

Positivity in Amplitudes from Quantum Entanglement

Rafael Aoude,¹ Gilly Elor,² Grant N. Remmen,³ and Olcyr Sumensari⁴

¹*Higgs Centre for Theoretical Physics, School of Physics and Astronomy,
The University of Edinburgh, Edinburgh EH9 3JZ, Scotland, United Kingdom*

²*Weinberg Institute, Department of Physics, University of Texas at Austin, Austin, TX 78712, United States*

³*Center for Cosmology and Particle Physics, Department of Physics,
New York University, New York, NY 10003, United States*

⁴*IJCLab, Pôle Théorie (Bat. 210), CNRS/IN2P3 et Université, Paris-Saclay, 91405 Orsay, France*

We show that positivity of the imaginary part of forward elastic amplitudes for perturbative scattering is equivalent to consistency of the entanglement generated by the S-matrix, for states with arbitrary internal quantum numbers such as flavor. We also analyze “disentanglers,” certain highly entangled initial states for which the action of the S-matrix is to decrease subsystem entanglement.

Introduction.—Scattering amplitudes are among the most foundational and compelling observables in physics. As the fundamental objects of study in quantum field theory, their structure underpins much of modern high-energy physics, from phenomenology to string theory. The mathematical structure of amplitudes has been studied for decades, and important theorems arising from unitarity and locality are well known, including constraints on their asymptotic growth and analytic structure, e.g., the bounds of Froissart, Martin, Lehmann, et al. [1–3]. Moreover, in the modern amplitudes program, it has been shown that unitarity and locality themselves can emerge from more primordial mathematical structures [4, 5].

Chief among the many useful and famous facts about scattering amplitudes is the optical theorem, which relates the imaginary part of an amplitude—at forward kinematics, where the outgoing and incoming states are identical and particles pass straight through each other—to its cross section. Systematic use of this fact, dubbed “positivity” of amplitudes, has especially in the last two decades allowed the laws of physics themselves to be constrained using analytic dispersion relations, implying numerous different bounds on the space of coefficients of higher-dimension operators in effective field theories (EFTs) [6–10], ranging from the standard model EFT [11–16] to quantum corrections to gravity [17–21].

Meanwhile, entanglement is the quintessentially novel property of quantum mechanics [22, 23]. Historically, it was experimentally verified in classic laboratory demonstrations of violations of Bell’s inequality [23, 24]. Excitingly, the highest-energy probes of entanglement can now be implemented in colliders, such as measurements of spin entanglement in top quark systems recently observed by the ATLAS Collaboration at the LHC [25] as proposed in Ref. [26]; see also Refs. [27–30]. These measurements open novel directions in constraining possible new physics using quantum information [31–33].

Quantum information theory also provides many positivity bounds on physics of an a priori different sort than dispersion relations. These bounds take the form of inequalities on quantum entanglement, that is, the von Neumann entropy $S(\hat{\rho}) = -\text{Tr}[\hat{\rho} \log \hat{\rho}]$ for a den-

sity matrix $\hat{\rho}$. Various such bounds can be derived from quantum mechanics alone as consequences of linear algebra, including monogamy of entanglement, subadditivity, strong subadditivity, the Araki-Lieb inequality, etc. [34]. By considering other quantum information theoretic measures, such as relative entropy, yet more relations can be found [35]. In addition, entanglement entropy for states of conformal field theories with holographic spacetime duals are subject to additional bounds, such as monogamy of mutual information [36] or reflected entropy inequalities [37, 38], which can be proved geometrically via the AdS/CFT correspondence. Underpinning all of these results is the basic requirement of positivity of entanglement entropy, $S(\hat{\rho}) \geq 0$.

An interesting question is whether these two notions of positivity—that is, the amplitudes’ positivity and entanglement entropy—can be related. Do quantum information bounds place additional constraints on EFTs? Or is the optical theorem somehow itself a statement about the entanglement structure of the S-matrix? While there are many realizations of the unitarity in quantum field theory, in this paper we aim to connect those associated with these two manifestations of positivity. The intersection of quantum entanglement and the S-matrix has not been fully investigated, though there have been interesting observations made about the relation between entanglement extremization and low-energy emergent symmetries [39–43], as well as between entanglement and the Yang-Mills/gravity double copy [44]. In notable work, Ref. [45] considered the quantum Tsallis entropy and showed that, subject to certain conditions on an unentangled initial state, it grows under perturbative scattering. The deeper question of whether positivity of the quantum entanglement itself can *imply* amplitudes’ positivity, in a broader context of states with general quantum numbers, has not been explored. This question—focusing on discrete Hilbert spaces and asking the converse of whether positivity can emerge from entanglement—is especially interesting from both a quantum computing and phenomenological standpoint, as many particles in the standard model possess flavor or polarization.

In this Letter, we address this problem and show that

amplitudes' positivity is *precisely equivalent* to positivity of entropy, for a general perturbative quantum theory. Specifically, we will show that the linearized entropy,¹ defined as

$$\begin{aligned}\mathcal{E}[\Omega] &= 1 - \text{Tr}_A [\hat{\rho}_A^2], \\ \hat{\rho}_A &= \text{Tr}_B \hat{\rho}_{AB}, \quad \hat{\rho}_{AB} = \hat{S}|\Omega\rangle\langle\Omega|\hat{S}^\dagger,\end{aligned}\tag{1}$$

for a two-to-two scattering process through S-matrix \hat{S} , is nonnegative *if and only if* the corresponding amplitude has positive imaginary part in the forward limit, i.e.,

$$\mathcal{E}[|\Omega^{\text{prod}}\rangle] \geq 0 \iff \text{Im } \mathcal{A} > 0.\tag{2}$$

Furthermore, we will show that the elastic optical theorem implies $\mathcal{E}[|\Omega^{\text{prod}}\rangle] \geq 0$. We will write a general initial two-particle state in Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ as $|\Omega\rangle = \sum_{ab} \Omega_{ab} |k_1, a\rangle_A \otimes |k_2, b\rangle_B$, with definite momenta k_i and summed over internal quantum numbers, and Ω^{prod} refers to a product (unentangled) initial state. For the basis vectors we use the notation $|k_1, a\rangle_A \otimes |k_2, b\rangle_B \equiv |k_1, a; k_2, b\rangle$, and write $\hat{\rho}_A$ for the reduced density matrix in \mathcal{H}_A after tracing out degrees of freedom in \mathcal{H}_B . The results of this work represent an exciting connection between entanglement and the S-matrix, opening the door to future insights that might be gained in the amplitudes program via the use of powerful results from quantum information theory. From a phenomenological perspective, the study of entanglement in EFTs could yield insights to model building principles in a similar spirit to Refs. [39–43].

