Ch 1. Curves on a Surface

X: surface.

§1 Topological invariants:

- Betti #: $b_i(X) = b_{4-i}(X)$.
- Poincaré duality: $H_2(X,\mathbb{Z}) \simeq H^2(X,\mathbb{Z})$
- Intersection pairing: $H^2(X,\mathbb{Z}) \otimes H_2(X,\mathbb{Z}) \to H^4(X,\mathbb{Z}) \simeq \mathbb{Z}$
  - over $\mathbb{R}$, the intersection pairing is specified by $b_2(X)$ and $b_2^+(X)$.
  - If $X = \mathbb{P}^2$, or the universal covering of X is the unit ball in $\mathbb{C}^2$, (there are finite # of examples)
    then $H^2(X,\mathbb{R}) = \mathbb{R}$.

    If X does not belong to the above list,
    it then follows from the classification of quadratic forms over $\mathbb{Z}$,
    the intersection pairing on $H_2(X,\mathbb{Z})$ mod torsion is specified by its rank, signature and type.

    Type: If $\exists\, \alpha \in H_2(X,\mathbb{Z})$, s.t. $\alpha^2 \equiv 1 \mod 2$, it is of Type I.
    If $\nexists$ ———, it is of Type II.

    Wu formula: $\alpha^2 \equiv \alpha \cdot [K_X] \mod 2$.

    $\rightsquigarrow \exists\, \alpha \in H_2(X,\mathbb{Z})$ s.t. $\alpha^2 \equiv 1 \mod 2 \iff$ the image of $[K_X]$ in $H_2(X;\mathbb{Z})$ mod torsion is not divisible by two.

Holomorphic invariants:

- Irregularity $q(X)$: $q(X) = \dim_{\mathbb{C}} H^0(X,\Omega_X^1) = \dim_{\mathbb{C}} H^1(X,\mathcal{O}_X)$.

  $\Rightarrow q(X)$ is the # of independent holomorphic 1-forms on X.

  The second equality comes from <u>Hodge theory</u>, thus does not hold for an arbitrary compact complex surface or a surface defined over a field of positive char.

  In this book X is required to be a holomorphic submanifold of $\mathbb{P}^N$ for some N. i.e. algebraic surface.

  The "correct" definition should be $\dim H^1(X,\mathcal{O}_X)$
- Geometric genus $p_g(X)$: $p_g(X) = \dim_{\mathbb{C}} H^0(X,\Omega_X^2) = \dim_{\mathbb{C}} H^2(X,\mathcal{O}_X)$.

  $\Rightarrow p_g(X)$ is the # of independent holomorphic 2-form of X.

  The second equality follows from Serre's duality. thus holds generally.
- Additional invariants: $h^{1,1}(X) = \dim H^1(X;\Omega_X^1)$

  $c_1^2(X) = [K_X]^2$

Relations:

- $b_1(X) = 2q(X)$.
- $b_2(X) = 2p_g(X) + h^{1,1}(X)$. } Hodge theory
- $b_2^+(X) = 2p_g(X) + 1$. — Hodge index theorem for a surface.
- Euler characteristic:
$$\chi(X) = 1 - b_1(X) + b_2(X) - b_3(X) + 1$$
$$= 2 - 2b_1(X) + b_2(X) = 2 - 4q + 2p_g(X) + h^{1,1}(X).$$
- Holomorphic Euler characteristic:
$$\chi(\mathcal{O}_X) = h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X) = 1 - q(X) + p_g(X).$$
- Noether's formula: $c_1^2(X) + c_2(X) = 12\,\chi(\mathcal{O}_X)$.
$$\iff [K_X]^2 + \chi(X) = 12(1 - q(X) + p_g(X))$$
- Hirzebruch signature theorem:
$$b_2^+(X) - b_2^-(X) = \tfrac{1}{3}(c_1^2(X) - 2c_2(X))$$

Exercise 1: $b_2^+(X) = 2p_g(X) + 1$

$b_2(X) = 2p_g(X) + h^{1,1}(X)$

$c_1^2(X) = [K_X]^2$. $c_2(X) = \chi(X)$.

