Mar 22

10.4 $G = \langle x, y \mid xyx^{-1}y^{-1} \rangle$. If u and v are elements of an abelian group A, there is a unique hom $\varphi: G \to A$. s.t. $\varphi(x) = u$, $\varphi(y) = v$.

Pf: $G = \{x^m y^n \mid m, n \in \mathbb{Z}\}$. $\leftarrow$

Define a map $\varphi: G \to A$. $x^m y^n \longmapsto u^m v^n$

$$\varphi(e) = 0$$
$$\varphi(x^{m+m'} y^{n+n'}) = u^{m+m'} v^{n+n'}$$
$$\varphi(x^{-m} y^{-n}) = u^{-m} v^{-n}$$

($u^m v^n u^{m'} v^{n'} \neq u^{m+m'} v^{n+n'}$ if A is not comm.)

$\blacksquare$

2.16 Let $\varphi: G \twoheadrightarrow G'$ $\twoheadrightarrow$ $\hookrightarrow$

surjective

C: conjugacy class of x in G.

C': conjugacy class of $\varphi(x)$ in G'.

To check: φ maps C surjectively to C'. $|C'|$ divides $|C|$.

Pf: $C = C(x)$ $\quad Z(x)$: centralizer of x.

This is a gp.

$$\left.\begin{aligned} |C||Z(x)| &= |G| \\ |C'||Z(\varphi(x))| &= |G'| \end{aligned}\right\} \Rightarrow \quad |C| = \frac{|G|}{|Z(x)|}, \quad |C'| = \frac{|G'|}{|Z(\varphi(x))|}$$

This is a group

To check $\frac{|C|}{|C'|} \in \mathbb{N}^*$.

$$\frac{|C|}{|C'|} = \frac{|G|}{|Z(x)|} \Big/ \frac{|G'|}{|Z(\varphi(x))|}$$
$$= \frac{|G|}{|G'|} \cdot \frac{|Z(\varphi(x))|}{|Z(x)|} \qquad \varphi(Z(x))$$

$\Uparrow$ $\mathbb{Z}$ $\qquad |Z(\varphi(x))| \,\big|\, |Z(x)|$

$$= \frac{|G|}{|G'|} \cdot \frac{1}{|Z(x)| / |\varphi(Z(x))|} \cdot \frac{|Z(\varphi(x))|}{|\varphi(Z(x))|}$$

① $\varphi(Z(x))$ is a subgp. of $Z(\varphi(x))$

gp. $\qquad$ gp.

② $|Z(x)| / |\varphi(Z(x))| = |\mathrm{Ker}(\varphi|_{Z(x)})|$ by 1st iso thm applied for $\varphi|_{Z(x)}: Z(x) \to \varphi(Z(x))$

① $\varphi(Z(x)) \subseteq Z(\varphi(x))$

Pick $\quad g \in Z(x). \quad gxg^{-1} = x \xrightarrow{\text{mapped by } \varphi} \varphi(g)\varphi(x)\varphi(g)^{-1}$

$$= \varphi(x)$$

$$\Rightarrow \varphi(g) \in Z(\varphi(x)).$$

$$|\mathrm{Ker}\varphi| = \frac{|G|}{|G'|} \cdot \frac{1}{|Z(x)| / |\varphi(Z(x))|} \cdot \frac{|Z(\varphi(x))|}{|\varphi(Z(x))|}$$

$$= \underbrace{\frac{|\mathrm{Ker}\varphi|}{|\underbrace{\mathrm{Ker}\varphi|_{Z(x)}}_{= \mathrm{Ker}\varphi \cap Z(x)}|}}_{\in \mathbb{Z}} \cdot \frac{|Z(\varphi(x))|}{|\varphi(Z(x))|} \in \mathbb{Z} \quad \Rightarrow \quad \frac{|C|}{|C'|} \in \mathbb{Z}$$

∎