Entanglement from Two-to-Two Scattering.—The entanglement entropy can be computed perturbatively, order by order, in a quantum field theory [46]. Furthermore, such calculations have been shown to be valid in a Wilsonian EFT [47, 48] where one deals with effective operators in the infrared. In practice, this amounts to tracing over continuous degrees of freedom. For example, Refs. [49–52] built upon the results of Ref. [47] to compute momentum-space entanglement generated in scattering (with and without spin entanglement). In this paper, we are interested in the entanglement generated between scalar fields with an internal quantum number, e.g., flavor; we will address the generalization to fermions or vectors in future work. We thus compute the traces of operators $\hat{\mathcal{O}}$ as $\text{Tr}_B[\hat{\mathcal{O}}] = \sum_c \int_u \langle u, c|_B \hat{\mathcal{O}}|u, c\rangle_B$. We use the shorthand $\int_p \equiv \int d^3 p / [(2\pi)^3 2E_p]$ throughout. The

reduced density matrix in Eq. (1) of a scalar particle in \mathcal{H}_A is computed by tracing out degrees of freedom in \mathcal{H}_B ,

$$\hat{\rho}_A = \frac{1}{N} \sum_c \int_u \langle u, c|_B \hat{S}|\Omega\rangle\langle\Omega|\hat{S}^\dagger|u, c\rangle_B.\tag{3}$$

Here N is a factor that ensures that the incoming state (for which \hat{S} is replaced by $\hat{1}$) is properly normalized.

We now compute the linearized entanglement generated by two-to-two scalar scattering $\phi_i(k_1)\phi_j(k_2) \rightarrow \phi_m(p_1)\phi_n(p_2)$, induced by a perturbative S-matrix $\hat{S} = \hat{1} + i\hat{T}$ but *without assuming unitarity*. The indices i, j, m, n denote internal quantum numbers and p/k are outgoing/incoming momentum. This computation holds both for the case where scattering proceeds through a four-scalar contact term or where the scattering is generated by a mediating particle.

As stated above we will *not* assume unitarity. Our assumption is simply that there exists an S-matrix, which we will take to be a *state-independent general linear operator* acting on the Hilbert space. We will *not* assume a priori that \hat{S} preserves normalizations, i.e., we will not use $\hat{S}^\dagger \hat{S} = \hat{1}$.² In this way, failure of \hat{S} to square to unity can be thought of as modeling an open quantum system, in which the initial state evolves into some inaccessible degrees of freedom. However, we need not invoke this picture and can instead consider a closed quantum system of fixed Hilbert space with a (not necessarily unitary) S-matrix. We will assume that \hat{S} can, at worst, decrease the normalization of a state, so that the linearized entropy (1) is by definition positive. Perturbative scattering amplitudes are defined by acting on state vectors with the transition matrix \hat{T} , i.e., $\langle p_1, m; p_2, n|\hat{T}|k_1, a; k_2, b\rangle = (2\pi)^4 \delta^4(k_i - p_f) \mathcal{A}_{ab}^{mn}(k_1 k_2 \rightarrow p_1 p_2)$. Likewise, we define the Hermitian conjugate as $\langle p_1, m; p_2, n|\hat{T}^\dagger|k_1, a; k_2, b\rangle = (2\pi)^4 \delta^4(k_i - p_f) \mathcal{A}_{ab}^{\dagger mn}(k_1 k_2 \rightarrow p_1 p_2)$. While momentum conservation is enforced through the delta functions in the definitions above, *we do not demand conservation of the internal symmetry*. Note that in defining \mathcal{A}^\dagger we have simply taken the Hermitian conjugate of the S-matrix element. We need not assume, e.g., that \mathcal{A}^\dagger is the CPT conjugate of \mathcal{A} , nor any other assumption that would require unitarity. Furthermore, since we have not assumed unitarity, we also do not a priori have the optical theorem, and so the sign of $\text{Im } \mathcal{A}$ is as yet unfixed. The two-particle incoming pure state $|\Omega\rangle$ is defined below Eq. (1). The internal quantum numbers of the incoming states live in $\mathcal{H}_A \otimes \mathcal{H}_B$, which we take as discrete, e.g., $\mathbb{C}^n \otimes \mathbb{C}^n$, so that we can use the above expressions to compute the entanglement generated in scattering.

¹ The linearized entropy equals the von Neumann entropy at leading order, linearized around a maximally mixed state. The quantity $\gamma \equiv \text{Tr}_A [\hat{\rho}_A^2]$ is known as the purity and ranges over $1/d \leq \gamma \leq 1$ where d is the dimension of \mathcal{H}_A . The case $\gamma = 1$ corresponds to a pure state and $\gamma = 1/d$ to a maximally mixed state.

² Here, \hat{S}^\dagger is defined as simply the Hermitian conjugate of \hat{S} , i.e., where \hat{S} acts with left multiplication on vectors $|\psi\rangle$ in Hilbert space, \hat{S}^\dagger acts with right multiplication on covectors $\langle\psi|$.

We compute the entanglement for a general S-matrix in Eq. (1) by inserting into Eq. (3) the identity $\hat{1} = \sum_n \hat{1}_n$, where $\hat{1}_n = (\prod_{i=1}^n \int_{q_i}) |\{q_i\}\rangle\langle\{q_i\}|$ is the n -particle identity operator, and the sum is over all n -particle intermediate states, $\{q_i\} \equiv \{q_1, q_2, \dots, q_n\}$. One-particle states are normalized as $\langle k, a | l, b \rangle = (2\pi)^3 2E_k \delta^3(k - l) \delta_{ab}$ following the conventions in Ref. [52]. The linearized entanglement in Eq. (1) may then be compactly written as

$$\mathcal{E}[\Omega] = 1 - \frac{1}{N^2} \sum_{ab} \int_{q_1} \int_{q_2} \mathcal{F}(q_1, q_2)_{ab} \mathcal{F}(q_2, q_1)_{ba} \quad (4)$$

in terms of a scalar function,

$$\mathcal{F}(q_1, q_2)_{ab} \equiv \sum_m \int_l \langle q_1, a; l, m | \hat{S} |\Omega\rangle \langle \Omega | \hat{S}^\dagger | q_2, b; l, m \rangle. \quad (5)$$

Again, we emphasize that the \hat{S}^\dagger appearing above does not arise from any assumption of unitarity, but instead simply from the definition of bras, kets, and the inner product.

