$$\mathbb{C}[x_1, \dots, x_n]$$

$$\Rightarrow f/g \in \mathbb{C}[x_1, \dots, x_n] \Rightarrow f/g \in A_{\tau_i}$$

$$A_\sigma = \bigcap A_{\tau_i} \Rightarrow A_\sigma \text{ is int. closed.}$$

■

## §3.1 Orbits

Review:

def of distinguished points: $x_\sigma \in \mathrm{Hom}_{sg}(S_\sigma, \mathbb{C})$ defined by the rule:

$$u \mapsto \begin{cases} 1 & \text{if } u \in \sigma^\perp \\ 0 & \text{otherwise} \end{cases}$$

Torus action on $U_\sigma$: If $\sigma$ is a cone in $N$.

then the action of $T_N$ on $U_\sigma$ is interpreted as

$$\mathrm{Hom}_{gp}(M, \mathbb{C}^*) \times \mathrm{Hom}_{sg}(S_\sigma, \mathbb{C}) \longrightarrow \mathrm{Hom}_{sg}(S_\sigma, \mathbb{C})$$

Claims:

- $\Delta$ is a fan in $N$.
  With the torus action. $X(\Delta) = \coprod_{\tau \prec \Delta} O_\tau$. where $O_\tau$ is the orbit of the distinguished point $x_\tau$.
- Let the closure of $O_\tau$ be denoted by $V(\tau)$.
  Then $V(\tau)$ is a closed subvariety of $X(\Delta)$. and it's also a toric variety.
  $V(\tau)$ is the disjoint union of those orbits $O_\gamma$ for which $\gamma$ contains $\tau$ as a face.

Examples:

① $X = \mathbb{C}^n$. $T = (\mathbb{C}^*)^n$.

$$T \times X \longrightarrow X$$
$$(t, x) \longmapsto tx$$

Orbits are the sets:

$\{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i = 0 \text{ for } i \in I.\ z_i \neq 0 \text{ for } i \notin I\}$, $I$ ranges over all subsets of $\{1, \dots, n\}$

This is the orbit $O_\tau$ of $x_\tau$, where $\tau$ is the cone gen. by $\{e_i\}_{i \in I}$.

Consider the cone $\tau$ gen by $\{e_1, \dots, e_k\}$ in $N$.

$\tau^\perp = \langle e^*_{k+1}, \dots, e^*_n \rangle$. $\Rightarrow x_\tau = (0, \dots, 0, 1, \dots, 1) \in U_\tau = \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$

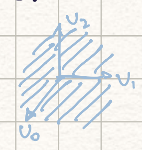
$O_\tau = T_N \cdot \{x_\tau\} = \{(z_1, \dots, z_n) \mid z_i = 0,\ 1 \le i \le k;\ z_i \neq 0,\ k < i \le n\}$

The case of $\tau_I = \langle e_i \rangle_{i \in I}$ is similar

② $\underline{X = \mathbb{P}^n}$.

The corresponding fan consists of cones gen. by all proper subsets of $\{v_0, v_1, \cdots, v_n\}$ where $\{v_1, \cdots, v_n\}$ gen. the lattice and $v_0 + v_1 + \cdots + v_n = 0$.

If $\tau$ is the cone gen. by $\{v_i\}_{i \in I}$. $I \subset \{0, 1, \cdots, n\}$ $\quad \mathbb{P}^2$




then $V(\tau) = \bigcap_{i \in I} \{z_i = 0\}$

$O_\tau$ = pts in $V(\tau)$ with other coordinates nonzero.

Verification of the construction of corresponding fan:

WLOG. Let $v_i = e_i$ for $i = 1, 2, \cdots, n$. then $v_0 = -(v_1 + \cdots + v_n)$
$= -(e_1 + \cdots + e_n)$.

$$\begin{tikzcd} & U_i \arrow[r, "\varphi_i"] & \mathbb{C}^n \\ U_i \cap U_j \arrow[ur] \arrow[dr] & & \\ & U_j \arrow[r, "\varphi_j"] & \mathbb{C}^n \end{tikzcd}$$

$\mathcal{O}(\varphi_i(U_i)) = \mathbb{C}[X_1, \cdots, X_n]$

$\varphi_{ij}: \varphi_i(U_i \cap U_j) \longrightarrow \varphi_j(U_i \cap U_j)$

$\mathcal{O}(\varphi_j(U_j)) = \mathbb{C}[\varphi_{ij}^1(x_1, \cdots, x_n), \cdots, \varphi_{ij}^n(x_1, \cdots, x_n)]$

Calculate the cone $\tau = \langle v_0, v_1, \cdots, v_{n-1} \rangle$ as an example.

