Mar 1

Section 2.3.4 of Ch2 in HW.
Section 5.6.

2.2. 2.3. 2.5 2.6

2.2 Let S be a set with an associative law of comp. and with an id. Prove the subset consisting of inv. elements in S is a group.

Pf: $S^\times = \{$ all invertible elements in $S\}$.

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① id. $\in S^\times$
② associativity
③ inverse exists
④ $S^\times$ is closed under the composition (closure).

$\blacksquare$

2.3 Let x, y, z and w be elements of a gp. G.

a) Solve y. $xyz^{-1}w = 1$. Ans: $x^{-1}w^{-1}z$

b) Suppose $xyz = 1$. If $yzx = 1$. $yxz = 1$ false.

$x(yz) = 1 \Leftrightarrow (yz)x = 1$

2.5. verify the identity and inverse of a subgp. H coincide with the original gp. G.

Pf: e_H, e_G. Consider in G. $e_H \cdot e_G = e_H$
then consider in H. $\Rightarrow e_H = e_G$.

$\blacksquare$

2.6. (Opposite group G°) $a * b = ba$. Show that G° is a group.

6.8 $A \mapsto (A^t)^{-1}$ is an aut. of $GL_n(\mathbb{R})$.

What if we just consider $A \mapsto A^{-1}$ or $A \mapsto A^t$.

Given A, B. $(AB)^{-1} = B^{-1}A^{-1} = A^{-1} * B^{-1}$

$A \mapsto A^{-1}$ is a hom of $GL_n(\mathbb{R})$ to $GL_n(\mathbb{R})^\circ$

3.2. Prove that if a, b are positive integers whose sum is a prime p, their gcd is 1.

Pf: $\gcd(a, b) = r$. $r \mid a+b = p$. $a, b \neq 0$. $\Rightarrow r = 1$. ▨

3.3. (a) $\boxed{\text{Define}}$ gcd of a set $\{a_1, \cdots, a_n\}$ of n integers. Prove that it exists and that it is an integer comb. of $a_1, \cdots, a_n$.

(b) Prove if $\gcd\{a_1, \cdots, a_n\} = d$. then $\gcd\{\frac{a_1}{d}, \cdots, \frac{a_n}{d}\} = 1$.

Pf: (a) ① $\gcd \hat{=}$ the greatest positive integer that divides all of $\{a_1, \cdots, a_n\}$.

② $S = \mathbb{Z}a_1 + \cdots + \mathbb{Z}a_n$, S is a subgp. of $(\mathbb{Z}, +)$. $\mathbb{Z}^+$

Thm 2.3.3. $\Rightarrow S = d\mathbb{Z}$ ← gcd. ← can be positive or negative

First way: Verify $\begin{cases} \text{existence} \\ \text{Integer combination property} \end{cases}$

Second way: ↳ natural here

Verify what you define is a common divisor.

(b) ① $\Rightarrow$ direct ▨

② $\Rightarrow \mathbb{Z}\frac{a_1}{d} + \cdots + \mathbb{Z}\frac{a_n}{d} = \mathbb{Z}$

4.2 n-th root of unity.

(a) Prove it's a cyclic gp. of order n.

(b) Determine the product of all n-th root of unity.

↳ Two ways: ① $\{e^{\frac{2\pi i}{n}k},\ k = 1, \cdots, n\}$.

$$\prod_{k=1}^{n} e^{\frac{2\pi i}{n}k} = e^{\frac{2\pi i}{n} \cdot \frac{1}{2}n(n+1)} = e^{(n+1)\pi i} = (-1)^{n+1}$$

② $\langle \mu \rangle = \{1, \mu, \cdots, \mu^{n-1}\}$

They are roots of the polynomial $z^n - 1$.

$$z^n - 1 = (z-1)(z-\mu)\cdots(z-\mu^{n-1})$$

$$= z^n + \cdots + (-1)^n \prod_{k=0}^{n-1}\mu^k$$

$$\Rightarrow \prod_{k=0}^{n-1} \mu^k = (-1)^{n+1}$$ ▨

4.4. Describe all groups G that contain no proper subgp.

G is a cyclic group of prime order.