$b^-(X) = b_2(X) - b_2^+(X) = h^{1,1}(X) - 1$.

$\Rightarrow b_2^+(X) - b_2^-(X) = 2p_g(X) - h^{1,1}(X) + 2$

$$\tfrac{1}{3}(c_1^2(X)-2c_2(X)) = \tfrac{1}{3}(12\chi(\mathcal{O}_X) - 3\chi(X)) \quad \text{by Noether's formula}$$
$$= 4\chi(\mathcal{O}_X) - \chi(X)$$
$$= 4 - 4q(X) + 4p_g(X) - (2 - 4q + 2p_g(X) + h^{1,1}(X))$$
$$= 2 + 2p_g(X) - h^{1,1}(X).$$
$$\Rightarrow b_2^+(X) - b_2^-(X) = \tfrac{1}{3}(c_1^2(X) - 2c_2(X))$$

- Plurigenera: $P_n(X) = \dim H^0(X, K_X^{\otimes n})$, for $n \geq 1$. $P_1(X) = p_g(X)$.

§2:

|| Smooth curve : - a reduced irreducible (rd.ir.) curve C on X is an ir. holo. subvariety of complex dimension 1.
$\Rightarrow$ locally C is described as $\{f(z_1, z_2) = 0\}$, where f is a holo. fcn. of z_1 & z_2.
- C is said to be a smooth curve if it is a holo. submanifold of X. i.e. no singularities.

|| Divisor : a finite formal sum $\sum_i n_i C_i$ of distinct ir. curves (not nec. smooth).
The divisor is effective if $n_i \geq 0$ for all i. write $D \geq 0$ if D is eff.
$D_1 \geq D_2$ if $D_1 - D_2 \geq 0$.

|| Line bundle $\mathcal{O}_X(D)$:
the associated sections on U are mero. fcn.s g on U s.t. $(g) + D|_U \geq 0$.
If $D \geq 0$, then 1 defines a global section $\mathcal{O}_X(D)$.

|| Linearly equivalent :
D_1 & D_2 is said to be linearly equivalent if $\mathcal{O}_X(D_1) \simeq \mathcal{O}_X(D_2)$
$\Leftrightarrow D_1 - D_2 = (f)$, f is a global mero. fcn. on X.

|| Some complex linear algebra. V: real vector space.
- Almost complex structure : $I \in \mathrm{End}_{\mathbb{R}}(V)$ s.t. $I^2 = -\mathrm{id}$.
- Compatible with the scalar product $\langle\,,\,\rangle$: if $\langle I(v), I(w)\rangle = \langle v, w\rangle$ for all $v, w \in V$.
- Fundamental form on $(V, \langle\,,\,\rangle, I)$: $\omega := -\langle \cdot, I(\cdot)\rangle$.
$V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$
- $(V^*)^{1,0} = \{f \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \mid f(I.v) = i f(v)\} = (V^{1,0})^*$
$(V^*)^{0,1} = \{f \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \mid f(I.v) = -i f(v)\} = (V^{0,1})^*$.
$$\left.\begin{array}{l} V^{1,0} = \{v \in V_{\mathbb{C}} \mid I(v) = i \cdot v\} \\ V^{0,1} = \{v \in V_{\mathbb{C}} \mid I(v) = -i \cdot v\} \end{array}\right\} \Rightarrow V = V^{1,0} \oplus V^{0,1}$$
- $\dim V = d$. $\Lambda^* V = \bigoplus_{k=0}^{d} \Lambda^k V$
$\Lambda^* V_{\mathbb{C}} = \bigoplus_{k=0}^{d} \Lambda^k V_{\mathbb{C}}$
$\Lambda^{p,q} V := \Lambda^p V^{1,0} \otimes_{\mathbb{C}} \Lambda^q V^{0,1}$, for V endowed with an almost complex structure I. thus $\dim V$ is even. say $d = 2n$.

