

MEAM 620

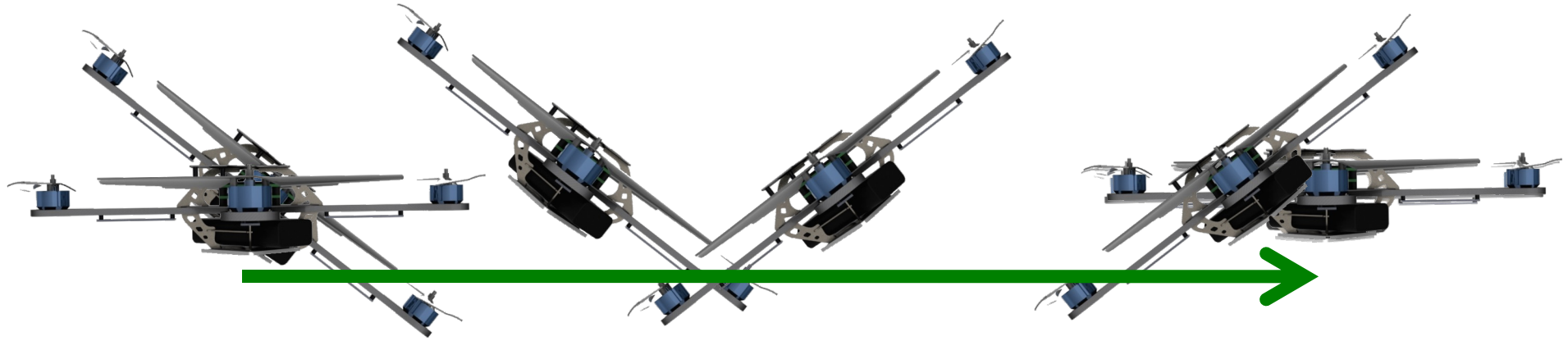
QUADROTOR DYNAMICS &
STATE-SPACE SYSTEM MODELING



What we'll Cover Today

- Newton-Euler equations of motion
 - Equations of motion for the quadrotor
- State-space system modeling & stability
 - State-space model for the quadrotor
- First assignment

Motivating Example



Newton-Euler Equations of Motion

Forces & Linear Momentum For Rigid Bodies

The rate of change of linear momentum \mathbf{L} in an inertia frame A for a rigid body B equals the net applied force \mathbf{F} .

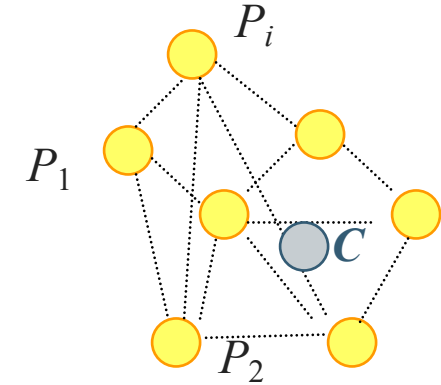
$$\mathbf{F} = \frac{{}^A d\mathbf{L}}{dt} \quad \mathbf{L} = m {}^A \mathbf{v}^C$$

Where m is the total mass and ${}^A \mathbf{v}^C$ is the velocity of the center of mass, a point C located at \mathbf{r}_C .

center of mass:
$$\mathbf{r}_C = \frac{1}{m} \sum_{i=1}^N m_i \mathbf{p}_i$$

Net force:
$$\mathbf{F} = \sum_{i=1}^N \mathbf{F}_i$$

then:
$$\mathbf{F} = m \frac{{}^A d\mathbf{v}^C}{dt} \quad \longrightarrow \quad \mathbf{F} = m {}^A \mathbf{a}^C$$



The center of mass of a rigid body moves like a point mass m driven by a force \mathbf{F} .