For $k_1 e_1^* + \cdots + k_n e_n^*$ in $M$

$\langle v_0, k_1 e_1^* + \cdots + k_n e_n^* \rangle = -(k_1 + \cdots + k_n) \geq 0$

$\langle v_i, k_1 e_1^* + \cdots + k_n e_n^* \rangle = k_i \geq 0$. $i = 1, \cdots, n-1$

$\Rightarrow \tau^\vee = \langle -e_n^*, e_1^* - e_n^*, \cdots, e_{n-1}^* - e_n^* \rangle$

$\mathbb{C}[S_\tau] = \mathbb{C}[\frac{X_1}{X_n}, \cdots, \frac{X_{n-1}}{X_n}, \frac{1}{X_n}]$

$X(\Delta)$ is to glue $\mathbb{C}[\frac{1}{X_i}, \frac{X_1}{X_i}, \cdots, \frac{X_{n-1}}{X_i}, \frac{X_n}{X_i}]_{i=1,2,\cdots,n}$ and $\mathbb{C}[X_1, \cdots, X_n]$ together.

In $\mathbb{P}^2$ case, they're $\mathbb{C}[X^{-1}, X^{-1}Y]$. $\mathbb{C}[Y^{-1}, XY^{-1}]$. $\mathbb{C}[X, Y]$
$\qquad\qquad\qquad\qquad \updownarrow \qquad\qquad \updownarrow \qquad\qquad \updownarrow$
$\qquad\qquad\qquad\qquad U_1 \qquad\qquad U_2 \qquad\qquad U_0$

$$\boxed{\begin{array}{l} \mathrm{Spec}\, \mathbb{C}[\frac{1}{X_i}, \frac{X_1}{X_i}, \cdots, \frac{X_{n-1}}{X_i}, \frac{X_n}{X_i}] \text{ is identified with } \{(z_0, \cdots, \underset{i\text{-th}}{1}, \cdots, z_n) \mid z_i \in \mathbb{C}\} \subset \mathbb{C}^{n+1} \\ \mathrm{Spec}\, \mathbb{C}[X_1, \cdots, X_n] \text{ with } z_0 = 1 \text{ in } \mathbb{C}^{n+1}. \end{array}}$$

## General Case:

$\tau$: a cone of a fan $\Delta$.

$N_\tau :=$ sublattice of $N$ gen. by $\tau \cap N$.

$N(\tau) := N/N_\tau$. the quotient lattice.

$M(\tau) := \tau^\perp \cap M$. the dual lattice of $N(\tau)$

$O_\tau = T_{N(\tau)} = \mathrm{Hom}_{gp}(M(\tau), \mathbb{C}^*) = \mathrm{Spec}(\mathbb{C}[M(\tau)]) = N(\tau) \otimes_{\mathbb{Z}} \mathbb{C}^*$ $\quad$ ($X_\tau$ is "perpendicular" to $\tau$)

$O_\tau$ is a torus of dim. $n-k$. $k = \dim(\tau)$.

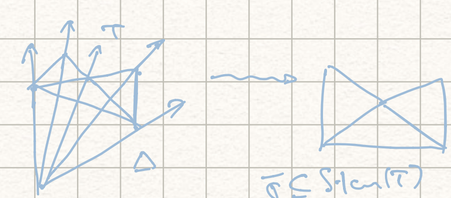
$\boxed{\bar{\sigma}} = \sigma + (N_\tau)_{\mathbb{R}} / (N_\tau)_{\mathbb{R}} \subset N(\tau)_{\mathbb{R}}$.

Def (Star($\tau$)): $\{\bar{\sigma} \mid \tau < \sigma\}$ forms a fan in $N(\tau)$, denoted by Star($\tau$).

$V(\tau) := X(\mathrm{Star}(\tau))$.

$V(\tau) = \bigcup_{\tau < \sigma} U_\sigma(\tau)$ is an open covering of $V(\tau)$

The open subset cor. to $\bar{\sigma}$ in $N(\tau)$, by def.