$G = \{e\}$ is trivial group. 1 is not prime!

4.5 Prove that every subgp. of a cyclic gp. is cyclic.

Pf: Suppose the gp. is gen. by x.

Define a hom. $\varphi: \mathbb{Z} \to \langle x\rangle \hat{=} G$.

$$0 \mapsto e$$
$$1 \mapsto x$$

For a subgp. H of G. $\varphi^{-1}(H)$ is a subgroup of $\mathbb{Z}$

By $\boxed{\text{Thm } 2.3.3}$ $\varphi^{-1}(H) = d\mathbb{Z}$ for some integer $d \in \mathbb{Z}$.

We need to show that $H = \langle \varphi(d) \rangle$. ▨

4.6

4.8. (a) Prove that the elementary matrices of the 1st and 3rd types gen. $GL_n(\mathbb{R})$

(b) Prove that ——— of the 1st type gen. $SL_n(\mathbb{R})$.

Pf: (a) Use Gaussian elimination to reduce $A \in GL_n(\mathbb{R})$ to identity I_n.

- Show that elem. mat. of 2nd type can be expressed as product of 1st type & 3rd type.

$$\begin{array}{c} \\ i \\ \\ j \\ \\ \end{array}\begin{bmatrix} 1 & & & & \\ & 0 & \cdots & 1 & \\ & \vdots & \ddots & \vdots & \\ & 1 & \cdots & 0 & \\ & & & & 1 \end{bmatrix} \xrightarrow[\text{1st}]{} \begin{bmatrix} 1 & & & & \\ & 1 & \cdots & 1 & \\ & \vdots & & \vdots & \\ & 1 & \cdots & 0 & \\ & & & & 1 \end{bmatrix}$$

ith row $\times(-1)+j$th row $\xrightarrow[\text{1st}]{}$ $\begin{bmatrix} 1 & & & & \\ & 1 & \cdots & 1 & \\ & \vdots & & \vdots & \\ & 0 & \cdots & -1 & \\ & & & & 1 \end{bmatrix}$

jth $+$ ith $\xrightarrow[\text{1st}]{}$ $\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix}$ $\xrightarrow[\text{3rd.}]{\times(-1) \text{ on } j\text{th row}}$ I_n

(b) $n=2$. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$. $ad-bc=1$

① $b=c=0$. $ad=1$. $\begin{pmatrix} a & \\ & \frac{1}{a} \end{pmatrix} \longrightarrow \begin{pmatrix} a & a \\ 0 & \frac{1}{a} \end{pmatrix}$.

$\times \frac{1-a}{a}$ on 2nd Col. + 1st Col.

$\begin{pmatrix} 1 & a \\ (*) & \frac{1}{a} \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$

$\longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

② One of b, c is nonzero

general case. by induction.

$$\left(\begin{array}{c|ccc} a_{11} & a_{12} & \cdots & a_{1n} \\ \hline a_{21} & & & \\ \vdots & & A & \\ a_{n1} & & & \end{array}\right)$$

If one of $a_{21}, \cdots, a_{n1}$, $a_{12}, \cdots, a_{1n}$ is nonzero.

$a_{11} \longrightarrow 1$.

$$\longrightarrow \left(\begin{array}{c|c} 1 & * \\ \hline * & A \end{array}\right)$$

$$\xrightarrow{\text{by 1st type elem. mat.}}$$

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & A' \end{array}\right) \qquad A' \in SL_{n-1}(\mathbb{R}).$$

By induction hypothesis, the conclusion holds.

If all of $a_{21}, \cdots, a_{n1} = 0$
$a_{12}, \cdots, a_{1n} = 0$.
then a_{11} can not be zero.

$$\left(\begin{array}{c|c} a_{11} & 0 \\ \hline 0 & A \end{array}\right) \longrightarrow \left(\begin{array}{c|c} a_{11} & a_{11} \cdots \\ \hline & A' \end{array}\right)$$

Follow the first case. ∎

4.9. Ans is 9.