After various manipulations, we find the leading-order change in entanglement of the two scalars generated by scattering:

$$\begin{aligned} \Delta\mathcal{E}[\Omega] &= 4 \left(\frac{1}{2E_{k_1} 2E_{k_2}} \frac{T}{V} \right) \times \\ &\quad \times \text{Im} \left[\sum_{abij} \Omega_{ij} \mathcal{A}_{ab}^{ij}(k_1 k_2 \rightarrow k_1 k_2) (\Omega^\dagger \cdot \Omega \cdot \Omega^\dagger)_{ab} \right], \end{aligned} \quad (6)$$

through $O(g^2)$, where g is a small coupling in the perturbative expansion $\mathcal{A} \sim g^2$. Here $\Delta\mathcal{E} = \mathcal{E}_f - \mathcal{E}_i$ and $\mathcal{E}_i[\Omega] = 1 - \text{Tr}[(\Omega^\dagger \Omega)^2]$, which vanishes for a pure initial state. Divergences in spacetime volume are defined as $V \equiv (2\pi)^3 \delta_V^3(0)$ and $T \equiv 2\pi \delta_T(0)$. The seemingly divergent prefactor in Eq. (6), which we henceforth denote as $\mathcal{N} \equiv \left(\frac{1}{2E_{k_1} 2E_{k_2}} \frac{T}{V} \right)$, is an artifact of using plane waves in our derivation and is shown to be equal to unity in the Supplemental Material with the aid of a wave-packet formulation. For properly normalized initial states Ω , with $\text{Tr}[\Omega^\dagger \Omega] = 1$, we have $N = 2E_{k_1} 2E_{k_2} V^2$. Interestingly, the entanglement computation has automatically selected amplitudes with forward kinematics from the integral in Eq. (4). See the Supplemental Material for the full derivation, including details on how forward kinematics arise due to momentum conservation and the tracing out states in the $O(g^2)$ term.³

Amplitudes' Positivity from Entanglement.—Using our calculations from the previous section, we are

now poised to demonstrate the primary result of this paper summarized in Eq. (2), i.e., that positivity of entanglement, computed without assuming unitarity, implies the positivity of elastic amplitudes.

We are interested in entanglement that is generated by the S-matrix, so as such we will consider a state where the two initial particles are unentangled, so that $\mathcal{E}_i = 0$. By Schmidt decomposition, this means that there exist unique vectors $|\alpha\rangle_A \in \mathcal{H}_A$ and $|\beta\rangle_B \in \mathcal{H}_B$ —given by some sums over the basis vectors $|a\rangle_A$ and $|b\rangle_B$, respectively—such that $|\Omega^{\text{prod}}\rangle = |k_1, \alpha\rangle_A \otimes |k_2, \beta\rangle_B$ is a direct product state. In that case, $(\Omega \cdot \Omega^\dagger \cdot \Omega)_{ab}$ is simply Ω_{ab} , so from Eq. (6) the leading-order entanglement generated by scattering is

$$\Delta\mathcal{E}[\Omega^{\text{prod}}] = 4\mathcal{N} \text{Im} \mathcal{A}_{\alpha\beta}^{\alpha\beta}(k_1 k_2 \rightarrow k_1 k_2). \quad (7)$$

The amplitude on the right-hand side is kinematically forward, as well as elastic in the internal quantum numbers. Critically, $\Delta\mathcal{E}[\Omega^{\text{prod}}] \geq 0$ since $\mathcal{E}_i = 0$ and $\mathcal{E}_f \geq 0$, i.e., the entanglement is necessarily nonnegative. Thus, positivity of entanglement also implies positivity of the imaginary part of the forward amplitude (which need not have been positive a priori without unitarity, $\hat{S}^\dagger \hat{S} = \hat{1}$). If we were to invoke the optical theorem at this point, we would have $\text{Im} \mathcal{A}_{\alpha\beta}^{\alpha\beta}(k_1 k_2 \rightarrow k_1 k_2) = 2E_{\text{cm}} p_{\text{cm}} \sigma_{\alpha\beta}$, where $\sigma_{\alpha\beta}$ is the cross section for $|\alpha\rangle_A \otimes |\beta\rangle_B \rightarrow$ anything and which as a physical area is by definition positive, E_{cm} is the energy of the two-particle initial state in the center-of-mass frame, and $p_{\text{cm}} = v_1 E_{k_1} = v_2 E_{k_2}$ is the momentum of either of the initial particles. Thus, the optical theorem implies positivity of the entanglement for elastic scattering,

$$\Delta\mathcal{E}[\Omega^{\text{prod}}] = 2(T/V)(v_1 + v_2) \sigma_{\alpha\beta} \geq 0. \quad (8)$$

Note that for this product initial states, this leading-order tree-level entanglement is only nonvanishing when it has support on the resonance (as shown in the Supplemental Material). However, this conclusion generalizes to all weakly coupled theories, as one can view any loop-level process as an infinite sum over trees, as shown in App. B of Ref. [10]. We leave a detailed study of higher-order contributions to future work.

These conclusions generalize to the quantum Tsallis entropy $\mathcal{E}_n = \frac{1}{n-1} (1 - \text{Tr}_A[\hat{\rho}_A^n])$, for which the linearized entanglement is the $n=2$ case. As above, \mathcal{E}_n is non-negative and vanishes for a pure state.⁴ For an initial product state, as shown in the Supplemental Material, we have $\Delta\mathcal{E}_n[\Omega^{\text{prod}}] = \frac{2n}{n-1} \mathcal{N} \text{Im} \mathcal{A}_{\alpha\beta}^{\alpha\beta}(k_1 k_2 \rightarrow k_1 k_2)$ for all integers $n \geq 2$.

³ A related computation carried out in Ref. [45] did not fix forward kinematics, precisely because their derivation involved evoking the optical theorem throughout, setting $i(\hat{T} - \hat{T}^\dagger)$ to $-\hat{T} \hat{T}^\dagger$.

⁴ The Tsallis entropy is bijectively related to the n th Rényi entropy, $S_n(\hat{\rho}) = \frac{1}{1-n} \log \text{Tr}[\hat{\rho}^n]$, and both reduce to the von Neumann entropy in the limit $n \rightarrow 1$.

