25 Components and Local Connectedness

Exercise 25.1. What are the components and path components of $\mathbb{R}_\ell$? What are the continuous functions $f : \mathbb{R} \to \mathbb{R}_\ell$?

Solution. For any points $a < b \in \mathbb{R}_\ell$ and a set S containing a and b, S has the separation into $[-\infty, b) \cap S$ and $[b, \infty) \cap S$. Therefore, the only connected sets are single point sets. This is also clearly true of path connected sets.

Since $\mathbb{R}$ is connected, any continuous function $f : \mathbb{R} \to \mathbb{R}_\ell$ must be constant. $\square$

Exercise 25.2. (a) What are the components and path components of $\mathbb{R}^\omega$ in the product topology?

(b) Consider $\mathbb{R}^\omega$ in the uniform topology. Show that $\mathbf{x}$ and $\mathbf{y}$ lie in the same component if and only if the sequence $\mathbf{x} - \mathbf{y}$ is bounded.

(c) Consider $\mathbb{R}^\omega$ in the box topology. Show that $\mathbf{x}$ and $\mathbf{y}$ lie in the same component if and only if the sequence $\mathbf{x} - \mathbf{y}$ is nonzero for only finitely many indices.

Solution. (a) By Exercise 24.8, the space $\mathbb{R}^\omega$ is path connected, so only has one path component.

(b) For any $x \in \mathbb{R}^\omega$, let B_x be the collection $\{y \mid x - y \text{ is bounded}\}$. I claim that every B_x is path connected. WLOG assume that $x = 0$, and let y be a bounded sequence such that $|y_n| < M$.

Consider the function $p(t) = ty$ with domain $[0, 1]$. Fix $\epsilon > 0$. Then if $|t_1 - t_2| < \epsilon/M$, then

$$\bar{\rho}(p(t_1), p(t_2)) = \sup_{n \in \mathbb{N}} \min\{|t_1 - t_2||y_n|, 1\} < \epsilon.$$

This shows that p is continuous (uniformly), and is thus a path from 0 to y. This shows that B_0 is connected, and more generally that B_x is connected for all x by the distance-preserving homeomorphism $a \mapsto a - x$.

It's easy to see that each B_x is open and they form a partition of $\mathbb{R}^\omega$, thus these are the components.

(c) This is similar to (b). Let B_x instead be the collection

$$B_x = \{y \mid x - y \text{ is almost all zeros}\}.$$

Again, WLOG assume that $x = 0$ and let y be a sequence with almost all zeros. Consider the function $p(t) = ty$ with domain $[0, 1]$. Let $t_0 \in [0, 1]$; let $U = \prod U_n$ be a neighborhood of t_0, where $U_n = \mathbb{R}$ for almost all n.

Let J be the finite set of indices such that $U_n \neq \mathbb{R}$. We have

$$p^{-1}(U) = \bigcap_{n \in J} (\pi_n \circ p)^{-1}(U_n),$$

which is open since $\pi_n \circ p$ is continuous for each n. This shows that p is a path from 0 to y, therefore the set B_0 is path connected. More generally, B_x is path connected for each x.

Now suppose that y instead has infinitely many nonzero terms. Consider the function $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ defined as

$$h(a_1, a_2, \dots) = (0 \text{ if } y_n = 0, \text{ otherwise } na_n/y_n)_{n \in \mathbb{N}}.$$

It's easy to check that h is a homeomorphism, and that $h(0) = 0$ and $h(y)$ is unbounded. Therefore, 0 and y lie in different components. This shows that B_0 is a connected component, and similarly B_x also is for each $x \in \mathbb{R}^\omega$.

□

Exercise 25.3. Show that the ordered square is locally connected, but not locally path connected.

Solution. Every basis element of the ordered square is connected since it's a linear continuum, which is sufficient to show that the ordered square is locally connected.

