17 Closed Sets and Limit Points

Exercise 17.1. Let $\mathcal{C}$ be a collection of subsets of X. Suppose that $\emptyset$ and X are in $\mathcal{C}$, and finite unions and arbitrary intersections of elements in $\mathcal{C}$ are also in $\mathcal{C}$. Show that

$$\{X - C \mid C \in \mathcal{C}\}$$

is a topology on X.

Solution. Trivial application of DeMorgan's laws. $\square$

Exercise 17.2. If A is closed in Y and Y is closed in X, then A is closed in X.

Solution. Since A is closed in Y, we have $A = Y \cap B$ for some B which is closed in X. Now A is the intersection of closed sets, so A is closed in X. $\square$

Exercise 17.3. If A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.

Solution. We have

$$A \times B = (X \times Y) - (((X - A) \times Y) \cap (X \times (Y - B))),$$

so we've written $A \times B$ as the complement of an open set. $\square$

Exercise 17.4. If U is open in X and A is closed, then $U - A$ is open and $A - U$ is closed.

Solution. We have $U - A = U \cap (X - A)$ and $A - U = A \cap (X - U)$. $\square$

Exercise 17.5. Let X be an ordered set in the order topology. Show that $\overline{(a, b)} \subseteq [a, b]$. Under what conditions does equality hold?

Solution. If $x < a$, then $x \in (-\infty, a)$, which is open, thus $x \notin \overline{(a, b)}$. A similar argument applies if $x > b$, so we have $\overline{(a, b)} \subseteq [a, b]$.

Equality holds if and only if both a and b are limit points of (a, b). This happens exactly when a has no immediate successor and b has no immediate predecessor. $\square$

Exercise 17.6. Let A, B, A_α denote subsets of a space X.

(a) If $A \subseteq B$, then $\bar{A} \subseteq \bar{B}$

(b) $\overline{A \cup B} = \bar{A} \cup \bar{B}$

Solution. (a) If $x \in A$, then $x \in B$. If x is a limit point of A, then x is a limit point of B. Therefore, $\bar{A} \subseteq \bar{B}$.

(b) $\overline{A \cup B}$ is a closed set including both A and B, therefore $\bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$. This argument also applies with (c). In (b), we have that $\bar{A} \cup \bar{B}$ is closed because it's the finite union of closed sets. Also, $\bar{A} \cup \bar{B}$ contains $A \cup B$, thus we have $\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$.

(c) An example where equality fails is $A_n = [1/n, \infty]$ for each $n \in \mathbb{N}$. $\square$

Exercise 17.7. Criticize the following "proof" that $\overline{\bigcup A_\alpha} \subseteq \bigcup \overline{A_\alpha}$. If $\{A_\alpha\}$ is a collection of sets in X and if $x \in \overline{\bigcup A_\alpha}$, then every neighborhood U of x intersects $\bigcup A_\alpha$. Thus U must intersect some A_α, so that x must belong to the closure of some A_α. Therefore, $x \in \bigcup \overline{A_\alpha}$.

Solution. U does intersect some A_α, but this isn't true for this particular A_α for all U. This argument can be made to work if $\{A_\alpha\}$ is finite. $\square$

Exercise 17.8. Let A, B, A_α denote subsets of a space X. Prove or disprove the following, and state whether an inclusion applies instead.

(a) $\overline{A \cap B} = \bar{A} \cap \bar{B}$

(b) $\overline{\bigcap A_\alpha} = \bigcap \overline{A_\alpha}$

(c) $\overline{A - B} = \bar{A} - \bar{B}$

Solution. (a)(b) Equality does not hold. For example, if $A = (0, 1)$ and $B = (1, 2)$, then $\overline{A \cap B} = \emptyset$, but $\overline{A} \cap \overline{B} = \{1\}$. We will instead show that

$$\overline{\bigcap A_\alpha} \subseteq \bigcap \overline{A_\alpha}.$$

If $x \in \overline{\bigcap A_\alpha}$, then every neighborhood U of x intersects $\bigcap A_\alpha$, so it must intersect every A_α (this is where the converse fails). Therefore, we have $x \in \overline{A_\alpha}$ for all α.