Prop: For real vector space V endowed with an almost complex structure I. one has:
i) $\Lambda^{p,q} V$ is in a canonical way a subspace of $\Lambda^{p+q} V_{\mathbb{C}}$.
ii) $\Lambda^k V_{\mathbb{C}} = \bigoplus_{p+q=k} \Lambda^{p,q} V$.
iii) Complex conjugation on $\Lambda^* V_{\mathbb{C}}$ defines a $\mathbb{C}$-antilinear isomorphism $\Lambda^{p,q} V \cong \Lambda^{q,p} V$, i.e. $\overline{\Lambda^{p,q} V} = \Lambda^{q,p} V$.
iv) The exterior product is of bidegree $(0,0)$, i.e. $(\alpha, \beta) \mapsto \alpha \wedge \beta$ maps $\Lambda^{p,q} V \times \Lambda^{r,s} V$ to the subspace $\Lambda^{p+r, q+s} V$.

If $\{x_i, y_i = I(x_i)\}$ is a basis for $\mathbb{R}$-space V, then $\{z_j = \frac{1}{2}(x_j - iy_j)\}$ is a basis for $\mathbb{C}$-space $V^{1,0}$.

Def: There're natural projections $\Pi^k : \Lambda^* V_{\mathbb{C}} \to \Lambda^k V_{\mathbb{C}}$.
$\Pi^{p,q} : \Lambda^* V_{\mathbb{C}} \to \Lambda^{p,q} V$.

$$\mathbf{I} := \sum_{p,q} i^{p-q} \cdot \Pi^{p,q} : \Lambda^* V_{\mathbb{C}} \to \Lambda^* V_{\mathbb{C}}.$$

\$\textbf{I}\$

$\Pi^k, \Pi^{p,q}, \mathbf{I}$ are also defined on $\Lambda^* V_{\mathbb{C}}^*$. there's the relation:
$$\mathbf{I}(\alpha)(v_1, \cdots, v_k) = \alpha(\mathbf{I}(v_1), \cdots, \mathbf{I}(v_k)) \quad \text{for } \alpha \in \Lambda^k V_{\mathbb{C}},\ v_i \in V_{\mathbb{C}}.$$

- For $(V, \langle , \rangle)$ equipped with an almost complex structure, the fundamental form ω is real and of type $(1,1)$, i.e. $\omega \in \Lambda^2 V^* \cap \Lambda^{1,1} V^*$.

Pf: ω is in $(V^*)^{\otimes 2}$, then for $v, w \in V$. $\omega(v,w) = -\omega(w,v) \Rightarrow \omega \in \Lambda^2 V^*$
$$(\mathbf{I}\omega)(v,w) = \omega(\mathbf{I}v, \mathbf{I}w)$$
$$= (I(I(v)), I(w)) = \omega(v,w)$$

For $v \in V$, $v = v_{1,0} + v_{0,1}$, $\mathbf{I}v = i \cdot v_{1,0} - i v_{0,1}$ $\Rightarrow \mathbf{I}\omega = \omega$.
$= I(v)$. $\Rightarrow \omega \in \Lambda^{1,1} V^*$. ∎

- Two of the three structures $\{\langle , \rangle, I, \omega\}$ will determine the remaining one.
- Lemma: let $(V, \langle , \rangle)$ be an euclidean vector space endowed with a compatible complex structure. The form $(,) := \langle , \rangle - i\cdot\omega$ is a positive hermitian form on (V, I).

Need to verify $(v, v) > 0$ for $v \neq 0$ in V
$(v, w) = \overline{(w, v)}$
$(I(v), w) = i(v, w)$.

Except $(,)$ on (V, I), we also have the extension of scalar product $\langle , \rangle$ on $V_{\mathbb{C}}$ by
$$\langle v \otimes \lambda, w \otimes \mu \rangle_{\mathbb{C}} = \lambda\bar{\mu} \langle v, w \rangle, \quad v, w \in V, \ \lambda, \mu \in \mathbb{C}.$$

Under the canonical isomorphism $(V, I) \simeq (V^{1,0}, i)$, there is $\frac{1}{2}(,) = \langle , \rangle_{\mathbb{C}}\big|_{V^{1,0}}$

- Given $(V, \langle , \rangle, I)$, and the associated fundamental form ω, the Lefschetz operator $L : \Lambda^* V_{\mathbb{C}}^* \to \Lambda^* V_{\mathbb{C}}^*$ is given by $\alpha \mapsto \omega \wedge \alpha$.