Thus, we have shown that positivity of the linear entropy and positivity of the amplitudes are *equivalent*, as advertised in Eq. (2). Both are manifestations of unitarity, but in a priori very different contexts, namely dispersion relations and consistency of quantum information. From this perspective, all positivity bounds on EFTs—including their many consequences including applications to quantum gravity, the EFT-hedron [10], etc.—are all quantum information theoretic statements, demanding consistency of the entanglement produced among the decay products.

Disentanglers.—For entangled initial states, $\Delta\mathcal{E}$ in Eq. (6) need not always be positive. Such “disentangler states” occasionally arise when the S-matrix itself acts as a “Maxwell’s demon” or “disentangler,” decreasing the entanglement of the final state relative to the initial one. Similar states were explored in Ref. [44], and their existence can be anticipated simply from the fact that in any quantum theory the action of the S-matrix on a state may be reversed through acting with \hat{S}^\dagger ; such objects have found extensive use in tensor network contexts [53, 54].

We now demonstrate the existence of such disentangler states and highlight their role in hindering us from deriving the *generalized* optical theorem from entanglement.⁵ Recall that the generalized optical theorem is the statement that \mathcal{A}_{ab}^{ij} is a positive definite matrix, a property that is preserved under unitary transformations.

Let us Schmidt-decompose the initial state,

$$|\Omega\rangle = \sum_k \lambda_k |\alpha_k\rangle \otimes |\beta_k\rangle, \text{ i.e., } \Omega_{ij} = \sum_k \lambda_k \alpha_i^{(k)} \beta_j^{(k)}, \quad (9)$$

for some real $\lambda_k \geq 0$ and orthonormal vectors $|\alpha_k\rangle$ and $|\beta_k\rangle$, so Eq. (6) now becomes

$$\Delta\mathcal{E}[\Omega] = 4\mathcal{N} \operatorname{Im} \left[\sum_{kk'} \lambda_k \alpha_j^{*(k)} \beta_i^{*(k)} \mathcal{A}_{ab}^{ij} \lambda_{k'}^3 \alpha_a^{(k')} \beta_b^{(k')} \right]. \quad (10)$$

Note that without internal flavor indices, we would have $\Delta\mathcal{E} \propto \operatorname{Im} \mathcal{A}(k_1 k_2 \rightarrow k_1 k_2) \propto \sigma \geq 0$, in which case disentangler states would not arise. However, in the presence of the internal degrees of freedom considered here, this is no longer the case as we will now demonstrate.

Let us write the basis of $|i\rangle \otimes |j\rangle$ states as $|I\rangle$, each denoted by some vector v_I

$$\Delta\mathcal{E}[\Omega] = 4\mathcal{N} \operatorname{Im} \left[\sum_{k,k'} \lambda_k \lambda_{k'}^3 v_I^{(k)*} \mathcal{A}_{I'I'}^I v_{I'}^{(k')} \right]. \quad (11)$$

Without loss of generality, we take each v_I to be real (a complex v_I can be made real via a unitary transformation,

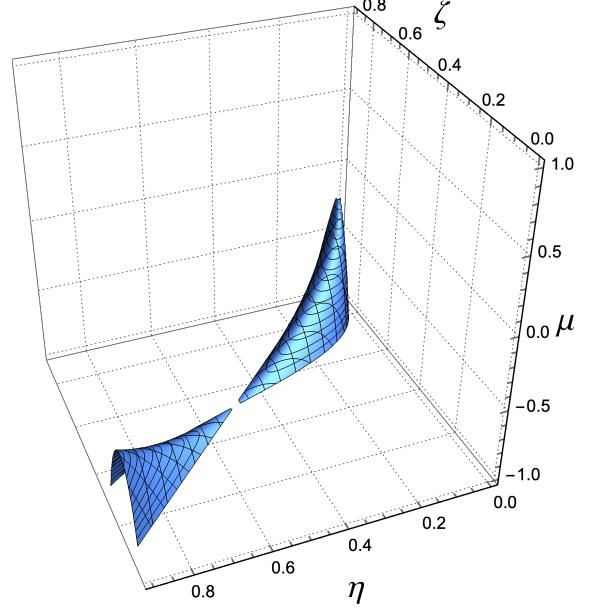


FIG. 1. The blue surface and volume below it corresponds to the parameter space described in text for a disentangler state, for which $\Delta\mathcal{E} < 0$.

which we absorb into \mathcal{A}). Thus, we can pull everything but the amplitude out of the imaginary part of Eq. (10), yielding

$$\Delta\mathcal{E}[\Omega] = 4\mathcal{N} \sum_{k,k'} \lambda_k \lambda_{k'}^3 x^{(k)} \cdot x^{(k')}, \quad (12)$$

where we have used the generalized optical theorem to decompose $\operatorname{Im} \mathcal{A}_{I'I'}^I = M^{IJ} M_{JI'}$ for some matrices M and defined the vectors $x^{(k)} = M \cdot v^{(k)}$. The M matrices—which can be taken to be real [15, 21]—describe the cuts of the four-particle amplitudes, that is, the (real and imaginary parts of) amplitudes for a given two-particle state described by an element of the Schmidt basis to go to some exchanged state. By simple algebra and relabeling of the indices k, k' , we have:

$$\Delta\mathcal{E}[\Omega] = \frac{\mathcal{N}}{4} \sum_{k,k'} [(\lambda_k + \lambda_{k'})^4 - (\lambda_k - \lambda_{k'})^4] x^{(k)} \cdot x^{(k')}. \quad (13)$$

Since the generalized optical theorem is both necessary and sufficient for a healthy perturbative ultraviolet completion, it is possible to design a completion with arbitrary $x^{(k)} \cdot x^{(k')}$. One can think of $x^{(k)} \cdot x^{(k')}$ as defining a Riemannian metric $g_{kk'}$, i.e., $g_{kk'}$ is positive definite. For an *un*-entangled initial state, the Schmidt decomposition has only one nonzero value of λ , so that $\Delta\mathcal{E}$ is manifestly nonnegative since all the $\lambda_k - \lambda_{k'}$ differences vanish. However, for general $|\Omega\rangle$, we now demonstrate that disentangler states for which $\Delta\mathcal{E}[\Omega] < 0$ can arise.