$U_\sigma(\tau) = \mathrm{Spec}(\mathbb{C}[\bar{\sigma}^\vee \cap M(\tau)]) = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap \tau^\perp \cap M])$

If $\sigma = \tau$. $U_\tau(\tau) = O_\tau$.

the closed embedding $V(\tau) \hookrightarrow X(\Delta)$:

(First the affine open case) $\quad U_\sigma(\tau) = \mathrm{Hom}_{sg}(\sigma^\vee \cap \tau^\perp \cap M, \mathbb{C}) \hookrightarrow \mathrm{Hom}_{sg}(\sigma^\vee \cap M, \mathbb{C}) = U_\sigma$. $\Big\}$ $U_\sigma(\tau)$ is closed in $U_\sigma$

extended by zero.

The cor. surj. of rings:

$$\mathbb{C}[\sigma^\vee \cap M] \twoheadrightarrow \mathbb{C}[\sigma^\vee \cap \tau^\perp \cap M]$$
$$\chi^u \mapsto \chi^u \text{ if } u \in \sigma^\vee \cap \tau^\perp \cap M.$$
$$\mapsto 0 \text{ otherwise.}$$

surjective hom. of rings.

**Compatibility:** $\tau < \sigma < \sigma'$

$$\begin{tikzcd} U_\sigma(\tau) \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowleft"] & U_{\sigma'}(\tau) \arrow[d, hook] \\ U_\sigma \arrow[r, hook] & U_{\sigma'} \end{tikzcd}
\qquad
\begin{tikzcd} \mathrm{Hom}_{sg}(\sigma^\vee \cap \tau^\perp \cap M, \mathbb{C}) \arrow[r, "\text{restriction}"] \arrow[d, "\text{extended by zero}"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Hom}_{sg}(\sigma'^\vee \cap \tau^\perp \cap M, \mathbb{C}) \arrow[d] \\ \mathrm{Hom}_{sg}(\sigma^\vee \cap M, \mathbb{C}) \arrow[r, "\text{res}"] & \mathrm{Hom}_{sg}(\sigma'^\vee \cap M, \mathbb{C}) \end{tikzcd}$$

Glue them together: $V(\tau) \hookrightarrow X(\Delta)$.

If $\tau < \tau'$, we have $V(\tau') \hookrightarrow V(\tau)$.
by ext. by zero in $\mathrm{Hom}_{sg}(\sigma^\vee \cap \tau'^\perp \cap M, \mathbb{C}) \hookrightarrow \mathrm{Hom}_{sg}(\sigma^\vee \cap \tau^\perp \cap M, \mathbb{C})$.

$\rightsquigarrow$ Order-reversing cor. of cones $\tau \subset \Delta$ with orbit closures $V(\tau) \subset X(\Delta)$.

Ideal of $V(\tau) \cap U_\sigma$ in $A_\sigma$ is $\bigoplus \mathbb{C}\cdot\chi^u$. of $u \in S_\sigma$ s.t. $\langle u, v \rangle > 0$, $\forall v$ in the relative interior of $\tau$.

This point follows from the surjective hom:

$$\boxed{\begin{array}{l} \mathbb{C}[\sigma^\vee \cap M] \twoheadrightarrow \mathbb{C}[\sigma^\vee \cap \tau^\perp \cap M] \\ \chi^u \mapsto \chi^u \text{ if } u \in \sigma^\vee \cap \tau^\perp \cap M. \\ \quad \mapsto 0 \text{ otherwise.} \end{array}}$$

A singular example:


$v_1 = 2e_1 - e_2$
$v_2 = e_2$

$S_\sigma = \mathbb{C}[X, XY, XY^2]$

$U_\sigma$ is a cone of conic in $\mathbb{C}^3$ defined by $V^2 - UW$

$$U_\sigma = \mathrm{Spec}\left(\mathbb{C}[U, V, W]/(V^2 - UW)\right)$$

$V(\tau_1) = \{U = V = 0\}$
$V(\tau_2) = \{V = W = 0\}$
$V(\sigma) = \{0\}$

**Prop:** (a) $U_\sigma = \coprod_{\tau < \sigma} O_\tau$

(b) $V(\tau) = \coprod_{\gamma > \tau} O_\gamma$.

