An introduction to quiver moduli

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Seminar on Nonlinear Algebra - MPI MIS Leipzig

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[0] Slides at github.com/Catullo99/quiver-moduli-leipzig

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1. Construction of moduli spaces of representations of quivers.

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Aknowledgements

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Joint work with P. Belmans, A. Brecan, H. Franzen, M. Reineke

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Two representations of the same *dimension vector* V, W are isomorphic if there is a common base change $M_i : V_i \xrightarrow{\sim} W_i$ such that for all vertices i, j and all arrows $\alpha : i \to j$,

$$M_j \circ V_\alpha = W_\alpha \circ M_i, \quad \text{that is,} \quad \begin{array}{ccc} V_i & \xrightarrow{V_{i\to j}} & V_j \\ \downarrow{\scriptstyle M_i} & & \downarrow{\scriptstyle M_j} \\ W_i & \xrightarrow{W_{i\to j}} & W_j. \end{array}$$

Quiver moduli 1/2

Once a dimension vector $\mathbf{d}$ is fixed, a representation is determined by a point in the *parameter space*

$$\mathsf{R}(Q, \mathbf{d}) := \bigoplus_{i \to j \in Q_1} \mathrm{Mat}_{d_j \times d_i}(k).$$

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which for $g=(g_i)_{i\in Q_0}$ and $M=(M_a)_{a:i\to j\in Q_1}$ is defined by

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Problem: The GIT quotient is just a bunch of points!

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The stable locus $\mathrm{R}^{\theta-\mathrm{st}}(Q, \mathbf{d})$ *(respectively the semistable locus* $\mathrm{R}^{\theta-\mathrm{sst}}(Q, \mathbf{d})$*) is a* $\mathrm{GL}_{\mathbf{d}}$*-invariant Zariski open which admits a geometric quotient, denoted by* $\mathrm{M}^{\theta-\mathrm{st}}(Q, \mathbf{d})$ *(respectively by* $\mathrm{M}^{\theta-\mathrm{sst}}(Q, \mathbf{d})$*).*

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1. The GIT quotient of the semistable locus is projective.
2. The GIT quotient of the stable locus is smooth.

"Holy diagram" of quiver moduli

$$\begin{array}{ccccc}
\mathrm{R}^{\theta-\mathrm{st}}(Q,\mathbf{d}) & \xhookrightarrow{\textit{open}} & \mathrm{R}^{\theta-\mathrm{sst}}(Q,\mathbf{d}) & \xhookrightarrow{\textit{open}} & \mathrm{R}(Q,\mathbf{d}) \\
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\mathrm{R}^{\theta-\mathrm{st}}(Q,\mathbf{d})/_{\theta}\,\mathrm{GL}_{\mathbf{d}} & \hookrightarrow & \mathit{Proj}\left(\bigoplus_n \mathcal{O}^{n\theta}_{\mathrm{R}(Q,\mathbf{d})}\right) & \longrightarrow & \mathit{Spec}(\mathcal{O}^{\mathrm{GL}_{\mathbf{d}}}_{\mathrm{R}(Q,\mathbf{d})})
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Lemma (Adriaenssens–Le Bruyn [1])

The ring of invariants $\mathcal{O}^{\mathsf{GL}_{\mathbf{d}}}_{\mathsf{R}(Q,\mathbf{d})}$ *is generated by elements in bijection with oriented cycles in the quiver.*

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From now on, our quivers are acyclic.

Intermezzo: quiver moduli vs moduli of bundles on curves

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- Abelian categories of cohomological dimension 1;
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ask me about these later if you are interested! :)

Universal families 1/2

Definition

A *universal family* for a moduli space M is a $\mathcal{U} \to M$ such that for any family $W \to B$ of the objects being parametrized over an arbitrary B, there exists a unique morphism $B \to M$ for which W fits in a cartesian diagram

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Lemma

For all $x \in \mathrm{R}(Q, \mathbf{d})$, the fiber of $U := \oplus_i U_i$ on x, together with $\{U_a|_x \mid a \in Q_1\}$, is equal to the representation encoded by x.

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Lemma (Kempf, Théorème 2.3 [4])

Any equivariant vector bundle E descends to $\mathrm{M}^{-\mathrm{st}}$ if and only if for all $x \in R^{\theta-\mathrm{st}}$ the stabilizer of x acts trivially on the fiber E_x.

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$$g \cdot (x, s) := (g \cdot x, \prod_{v \in Q_0} \det(g_v)^{-a_v} g_i \cdot s).$$

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The resulting bundles on M are denoted by $\mathcal{U}_i$.

The fundamental exact sequence

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Applications include

1. Deformation theory [3];
2. Presentation of Chow rings [5] and intersection theory [2];
3. Understanding of some Brauer groups [11];
4. Fano-ness of quiver moduli [6];
5. ...

Rigidity for quiver moduli 1/n

Theorem (Belmans–Brecan–Franzen–P.–Reineke)

If $\mathbf{d}$ is θ-coprime[1]*, for any $i, j \in Q_0$ and for all $k \geq 1$ we have*

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Corollary

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Note: the group $\mathrm{H}^1(X, \mathcal{T}_X)$ parametrizes infinitesimal deformations of a projective variety X.

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Proof background - GIT stratification

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Each stratum S_ℓ is G-invariant and is indexed by a one-parameter subgroup of G, that is, by an embedded copy of $\lambda_\ell \colon \mathbb{G}_m \hookrightarrow G$.