As an illustration that $\Delta\mathcal{E}$ can be negative, consider for simplicity a scenario with only two nonzero λ_k values

⁵ Understanding the connection between entanglement and the generalized optical theorem is particularly interesting in light of recent positivity bounds that make use of it [15, 16, 21, 55].

in the Schmidt basis. For example, each particle could be a qubit, leaving us with a four-dimensional joint Hilbert space, and we can choose to scatter an incoming Bell state, e.g., $(|1\rangle|1\rangle+|2\rangle|2\rangle)/\sqrt{2}$ or $(|1\rangle|2\rangle-|2\rangle|1\rangle)/\sqrt{2}$. Let us define $\xi = \lambda_2/\lambda_1 > 0$ and write $x^{(1)} = (\cos\theta_1, \sin\theta_1)$ as a general unit vector and $x^{(2)} = \nu(\cos\theta_2, \sin\theta_2)$ for some $\nu > 0$. (Roughly speaking, ν describes the ratio in the overall size of the couplings of the two elements of the Schmidt basis to the ultraviolet states.) Writing $\cos(\theta_1 - \theta_2) = \mu$, $\xi = \zeta/(1 - \zeta)$, and $\nu = \eta/(1 - \eta)$, the parameter space is given by $\mu \in [-1, 1]$, $\zeta \in [0, 1]$, and $\eta \in [0, 1]$. Note that $\zeta = 1$ or 0 corresponds to a pure state and $\zeta = 1/2$ to a maximally mixed state. We indeed find that a fraction of the parameter space leads to disentanglers, as shown by the region on and under the blue surface in Fig. 1. Note that disentanglers only appear when $\mu < 0$, corresponding to the physical picture where vectors x^1 and x^2 are pointing away from each other such that the element $g_{12} < 0$ is possible.

We observe in passing that if $x^{(k)} \cdot x^{(k')}$ is diagonal, equal to $\gamma_k \delta_{kk'}$ for $\gamma_k > 0$, then from Eq. (13) $\Delta\mathcal{E}[\Omega] = 4\mathcal{N} \sum_k \gamma_k \lambda_k^4 \geq 0$ for any λ_k . This situation corresponds to the case where there is no mixing among the exchanged states to which each two-particle state in the Schmidt basis couples, that is, the scattering describes a sum over disjoint superselection sectors.

Moreover, for projector states,⁶ where $\lambda_k = c$ for $k \leq n$ for some n and $\lambda_k = 0$ for $k > n$, $\Delta\mathcal{E}$ is positive for all S-matrices, since $\Delta\mathcal{E} = 2\mathcal{N}c^4|X^{(n)}|^2 \geq 0$, defining the vector $X^{(n)} \equiv \sum_{k=1}^n x^{(k)}$. The converse is also true: if λ_k does not take the projector form, it is a disentangler state for some S-matrix. Without loss of generality, for λ_k not a projector, there exist $c_1 > c_2$ for which we can write $\lambda_k = c_1$ for $k \leq n$ for some n and $\lambda_{k+1} = c_2$. We pick the S-matrix such that $x^{(k)} = 0$ for $k > n+1$ and $X^{(n)} \cdot x^{(n+1)} = -\sqrt{|X^{(n)}|^2|x^{(n+1)}|^2}$, as permitted by Cauchy's inequality. Defining $q = \sqrt{|x^{(n+1)}|^2/|X^{(n)}|^2}$, we have $\Delta\mathcal{E}/|X^{(n)}|^2 = 2(c_1 - qc_2)(c_1^3 - qc_2^3)$. Since $c_1 > c_2 > 0$, it is always possible to choose q such that $c_1/c_2 < q < c_1^3/c_2^3$, in which case $\Delta\mathcal{E} < 0$. This construction completes the proof that the projector form for the Schmidt coefficients is both necessary and sufficient to prevent disentangler states—for which scattering acts as a Maxwell's demon—for arbitrary S-matrices.

Discussion and Future Directions.—In this work we have shown that amplitudes' positivity—a consequence of unitarity in the form of the optical theorem—implies positivity of entanglement, and vice versa. Could there be other consequences of unitarity that are related to entanglement? Could entanglement be used as a “guiding

principle” of sorts for effective field theories? This Letter paves the way to such explorations.

As discussed above, constraints on Wilson coefficients of EFT operators are traditionally derived from unitarity. Positivity of entanglement can be used to replicate (and perhaps even go beyond) such constraints. Consider as a simple example collinear two-to-two photon scattering arising from the EFT operator $\frac{C}{M_\Phi^2}(F_{\mu\nu}F^{\mu\nu})^2$, which can be generated via a heavy scalar Φ coupled to $F_{\mu\nu}F^{\mu\nu}$. In Ref. [6], the bound on the Wilson coefficient was shown to be simply $C > 0$. In the forward limit the photon has only transverse modes and behaves as a complex scalar. As shown in the Supplemental Material, positivity of entanglement

$$\Delta\mathcal{E}_{\gamma\gamma}[\Omega^{\text{prod}}] \propto Cs^2\delta(s - M_\Phi^2) \quad (14)$$

implies the positivity bound. We leave it to future work to extend such a study beyond this simplest of cases. For instance, one could consider a theory with two scalars, allowing for a CP phase, as well scattering through other operators considered in Ref. [11]. Such studies could also lead to new insights for the emergence of symmetries, in a similar spirit as Refs. [39, 40]. It would also be interesting to explore more generally the entanglement structure of theories with decays (where the leading-order entanglement will go as $\mathcal{A}_{1 \rightarrow 2}\mathcal{A}_{1 \rightarrow 2}^\dagger$), mass mixing, and particle-antiparticle oscillations. Another avenue for future work is to generalize the results of this letter to theories with fermions or vectors where the interplay between spin, momentum, and internal degrees of freedom can be studied. Finally, extending these results beyond leading order in perturbation theory could perhaps allow us to derive the generalized optical theorem.

ACKNOWLEDGMENTS

We thank Cliff Cheung, Ian Low, and Zhewei Yin for comments. This work was supported in part by the European Union’s Horizon research and innovation programme under the Marie Skłodowska-Curie grant agreements No. 860881-HIDDeN and No. 101086085-ASYMMETRY. R.A. was funded by the STFC grant “Particle Physics at the Higgs Centre.” The research of G.E. is supported by the National Science Foundation (NSF) Grant Number PHY-2210562, by a grant from University of Texas at Austin, and by a grant from the Simons Foundation. G.N.R. is supported by the James Arthur Postdoctoral Fellowship at New York University. G.E. thanks the Aspen Center for Physics (supported by NSF grant PHY-2210452) for their hospitality while this work was in progress.

