Derivation of Coupled-Channel Lippmann-Schwinger Equation

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1 Problem Setup

We consider a quantum scattering problem where a particle has both translational motion (coordinate x) and rotational motion (angle θ) with coupling between angular momentum channels. The time-independent Schrödinger equation is:

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}-\frac{\hbar^2}{2I_0}\frac{\partial^2}{\partial\theta^2}+V_0\cos\theta\sum_{n=-1}^{+1}e^{-(x-n\xi)^2}\right]\psi(x,\theta)=E\psi(x,\theta) \tag{1}$$

The potential consists of Gaussian barriers at positions $x = n\xi$ with angular coupling through $\cos\theta$.

2 Channel Decomposition

We expand in angular momentum eigenstates:

$$\psi(x,\theta)=\sum_{l=-\infty}^{\infty}\frac{\psi_l(x)}{\sqrt{2\pi}}e^{il\theta} \tag{2}$$

The angular kinetic energy operator acts as:

$$-\frac{\hbar^2}{2I_0}\frac{\partial^2}{\partial\theta^2}\frac{e^{il\theta}}{\sqrt{2\pi}}=\frac{\hbar^2 l^2}{2I_0}\frac{e^{il\theta}}{\sqrt{2\pi}} \tag{3}$$

3 Coupled-Channel Equations

To obtain equations for each channel $\psi_l(x)$, we project the Schrödinger equation onto angular momentum states by multiplying by $\frac{e^{-il'\theta}}{\sqrt{2\pi}}$ and integrating over θ.

The coupling matrix element is:

$$\int_0^{2\pi}\frac{e^{-il'\theta}}{\sqrt{2\pi}}\cos\theta\frac{e^{il\theta}}{\sqrt{2\pi}}d\theta=\frac{1}{2}(\delta_{l',l+1}+\delta_{l',l-1}) \tag{4}$$

since $\cos\theta=\frac{1}{2}(e^{i\theta}+e^{-i\theta})$.

This gives the coupled-channel equations:

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\frac{\hbar^2 l^2}{2I_0}\right)\psi_l(x)+\frac{V_0}{2}\sum_n e^{-(x-n\xi)^2}[\psi_{l+1}(x)+\psi_{l-1}(x)]=E\psi_l(x) \tag{5}$$

4 Green's Function Formulation

Define the channel energies and wave numbers:

$$E_l = E - \frac{\hbar^2 l^2}{2I_0}, \quad k_l^2 = \frac{2mE_l}{\hbar^2} \tag{6}$$

The free Green's function for channel l satisfies:

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - E_l\right) G_l^{(0)}(x, x') = -\delta(x - x') \tag{7}$$

With outgoing boundary conditions, this gives:

$$G_l^{(0)}(x, x') = \frac{im}{\hbar^2 k_l} e^{ik_l|x-x'|} \tag{8}$$

5 Lippmann-Schwinger Equation

The coupled-channel equations can be written as integral equations:

$$\psi_l(x) = \psi_l^{(0)}(x) + \int_{-\infty}^{\infty} G_l^{(0)}(x, x') V_{l,l'}(x') \psi_{l'}(x') dx' \tag{9}$$

where $\psi_l^{(0)}(x) = e^{ik_l x}\delta_{l,l_{\text{in}}}$ is the incident wave in channel l_{in}, and:

$$V_{l,l'}(x') = \frac{V_0}{2} \sum_n e^{-(x'-n\xi)^2} [\delta_{l',l+1} + \delta_{l',l-1}] \tag{10}$$

Explicitly:

$$\begin{aligned} \psi_l(x) = e^{ik_l x}\delta_{l,l_{\text{in}}} + \frac{im}{\hbar^2 k_l} \int_{-\infty}^{\infty} & e^{ik_l|x-x'|} \frac{V_0}{2} \sum_n e^{-(x'-n\xi)^2} \\ & \times [\psi_{l+1}(x') + \psi_{l-1}(x')] dx' \end{aligned} \tag{11}$$

6 Asymptotic Behavior and Scattering Amplitudes

For large $|x|$, the Green's function behaves as:

$$G_l^{(0)}(x, x') \approx \frac{im}{\hbar^2 k_l} e^{ik_l|x|} e^{-ik_l x' \text{sgn}(x)} \tag{12}$$

This gives the asymptotic form:

$$\psi_l(x) \xrightarrow{x\to\pm\infty} e^{ik_l x}\delta_{l,l_{\text{in}}} + e^{ik_l|x|} f_{l,l_{\text{in}}} \tag{13}$$

where the scattering amplitudes are:

$$f_{l,l_{\text{in}}} = -\frac{im}{\hbar^2 k_l} \int_{-\infty}^{\infty} e^{-ik_l x' \text{sgn}(x)} \frac{V_0}{2} \sum_n e^{-(x'-n\xi)^2} [\psi_{l+1}(x') + \psi_{l-1}(x')] dx' \tag{14}$$

7 Born Approximation

In the first Born approximation, we replace $\psi_{l'}(x')$ with the incident wave $\psi_{l'}^{(0)}(x') = e^{ik_{l'}x'}\delta_{l',l_{\text{in}}}$:

$$f_{l,l_{\text{in}}}^{\text{Born}} = -\frac{imV_0}{2\hbar^2 k_l}\int_{-\infty}^{\infty} e^{-ik_l x' \text{sgn}(x)} \sum_n e^{-(x'-n\delta)^2}[\delta_{l+1,l_{\text{in}}} + \delta_{l-1,l_{\text{in}}}]e^{ik_{l_{\text{in}}}x'}dx' \tag{15}$$

This shows that scattering only occurs between adjacent angular momentum channels ($l = l_{\text{in}} \pm 1$) due to the dipole selection rule from $\cos\theta$ coupling.

For the specific case $l = l_{\text{in}} + 1$:

$$f_{l_{\text{in}}+1,l_{\text{in}}}^{\text{Born}} = -\frac{imV_0}{2\hbar^2 k_{l_{\text{in}}+1}}\sum_n \int_{-\infty}^{\infty} e^{i(k_{l_{\text{in}}} - k_{l_{\text{in}}+1}\text{sgn}(x))x'} e^{-(x'-n\xi)^2} dx' \tag{16}$$

The integral over each Gaussian barrier can be evaluated analytically, giving contributions that depend on the momentum transfer $q = k_{l_{\text{in}}} - k_{l_{\text{in}}+1}\text{sgn}(x)$ and the barrier spacing ξ.

8 Summary

The key steps in this derivation were:

1. Angular momentum decomposition of the wavefunction
2. Projection onto channel states to obtain coupled differential equations
3. Introduction of Green's functions to transform to integral equations
4. Asymptotic analysis to extract scattering amplitudes
5. Born approximation for practical calculations

The result provides a systematic framework for calculating scattering between different angular momentum channels in systems with localized coupling potentials.