Moments & Angular Momentum for a Rigid Body

The rate of change of angular momentum \mathbf{H} in an inertial frame A for a rigid body B relative to point C equals the net moment \mathbf{M} about C due to applied forces relative to C.

$$\mathbf{M}_C^B = \frac{{}^A d {}^A \mathbf{H}_C^B}{dt}$$

*Net moment on
body B around
point C from all
external forces
and torques*

$${}^A \mathbf{H}_C^B = \mathbf{I}_C \cdot {}^A \boldsymbol{\omega}^B$$

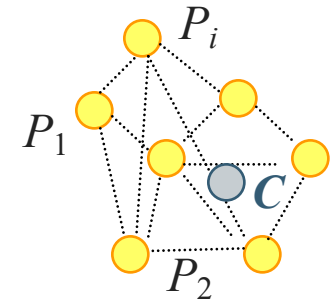
*Angular
momentum of
body B about
point C in frame A*

inertia tensor

$$\mathbf{I}_C = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix}$$

$$I_{xx} = \sum_{k=1}^N m_k (y_k^2 + z_k^2)$$

$$I_{xy} = -\sum_{k=1}^N m_k (x_k y_k)$$



Principal Axes

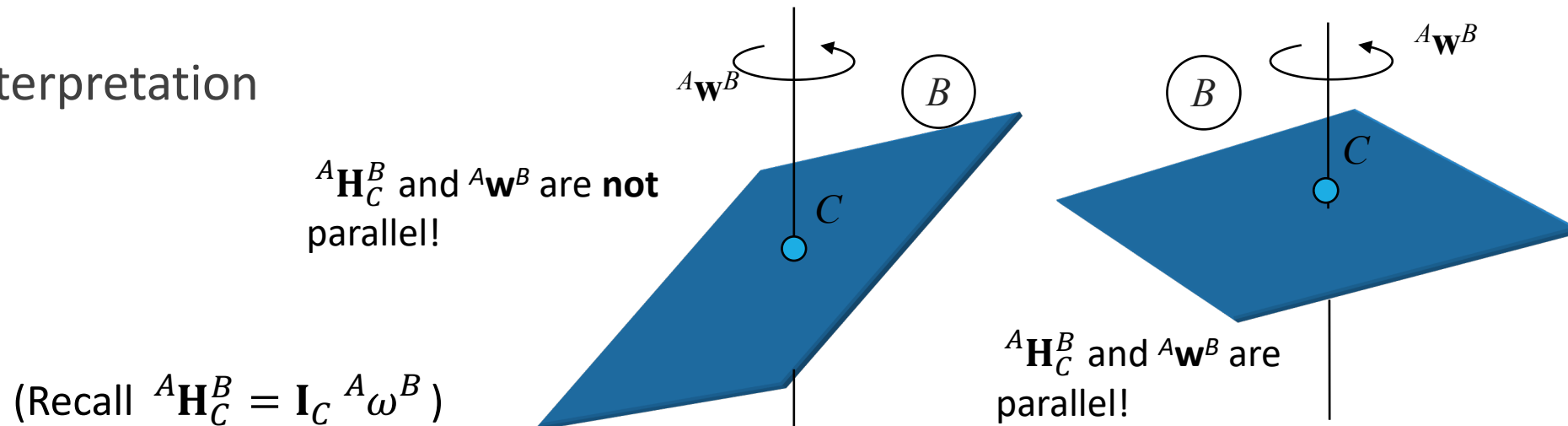
Principal axis of inertia

- \mathbf{u} is a unit vector along a principal axis if $\mathbf{I}\mathbf{u}$ is parallel to \mathbf{u}
- You can always find 3 independent principal axes!
- Axes of symmetry are always principle axes.

Principal moment of inertia

- The moment of inertia with respect to a principal axis, $\mathbf{u}^T \mathbf{I} \mathbf{u}$, is called a principal moment of inertia

Physical interpretation



Euler's Equations

What is the correct rotational analog to $\mathbf{F} = m\mathbf{a}$?

$$\mathbf{M}_C^B = \frac{{}^A d \mathbf{H}_C^B}{dt} = \frac{{}^B d \mathbf{H}_C^B}{dt} + \underbrace{{}^A \boldsymbol{\omega}^B \times \mathbf{H}_C^B}_{\text{differentiating in a moving frame}}$$

$$\mathbf{H}_C^B = \mathbf{I}_C \cdot \boldsymbol{\omega}^B$$

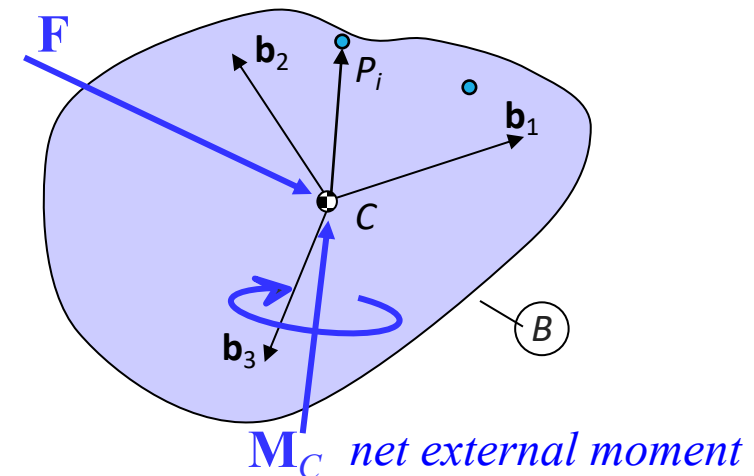
Not constant in inertial coordinates!