⁶ The result about projector states and Maxwell's demon was also found in Ref. [45] in the context of a product initial two-particle state that is not necessarily pure.

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Supplemental Material for Positivity in Amplitudes from Quantum Entanglement

Rafael Aoude, Gilly Elor, Grant N. Remmen, and Olcyr Sumensari

In this Supplemental Material, we first provide a detailed derivation of $\Delta\mathcal{E}[\Omega]$ in Eq. (6) of the main text of this Letter. We begin by deriving an expression for the initial state entanglement, followed by a derivation of Eq. (6) generated by both a quartic contact term and by exchange of a heavy mediator. Critically, this exercise demonstrates the validity of our result to leading order in perturbation theory. We will also see the details of how forward kinematics are automatically selected. We include a computation of the next-highest order term ($O(g^4)$) in the perturbation expansion. We conclude with a section providing an alternative derivation of $\Delta\mathcal{E}[\Omega]$ using a wave function formulation. This approach will also provide justification for setting the seemingly divergent prefactor \mathcal{N} in Eq. (6) to one. Finally we include some details on extending these results to a more general case considering the Tsallis entropy.

$\Delta\mathcal{E}[\Omega]$ Derivation

We now present details of the computation of $\Delta\mathcal{E}[\Omega]$, the linearized entanglement in Eq. (6) generated in $\phi_i(k_1)\phi_j(k_2) \rightarrow \phi_k(p_1)\phi_l(p_2)$ scattering. First, we will compute the entanglement for a general initial state $|\Omega\rangle$ using the formalism discussed in this paper. We then compute the leading-order entanglement generated by the final state through the action of a perturbative S-matrix. We do this first in the case where the scattering proceeds through a four-point scalar contact term. We then repeat this computation in the case where the scattering arises through the exchange of a heavy particle. In both cases, we find that Eq. (6) describes the leading-order linearized entanglement generated through the scattering, as must have been the case. We present our derivation in terms of a toy model, but the result holds for, e.g., photon scattering $\gamma\gamma \rightarrow \Phi \rightarrow \gamma\gamma$ mediated by a heavy scalar Φ , where transverse photon helicities can be treated as an internal symmetry in the forward limit.

Throughout this derivation, our starting point will be a general two-particle incoming state,

$$|\Omega\rangle = \sum_{ij} \Omega_{ij} |k_1, i\rangle_A \otimes |k_2, j\rangle_B \equiv \sum_{ij} \Omega_{ij} |k_1, i; k_2, j\rangle, \quad (\text{S1})$$

where $\phi_i \in \mathcal{H}_A$ and $\phi_j \in \mathcal{H}_B$. Initial states are normalized such that $\text{Tr}[\Omega^\dagger \Omega] = 1$, as discussed in the main text. In the absence of a scalar mass mixing term, we may normalize the one-particle inner product as follows:

$$\langle p_1, i | p_2, j \rangle = (2\pi)^3 2E_{p_1} \delta_{ij} \delta^3(p_1 - p_2). \quad (\text{S2})$$

Throughout, as in the main text, we use the shorthand $\int_u \equiv \int d^3u/(2\pi)^3(2E_u)$, and define spacelike and timelike divergences as $V \equiv (2\pi)^3 \delta_V^3(0)$ and $T \equiv (2\pi) \delta_T(0)$.

Entanglement of the initial state

We begin with the straightforward derivation of the initial state entanglement in Eq. (S1). The full system density matrix of the two-particle initial state is constructed as follows:

$$\hat{\rho}_{AB}^i = |\Omega\rangle\langle\Omega| = \sum_{ijab} \Omega_{ij} \Omega_{ab}^\dagger |k_1, i\rangle_A \otimes |k_2, j\rangle_B \langle k_1, a|_A \otimes \langle k_2, b|_B. \quad (\text{S3})$$

The reduced density matrix is computed by tracing over particle B , and using Eq. (S2) this yields

$$\begin{aligned} \text{Tr}_B[\hat{\rho}_{AB}] &= \sum_{ijabc} \Omega_{ij} \Omega_{ab}^\dagger \int_u \left(\langle u, c | k_2, j \rangle_B \langle k_2, b | u, c \rangle_B \right) |k_1, i\rangle_A \langle k_1, a|_A \\ &= 2E_{k_2} V \sum_{ija} \Omega_{ij} \Omega_{aj}^\dagger |k_1, i\rangle_A \langle k_1, a|_A. \end{aligned} \quad (\text{S4})$$

The density matrix must be properly normalized so that $\text{Tr}_A \hat{\rho}_{nA} = 1$:

$$\begin{aligned} N &= \text{Tr}_A[\text{Tr}_B \hat{\rho}_{AB}^i] = 2E_{k_2} V \sum_{ijac} \int_c \Omega_{ij} \Omega_{aj}^\dagger \langle u, c | k_1, i \rangle \langle k_1, a | u, c \rangle \\ &= 2E_{k_2} 2E_{k_1} V^2 \text{Tr}[\Omega^\dagger \Omega] = 2E_{k_2} 2E_{k_1} V^2. \end{aligned} \quad (\text{S5})$$

The linear entanglement is then

$$\begin{aligned}\mathcal{E}_i[\Omega] &= 1 - \frac{1}{N^2} \text{Tr}_A[\hat{\rho}_A^2] \\ &= 1 - \frac{1}{(2E_{k_1})^2 V} \sum_c \sum_{ija} \sum_{i'j'a'} \Omega_{ij} \Omega_{aj}^\dagger \Omega_{i'j'} \Omega_{a'j'}^\dagger \int_u \langle u, c | k_1, i \rangle \langle k_1, a | k_1, i' \rangle \langle k_1, a' | u, c \rangle \\ &= 1 - \sum_{ija} \sum_{i'j'a'} \Omega_{ij} \Omega_{aj}^\dagger \Omega_{i'j'} \Omega_{a'j'}^\dagger \delta_{ai'} \delta_{a'i} = 1 - \sum_{ijj'a} \Omega_{ij} \Omega_{aj}^\dagger \Omega_{aj'} \Omega_{ij'}^\dagger = 1 - \text{Tr}[(\Omega^\dagger \Omega)^2].\end{aligned}\quad (\text{S6})$$

Note that for a pure state $\text{Tr}[(\Omega^\dagger \Omega)^2] = 1$ and the linear entanglement vanishes, $\mathcal{E}_i[\Omega^{\text{pure}}] = 0$, as must be the case.