Rmk: i) L is the $\mathbb{C}$-linear extension of the real operator
$$\Lambda^* V^* \to \Lambda^* V^*, \quad \alpha \mapsto \omega \wedge \alpha$$
ii) The Lefschetz operator L induces bijections
$$L^{n-k} : \Lambda^k V^* \xrightarrow{\simeq} \Lambda^{2n-k} V^* \quad \text{for } k \leq n, \ \dim_{\mathbb{R}} V = 2n.$$
proven by $\mathfrak{sl}_2$-representation theory.

- Hodge $*$-operator

Def: $\alpha, \beta \in \Lambda^* V$, $\alpha \wedge *\beta = \langle \alpha, \beta \rangle \text{ vol}$.
It is easy to see that $* : \Lambda^k V \to \Lambda^{d-k} V$.

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The line bundle $\mathcal{O}_X(D)$, again:
A second def.: For an open cover $\{U_i\}$ of X s.t.
(cf. Huybrechts) D is defined by invertible meromorphic function f_i on U_i.
then $\mathcal{O}_X(D)$ is the line bundle defined by

transition functions $\psi_{ij} = f_i \cdot f_j^{-1}$

$\mathcal{O}_X(D)$ is subsheaf to the sheaf K_X of meromorphic sections.

A third def: (cf. Hartshorne) Let X be a scheme. For each open affine subset $U = \operatorname{Spec} A$, let S be the set of elements of A which are not zero divisors, and let $K(U) = S^{-1}A$. We call $K(U)$ the total quotient ring of A.

For each open set U, let $S(U)$ denote the set of elements of $\Gamma(U, \mathcal{O}_X)$ which are not zero divisors in each local ring $\mathcal{O}_x$ for $x \in U$. Then $U \mapsto S(U)^{-1}\Gamma(U, \mathcal{O}_X)$ form a presheaf, whose associated sheaf of rings $\mathcal{K}$ we call the sheaf of total quotient rings of $\mathcal{O}$.

$\Rightarrow \mathcal{K}_x = S_x^{-1}\mathcal{O}_x$

Let D be a Cartier divisor on a scheme X, represented by $\{(U_i, f_i)\}$. We define the subsheaf $\mathcal{O}_X(D)$ of the sheaf of total quotient rings by taking $\mathcal{O}_X(D)$ to be the sub-$\mathcal{O}_X$-mod of $\mathcal{K}$ generated by f_i^{-1} on U_i. This is well-defined, since f_i/f_j is invertible on $U_i \cap U_j$, so f_i^{-1} and f_j^{-1} generate the same $\mathcal{O}_X$-mod.

Prop: Let $f: X \to Y$ be a holomorphic map of connected complex manifolds and suppose that f is dominant, i.e. $f(X)$ is dense in Y. Then the pull-back defines a group hom.

$$f^*: \operatorname{Div}(Y) \to \operatorname{Div}(X).$$

Cor: Let $f: X \to Y$ be a holo. map. If $D \in \operatorname{Div}(Y)$ is a divisor s.t. f^*D is defined then $\mathcal{O}(f^*D) = f^*\mathcal{O}(D)$.

(irrelevant content)

null eins zwei drei vier fünf sechs sieben /ziːbən/ acht neun zehn /ʦ'in/?

elf zwölf dreizehn vierzehn fünfzehn sechzehn siebzehn achtzehn neunzehn zwanzig

einundzwanzig sechsundzwanzig siebenundzwanzig dreißig vierzig

fünfzig sechzig siebzig achtzig neunzig (ein)hundert

Ample / Very ample line bundle:

$f: (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, $\mathcal{G}$ is an $\mathcal{O}_Y$-mod. then $f^*\mathcal{G} \triangleq f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$ is an $\mathcal{O}_X$-mod [inverse image]

Def: An invertible sheaf $\mathcal{L}$ on a nt. scheme X is said to be ample if for every coherent sheaf $\mathcal{F}$ on X, there is an integer $n_0 > 0$ (depending on $\mathcal{F}$) such that for every $n \geq n_0$, the sheaf $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections.

