Fermi's golden rule – derivation and application

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Fermi's Golden Rule is a simple formula for the constant transition rate from one energy eigenstate of a quantum system into other ennergy eigenstates in a continuum,effected by a perturbation. It is always used calculating lifetime.

1 Derivation of Fermi's Golen Rule

Consider the system to begin in an eigenstate, $|i\rangle$, of a given Hamiltonian, H_0. Consider the effect of a (possibly time-dependent) perturbing Hamiltonian, $H^{'}$. If $H^{'}$ is time-independent, the system goes only into those states in the continuum that have the same energy as the initial state. If $H^{'}$ is oscillating as a function of time with an angular frequency ω, the transition is into states with energies that differ by $\hbar\omega$ from the energy of the initial state.

1.1 Time dependent perturbation theory

Hamiltonian with perturbation is like

$$H = H_0 + H^{'}(t) \tag{1}$$

The unperturbed Time independent *Schrödinger* equation is satisfied

$$H_0|\phi_n\rangle = E_n|\phi_n\rangle \tag{2}$$

The time dependent wave function can be expressed as

$$|\Psi(t)\rangle = |\phi_n\rangle e^{-iE_nt/\hbar} \tag{3}$$

Some derivation

$$\begin{aligned}
&H|\Psi(t)\rangle = [H_0 + H^{'}(t)]|\Psi(t)\rangle = i\hbar\frac{\partial|\Psi(t)}{\partial t}\\
&|\Psi(t)\rangle = \Sigma_n c_n(t)|\psi_n(t)\rangle = \Sigma_n c_n(t)\phi_n\rangle e^{-iE_nt/\hbar}\\
&H_0\Sigma_k c_k(t)|\phi_k\rangle e^{-iE_kt/\hbar} + H^{'}(t)\Sigma_k c_k(t)|\phi_k\rangle e^{-iE_kt/\hbar} =\\
&\qquad\qquad i\hbar\frac{\partial}{\partial t}\Sigma_k c_k(t)|\phi_k\rangle e^{-iE_kt/\hbar}\\
&\Sigma c_k(t)E_k|\phi_k\rangle e^{-iE_kt/\hbar} + \Sigma c_k(t)H^{'}(t)|\phi_k\rangle e^{-iE_kt/\hbar} =\\
&\qquad\qquad i\hbar\Sigma\frac{\partial c_k(t)}{\partial t}|\phi_k\rangle e^{-iE_kt/\hbar} + \Sigma_k c_k(t)|\phi_k\rangle(-\frac{iE_k}{\hbar})e^{-iE_kt/\hbar}\\
&\Sigma_k c_k(t)W_{nk}(t)e^{-iE_kt/\hbar} = i\hbar\frac{\partial c_n(t)}{\partial t}e^{-iE_nt/\hbar}
\end{aligned} \tag{4}$$

where $W_{nk}(t) = \langle\phi_n|H^{'}|\phi_k\rangle$. At last, we got the equation.

$$\frac{\partial c_n(t)}{\partial t} = \frac{1}{i\hbar}c_i(t)W_{ni}(t)e^{i\omega_{ni}t} \tag{5}$$

In which, i represents the initial state $|i\rangle$, and $\omega = (E_n - E_i)/\hbar$. Solving this, we can get the final result.

$$c_f(t) = \frac{1}{i\hbar}\int_0^t W_{fi}(t^{'})e^{i\omega_{fi}t^{'}}dt^{'} \tag{6}$$

The probability of finding the system in the eigenstate $|\phi_f$ is

$$\begin{aligned}\mathscr{P}(t)_{if} &= |\langle\phi_f|\psi(t)|^2 \\ &= \frac{1}{\hbar^2}|\int_0^t e^{i\omega_{fi}(t^{'})}dt^{'}|^2\end{aligned} \tag{7}$$

1.2 High frequency harmonic perturbation

We could define a time dependent perturbation,

$$H^{'}(t) = 2Hcos(\omega t) = H(e^{i\omega t} + e^{-i\omega t}) \tag{8}$$

In this case, the probability becomes

$$\mathscr{P}_{if}(t) = \frac{W_{fi}^2}{\hbar^2}|\frac{e^{i(\omega_{fi}+\omega)t}-1}{i(\omega_{fi}+\omega)} + \frac{e^{i(\omega_{fi}-\omega)t}-1}{i(\omega_{fi}-\omega)}|^2 \tag{9}$$

Assuming that the oscilating angular frequency of the perturbation has a value near the Bohr angular frequency of the initial and final eigenstates, $\omega \sim \omega_{fi}$. The first term in Eq.9 becomes negliable, and the second term

$$\begin{aligned}A_- &= \frac{e^{i(\omega_{fi}-\omega)t}-1}{i(\omega_{fi}-\omega)} \\ &= e^{i(\omega_{fi}-\omega)t/2}\frac{e^{i(\omega_{fi}-\omega)t/2} - e^{-i(\omega_{fi}-\omega)t/2}}{i(\omega_{fi}-\omega)} \\ &= e^{i(\omega_{fi}-\omega)t/2}\frac{sin[(\omega_{fi}-\omega)t/2]}{(\omega_{fi}-\omega)/2}\end{aligned} \tag{10}$$

The probability becomes

$$\mathscr{P}_{fi}(t) = \frac{W_{fi}^2}{\hbar^2}\frac{sin^2[(\omega_{fi}-\omega)t/2]}{[(\omega_{fi}-\omega)/2]^2} \tag{11}$$

when $\omega \sim \omega_{fi}$, the probability sharply increases. This is call resonant point. In A_+, the resonant point is located on $\omega = -\omega_{fi}$. The resonant approximation is justified when $|A_+|^2$ and $|A_-|^2$ is far apart. This means that the matrix elements of the perturbation must be much smaller than the energy seperation between the initial and final states.

1.3 Continuum

$$\mathscr{P}(t) = \int \mathscr{P}_{fi}(t)\rho(E)dE \tag{12}$$

Because the probability is very sharp at the resonant point just like a delta function, the density $\rho(E)$ can be considered as a constant.

$$\begin{aligned}\mathscr{P}(t) &= \frac{W_{fi}^2}{\hbar^2}\rho(E_{fi})\int\frac{sin^2[(\omega_{fi}-\omega)t/2]}{[(\omega_{fi}-\omega)/2]^2}\hbar d\omega \\ &= \frac{2\pi}{\hbar}W_{fi}^2\rho(E_{fi})t\end{aligned} \tag{13}$$

the transition rate is

$$\begin{aligned}\mathscr{W}(t) &= \frac{d\mathscr{P}(t)}{dt} \\ &= \frac{2\pi}{\hbar}W_{fi}^2\rho(E_{fi})\end{aligned} \tag{14}$$

This is called Fermi's Golden Rule.