Using body-fixed coordinates, \mathbf{I}_C is constant.

$$\frac{{}^B d \mathbf{H}_C^B}{dt} = \mathbf{I}_C \cdot {}^A \dot{\boldsymbol{\omega}}^B$$

$${}^A \boldsymbol{\omega}^B \times \mathbf{H}_C^B = {}^A \boldsymbol{\omega}^B \times \mathbf{I}_C \cdot \boldsymbol{\omega}^B$$

Euler's Equations: $\mathbf{I}_C {}^A \dot{\boldsymbol{\omega}}^B + {}^A \boldsymbol{\omega}^B \times \mathbf{I}_C \boldsymbol{\omega}^B = \mathbf{M}_C^B$



Euler's Equations

Define a body fixed frame with $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ all along principal axes.

From Euler's Equation

$$\mathbf{I}_C \cdot {}^A\dot{\boldsymbol{\omega}}^B + {}^A\boldsymbol{\omega}^B \times \mathbf{I}_C \cdot {}^A\boldsymbol{\omega}^B = \mathbf{M}_C^B$$

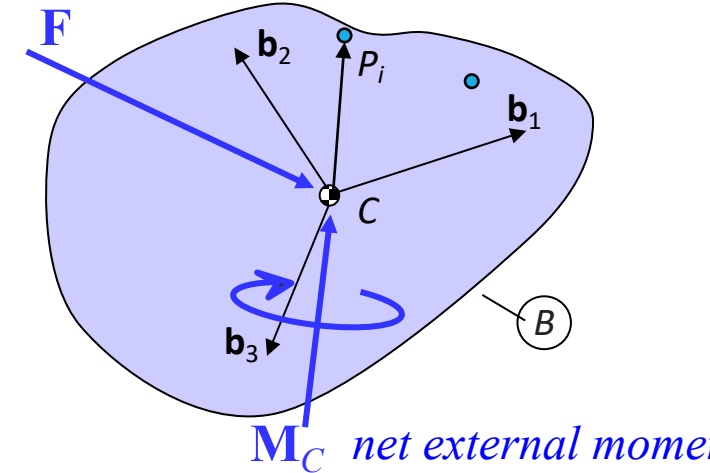
$${}^A\boldsymbol{\omega}^B = \omega_1 \mathbf{b}_1 + \omega_2 \mathbf{b}_2 + \omega_3 \mathbf{b}_3$$

$$\mathbf{M}_C^B = M_{C1} \mathbf{b}_1 + M_{C2} \mathbf{b}_2 + M_{C3} \mathbf{b}_3$$

$$\mathbf{I}_C = \text{diag}(I_{11}, I_{22}, I_{33})$$

Then, traditional matrix form for Euler's Equations

$$\begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} + \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix} \begin{bmatrix} \overset{p}{\omega}_1 \\ \overset{q}{\omega}_2 \\ \overset{r}{\omega}_3 \end{bmatrix} = \begin{bmatrix} M_{C,1} \\ M_{C,2} \\ M_{C,3} \end{bmatrix}$$



Examples

Football thrown expertly.



Is the angular momentum constant?

$$\overset{0}{\cancel{\mathbf{M}_C^B}} = \frac{d}{dt} {}^A\mathbf{H}_C^B$$

Is the angular velocity constant?

$$\mathbf{I}_C {}^A\dot{\boldsymbol{\omega}}^B + \left\{ \overset{0}{\cancel{{}^A\boldsymbol{\omega}^B \times (\mathbf{I}_C {}^A\boldsymbol{\omega}^B)}} \right\} \overset{0}{\cancel{= \mathbf{M}_C^B}}$$

Examples

Football thrown poorly.