The ordered square is path connected at points of the form $x \times y$ such that $y \notin \{0, 1\}$. On the other hand, consider the point (0×1). Suppose there is some neighborhood V of (0×1) which is path connected. Choose some $\epsilon > 0$ such that $(\epsilon \times 1) \in V$. Since V is path connected, there exists a path $f : [0, 1] \to V$ from (0×1) to $(\epsilon \times 1)$. The image $f([0, 1])$ is then connected, so by the intermediate value theorem it contains all of $(0, \epsilon] \times [0, 1]$. In particular, this means $(0, \epsilon] \times (0, 1) \subseteq V$.

We've fit uncountably many disjoint open intervals into V, thus the inverse images form uncountably many disjoint open sets in $[0, 1]$. This is impossible by the density of $\mathbb{Q}$ in $\mathbb{R}$. □

Exercise 25.4. Let X be locally path connected. Show that every open connected set in X is path connected.

Solution. By Theorem 25.4, every path component of U is open. Since U is connected, there can be only one component. $\square$

Exercise 25.5. Let X denote the rational points of the interval $[0,1] \times \{0\}$ of $\mathbb{R}^2$. Let T denote the union of all line segments joining the point $P = 0 \times 1$ to points of X.

(a) Show that X is path connected, but only locally connected at P.

(b) Find a subset of $\mathbb{R}^2$ which is path connected but not locally connected at any of it's points.

Solution. (a) Each segment is path connected, so X is path connected because of the common point P. Similarly, X is locally connected at P.

For each $x \neq P$, choose a neighborhood not intersecting P. Then each subneighborhood U intersects with infinitely many line segments corresponding to a set of rational numbers A. Choose some irrational r such that $p < r < q$ for some $p, q \in A$. Then we can separate U into

$$U \cap \{L_p \mid p < r\} \qquad \text{and} \qquad U \cap \{L_q \mid q > r\}.$$

These are both open in U.

(b) Defining X as before, let $X_n = X + (0 \times n)$ for each $n \in \mathbb{Z}_{\geq 0}$. The space $\bigcup X_n$ is then path connected but not locally connected at any point. $\square$

Exercise 25.6. A space X is said to be **weakly locally connected** at x if for every neighborhood U of x, there is a connected subspace of X contained in U that contains a neighborhood of x. Show that if X is weakly locally connected at each of it's points, then X is locally connected.

Solution. Suppose X is weakly locally connected, and let U be an open set in X. Let C be a component of U, and let $x \in C$. By hypothesis, there exists a connected subspace A contained in U which contains a neighborhood V of x. In particular,

$$x \in V \subseteq A \subseteq C,$$

thus C is open. Therefore, every component of an open set in X is open, so X is locally connected. $\square$

Exercise 25.7. Consider the "infinite broom" X pictured in Figure 25.1. Show that X is not locally connected at p, but is weakly locally connected at p.

Solution. Any neighborhood of a point p must contain at least one of the points a_n for some n. But then it must contain a_{n-1}, and so on, so that it contains every point (a_i) to be connected. We can just choose a neighborhood of p that doesn't contain a_1, and no subneighborhood can be connected.

On the other hand, let U be a neighborhood of p. Then U contains all broom fragments of sufficiently large index, so we can let C be this collection of broom fragments. This will indeed contain a neighborhood of x. Note that it doesn't matter that **this** neighborhood is disconnected. □

Exercise 25.8. Let $p : X \to Y$ be a quotient map. Show that if X is locally connected, then Y is locally connected.

Solution. Intuitively, p glues together points of X, so it should also glue together the components.

Let $U \subseteq Y$ be open; let C be a component of U. We will show that $p^{-1}(C)$ is the union of components of $p^{-1}(U)$, and thus is open because X is locally connected. It suffices to prove that if $x \in p^{-1}(C)$, then the connected component A of $p^{-1}(U)$ containing x is contained in $p^{-1}(C)$.

If $x \in p^{-1}(C)$, then $p(x) \in C$. Also, $p(A)$ is connected, and so $p(A) \subseteq C$, which implies $A \subseteq p^{-1}(C)$.