(c) $O_\tau = V(\tau) \setminus \bigcup_{\gamma \gneq \tau} V(\gamma)$

Pf: (a) A pt. $x \in U_\sigma$ is given by a semigp. hom. $x : \sigma^\vee \cap M \to \mathbb{C}$.

- $x \in T_N = O_{\{0\}}$ iff $x$ does not take on the value $0$.
since pt.s in $T_N$ can be extended to hom. $: M \to \mathbb{C}^*$

In general: sum of two elements of $\sigma^\vee$ cannot be in $x^{-1}(\mathbb{C}^*)$ unless both are in $x^{-1}(\mathbb{C}^*)$

For $p, q \in \sigma^\vee$. $x(p+q) = x(p)x(q) \Rightarrow$ the assertion holds.

By ex. in §1.2 $\Rightarrow x^{-1}(\mathbb{C}^*) = \sigma^\vee \cap \tau^\perp \cap M$ for some face $\boxed{\tau < \sigma}$

$\Rightarrow x^{-1}(\mathbb{C}^*)$ is a pt. of $O_\tau \subset U_\sigma(\tau)$ $(M(\tau) = \sigma^\vee \cap \tau^\perp \cap M)$

$\Rightarrow U_\sigma = \bigsqcup_{\tau < \sigma} O_\tau$

(c) When $\tau = 0$.

we need to show $T_N = X(\Delta) \setminus \bigcup_{\gamma \neq \{0\}} V(\gamma)$

By intersecting with $U_\sigma$

$$\Big(X(\Delta) \setminus \bigcup_{\gamma \neq \{0\}} V(\gamma)\Big) \cap U_\sigma = U_\sigma \setminus \bigcup_{\gamma \neq \{0\}} (V(\gamma) \cap U_\sigma)$$

$$= U_\sigma \setminus \bigcup_{\substack{\gamma \neq \{0\} \\ \gamma < \sigma}} V(\gamma)$$

By (a): $U_\sigma = \bigsqcup_{\gamma < \sigma} O_\gamma = T_N \sqcup \Big(\boxed{\bigsqcup_{\substack{\gamma \neq \{0\} \\ \gamma < \sigma}} O_\gamma}\Big)$, $T_N$ is open in $U_\sigma \Rightarrow \boxed{\bigcup_{\substack{\gamma \neq \{0\} \\ \gamma < \sigma}} V(\gamma)} = \boxed{\bigsqcup_{\substack{\gamma \neq \{0\} \\ \gamma < \sigma}} O_\gamma}$

$\Rightarrow T_N = U_\sigma \setminus \bigcup_{\gamma \neq \{0\}} V(\gamma)$ $\quad U_\sigma \setminus \bigcup_{\substack{\gamma \neq \{0\} \\ \gamma < \sigma}} V(\gamma) = U_\sigma \setminus \bigsqcup_{\substack{\gamma \neq \{0\} \\ \gamma < \sigma}} O_\gamma = T_N$

patch $U_\sigma$ together, we know that $T_N = X(\Delta) \setminus \bigcup_{\gamma \neq \{0\}} V(\gamma)$

For general $V(\tau)$, we consider in $N(\tau)$ instead of $N$.

(b) Consider the intersection with $U_\sigma$.

$V(\sigma) \cap U_\sigma = O_\sigma \cap U_\sigma = \{0\}$, the formula is verified.

For $\tau < \sigma$, suppose the formula is true for faces containing $\tau$.

by (c). $\boxed{O_\tau = V(\tau) \setminus \bigcup_{\gamma \supsetneq \tau} V(\gamma)}$.

and $V(\gamma) = \bigsqcup_{\gamma' \supseteq \gamma} O_{\gamma'}$

$\Rightarrow \bigcup_{\gamma \supsetneq \tau} V(\gamma) = \bigsqcup_{\gamma \supsetneq \tau} O_\gamma$

$\Rightarrow O_\tau = V(\tau) \setminus \bigsqcup_{\gamma \supsetneq \tau} O_\gamma$

$\Rightarrow V(\tau) = \bigsqcup_{\gamma \succ \tau} O_\gamma.$

■

## §3.2 Fundamental groups and Euler characteristics

Main fact: complete toric varieties are simply connected.

Prop: Let $\Delta$ be a fan that contains an $n$-dimensional cone. Then $X(\Delta)$ is simply connected.

