22 Quotient Topology

Exercise 22.1. Let p be the map from $\mathbb{R}$ to the 3 point set $A = \{a, b, c\}$ defined by

$$p(x) = \begin{cases} a & x > 0 \\ b & x < 0 \\ c & x = 0. \end{cases}$$

The quotient topology on A induced by p is $\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$.

Solution. Just check whether the inverse image of each subset of A is open in $\mathbb{R}$. $\square$

Exercise 22.2. (a) Let $p : X \to Y$ be a continuous map. Show that if there is a continuous map $f : Y \to X$ such that $p \circ f$ equals the identity map of Y, then p is a quotient map.

(b) If $A \subseteq X$, a retraction of X onto A is a continuous map $r : X \to A$ such that $r(a) = a$ for each $a \in A$. Show that a retraction is a quotient map.

Solution. (a) Suppose U is a subset of Y and that $p^{-1}(U)$ is open. Then we have

$$U = (p \circ f)^{-1}(U) = f^{-1}(p^{-1}(U)),$$

which is open because f is continuous.

(b) Let U be a subset of A such that $r^{-1}(U)$ is open. Then $r^{-1}(U) \cap A$ is open in A by definition. But by the retraction property, $r^{-1}(U) \cap A = U$, so U must be open. $\square$

Exercise 22.3. Let $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be projection onto the first coordinate. Let A be the subspace of $\mathbb{R} \times \mathbb{R}$ consisting of all points $x \times y$ for which $x \geq 0$ or $y = 0$; let $q : A \to \mathbb{R}$ be obtained by restricting π_1. Show that q is a quotient map and is neither open nor closed.

Solution. q is a quotient map because A is closed. We have

$$f(A \cap (\mathbb{R} \times (0, \infty))) = [0, \infty)$$
$$f(\{(1/n, n) \mid n \in \mathbb{N}\}) = \{1/n \mid n \in \mathbb{N}\}$$

□