Def: A sheaf $\mathcal{L}$ on X is said to be very ample relative to Y if there is an immersion $i: X \to \mathbb{P}_Y^n$ for some n such that $\mathcal{L} \cong i^*\mathcal{O}(1)$.

$\Rightarrow$ If $Y = \operatorname{Spec} A$, this is the same thing as saying that $\mathcal{L}$ admits a set of global sections $s_0, \dots, s_n$ such that the cor. morphism

$X \to \mathbb{P}^n_A$ is an immersion.

Thm 5.17 in Hartshorne by Serre: Let X be a projective scheme over a nt. ring A. Let $\mathcal{O}(1)$ be a very ample invertible sheaf on X, and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-mod. Then there is an integer n_0 such that for all $n \geq n_0$, the sheaf $\mathcal{F}(n)$ can be generated by a finite number of global sections.

|| Intersection of divisors

Def: Let C_1, C_2 be two curves with no component in common and let $x \in X$. Define $C_1 \cdot_x C_2 = \dim_{\mathbb{C}} \mathcal{O}_{X,x}/(f_1, f_2)$, where f_i is a local equation for C_i at x.

Lemma: The curves C_1 and C_2 meet transversally at the point x if and only if $C_1 \cdot_x C_2 = 1$.

This coincides with the definition in differential topology i.e. $T_xC_1 + T_xC_2 = T_xX$.

Def: $C_1 \cdot C_2 \triangleq \sum_{x \in X} C_1 \cdot_x C_2$.

$= \sum_{x \in C_1 \cap C_2} C_1 \cdot_x C_2$.

C_1 and C_2 have no component in common $\Rightarrow$ There are only finite points in $C_1 \cap C_2$.

Lemma: If C_1 is a smooth irreducible curve and C_1 is not a component of C_2, then $C_1 \cdot C_2 = \deg \mathcal{O}_X(C_2)|_{C_1}$ (degree of complex line bundle on C_1)

Pf: There is the exact sequence:

$$0 \to \mathcal{O}_X(-C_2) \to \mathcal{O}_X \to \mathcal{O}_{C_2} \to 0$$

Suppose C_2 is defined by f_2 at $x \in X$ ($f_2 = 0$ for $x \notin C_2$)

then $\mathcal{O}_{C_2,x} = \mathcal{O}_{X,x}/f_2\mathcal{O}_{X,x}$

and $\mathcal{O}_X(-C_2)_{,x} = f_2\mathcal{O}_{X,x}$.

$\Rightarrow 0 \to \mathcal{O}_X(-C_2) \to \mathcal{O}_X \to \mathcal{O}_{C_2} \to 0$.

Tensor by $\mathcal{O}_{C_1}$ (apply the functor $-\otimes_{\mathcal{O}_X}\mathcal{O}_{C_1}$), the book claims that $-\otimes_{\mathcal{O}_X}\mathcal{O}_{C_1}$ is exact instead of merely right exact, so why?

we get $0 \to \mathcal{O}_X(-C_2)\otimes\mathcal{O}_{C_1} \to \mathcal{O}_{C_1} \to \mathcal{O}_{C_2}\otimes\mathcal{O}_{C_1} \to 0$.

$(\mathcal{O}_{C_2}\otimes\mathcal{O}_{C_1})_x = \mathcal{O}_{C_2,x}\otimes\mathcal{O}_{C_1,x}$ is not null if and only if $x \in C_1 \cap C_2$.

For $x \in C_2 \cap C_1$, $\mathcal{O}_{C_2,x}\otimes\mathcal{O}_{C_1,x} = \mathcal{O}_{X,x}/(f_1, f_2)$, where C_i is defined by f_i at x.