Pf:

- The inclusion $T_N \hookrightarrow X(\Delta)$ gives a surjection $\pi_1(T_N) \longrightarrow \pi_1(X(\Delta))$

  Lemma 1: Let $X$ be a normal variety and take $x \in X$. Then there is a basis $\{V_\alpha \mid \alpha \in A\}$ of open nbhd of $x$ in $X$, such that $V_\alpha \setminus (V_\alpha \cap \operatorname{Sing} X)$ is connected for all $\alpha$.

  Lemma 2: If $X$ is a normal variety. and $i: U \to X$ is the inclusion of an open subvariety $U$. then $U \times_X \tilde{X}$ is path connected. $\tilde{X}$ the universal covering of $X$.

  Lemma 3: Let $X, Z$ be topological spaces and assume $X$ has a universal covering space $p: \tilde{X} \to X$. Let $f: Z \to X$ be continuous. If $Z \times_X \tilde{X}$ is path connected. then $f_*: \pi_1(Z) \longrightarrow \pi_1(X)$ is a surjection.

- There's a canonical isomorphism $N \xrightarrow{\cong} \pi_1(T_N)$ given by $v \longmapsto (\lambda_v : \mathbb{C}^* \to T_N)$ restrict $\lambda_v$ on $S^1 \subset \mathbb{C}^*$

  $T_N \cong \operatorname{Hom}(M, \mathbb{C}^*) \cong N \otimes_{\mathbb{Z}} \mathbb{C}^*$, $\mathbb{C}^* = \mathbb{R}_{>0} \times S^1$ $\Big\} \Rightarrow (N \otimes \mathbb{R}_{>0}) \times (N \otimes \mathbb{R})$ is a universal covering of $T_N$. denoted by $Y$.

  $\lambda_v : \mathbb{C}^* \to T_N$ $\quad \lambda_v(z)(e_i^*) = z^{\langle e_i^*, v\rangle} \Rightarrow \lambda_v(z)$ is $\sum_i z^{\langle e_i^*, v\rangle} e_i$ in $N \otimes \mathbb{C}^*$, if $v = (a_1, \dots, a_n)$ $\quad \lambda_v(S^1) = \{(e^{ia_1\theta}, \dots, e^{ia_n\theta})\}$

  For a loop $\gamma$ in $T_N$ based at $x_0$

  consider the lifting of $\gamma$ in $Y$, suppose the start point and final point are $\tilde{x}_0$ and $\tilde{x}_1$ resp.

  $v \triangleq \tilde{x}_1 - \tilde{x}_0$ must be in $N \otimes \mathbb{Z} = N$, and $\lambda_v(S^1)$ as a loop in $T_N$ is homotopic to $\gamma$. since their liftings are homotopic in $Y$. Therefore $N \cong \pi_1(T_N)$

- In $U_\sigma$, the loops defined by $v \in \sigma \cap N$ are all contractable.

  If $v$ is in $\sigma \cap N$ for some cone $\sigma$, the loop can be contracted in $U_\sigma$.
  since $\lim_{z \to 0} \lambda_v(z) = x_\sigma$ exists in $U_\sigma$.
  The contraction is given by $\boxed{\lambda_{v,t}(z) = \begin{cases} \lambda_v(tz), & z \in S^1,\ 0 < t \leq 1. \\ x_\sigma & z \in S^1,\ t = 0 \end{cases}}$

- $\pi_1(X(\Delta))$ is generated by classes of loops in $U_\sigma$ since $N$ is generated by $\sigma \cap N$ as a group

$\Rightarrow \pi_1(X(\Delta))$ is trivial. ■

Cor: If $\sigma$ is a $k$-dimensional cone, then $\pi_1(U_\sigma) \cong \mathbb{Z}^{n-k}$

Pf: $U_\sigma = U_{\sigma'} \times (\mathbb{C}^*)^{n-k}$

or use the fibration $\underset{\text{fibre}}{U_{\sigma'}} \to U_\sigma \xrightarrow{p} \underset{\text{base space}}{T_{N(\sigma)}}$