(c) If $A = [0, 1]$ and $B = (0, 1)$, then $\overline{A - B} = \{0, 1\}$ and $\bar{A} - \bar{B} = \emptyset$, so equality doesn't hold. We will show that the $\supseteq$ direction holds though.

Assume $x \in \bar{A} - \bar{B}$. Then every neighborhood of x intersects A, but at least one neighborhood U of x does not intersect B. Now fix any neighborhood V of x. Then we have $U \cap V$ does not intersect B, which means V intersects some point in $A - B$. Since V was arbitrary, this means $x \in \overline{A - B}$. $\square$

Exercise 17.9. Let $A \subseteq X$ and $B \subseteq Y$. Show that in the space $X \times Y$,

$$\overline{A \times B} = \bar{A} \times \bar{B}$$

Solution. Suppose $x \times y \in \overline{A \times B}$. Then for all neighborhoods U of x and V of y, we have $U \times V$ intersects $A \times B$. Therefore, U intersects A and V intersects B, thus $x \in \bar{A}$ and $y \in \bar{B}$. All steps are reversible. $\square$

Exercise 17.10. Show that every order topology is Hausdorff

Solution. Let X be equipped with an order topology. Let a, b be distinct elements of X and assume WLOG that $a < b$. If b is the immediate successor of a, then the sets $(-\infty, b)$ and (a, ∞) are disjoint and contain a and b respectively. Otherwise, choose x such that $a < x < b$. Then we have $(-\infty, x)$ and (x, ∞) are disjoint and contain a and b. $\square$

Exercise 17.11. Show that the product of two Hausdorff spaces is Hausdorff.

Solution. Let X and Y be Hausdorff spaces, and pick two distinct points (a, b) and (c, d) in $X \times Y$. WLOG assume that $a \neq c$. Since X is Hausdorff, we can pick disjoint neighborhoods $U, V \subseteq X$ containing a and c. Then the neighborhoods $U \times Y$ and $V \times Y$ of (a, b) and (c, d) respectively are disjoint.

If $a = c$, then $b \neq d$ and we can use the fact that Y is Hausdorff instead. $\square$

Exercise 17.12. Show that a subspace of a Hausdorff space is Hausdorff.

Solution. Let X be a Hausdorff space and $Y \subseteq X$. Choose two distinct point $x, y \in Y$. Since X is Hausdorff, choose disjoint open sets U and V containing x and y respectively. Then $x \in U \cap Y$ and $y \in V \cap Y$ are disjoint neighborhoods in Y, so Y is Hausdorff. $\square$

Exercise 17.13. Show that X is Hausdorff if and only if $\Delta = \{x \times x \mid x \in X\}$ is closed in $X \times X$.

Solution. Pick a point (x, y) not on the diagonal. Then $x \neq y$, so since X is Hausdorff, we can find disjoint neighborhoods U and V of x and y. Then we have $(x, y) \in U \times V$ and $U \times V$ doesn't intersect Δ, so Δ is closed. These steps work in reverse for the other direction. $\square$

Exercise 17.14. How does convergence of sequences work in $\mathbb{R}$ with the finite complement topology.

Solution. If no term appears infinitely often, the sequence converges to every point. If a single term appears infinitely often, the sequence converges to that point. If multiple terms appear infinitely often, the sequence doesn't converge. $\square$

Exercise 17.15. Show that the T_1 axiom is equivalent to the condition that for each pair of distinct points in X, each has a neighborhood not containing the other.

