Outline of Maxwell Equations

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The electromagnetic theory of optics begins with Maxwell Equation.

1 Maxwell Equations

$$\begin{aligned}\nabla\cdot\boldsymbol{D}&=\rho\\ \nabla\cdot\boldsymbol{B}&=0\\ \nabla\times\boldsymbol{E}&=-\frac{\partial\boldsymbol{B}}{\partial t}\\ \nabla\times\boldsymbol{H}&=\frac{\partial\boldsymbol{D}}{\partial t}+\boldsymbol{J}\end{aligned}\tag{1}$$

2 Constitutive Relations

$$\begin{aligned}\boldsymbol{D}&=\epsilon_0\boldsymbol{E}+\boldsymbol{P} \qquad &\epsilon_0=8.854\times10^{-12}\\ \boldsymbol{B}&=\mu_0\boldsymbol{H}+\boldsymbol{M} \qquad &\mu_0=4\pi\times10^{-7}\end{aligned}$$

3 Boundary conditions

$$\begin{aligned}E_{1t}-E_{2t}&=0\\ H_{1t}-H_{2t}&=\boldsymbol{J}_s\times\boldsymbol{n}\\ D_{1n}-D_{2n}&=\rho_s\\ B_{1n}-B_{2n}&=0\end{aligned}\tag{2}$$

4 Charge conservation

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\boldsymbol{J}=0 \qquad\to\qquad \rho(r,t)=\rho_0(r)e^{-\sigma t/\tau}$$

5 Energy flux and energy conservation

Because energy can be converted into different forms, the corresponding conservation equation should have a non-zero term in the right-hand side corresponding to the rate by which energy is being lost from the fields into other forms.

$$
\begin{aligned}
\frac{\partial \rho_{en}}{\partial t} + \nabla \cdot \boldsymbol{J}_{en} &= rateofenergyloss \\
\rho_{en} = w &= \frac{1}{2}\epsilon |E|^2 + \frac{1}{2}\mu |H|^2 \\
\boldsymbol{J}_{en} = \boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H} \qquad & poyntingvector \\
\frac{\partial w}{\partial t} + \nabla \cdot \boldsymbol{S} &= -\boldsymbol{J}_{en} \cdot \boldsymbol{E}
\end{aligned} \tag{3}
$$

6 Harmonic Time dependence

Maxwells equations simplify considerably in the case of harmonic time dependence. We assume that all fields have a time dependence $e^{i\omega t}$.

$$
\boldsymbol{E}(r,t) = \boldsymbol{E}(r)e^{i\omega t} \qquad \boldsymbol{H}(r,t) = \boldsymbol{H}(r)e^{i\omega t}
$$

We may rewrite Maxwell Equations in the form

$$
\begin{aligned}
\nabla \cdot \boldsymbol{D} &= \rho \\
\nabla \cdot \boldsymbol{B} &= 0 \\
\nabla \times \boldsymbol{E} &= -i\omega \boldsymbol{B} \\
\nabla \times \boldsymbol{H} &= i\omega \boldsymbol{D} + \boldsymbol{J}
\end{aligned} \tag{4}
$$

7 Dielectrics

A simple model for the dynamics of the displacement x of the bound electron is as follows

$$
m\ddot{x} = eE - kx - m\gamma\dot{x} \tag{5}
$$

spring-like restoring force to the binding of the electron to nucleus.
friction-type force proportional to the velocity of the electron

$$
\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{e}{m}E \tag{6}
$$

suppose the solution has the form as $x(t) = xe^{i\omega t}$

$$
-\omega^2 x + i\omega\gamma x + \omega_0^2 x = \frac{e}{m}E \tag{7}
$$

solution

$$
x = \frac{\frac{e}{m}E}{\omega_0^2 - \omega^2 + i\omega\gamma} \tag{8}
$$

the corresponding velocity

$$
v = \frac{i\omega\frac{e}{m}E}{\omega_0^2 - \omega^2 + i\omega\gamma} \tag{9}
$$

the effective permittivity

$$
\epsilon(\omega) = \epsilon_0 + \frac{\frac{Ne^2}{m}}{\omega_0^2 - \omega^2 + i\omega\gamma} = \epsilon_0 + \frac{\epsilon_0\omega_p^2}{\omega_0^2 - \omega^2 + i\omega\gamma} \tag{10}
$$

where ω_p is so-called plasma frequency of material. The real and imaginary parts of $\epsilon(\omega)$ characterize the refractive and absorptive properties of the material. By convention, we define the imaginary part with the negative sign:

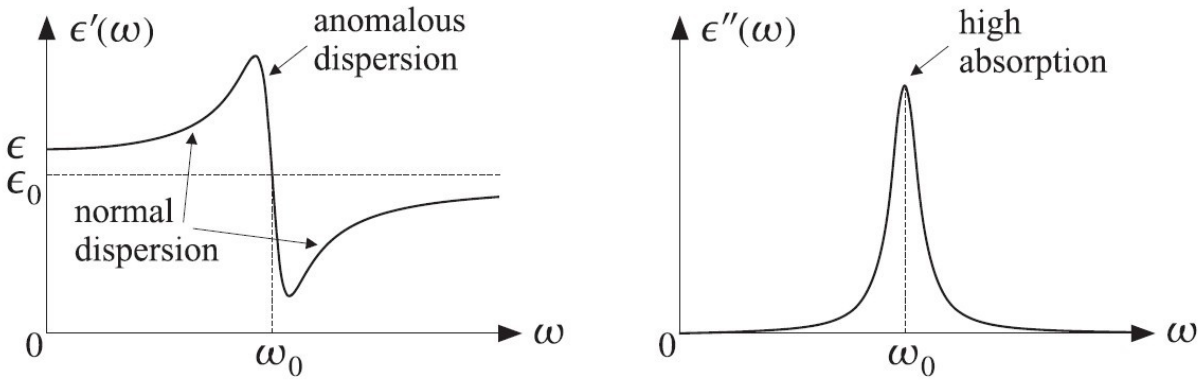

Figure 1: Real and imaginary parts of the effective permittivity

$$\epsilon(\omega) = \epsilon_1(\omega) - i\epsilon_2(\omega)$$
$$\epsilon_1 = \epsilon_0 + \frac{\epsilon_0\omega_p^2(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2} \qquad \epsilon_2 = \frac{\epsilon_0\omega_p^2\omega\gamma}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2} \tag{11}$$

Real dielectric materials exhibit serveral such resonant frequency corresponding to various vibrational modes and polarization mechanisms.

$$\epsilon(\omega) = \epsilon_0 + \epsilon_0\Sigma_i \frac{\frac{N_i e_i^2}{m_i\epsilon_0}}{\omega_i^2-\omega^2+i\omega\gamma_i} \tag{12}$$

A more correct quantum-mechenical treatment leads essentially to the formula

$$\epsilon(\omega) = \epsilon_0 + \epsilon_0\Sigma_{j>i} \frac{\frac{f_{ji}(N_i-N_j)e^2}{m\epsilon_0}}{\omega_{ji}^2-\omega^2+i\omega\gamma_{ji}} \tag{13}$$

where ω_{ji} is transition frequency, $\omega_{ji} = (E_j - E_i)/\hbar$.

8 Conductors

The conductivity properties of a material are described by Ohms law. Define the conductivity $\sigma(\omega)$.

$$J = \rho v = Nev = \frac{i\omega\frac{Ne^2}{m}E}{\omega_0^2-\omega^2+i\omega\gamma} = \sigma(\omega)E \tag{14}$$

This is related to the expression of permittivity.

$$\epsilon(\omega) = \epsilon_0 - i\frac{\sigma(\omega)}{\omega} \tag{15}$$

Since metal conduction charges are unbound, we take $\omega_0 = 0$,

$$\sigma(\omega) = \frac{\epsilon_0\omega_p^2}{\gamma+i\omega} \qquad (Drudemodel) \tag{16}$$