$\Rightarrow \mathcal{O}_{C_2}\otimes\mathcal{O}_{C_1} = \bigoplus_{x \in C_1 \cap C_2} \mathcal{O}_{X,x}/(f_1,f_2)$.

$$\Rightarrow 0 \to \mathcal{O}_X(-C_2)\otimes\mathcal{O}_{C_1} \to \mathcal{O}_{C_1} \to \bigoplus_{x \in C_1 \cap C_2} \mathcal{O}_{X,x}/(f_1,f_2) \to 0.$$

tensor this 👆 sequence by $\mathcal{O}_X(C_2)|_{C_1}$.

we obtain: $0 \to \mathcal{O}_{C_1} \to \mathcal{O}_{C_1}(C_2) \to \bigoplus_{x \in C_1 \cap C_2} \mathcal{O}_{X,x}/(f_1, f_2) \to 0$.

$\Rightarrow$ $\mathcal{O}_{C_1}(C_2)$ has a section vanishes exactly on $C_1 \cdot C_2$ pts counted by multiplicity. (lies in $\ker(\mathcal{O}_{C_1}(C_2) \to \bigoplus_{x \in C_1 \cap C_2} \mathcal{O}_{X,x}/(f_1,f_2))$)

$\Rightarrow$ $\deg \mathcal{O}_{C_1}(C_2) = C_1 \cdot C_2$. ▨

Lemma: Let L be a line bundle on X. Then there exists two very ample divisor H' and H'' on X such that $L = \mathcal{O}_X(H' - H'')$.

If H is very ample, then $\mathcal{O}_X(H) = i^*\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(1)$ for some immersion $i: X \to \mathbb{P}^n_{\mathbb{C}}$. There is $i^*\mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{O}_X(i^*1)$ $\Rightarrow$ H is linear equivalent to i^*1 on X.

Thm: There is a unique symmetric bilinear pairing from $\operatorname{Div} X$ to $\mathbb{Z}$, denoted by $\langle D_1, D_2 \rangle$ which factors through linear equivalence and has the property that $\langle C_1, C_2 \rangle = C_1 \cdot C_2$ for C_1 and C_2 distinct smooth curves meeting transversally.

$D_1 = H_1' - H_1''$, $D_2 = H_2' - H_2''$

$\langle D_1, D_2 \rangle = H_1' \cdot H_2' - H_1' \cdot H_2'' - H_1'' \cdot H_2' + H_1'' \cdot H_2''$

By lemma above, $\langle D_1, D_2 \rangle = \deg \mathcal{O}_X(D_1)|_{H_2'} - \deg \mathcal{O}_X(D_1)|_{H_2''}$ since H_2', H_2'' are smooth.

👆 This is the general formula for $\langle D_1, D_2 \rangle$, denoted by $D_1 \cdot D_2$ later.

Question: For a smooth irr. curve C on X, is there C' linearly equivalent with C but intersects transversally with C ?

i.e. $\exists$ globally defined meromorphic function f such that $(f) + C \pitchfork C$.

$\Rightarrow$ (f) needs to intersect transversally with C at everywhere

Otherwise, how to define $\langle C, C \rangle$ or $C \cdot C$ in the proof ?

The degree of a line bundle L on an irr. curve C is defined equivalently in the following ways: (may not be smooth)

i) As deg. of pullback of L to the normalization $\tilde{C}$.
ii) By writing $L = \mathcal{O}_C(\sum_i n_i p_i)$ where p_i are points in the smooth part of C.
iii) By the exp. sheaf seq. $0 \to \mathbb{Z} \to \mathcal{O}_C \to \mathcal{O}_C^* \to 0$ and the fact that for an irr. curve $H^2(C;\mathbb{Z}) \simeq \mathbb{Z}$.

$\Rightarrow$ If D_2 is irr. curve, $\deg \mathcal{O}_X(D_1)|_{D_2}$

Rmk: ① if H is ample, D is effective and nonzero, then $H \cdot D > 0$.
② if C_1, C_2 are distinct irr. curves, then $C_1 \cdot C_2 \geq 0$. $C_1 \cdot C_2 = 0$ if and only if C_1 and C_2 are disjoint.