Is the angular momentum constant?

$$\overset{0}{\cancel{\mathbf{M}_C^B}} = \frac{{}^A d \mathbf{{}^A H}_C^B}{dt}$$

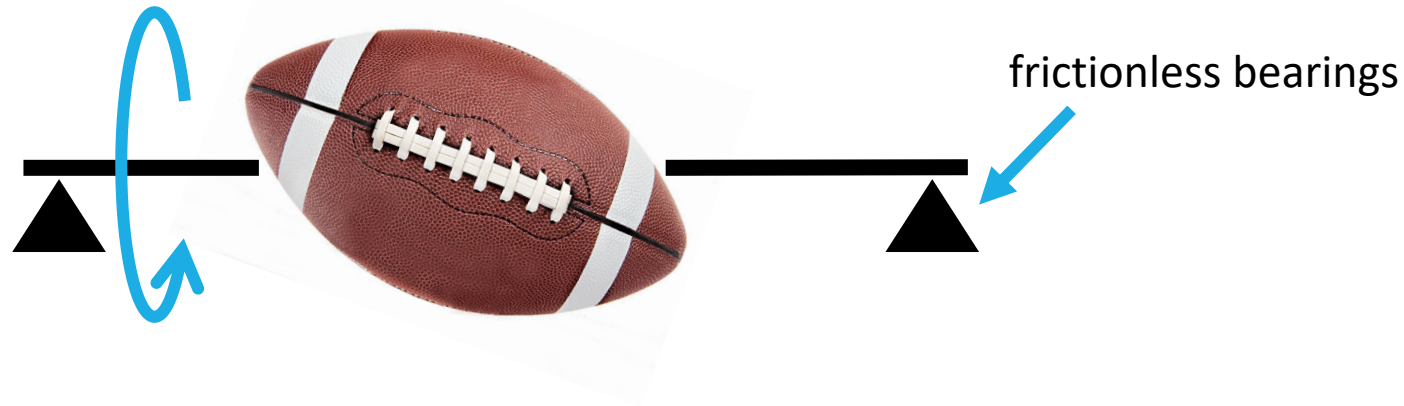
Is the angular velocity constant?

$$\mathbf{I}_C \mathbf{{}^A \dot{\omega}^B} + \left\{ \mathbf{{}^A \omega^B} \times (\mathbf{I}_C \mathbf{{}^A \omega^B}) \right\} \overset{0}{\cancel{= \mathbf{M}_C^B}}$$

not zero!

Examples

Football skewered on a stick.



Is the angular momentum constant?

not zero!

