Mar 8

Iso. thm.:

1st: $\phi: G \to H$ gp. hom. $K \triangleq \operatorname{Ker}\phi$.

If we have a normal subgp. $N \trianglelefteq G$. $N \subseteq K$.

then $\exists!$ gp. hom. $\Phi: G/N \to H$ s.t. $\Phi \circ \pi = \phi$ where $\pi: G \to G/N$.

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[dr, "\pi"'] & H \\ & G/N \arrow[u, dashed, "\exists!\Phi"'] \end{tikzcd}$$

Pf: $gN \in G/N$ $\qquad \Phi(gN) := \phi(g)$

Well-definedness. We can have more than 1 rep. for a coset.

If $gN = g'N$, we need to check that $\Phi(g) = \Phi(g')$

Since $gN = g'N$. $\exists\, n \in N$ s.t. $g = g'n$.

$$\Phi(\bar{g}) \underset{\text{def}}{=} \phi(g) = \phi(g'n) \underset{\text{def of gp hom}}{=} \phi(g')\phi(n) \underset{n \in N \subseteq \operatorname{Ker}\phi}{=} \phi(g') \overset{\text{def}}{=} \Phi(\bar{g}').$$

$\Rightarrow \Phi: G/N \to H$ is well-defined.

$\forall\, gN, hN \in G/N$. $\quad \Phi(gN \cdot hN) = \Phi(gN)\,\Phi(hN) \leftarrow$ to check.

By normality of N. $gN \cdot hN = ghN$.

By def. $\Phi(ghN) = \phi(gh) = \phi(g)\phi(h) = \Phi(gN)\cdot\Phi(hN)$.

$\Rightarrow \Phi$ is a gp. hom.

For the uniqueness. since $\pi: G \to G/N$ is surjective.

the image of elements of G/N is determined by ϕ.

so the induced Φ is unique. $\blacksquare$

Another form: $\phi: G \to H$ gp hom. $\quad G/\operatorname{Ker}\varphi \simeq \operatorname{Im}\varphi$.

2nd: $H \leq G$. $N \trianglelefteq G$. $\qquad H/H\cap N \simeq HN/N$

Pf: $\begin{tikzcd} H \arrow[r] \arrow[rr, bend left, "\tau"] & HN \arrow[r] & HN/N \end{tikzcd}$

gp hom.

Injectivity: If $h \in H$ is in the kernel, then $hN = N$ in $\underline{HN}$.

$\Rightarrow \exists\, n \in N$ s.t. $h = n$. $\Rightarrow h \in N$. $\Rightarrow h \in H \cap N$.

$\operatorname{Ker}\tau \subseteq H \cap N$

Conversely. if $h \in H \cap N \subseteq N$. $\tau(h) = \bar{e}$ in HN/N.

$\Rightarrow \tau$ is injective

Surjectivity: Elements in HN/N are cosets xN, where $x \in HN$.

x can be written into the form $x = hn$. for some $h \in H$, $n \in N$.

$$\Rightarrow \quad xN = hN = \pi(h).$$

■

3rd: $N \triangleleft G$, $H \triangleleft G$, $N \subseteq H$. $\Rightarrow$ $(G/N)\big/(H/N) \simeq G/H$

$\cancel{N \triangleleft H \triangleleft G}$ $\not\Rightarrow$ $N \triangleleft G$.

Pf: $G/N \to G/H$ $\Rightarrow$ $\mathrm{Ker} = H/N$. (to check)

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[rr, bend left, "\tau"] & G/N \arrow[r, "\varphi"] & (G/N)\big/(H/N) \end{tikzcd} \quad \Rightarrow \text{ check } \mathrm{Ker} = H.$$

Firstly it's nat. a surjective hom.

If $g \in \mathrm{Ker}\,\tau$. $gN \in \underbrace{H/N}_{= \mathrm{Ker}(G/N \to (G/N)/(H/N))} \subseteq G/N$

$\exists\, hN \in H/N$ s.t. $gN = hN$. $\Rightarrow \exists\, n \in N$. s.t. $g = hn$

Since $N \subseteq H$. $g \in H$.

$\mathrm{Ker}\,\tau \subseteq H$

Conversely, it is clear that $H \subseteq \mathrm{Ker}\,\tau$. $\Rightarrow \mathrm{Ker}\,\tau = H$.

$$\Rightarrow \quad G/H \simeq (G/N)\big/(H/N).$$

■

Semidirect product.

(I) $N \triangleleft G$, $H \leq G$.

$G = NH$, $N \cap H = \{e\}$ $\Leftrightarrow$ $\pi: G \to G/N$ induces $H \xrightarrow{\sim} G/N$ by restriction $\pi|_H$.

Pf: "$\Rightarrow$" To check: $\pi|_H : H \to G/N$ is iso.

Injective: $h \mapsto \bar{e} \in G/N$. $\Rightarrow h \in N \Rightarrow h \in H \cap N = \{e\}$.
$\Rightarrow \pi|_H$ is injective.

Surjective:

Given $g \in G$, it can be uniquely written as $g = \underset{\in H}{h}\,\underset{\in N}{n}$.

The existence is given by $G = NH$, and N is normal.

$$g = h'n' = hn. \Leftrightarrow \underbrace{h^{-1}h'}_{\in H} = n(n'^{-1}) \in N$$

$$\Rightarrow \quad h^{-1}h' = n(n'^{-1}) \in H \cap N = \{e\} \Rightarrow h = h',\ n = n'.$$

$\forall gN \in G/N$. $\exists h \in H$ s.t. $gN = hN$. $\Rightarrow \pi|_H$ is surjective.