As the union of components in a locally connected space, $p^{-1}(C)$ must be open, thus C is open because p is a quotient map. $\square$

Exercise 25.9. Let G be a topological group; let C be the connected component of G containing the identity element e. Show that C is a normal subgroup of G.

Solution. Because $x \mapsto ax$ is a homeomorphism, the set aC is a connected component for each $a \in G$. A similar argument applies to Ca, which means the left and right cosets of C match. $\square$

Exercise 25.10. Let X be a space. Let us define $x \sim y$ if there is no separation $X = A \cup B$ of X into disjoint open sets such that $x \in A$ and $y \in B$.

(a) Show this relation is an equivalence relation. The equivalence classes are called the **quasicomponents** of X.

(b) Show that each component of X lies in a quasicomponent of X, and that the components and quasicomponents are the same if X is locally connected.

(c) Let K denote the set $\{1/n \mid n \in \mathbb{Z}_+\}$. Determine the components, path components, and quasicomponents of the following subspaces of $\mathbb{R}^2$.

$$A = (K \times [0,1]) \cup \{0 \times 0\} \cup \{0 \times 1\}$$
$$B = A \cup ([0,1] \times \{0\})$$
$$C = (K \times [0,1]) \cup (-K \times [-1,0]) \cup ([0,1] \times -K) \cup ([-1,0] \times K)$$

Solution. (a) Clearly $x \sim x$ and $x \sim y \implies y \sim x$. Suppose for a contradiction that $x \sim y$ and $y \sim z$, but $x \not\sim z$. Then there exists a separation $X = A \cup B$ such that $x \in A$ and $z \in B$. Since either $y \in A$ or $y \in B$ must be true, one of the hypotheses $x \sim y$ or $y \sim z$ is contradicted.

(b) Let C be a component of X. It suffices to show that $x \sim y$ for each $x, y \in C$. Indeed, if $x \not\sim y$, there would be a separation $X = A \cup B$ which goes against C being connected.

Now assume X is locally connected, which implies that all components of X are open. The openness of the components gives us separations of X which place the quasicomponents in the components, thus they are equal in this case.

(c) It helps to think this way:

- The component containing x is the largest connected set containing x. Similarly with path components.

- The quasicomponent containing x is the intersection of all clopen sets containing x.

Now for A, let L_n be the vertical line segment $\{1/n\} \times [0,1]$. Then each L_n is a component, as well as the two corner points. These are all path connected, so the components are path components. Also, each vertical line is open and closed, and thus each L_n is a quasicomponent.

Let S be a clopen set in A containing the point 0×0. Since S is open and contains every component it intersects, it contains the segments L_n for all $n \geq N$. In particular, S contains almost all the points $\frac{1}{n} \times 1$, so it's limit point 0×1 is also in S since S is closed. This means $\{0 \times 0, 0 \times 1\}$ is a quasicomponent of A!

B adds a horizontal line segment H along the bottom of A, which makes B connected. Therefore, B is the one component/quasicomponent.

Suppose that 0×0 and 0×1 could be joined by a path f in B. Then $g = \pi_2 \circ f$ is continuous and it's range is the entire interval $[0,1]$ by the intermediate value theorem. Now choose $t_0 < 1$ such that $g(t) \geq 1/2$ for all $t \geq t_0$. Then the function f can be restricted to the domain $[t_0, 1]$ to create a path from $f(t_0)$ to $f(1)$ which is contained in A, which is impossible since $\{0 \times 1\}$ is a path component in A. Therefore, $\{0 \times 1\}$ is a path component of B.

The set C is made up of segments I_n, J_n, K_n, L_n in quadrants 1,2,3,4 respectively. We will now show that C is connected, and thus quasiconnected. Suppose that C had a separation $C = A \cup B$, and assume $\{0, 1\} \in A$. Since A is open, it must contain all but finitely many segments L_n. Since A is closed, A must contain it's limit points of the form $\{0, 1/n\}$, and thus every segment I_n. The process can be repeated for the remaining quadrants.

On the other hand, the path components of C are the individual line segments. $\square$