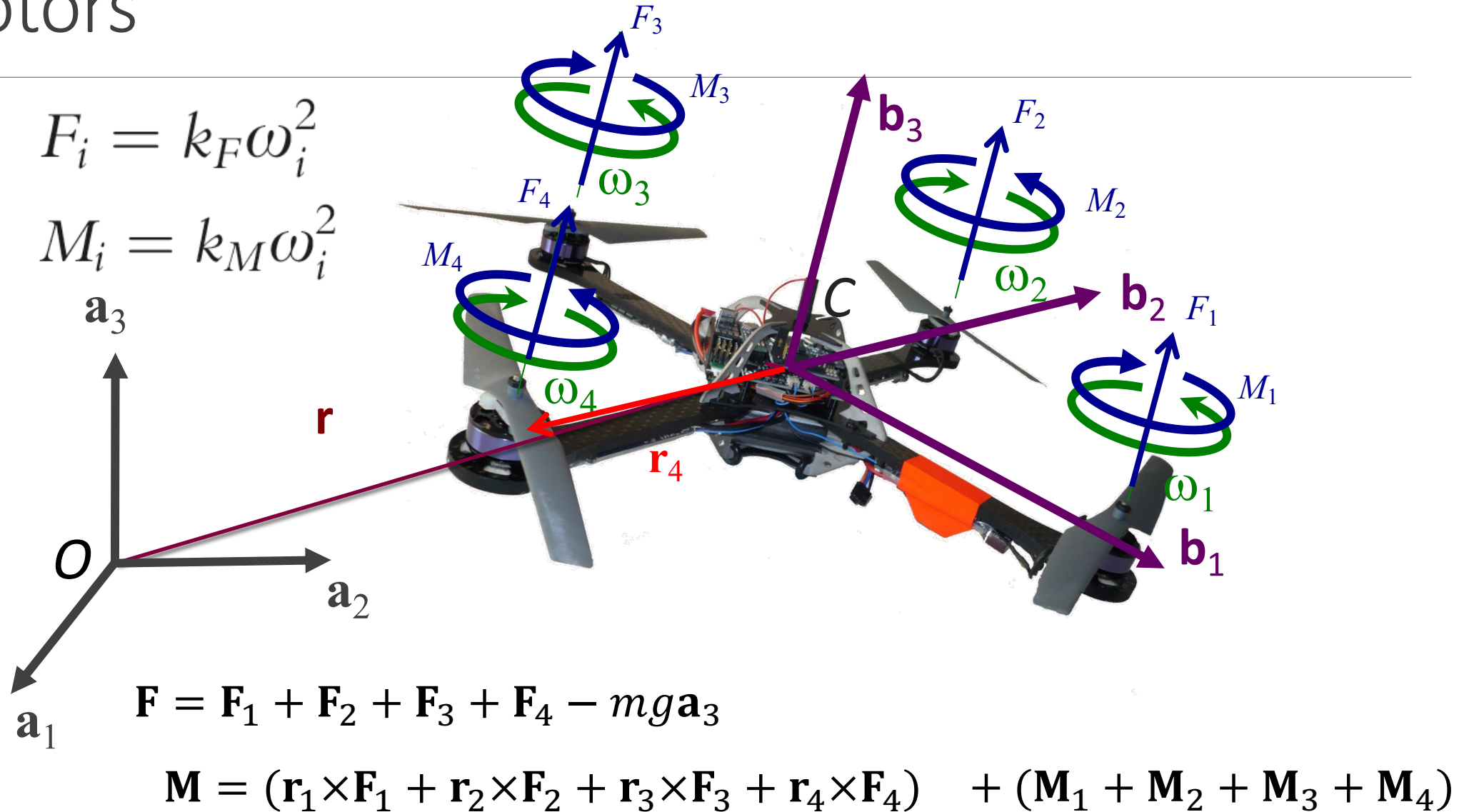
$$\mathbf{M}_C^B = \frac{d}{dt} {}^A\mathbf{H}_C^B$$

Is the angular velocity constant?

$$\mathbf{I}_C {}^A\dot{\omega}^B + {}^A\omega^B \times (\mathbf{I}_C \cdot {}^A\omega^B) = \mathbf{M}_C^B$$

Application to Quadrotors

Quadrotors



Newton-Euler Equations for a Quadrotor



$${}^A\boldsymbol{\omega}^B = p \mathbf{b}_1 + q \mathbf{b}_2 + r \mathbf{b}_3$$

In inertial frame

$$m\ddot{\mathbf{r}} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + R \begin{bmatrix} 0 \\ 0 \\ \boxed{F_1 + F_2 + F_3 + F_4} \end{bmatrix}$$

\mathbf{u}_1

In body frame

$$I \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} \boxed{L(F_2 - F_4)} \\ \boxed{L(F_3 - F_1)} \\ \boxed{M_1 - M_2 + M_3 - M_4} \end{bmatrix} - \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times I \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

\mathbf{u}_2

Newton-Euler Equations for a Quadrotor

Recall that $\mathbf{F}_i = k_F \omega_i^2$ and $\mathbf{M}_i = k_M \omega_i^2$

Let $\gamma = \frac{k_M}{k_F} = \frac{\mathbf{M}_i}{\mathbf{F}_i} \Leftrightarrow \mathbf{M}_i = \gamma \mathbf{F}_i$

$$I \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} L(F_2 - F_4) \\ L(F_3 - F_1) \\ M_1 - M_2 + M_3 - M_4 \end{bmatrix} - \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times I \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$I \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & L & 0 & -L \\ -L & 0 & L & 0 \\ \gamma & -\gamma & \gamma & -\gamma \end{bmatrix}}_{\mathbf{u}_2} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} - \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times I \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Inputs

Putting everything together, we have inputs:

$$\mathbf{u} = \begin{bmatrix} \underline{u_1} \\ u_2 \end{bmatrix} = \begin{bmatrix} \underline{\text{thrust}} \\ \text{moment about } x \\ \text{moment about } y \\ \text{moment about } z \end{bmatrix}$$

$$\mathbf{u} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & L & 0 & -L \\ -L & 0 & L & 0 \\ \gamma & -\gamma & \gamma & -\gamma \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} \quad F_i = k_F \omega_i^2$$



Note: All quantities are in the body frame!