Therefore $\pi|_H : H \xrightarrow{\sim} G/N$. ▨

$$n_1h_1 \cdot n_2h_2 = n_1 \underbrace{(h_1 n_2 h_1^{-1})}_{\in N\text{, by normality}} \cdot h_1h_2.$$

(II) General def. of Semidirect prod.:

Given two gp's H, N. & a gp. hom. $\phi : H \to N$.

$N \rtimes_\phi H = N \times H$ as sets. (with diff. product!)

$$(n_1, h_1) \times (n_2, h_2) \triangleq (\underbrace{n_1 \cdot \phi(h_1)(n_2)}_{\in N}, \underbrace{h_1h_2}_{\in H})$$

Prop: $N \rtimes_\phi H$ is a gp.

Pf: ① closure ✓

② associativity:
$$\big((n_1, h_1) \times (n_2, h_2)\big) \times (n_3, h_3)$$
$$= (n_1\phi(h_1)(n_2), h_1h_2) \times (n_3, h_3)$$
$$= (n_1 \underline{\phi(h_1)(n_2)\, \phi(h_1h_2)(n_3)}, h_1h_2h_3)$$

$$(n_1, h_1) \times \big((n_2, h_2) \times (n_3, h_3)\big)$$
$$= (n_1, h_1) \times (n_2\phi(h_2)(n_3), h_2h_3)$$
$$= (n_1 \underline{\phi(h_1)(n_2\phi(h_2)(n_3))}, h_1h_2h_3)$$

$=$

$\mathrm{Aut}\,N$

$$\underline{\phi(h_1)(n_2)} \cdot \big(\underbrace{\phi(h_1) \cdot \phi(h_2)}_{= \phi(h_1h_2)}\big)(n_3)$$ ✓

③ inverse $(n, h)^{-1} = (\phi(h^{-1})(n^{-1}), h^{-1})$.

④ id. (e, e). ▨

Examples:

① $G = NH$, $N \cap H = \{e\}$, $G \simeq N \rtimes_\phi H$. where $\phi : H \to \mathrm{Aut}\,N$
$h \mapsto (n \mapsto hnh^{-1})$
"inner auto"

$f : N \rtimes_\phi H \to G$
$(n, h) \mapsto nh$.

$(n_1, h_1) \times (n_2, h_2) = (n_1 \underline{h_1 n_2 h_1^{-1}}, h_1h_2)$

$$f((n_1,h_1)\times(n_2,h_2)) = n_1h_1n_2h_2 = f((n_1,h_1))\cdot f((n_2,h_2)).$$

② $D_{2n} = C_n \rtimes_\phi C_2$, where $C_2 = \langle\tau\rangle$, $C_n = \langle\sigma\rangle$

$$\phi : C_2 \longrightarrow \mathrm{Aut}(C_n)$$
$$\tau \longmapsto (\sigma \mapsto \sigma^{-1})$$
$$e \longmapsto (\sigma \mapsto \sigma)$$

$D_{2n} = \{(s,t) \mid s^n = t^2 = 1,\ tst^{-1} = s^{-1}\}$

$C_n \rtimes_\phi C_2 \longrightarrow D_{2n}$ $\quad (\sigma,\tau) \mapsto t$ $\quad (\sigma,1) \mapsto s.$

$$\underline{(\sigma,\tau)\times(\sigma,\tau)} = (\sigma\cdot\sigma^{-1},\ \tau^2) = (1,\ 1)$$

$$(\sigma,1)\times(\sigma,1) = (\sigma^2,1)$$

$$\underline{(\sigma,\tau)\times(\sigma,1)}\times(\sigma,\tau) = (1,\tau)\times(\sigma,\tau) = (\phi(\tau)(\sigma),\ 1) = (\sigma^{-1},1) = (\sigma,1)^{-1}$$

③ $\underline{\text{The isometry gp. on } \mathbb{R}^2} = \underset{\text{translation on } \mathbb{R}^2}{\underset{\uparrow}{(\mathbb{R}^2,+)}} \rtimes O(2) \leftarrow$ 2×2 Orthogonal real matrices

$\qquad \| $
$\qquad G$

$x \longmapsto Ax+b.$ $\quad \underline{A\in O(2)}$, $b \in \mathbb{R}^2$

Orthog. since it preserves the inner prod.

$G = \{(A,b) \mid A\in O(2),\ b\in\mathbb{R}^2\}.$

$$(A',b')\circ(A,b) := x \longmapsto Ax+b \longmapsto A'(Ax+b)+b' = A'Ax + (A'b+b')$$

$$(A',b')\circ(A,b) = \underline{(A'A,\ A'b+b')}$$

$$\phi: O(2) \longrightarrow \mathrm{Aut}(\mathbb{R}^2,+)$$
$$A \longmapsto (b \mapsto Ab)$$

$$f: (\mathbb{R}^2,+)\rtimes_\phi O(2) \longrightarrow G$$
$$(b,A) \longmapsto (A,b)$$

$$f(\underline{(b',A')\times(b,A)}) = \underline{f((b',A'))\cdot f((b,A))}$$

$$f((b'+A'b,\ A'A)) = (A'A,\ A'b+b')$$