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How does this apply to quiver moduli?

Harder–Nararasimhan stratification

For quiver moduli, the choice of norm on 1-PSs is given by $\alpha \in \mathbb{N}_0^{Q_0}$. Denote $\mu(x) := \frac{\theta \cdot x}{\alpha \cdot x}$ for any $x \in \mathbb{Z}^{Q_0}$.

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Given θ and α, every representation admits a unique *Harder–Narasimhan filtration*, i.e., $0 = V_0 \subsetneq V_1 \cdots \subsetneq V_s = V$ such that every successive quotient V_i / V_{i-1} is semistable.

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Separating representations in $\mathrm{R}(Q, \mathbf{d})$ *by their HN type gives a stratification into smooth, disjoint, locally closed subsets* $S_{\mathbf{d}^*}$*, ordered in a way such that* $S_{(\mathbf{d})} = X^{ss}$ *and*

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For the action of $\mathrm{GL}_{\mathbf{d}}$ *on* $\mathrm{R}(Q, \mathbf{d})$, *"twisted by* θ*", given a norm on 1-PSs* α, *the GIT stratification coincides with the Harder–Narasimhan stratification given by* θ *and* α.

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Corollary

The 1-PS λ corresponding to the HN type $\mathbf{d}^*$ is given by

$$\lambda_i(z) = \mathrm{diag}\,\big(\underbrace{z^{k_1}, \dots, z^{k_1}}_{d_i^1 \text{ times}}; \underbrace{z^{k_2}, \dots, z^{k_2}}_{d_i^2 \text{ times}}; \dots; \underbrace{z^{k_s}, \dots, z^{k_s}}_{d_i^s \text{ times}}\big),$$

where $k_t = \mu(\mathbf{d}^t)$.

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Theorem (Teleman quantization [7])

If on each stratum S_ℓ, the inequality $\max(W(\lambda_\ell, \mathcal{F})) < \eta_\ell$ holds, then there is an isomorphism

$$\mathrm{H}^k(X, \mathcal{F})^G \xrightarrow{\sim} \mathrm{H}^k(X^{ss}//G, \mathcal{F}_{desc})$$

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If we can apply Teleman quantization, we win, because $\mathrm{R}(Q, \mathbf{d})$ is affine, so

$$\mathrm{H}^k(\mathrm{R}(Q, \mathbf{d}), \mathcal{F}) = 0 \ \forall k \geq 1,$$

so we compute the weights we are interested in.

Proof of rigidity - weights

Lemma (Corollary 3.18 [3])

The weight $\eta_{\mathbf{d}^*}$ *of* $\det(\mathcal{N}^\vee_{S_{\mathbf{d}^*}/\mathrm{R}})|_{Z_{\mathbf{d}^*}}$ *is*

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Note: $\langle -, - \rangle$ is the so-called bilinear Euler form of a quiver.

Proof of rigidity - conclusion

Theorem

Under a technical assumption (that $\mathbf{d}$ *is strongly amply* θ*-stable), the Teleman inequality holds on every stratum: for all* $\mathbf{d}^*$*,*

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- In fact, these are implemented :)

Thank you for your attention!

Bibliography

[1] Jan Adriaenssens and Lieven Le Bruyn. "Local quivers and stable representations". In: *Comm. Algebra* 31.4 (2003), pp. 1777–1797. DOI: 10.1081/AGB-120018508.

[2] Pieter Belmans and Hans Franzen. *On Chow rings of quiver moduli*. 2023. arXiv: 2307.01711.

[3] Pieter Belmans et al. *Rigidity and Schofield's partial tilting conjecture for quiver moduli*. 2023. arXiv: 2311.17003.

[4] J.-M. Drezet and M. S. Narasimhan. "Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques". In: *Invent. Math.* 97.1 (1989), pp. 53–94. URL: https://hal.science/hal-01158986.

[5] H. Franzen. "Chow rings of fine quiver moduli are tautologically presented". In: *Mathematische Zeitschrift* 279.3-4 (Jan. 2015), pp. 1197–1223. DOI: 10.1007/s00209-015-1409-8.

[6] Hans Franzen, Markus Reineke, and Silvia Sabatini. "Fano quiver moduli". In: *Canadian Mathematical Bulletin* 64.4 (Dec. 2020), pp. 984–1000. DOI: 10.4153/s0008439520001009.

[7] Daniel Halpern-Leistner. "The derived category of a GIT quotient". In: *J. Amer. Math. Soc.* 28.3 (2015), pp. 871–912. DOI: 10.1090/S0894-0347-2014-00815-8.

[8] Victoria Hoskins. "Stratifications associated to reductive group actions on affine spaces". In: *Q. J. Math.* 65.3 (2014), pp. 1011–1047. DOI: 10.1093/qmath/hat046.

[9] Victoria Hoskins. "Stratifications for moduli of sheaves and moduli of quiver representations". In: *Algebr. Geom.* 5.6 (2018), pp. 650–685. DOI: 10.14231/AG-2018-017.

[10] Markus Reineke. "The Harder-Narasimhan system in quantum groups and cohomology of quiver moduli". In: *Invent. Math.* 152.2 (2003), pp. 349–368. DOI: 10.1007/s00222-002-0273-4.

[11] Markus Reineke and Stefan Schroeer. *Brauer groups for quiver moduli*. 2014. arXiv: 1410.0466.