Solution. Assuming the T_1 axiom is true, $X - \{y\}$ is a neighborhood of x not containing y. For the converse, we only need to show that single point sets are closed. Let $a \in X$. If b is a different point in X, there is a neighborhood of b not containing a by hypothesis. This immediately implies $\{a\}$ is closed. $\square$

Exercise 17.16. Consider the following topologies on $\mathbb{R}$:

$\mathcal{T}_1$ = the standard topology

$\mathcal{T}_2 = \mathbb{R}_K$

$\mathcal{T}_3$ = the finite complement topology

$\mathcal{T}_4$ = the upper limit topology

$\mathcal{T}_5 =$ the topology having all sets $(-\infty, a)$ as a basis.

Determine the closure of $K = \{1/n\}_{n\in\mathbb{N}}$ for each topology. Which topologies are Hausdorff? Which satisfy the T_1 axiom?

Solution. In $\mathcal{T}_1$, the closure of K is $K \cup \{0\}$. $\mathcal{T}_1$ is Hausdorff.

In $\mathcal{T}_2$, the closure of K is K. $\mathcal{T}_2$ is Hausdorff because $\mathcal{T}_2$ is finer than $\mathcal{T}_1$.

In $\mathcal{T}_3$, the closure of K is $\mathbb{R}$. $\mathcal{T}_3$ is not Hausdorff, but does satisfy the T_1 axiom.

In $\mathcal{T}_4$, the closure of K is K. $\mathcal{T}_4$ is Hausdorff.

In $\mathcal{T}_5$, the closure of K is $[0, \infty)$. $\mathcal{T}_5$ doesn't satisfy the T_1 axiom. □

Exercise 17.17. Consider the lower limit topology on $\mathbb{R}$ and the topology given by the basis

$$\mathcal{C} = \{[q, r) \mid q, r \in \mathbb{Q}\}.$$

Determine the closures of $A = (0, \sqrt{2})$ and $B = (\sqrt{2}, 3)$ in each topology.

Solution. In $\mathbb{R}_\ell$, we have $\bar{A} = [0, \sqrt{2})$ and $\bar{B} = [\sqrt{2}, 3)$.

In the topology generated by $\mathcal{C}$, we have $\bar{A} = [0, \sqrt{2}]$ and $\bar{B} = [\sqrt{2}, 3)$ □

Exercise 17.18. Determine the closures of the following subsets of the ordered square:

$$A = \{1/n \times 0 \mid n \in \mathbb{N}\}$$

$$B = \{(1 - 1/n) \times 1/2 \mid n \in \mathbb{N}\}$$

$$C = \{x \times 0 \mid 0 < x < 1\}$$

$$D = \{x \times 1/2 \mid 0 < x < 1\}$$

$$E = \{1/2 \times y \mid 0 < y < 1\}$$

Solution. Here are the results:

- $\bar{A} = A \cup \{0 \times 1\}$
- $\bar{B} = B \cup \{1 \times 0\}$
- $\bar{C} = \{x \times 0 \mid 0 < x \leq 1\} \cup \{x \times 1 \mid 0 \leq x < 1\}$
- $\bar{D} = D \cup \bar{C}$
- $\bar{E} = \{1/2 \times y \mid 0 \leq y \leq 1\}$

□

Exercise 17.19. If $A \subseteq X$, we define the *boundary* of A by the equation $\operatorname{Bd} A = \overline{A} \cap \overline{X - A}$.

(a) Show that the boundary and interior are disjoint, and $\bar{A} = \operatorname{Bd} A \cup \operatorname{Int} A$.

(b) Show that $\operatorname{Bd} A = \emptyset \iff A$ is open and closed.

(c) Show that U is open $\iff \operatorname{Bd} U = \bar{U} - U$.

(d) If U is open, is it true that $U = \operatorname{Int} \bar{U}$?

Solution. (a) We have $x \in \operatorname{Bd} A$ if and only if every neighborhood of x intersects both A and $X - A$. On the other hand, $x \in \operatorname{Int} A$ if and only if there is some neighborhood of x which lies completely in A. These can't happen at the same time, thus $\operatorname{Bd} A$ and $\operatorname{Int} A$ are disjoint. Also, either of these conditions occuring is equivalent to every neighborhood of x intersecting with A, which means $\operatorname{Bd} A \cup \operatorname{Int} A = \bar{A}$.