Entanglement derivation for a contact term

Consider four-scalar scattering through the following quartic contact term,

$$\mathcal{L}_{\text{contact}} = \frac{g^2}{M_X^2} c_{ij}^{kl} \phi_i \phi_j \phi_k \phi_l. \quad (\text{S7})$$

This can be generated in the UV through, for instance, the exchange of a heavy scalar X . The initial state (S1) and initial state entanglement (S5) are given above. We are now interested in the final state density matrix and entanglement, the definitions of which we repeat here for convenience:

$$\hat{\rho}_{AB}^f = \hat{S}|\Omega\rangle\langle\Omega|\hat{S}^\dagger, \quad \mathcal{E}^f[\Omega] = 1 - \frac{1}{N^2} \text{Tr}_A[\text{Tr}_B \hat{\rho}_{AB}^f]^2, \quad \text{Tr}_B \hat{\rho}_{AB}^f = \frac{1}{N} \sum_c \int_u \langle u, c | B \hat{S}|\Omega\rangle\langle\Omega|\hat{S}^\dagger|u, c \rangle_B. \quad (\text{S8})$$

Once again, the normalization is fixed by the initial state. Vector products are given by Eq. (S2).

To evaluate Eq. (S8), a useful trick is to insert the identity operator on Fock space, which we expand out to work at lowest order in perturbation theory: $\hat{S}|\Omega\rangle = \hat{1}_1 \hat{S}|\Omega\rangle + \hat{1}_2 \hat{S}|\Omega\rangle + \dots$, where $\hat{1}_1$ denotes the identity on the subspace of one-particle states, $\hat{1}_2$ on two-particle states, etc. (see Eq. (S18) below). At present, the $\hat{1}_1$ term does not contribute and the leading-order sum over states becomes (where the q denote final momenta and i, j are internal flavor indices):

$$\begin{aligned}\mathbb{1}_2 \hat{S}|\Omega\rangle &= \sum_{ijab} \int_{q_1} \int_{q_2} |q_1, i; q_2, j\rangle \langle q_1, i; q_2, j | (1 + i\hat{T}) \Omega_{ab} |k_1, a; k_2, b\rangle \\ &= |\Omega\rangle + i2\pi \sum_{ijab} \int_{q_2} \frac{1}{2E_{q_1}} \Omega_{ab} \delta(E_{k_1} + E_{k_2} - E_{q_1} - E_{q_2}) \mathcal{A}_{ij}^{ab}(k_1 k_2 \rightarrow q_1 q_2) |q_1, i; q_2, j\rangle \Big|_{q_1=q_2-k_1-k_2},\end{aligned}\quad (\text{S9})$$

and similarly for the conjugate $\langle\Omega|\hat{S}^\dagger\mathbb{1}_2$. Plugging into Eq. (S8), we compute the reduced density matrix:

$$\begin{aligned}N \hat{\rho}_A &= \sum_{ja a'} \int \frac{d^3 u}{2E_u} (2E_{k_2})^2 \Omega_{ac} \Omega_{a'j}^\dagger (2\pi)^3 (\delta^3(k_2 - u))^2 |k_1, a\rangle_A \langle k_1, a'|_A \\ &\quad - \left[i\pi \sum_{ia a' b j} \int \frac{d^3 u}{2E_u E_{q_1}} 2E_{k_2} \delta^3(k_2 - u) \Omega_{a'j} \Omega_{ab}^\dagger \delta(E_{k_1} + E_{k_2} - E_{q_1} - E_u) \mathcal{A}_{ij}^{ab\dagger}(q_1 u \rightarrow k_1 k_2) |k_1, a'\rangle_A \langle q_1, i|_A + \text{h.c.} \right]_{\substack{q_1=k_1+k_2-u}} \\ &\quad + \sum_{ia a' b b' j} \int \frac{d^3 u \pi^2 \Omega_{ab} \Omega_{a'b'}^\dagger}{(2\pi)^3 2E_u E_{q_1}^2} (\delta(E_{k_1} + E_{k_2} - E_{q_1} - E_u))^2 \mathcal{A}_{ij}^{ab}(k_1 k_2 \rightarrow q_1 u) \mathcal{A}_{ij}^{a'b'\dagger}(q_1 u \rightarrow k_1 k_2) |q_1, i\rangle_A \langle q_1, i|_A \Big|_{\substack{q_1=k_1+k_2-u}}.\end{aligned}\quad (\text{S10})$$

Critically, note that, at tree level, the fourth term is $O(g^4)$ as it arises from the product of two *four-point* amplitudes, while the second and third terms are $O(g^2)$. Computing the momentum integrals, we have to leading order,

$$\begin{aligned}N \hat{\rho}_A &= 2E_{k_2} V \sum_{ja a'} \Omega_{aj} \Omega_{a'j}^\dagger |k_1, a\rangle_A \langle k_1, a'|_A \\ &\quad - 2 \left[i \frac{T}{2E_{k_1}} \sum_{ia a' b j} \Omega_{a'j} \Omega_{ab}^\dagger \mathcal{A}_{ij}^{ab\dagger}(k_1 k_2 \rightarrow k_1 k_2) |k_1, a'\rangle_A \langle k_1, i|_A + \text{h.c.} \right] + O(g^4),\end{aligned}\quad (\text{S11})$$

where we have used $(2\pi)^3 \delta^3(0) \equiv V$ in the first line. We can now understand, pragmatically, how the kinematic limit arises; integrating over u momentum in Eq. (S10) selects $u = k_2$, so that $q_1 = k_1 + k_2 - u = k_1$. The energy delta function thus becomes $\delta(0)$ and we define $2\pi\delta(0) \equiv T$. The result in Eq. (S11) may now be squared and plugged into Eq. (S8) to compute the linearized entanglement to leading order $O(g^2)$, which after evaluating the integrals and summing of internal symmetry indices, becomes

$$\begin{aligned} N^2 \text{Tr}_A [\hat{\rho}_A^2] &= (2E_{k_1} 2E_{k_2} V^2)^2 \text{Tr}[(\Omega^\dagger \cdot \Omega)^2] \\ &\quad - 4 (2E_{k_2} 2E_{k_1} V^3 T) \text{Im} \left[\sum_{abij} \Omega_{ab} \mathcal{A}_{ij}^{ab}(k_1 k_2 \rightarrow k_1 k_2) (\Omega^\dagger \cdot \Omega \cdot \Omega^\dagger)_{ij} \right] + O(g^4), \end{aligned} \quad (\text{S12})$$

