding of X in Y.

Solution. That f is continuous follows directly from the distance-preserving property. Also, f is injective, since if $f(x_1) = f(x_2)$, then

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2) = 0,$$

thus $x_1 = x_2$. The inverse function f^{-1} is continuous because it's also distance-preserving. Therefore, f is an embedding. $\square$

Exercise 21.3. Let X_n be a metric space with metric d_n, for $n \in \mathbb{Z}_+$.

(a) Show that
$$\rho(x,y) = \max\{d_1(x_1,y_1),\ldots,d_n(x_n,y_n)\}$$
is a metric for the product space $X_1 \times \cdots \times X_n$.

(b) Let $\bar{d}_i = \min\{d_i, 1\}$. Show that
$$D(x,y) = \sup\{\bar{d}_i(x_i,y_i)/i\}$$
is a metric for the product space $\prod X_i$.

Solution. (a) Open balls are open in the product topology. Now suppose $U = \prod B_{d_i}(x_i, \epsilon_i)$ is a basis element of the product topology and $a \in U$. Then we have $a_i \in B_{d_i}(x_i, \epsilon)$, so we can find some δ_i such that
$$B_{d_i}(a_i, \delta_i) \subseteq B_{d_i}(x_i, \epsilon).$$
Let $\delta = \min\{\delta_i\}$, and we have $B_\rho(a,\delta) \subseteq U$.

(b) It's easy to check that any open set in the product topology contains an open ball with respect to the metric D. Now let $B_D(x,\epsilon)$ be the neighborhood of x with radius ϵ. Define the open set U by the equation
$$U = \prod_{i=1}^{\infty} B_{\bar{d}_i}(x_i, \epsilon i).$$
But we have $B_{\bar{d}_i}(x_i, \epsilon i) = \mathbb{R}$ for all $i > 1/\epsilon$, thus U is open in the product topology. By our chosen definition, we have $x \in U \subseteq B_D(x,\epsilon)$, so the metric D does indeed induce the product topology. □

Exercise 21.4. Show that $\mathbb{R}_\ell$ and the ordered square satisfy the first countability axiom.

Solution. Every neighborhood of x in $\mathbb{R}_n$ contains some set of the form $[x, x + 1/n)$.

If $x \times y$ is a point on the ordered square and $y \neq 1$, then every neighborhood of $x \times y$ in the ordered square topology contains some set

$$\{x\} \times (y - 1/n, y + 1/n).$$

If $y = 1$, then every neighborhood of $x \times 1$ contains some interval

$$((x \times y - 1/n), (x + 1/n \times y)).$$

$\square$

Exercise 21.5. Let $x_n \to x$ and $y_n \to y$ in the space $\mathbb{R}$. Then

$$\begin{aligned} x_n + y_n &\to x + y \\ x_n - y_n &\to x - y \\ x_n y_n &\to xy \\ x_n / y_n &\to x/y \qquad\qquad \text{if } y \neq 0 \text{ and } y_n \neq 0. \end{aligned}$$

Solution. Let $\epsilon > 0$. For addition, we have

$$|(x_n + y_n) - (x + y)| \leq |x_n - x| + |y_n - y| < \epsilon$$

whenever $|x_n - x| < \epsilon/2$ and $|y_n - y| < \epsilon$, which is true for almost all n. Subtraction is very similar.

For multiplication, let M be greater than all of $|y|, |x_1|, |x_2|, \dots$. Then we have

$$\begin{aligned} |x_n y_n - xy| &= |x_n y_n - x_n y + x_n y - xy| \\ &\leq |x_n||y_n - y| + |y||x_n - x| \\ &\leq M(|x_n - x| + |y_n - y|) < \epsilon \end{aligned}$$

when $|x_n - x| < \epsilon/2M$ and $|y_n - y| < \epsilon/2M$, which is true for almost all n.

