Ex 6.10 Find all automorphisms of
(a) a cyclic group of order 10
(b) the symmetric group S_3

Solution:

(a) Denote this group by $G = \langle \sigma \rangle$

An automorphism $\varphi: G \to G$ is determined by its image on σ.

$\varphi(\sigma) = \sigma^k \in G$ must have the same order with σ.

$\Rightarrow$ The index k should be coprime with 10.

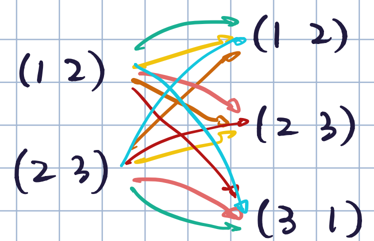
$\Rightarrow$ k can be $1, 3, 7, 9$.

(b) S_3 has 3 conjugacy classes: $\begin{cases} (1); \\ (12), (23), (31); \\ (123), (132). \end{cases}$

where $(123) = (12)(23)$, $(132) = (13)(23)$, $(31) = (12)(23)(12)$

So each automorphism $\varphi: S_3 \to S_3$ is determined by its image of (12) and (23).

Automorphisms preserve conjugacy classes, so here are all our choices:



In total 6 automorphisms on S_3. $\blacksquare$

Ex 8.3 Does every group whose order is a power of a prime p cotain an element of order p?

Pf: Denote this group by G. Suppose that $|G| = p^a$, $a \in \mathbb{N}$.

Pick an element $g \in G$. then there is $\mathrm{ord}(g) \mid |G|$, thus $\mathrm{ord}(g) = p^{a'}$. $1 \le a' \le a$.

If $a' = 1$, then we are done.

Otherwise. Let $g' = g^{p^{a'-1}}$. $(g')^p = e$. and $(g')^k = g^{kp^{a'-1}} \neq e$. for $0 < k < p$.

$\Rightarrow \mathrm{ord}(g') = p$. $\blacksquare$

Ex 8.6 Let $\varphi: G \to G'$ be a group homomorphism. Suppose that $|G| = 18$. $|G'| = 15$. and that φ is not the trivial homomorphism. What's the order of the kernel?

Solution: $\mathrm{Ker}\,\varphi$ is a subgroup of G. $\Rightarrow |\mathrm{Ker}\,\varphi| \mid |G| = 18$

$\mathrm{Im}\,\varphi$ is a subgroup of G' $\Rightarrow |\mathrm{Im}\,\varphi| \mid |G'| = 15$

And $|\mathrm{Ker}\,\varphi| \cdot |\mathrm{Im}\,\varphi| = |G| = 18$.

Therefore $|\mathrm{Im}\,\varphi| = 3$. $|\mathrm{Ker}\,\varphi| = 6$. $\blacksquare$

Ex 11.6 Let G be a group that contains normal subgroups of order 3 and 5, respectively. Prove that G contains an elements of order 15.

Pf: Let H be the normal subgp. of order 3.
K —— 5.

They must be cyclic. $\Rightarrow H \cap K = \{e\}$

Suppose that $H = \langle\sigma\rangle$. $K = \langle\tau\rangle$.

H is normal. By def. $\tau H \tau^{-1} = H$. so $\tau\sigma\tau^{-1}$ is either σ or σ^{-1}.

① $\tau\sigma\tau^{-1} = \sigma$. then $\sigma\tau = \tau\sigma$. $(\sigma\tau)^k = \sigma^k\tau^k$. $k \in \mathbb{Z}$. $\Rightarrow$ order$(\sigma\tau) = 15$.

② $\tau\sigma\tau^{-1} = \sigma^{-1}$. then $\sigma\tau\sigma = \tau$.

Since $\sigma K \sigma^{-1} = K$. $\Rightarrow \sigma\tau\sigma^{-1} = \tau^\lambda$ for some $\lambda \in \mathbb{Z}$.

$\tau^\lambda = \sigma\tau\sigma^{-1} = \sigma\tau\sigma^2 = (\sigma\tau\sigma)\sigma = \tau\sigma$.

$\Rightarrow \sigma = \tau^{\lambda-1} \in K$.

$H \cap K = \{e\} \Rightarrow \sigma = e$. which is impossible

∎

Ex 11.7 Let H be a subgroup of group G. let $\varphi: G \to H$ be a hom. whose res. to H is the id. map. and let N be its kernel. What can one say about the product map $H \times N \to G$?

Pf of $H \times N \cong G$:

There is naturally the map $H \times N \to G$. $(h, n) \mapsto hn$.

Kernel is trivial: If $(h, n) \mapsto e \in G$. i.e. $hn = e$. $n = h^{-1}$.
then $\varphi(n) = \varphi(h^{-1}) = \varphi(h)^{-1} = h^{-1}$
but $\varphi(n) = e$. since $n \in N$. $\Rightarrow h^{-1} = e = n$. $h = e$.

$H \times N \to G$ is surjective: $\forall g \in G$. $h = \varphi(g)$.
then $h^{-1}g \in N$.
$\Rightarrow$ The image of $(h, h^{-1}g)$ is g.

Therefore, $H \times N \cong G$

∎

Rmk: The general conclusion is:

For a short exact sequence

$$0 \to N \to M \to P \to 0.$$

if there is hom $M \to N$ s.t. the composition $N \to M \to N$ is identity on N.
or there is hom $P \to N$, s.t. $P \to M \to P$ is idp.
then we have $M \cong N \oplus P$.

The conditions can be written as.

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] & M \arrow[l, bend right] \arrow[r] & P \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] & M \arrow[r] & P \arrow[l, bend right] \arrow[r] & 0 \end{tikzcd}$$

(For Feb 8)