(b) Just realize that $\operatorname{Bd} A = \bar{A} - \operatorname{Int} A$ and the rest is trivial.

(c) Again, a simple application of $\operatorname{Bd} A = \bar{A} - \operatorname{Int} A$.

(d) No, for instance $U = (0,1) \cup (1,2)$ and $\operatorname{Int} \bar{U} = (0,2)$. □

Exercise 17.20. Find the boundary and interior of these subsets of $\mathbb{R}^2$:

(a) $A = \{x \times y \mid y = 0\}$

(b) $B = \{x \times y \mid x > 0 \text{ and } y \neq 0\}$

(c) $C = A \cup B$

(d) $D = \{x \times y \mid x \in \mathbb{Q}\}$

(e) $E = \{x \times y \mid 0 < x^2 - y^2 \leq 1\}$

(f) $F = \{x \times y \mid x \neq 0 \text{ and } y \leq 1/x\}$

Solution. (a) $\operatorname{Bd} A = A$ and $\operatorname{Int} A = \emptyset$.

(b) $\operatorname{Bd} B = \{x \times y \mid x = 0 \text{ or } x > 0, y = 0\}$ and $\operatorname{Int} B = B$.

(c) $\operatorname{Bd} C = \{x \times y \mid x = 0 \text{ or } x < 0, y = 0\}$ and $\operatorname{Int} C = \{x \times y \mid x > 0\}$.

(d) $\operatorname{Bd} D = \mathbb{R}^2$ and $\operatorname{Int} D = \emptyset$.

(e) $\operatorname{Bd} E = \{x \times y \mid x^2 - y^2 \in \{0,1\}\}$ and $\operatorname{Int} E = \{x \times y \mid 0 < x^2 - y^2 < 1\}$

(f) $\operatorname{Bd} F = \{x \times y \mid x = 0 \text{ or } y = 1/x\}$ and $\operatorname{Int} F = \{x \times y \mid x \neq 0 \text{ and } y < 1/x\}$. □

Exercise 17.21 (Kuratowski's Theorem)**.** Show that starting with a set $A \subseteq X$, 14 is the most number of unique sets can be reached by repeatedly applying closure and complement.

Solution. Let K denote closure and let C denote complement. We have $KK = K$ and $CC = \text{id}$, so every unique combination can be written by alternating K and C.

First notice that CKC is just the interior operator, which we will shorten to I. We need a lemma first.

Lemma 17.21.1. Suppose K is a closed set, and for every open set U intersecting K we have $\operatorname{Int}(U \cap K) \neq \emptyset$. Then $\overline{\operatorname{Int} K} = K$.

Proof. Since K is closed and includes it's interior, we have $\overline{\operatorname{Int} K} \subseteq K$.

Now assume $x \notin \overline{\operatorname{Int} K}$, so that there is some neighborhood U of x not intersecting $\operatorname{Int} K$. If $x \in K$, then clearly U intersects K, so by hypothesis there is some $y \in \operatorname{Int}(U \cap K)$. But this implies $y \in \operatorname{Int} K$ and $y \in U$, which is a contradiction. Therefore, $x \notin K$, and so we have $K \subseteq \overline{\operatorname{Int} K}$ □

I claim that $\overline{\operatorname{Int} A}$ is a closed set with the required property. If U is an open set intersecting $\overline{\operatorname{Int} A}$, then it also intersects $\operatorname{Int} A$, so $U \cap \operatorname{Int} A \subseteq U \cap \overline{\operatorname{Int} A}$ is part of the interior.

In particular, this means that $KIKI = KI$, which is the final reduction type we need. We can go up to $CKCKCKC$ or $CKCKCK$, and along with the starting set, this is 14 sets!

An example where 14 unique sets are achieved is $(0,1) \cup (1,2) \cup \{3\} \cup ([4,5] \cap \mathbb{Q})$ □