Fibration $U_{\sigma'} \to U_\sigma \to T_{N(\sigma)}$ gives
a long exact sequence of homotopy groups:

$$\cdots \to \pi_1(U_{\sigma'}) \to \pi_1(U_\sigma) \to \pi_1(T_{N(\sigma)}) \to \pi_0(U_{\sigma'}) \to 0 \ \cdots$$

both $\pi_1(U_{\sigma'})$ and $\pi_0(U_{\sigma'})$ are trivial. then $\pi_1(U_\sigma) \cong \pi_1(T_{N(\sigma)}) = N(\sigma)$

Prop: If $N'$ is the subgroup of $N$ gen. by all $\sigma \cap N$.
then $\pi_1(X(\Delta)) = N/N'$

Pf:
- $\pi_1(U_\sigma) = N/N_\sigma$
- $\{\pi_1(\bigcup U_\sigma)\}_{\bigcup U_\sigma}$ is a direct system by van Kampen theorem.

$$\begin{tikzcd} & \pi_1(U_\sigma) \arrow[dr, "j_1"] & \\ \pi_1(U_{\sigma\cap\tau}) \arrow[ur, "i_1"] \arrow[rr] \arrow[dr, "i_2"'] & & \pi_1(U_\sigma \cup U_\tau) = \pi_1(U_\sigma) * \pi_1(U_\tau) / \mathcal{N} \\ & \pi_1(U_\tau) \arrow[ur, "j_2"'] & \end{tikzcd}$$

$\mathcal{N}$ = the normal subgroup generated by $(j_1 \circ i_1[\omega])(j_2 \circ i_2[\omega])^{-1}$

$S$ = the set of all cones in $\Delta$. $I$ = the power set of $S$. $\alpha < \beta$ iff $\alpha \subseteq \beta$
$A_\alpha := \pi_1\left(\bigcup_{\sigma \in \alpha} U_\sigma\right)$ for $\alpha \in I$, $\varphi_{\alpha\beta} : A_\alpha \to A_\beta$ for $\alpha < \beta$ is defined as the natural hom induced by containing.
van Kampen theorem says that we can have $\varphi_{\alpha\gamma} = \varphi_{\alpha\beta} \circ \varphi_{\beta\gamma}$ for any $\alpha < \beta < \gamma$.

$\Rightarrow \pi_1(X(\Delta)) = \varinjlim \pi_1(U_\sigma) = \varinjlim N/N_\sigma = N/\sum N_\sigma = N/N'$. ■

Prop: If $\sigma$ is an $n$-dimensional cone, then $U_\sigma$ is contractible. Treat pts in $U_\sigma$ as elements in $\mathrm{Hom}_{sg}(S_\sigma, \mathbb{C})$

Pf: $\boxed{H(x \times t)(u) = \begin{cases} t^{\langle u, v\rangle} \cdot x(u), & t > 0. \\ x_\sigma(u) & t = 0 \end{cases}}$

$H(x \times t)$ defines a semigroup hom. for $x \in U_\sigma$.
$v$ is a fixed lattice point in the interior of $\sigma$.
$U_\sigma \times I \to U_\sigma$

For $u = 0$. $H(x \times t)(u) = 1$.
For $u \in S_\sigma \setminus \{0\}$. $\langle u, v\rangle > 0 \Rightarrow \lim_{t \to 0} H(x \times t)(u) = 0$
$\Rightarrow H(x \times t) \to x_\sigma$ as $t \to 0$.
$\Rightarrow H(x \times t)$ is continuous as a map on $U_\sigma \times [0,1] \subset \mathbb{C}^m \times [0,1]$ for some $m$.

$U_\sigma$ is contracted to $x_\sigma$.

Therefore. $U_\sigma$ is contractible. ■

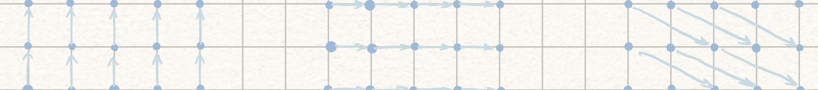
Cor: If $\dim(\sigma) = k$, then $O_\sigma \subset U_\sigma$ is a deformation retract.