For reciprocals, choose $\delta > 0$ such that $|x_n| > \delta$ for all n, and also $|x| > \delta$. Then we have

$$\left| \frac{1}{x_n} - \frac{1}{x} \right| = \frac{|x_n - x|}{|x_n||x|} \leq \frac{|x_n - x|}{\delta^2} < \epsilon$$

whenever $|x_n - x| < \delta^2 \epsilon$. Then convergence of division follows from this combined with multiplication. ☐

Exercise 21.6. Define $f_n : [0, 1] \to \mathbb{R}$ by the equation $f_n(x) = x^n$. Show that the sequence $(f_n(x))$ converges for each $x \in [0, 1]$, but that the sequence (f_n) doesn't converge uniformly.

Solution. The sequences $(f_n(x))$ all converge by the monotone convergence theorem. Also, we have

$$\lim x^n = \lim x^{n+1} = x \lim x^n,$$

thus $(f_n(x)) \to 0$ except when $x = 1$, where $(f_n(1)) \to 1$. This limit function is clearly not continuous, so the sequence (f_n) couldn't have converged uniformly. $\square$

Exercise 21.7. Let X be a set and let $f_n : X \to \mathbb{R}$ be a sequence of functions. Let $\bar{\rho}$ be the uniform metric on $\mathbb{R}^X$. Show that the sequence (f_n) converges uniformly to the function $f : X \to \mathbb{R}$ if and only if (f_n) converges to f as elements of the metric space $(\mathbb{R}^X, \bar{\rho})$.

Solution. (f_n) converging to f in the uniform metric is equivalent to the condition that for all $\epsilon > 0$ we have

$$\sup_{x \in X} |f_n(x) - f(x)| \leq \epsilon$$

for almost all n. This is equivalent to $|f_n(x) - f(x)| \leq \epsilon$ for all x and almost all n, which is the same as (f_n) uniformly converging to f. We can use nonstrict inequalities because $0 < \epsilon/2 < \epsilon$ for all $\epsilon > 0$. $\square$

Exercise 21.8. Let X be a topological space and let Y be a metric space. Let $f_n : X \to Y$ be a sequence of continuous functions. Let (x_n) be a sequence of points of X converging to x. Show that if the sequence (f_n) converges uniformly to f, then $(f_n(x_n))$ converges to $f(x)$.

Solution. Let $\epsilon > 0$. By uniform convergence, for almost all n, we have $d(f_n(a), f(a)) < \epsilon/2$ for all a. Also, since the limit f is continuous, we have $d(f(x_n), f(x)) < \epsilon/2$ for almost all n. Combining these,

$$d(f_n(x_n), f(x)| \leq d(f_n(x_n), f(x_n)) + d(f(x_n), f(x)) < \epsilon$$

for almost all n, thus $(f_n(x_n)) \to f(x)$. $\square$

Exercise 21.9. Let $f_n : \mathbb{R} \to \mathbb{R}$ be the function

$$f_n(x) = \frac{1}{n^3[x - (1/n)]^2 + 1}.$$

Let f be the zero function.

(a) Show that $f_n(x) \to f(x)$ for each $x \in \mathbb{R}$.

(b) Show that (f_n) does not converge uniformly to f.

Solution. (a) We have $f_n(x) \to 0$ by properties of the convergence of rational functions.

(b) For all n we have $f_n(1/n) = 1$, thus (f_n) doesn't converge uniformly to f. $\square$

Exercise 21.10. Using the closed set formulation of continuity, show that the following sets are closed in $\mathbb{R}^2$.

$$A = \{x \times y \mid xy = 1\}$$
$$S^1 = \{x \times y \mid x^2 + y^2 = 1\}$$
$$B^2 = \{x \times y \mid x^2 + y^2 \leq 1\}.$$

Solution. We will assume arithmetic operators are continuous, so the functions $f(x, y) = xy$ and $g(x, y) = x^2 + y^2$ are both continuous. Then we have

$$A = f^{-1}(\{1\})$$
$$S^1 = g^{-1}(\{1\})$$
$$B^2 = g^{-1}((-\infty, 1]).$$

$\square$