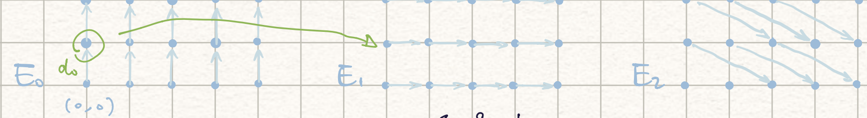
Cor: $H^i(U_\sigma; \mathbb{Z}) \simeq \Lambda^i(M(\sigma))$

$H^1(U_\sigma; \mathbb{Z}) \simeq \mathrm{Hom}(\underset{\| \atop N(\sigma)}{H_1(U_\sigma; \mathbb{Z})}, \mathbb{Z}) = M(\sigma)$ by universal coefficient theorem.
$H^i(U_\sigma; \mathbb{Z}) \simeq \Lambda^i(H^1(U_\sigma; \mathbb{Z}))$ by Künneth formula.

Def: (Spectral Sequence)

A spectral sequence is a sequence $\{E_r, d_r\}$ $(r \geq 0)$ of bigraded groups $E_r = \bigoplus_{p,q \geq 0} E_r^{p,q}$ together with differentials $d_r : E_r^{p,q} \to E_r^{p+r, q-r+1}$, $d_r^2 = 0$.
such that $H^\bullet(E_r) = E_{r+1}$. $E_r^{p,q}$ abuts to $\boxed{E_\infty^{p,q}}$ is denoted by $\boxed{E_r^{p,q} \Rightarrow E_\infty^{p,q}}$






$d^2=0,\ \delta^2=0$
$d\delta+\delta d=0$

If we have a complex of sheaves $\{\mathcal{K}^q, d\}$, then the double complex $\{\mathcal{C}^p(\mathfrak{U},\mathcal{K}^q);\delta,d\}$ defines a spectral sequence.

Čech complex for the sheaf $\mathcal{K}^q$.
$\delta$ is the diff. for Čech complex

such that
$$E_0^{p,q}=\mathcal{C}^p(\mathfrak{U},\mathcal{K}^q)$$
$$E_1^{p,q}=\mathcal{C}^p(\mathfrak{U},\mathcal{H}^q)$$
$$E_2^{p,q}=H^p(\mathfrak{U},\mathcal{H}^q),\quad \mathcal{H}^q_x=\varinjlim_{U\ni x}\frac{\operatorname{Ker}\{d:\mathcal{K}^q(U)\to\mathcal{K}^{q+1}(U)\}}{d\mathcal{K}^{q-1}(U)}$$

$$\begin{cases} C^n=\bigoplus_{p+q=n}\mathcal{C}^p(\mathfrak{U},\mathcal{K}^q)\\ D=d+\delta\end{cases}$$

Consider the single complex $(C^\bullet, D)$ related to $\{\mathcal{C}^\bullet(\mathfrak{U},\mathcal{K}^\bullet);\delta,d\}$.
the hypercohomology is defined to be
$$\mathbb{H}^n(X,\mathcal{K}^\bullet)=\varinjlim_{\mathfrak{U}}H^n(C^\bullet)$$
$$\boxed{E_r^{p,q}\Rightarrow\mathbb{H}^n(X,\mathcal{K}^\bullet)}$$

For $X$ paracompact and locally contractible, $H^q(X,A)\cong H^q_{\text{sing}}(X;A)$, where $A$ is an abelian group.
$H^q(X,A)$ is the sheaf cohomology with $A$ as a constant sheaf on $X$.
$H^q_{\text{sing}}(X;A)$ is the singular cohomology with coefficients in $A$.
We take the complex of sheaves to be a resolution of $A$. then there is $E_1^{p,q}=\bigoplus_{i_0<\cdots<i_p}H^q(U_{i_0}\cap\cdots\cap U_{i_p})$

- Consider the hypercohomology spectral sequence, we can compute the cohomology of $X$.
$$E_1^{p,q}=\bigoplus_{i_0<\cdots<i_p}H^q(U_{i_0}\cap\cdots\cap U_{i_p})\Rightarrow\boxed{H^{p+q}(X)}$$
$$H^{p+q}(X) = E_\infty^{p,q}(X)$$

Apply this to the affine open covering $\{U_{\sigma_i}\}$ where $\sigma_i$ are maximal cones in $X(\Delta)$
$$E_1^{p,q}=\bigoplus_{i_0<\cdots<i_p}\Lambda^q M(\sigma_{i_0}\cap\cdots\cap\sigma_{i_p})$$

- Euler characteristic $\chi(X(\Delta))=\boxed{\sum(-1)^{p+q}\operatorname{rank}E_1^{p,q}}$ $\left(=\sum(-1)^n\operatorname{rank}H^n(X;\mathbb{Z})\right.$, since this sum is invariant for $r\left.\right)$
$= \#\,n\text{-dim cones in }\Delta.$

$$\sum(-1)^{p+q}\operatorname{rank}E_r^{p,q}=\sum(-1)^{p+q}\operatorname{rank}E_\infty^{p,q}$$
$$=\sum(-1)^n\operatorname{rank}H^n(X;\mathbb{Z}).$$

Pf: $\sum(-1)^q\operatorname{rank}(\Lambda^q M(\tau))=\sum(-1)^q\binom{n-k}{q}$, where $k=\dim\tau<n$
$=0$

If $\dim\tau=n$, then $\Lambda^0M(\tau)=\mathbb{Z}$, $\Lambda^kM(\tau)=0$, $k>0$.
$$\Rightarrow\sum(-1)^q\operatorname{rank}\Lambda^qM(\tau)=1$$
$$\chi(X(\Delta))=\sum(-1)^{p+q}\operatorname{rank}E_1^{p,q}=\sum_p(-1)^p\sum_{i_0<\cdots<i_p}\left(\sum_q(-1)^q\operatorname{rank}M(\sigma_{i_0}\cap\cdots\cap\sigma_{i_p})\right)$$
$=\#\,n\text{-dim cone}.$ ∎

$E_r\to(E_r)_{\text{even}},\ (E_r)_{\text{odd}}.$
$p+q$ is even; $E_r^{p,q}$, $p+q$ is odd.
$\varphi:(E_r)_{\text{even}}\to(E_r)_{\text{odd}}.\quad \varphi\circ\psi=0$
$\psi:(E_r)_{\text{odd}}\to(E_r)_{\text{even}}.\quad \psi\circ\varphi=0$
$=\operatorname{rank}(E_r)_{\text{even}}-\operatorname{rank}(E_r)_{\text{odd}}$
$=\operatorname{rank}(\ker\varphi)+\operatorname{rank}(\operatorname{im}\varphi)-\operatorname{rank}(\ker\psi)-\operatorname{rank}(\operatorname{im}\psi)$
$=\operatorname{rank}(\ker\varphi)-\operatorname{rank}(\operatorname{im}\psi)$
$-(\operatorname{rank}(\ker\psi)-\operatorname{rank}(\operatorname{im}\varphi))$
$=\operatorname{rank}(E_{r+1})_{\text{even}}-\operatorname{rank}(E_{r+1})_{\text{odd}}.$

If all maximal cones are $n$-dim. i.e. $X(\Delta)$ is complete.
- $\boxed{E_1^{p,q}=0}$ for $q\geq 1$ $\quad H^i(U_\sigma)=0$ for $i\geq1$, $\sigma$ max. cone. $E_1^{p,q}=\bigoplus H^q(U_{\sigma_i})=0$. for $q\geq1$.
- The complex $E_1^{\bullet,0}$ is the cochain complex of a simplex $\Rightarrow E_2^{p,0}=0$. $p>0$.
$$E_1^{\bullet,0}:0\to\bigoplus_i\mathbb{Z}_{\sigma_i}\to\bigoplus_{i<j}\mathbb{Z}_{\sigma_i\cap\sigma_j}\to\bigoplus_{i<j<k}\mathbb{Z}_{\sigma_i\cap\sigma_j\cap\sigma_k}\to\cdots$$
$$\Rightarrow H^2(X(\Delta))=E_\infty^{1,1}=E_2^{1,1}=\operatorname{Ker}(E_1^{1,1}\to E_1^{2,1})$$
$$E_3^{1,1}=\frac{\operatorname{Ker}(E_2^{1,1}\to E_2^{3,0})}{\operatorname{Im}(0\to E_2^{1,1})}=\boxed{E_2^{1,1}}\Rightarrow\boxed{E_\infty^{1,1}=E_2^{1,1}}$$
$$E_\infty^{1,1} = H^2(X(\Delta))$$
$$=\operatorname{Ker}\left(\bigoplus_{i<j}M(\sigma_i\cap\sigma_j)\to\bigoplus_{i<j<k}M(\sigma_i\cap\sigma_j\cap\sigma_k)\right)$$
$\Rightarrow$ A cocycle in $H^2(X(\Delta))$ defines a line bundle on $X(\Delta)$.