using $i(X - X^\dagger) = -2 \text{Im}X$. The resulting change in linearized entanglement generated by scattering to leading order becomes Eq. (6), namely,

$$\Delta \mathcal{E}_{\text{contact}}[\Omega] = 4 \left(\frac{1}{2E_{k_1} 2E_{k_2}} \frac{T}{V} \right) \text{Im} \left[\sum_{abij} \Omega_{ab} \mathcal{A}_{ij}^{ab}(k_1 k_2 \rightarrow k_1 k_2) (\Omega^\dagger \cdot \Omega \cdot \Omega^\dagger)_{ij} \right] + O(g^4), \quad (\text{S13})$$

where we have subtracted off the initial state entanglement found in Eq. (S6), i.e., $\mathcal{E}_i = 1 - \text{Tr}[(\Omega^\dagger \Omega)^2]$. For a pure state $\Delta \mathcal{E}_{\text{contact}}[\Omega^{\text{pure}}] \propto \text{Im} \mathcal{A} \propto \text{Im}(g^2 c/M_X^2) = 0$ since the coupling is real, and the leading-order contribution arises at $O(g^4)$, i.e., at one-loop order. However $\mathcal{E}_{\text{contact}}[\Omega]$ will generally be nonvanishing for an impure initial state:

$$\Delta \mathcal{E}_{\text{contact}}[\Omega] = 4 \left(\frac{1}{2E_{k_1} 2E_{k_2}} \frac{T}{V} \right) \frac{g^2}{M_X^2} \text{Im} \left[\sum_{abij} \Omega_{ab} c_{ij}^{ab} (\Omega^\dagger \cdot \Omega \cdot \Omega^\dagger)_{ij} \right] + O(g^4), \quad (\text{S14})$$

where we have plugged in the amplitude from Eq. (S7). We could of course dress the contact term with derivatives, in which case we can view this result in an EFT context: at linear order in the Wilson coefficients, the entanglement generated by an EFT vanishes for a pure state, for scattering below the mass scale of the ultraviolet completion.

The $O(g^4)$ contribution to the entropy includes both the one-loop part of the amplitude $\mathcal{A}^{\text{1-loop}}$, as well as terms going like the square of the tree-level amplitude,

$$\begin{aligned} \Delta \mathcal{E}[\Omega]_{O(g^4)} &= 4\lambda \sum_{ijmn} \text{Im} [\Omega_{ij} \mathcal{A}_{mn}^{ij \text{ 1-loop}} (\Omega^\dagger \cdot \Omega \cdot \Omega^\dagger)_{nm}] \\ &\quad - 2\lambda^2 \sum_{ijmn} \sum_{i'j'm'n'} \left\{ \delta_{mm'} \Omega_{ij} (\Omega^\dagger)_{j'i'} \mathcal{A}_{mn}^{ij \text{ tree}} (\Omega^\dagger \cdot \Omega)_{nn'} \mathcal{A}_{m'n'}^{\dagger i'j' \text{ tree}} \right. \\ &\quad \left. + \delta_{nn'} \Omega_{ij} (\Omega^\dagger)_{j'i'} (\Omega \cdot \Omega^\dagger)_{m'm} \mathcal{A}_{mn}^{ij \text{ tree}} \mathcal{A}_{m'n'}^{\dagger i'j' \text{ tree}} \right. \\ &\quad \left. - \text{Re} [\Omega_{ij} \Omega_{i'j'} (\Omega^\dagger)_{n'm} \mathcal{A}_{mn}^{ij \text{ tree}} (\Omega^\dagger)_{nm'} \mathcal{A}_{m'n'}^{i'j' \text{ tree}}] \right\}, \end{aligned} \quad (\text{S15})$$

where $\lambda = (2\pi)^4 \delta^{(4)}(0)/\mathcal{N}$ and all amplitudes are in the forward limit.

Entanglement derivation for massive exchange

In the previous subsection, we computed the entanglement in two-to-two scalar scattering $\Delta \mathcal{E}[\Omega]$ that arises from a contact operator, namely Eq. (S7), and have shown that this is equivalent to Eq. (6). To verify the robustness of Eq. (6) we now compute $\Delta \mathcal{E}[\Omega]$ in the context of a theory with massive exchange, which reduces to Eq. (S7) in the low-energy limit:

$$\mathcal{L} = g f_{ijk} X_i \phi_j \phi_k. \quad (\text{S16})$$

Here X is a heavy scalar, and the whole Hilbert space is assumed to be decomposed as $\mathcal{H} = \mathcal{H}_X \otimes \mathcal{H}_A \otimes \mathcal{H}_B$. Once again the initial state is given by

$$|\Omega\rangle = \sum_{ij} \Omega_{ij} |\phi_i(k_1)\rangle_A \otimes |\phi_j(k_2)\rangle_B \equiv \sum_{ij} \Omega_{ij} |\phi_i(k_1) \phi_j(k_2)\rangle, \quad (\text{S17})$$

This is equivalent to Eq. (S3), but with a slight change of notation. We distinguish a state of the heavy scalar as $|X_m(Q)\rangle_X$. Since in the ultraviolet one can form on-shell X particles as intermediate states, we must sum over X when inserting the identity operator,

$$\hat{\mathbb{1}} = \sum_m \int_Q |X_m(Q)\rangle_X \langle X_m(Q)|_X + \sum_{ij} \int_{q_1} \int_{q_2} |\phi_i(q_1)\phi_j(q_2)\rangle \langle \phi_i(q_1)\phi_j(q_2)| + \dots, \quad (\text{S18})$$

which we write as $\hat{\mathbb{1}} = \hat{\mathbb{1}}_1 + \hat{\mathbb{1}}_2$. Since the \hat{T} matrix will now generate $2 \leftrightarrow 1$ processes involving X , we define amplitudes generically now as

$$\langle p_f, a_f | \hat{T} | p_i, a_i \rangle = (2\pi)^4 \delta^4(p_f - p_i) \mathcal{A}_{a_f}^{a_i}(p_i \rightarrow p_f), \quad (\text{S19})$$

where $p_{f/i}$ and $a_{f/i}$ stand for any number of final and initial state momenta (X or ϕ) and internal symmetry indices. Inner products between states are $\langle \phi_i(p_1)|_A|\phi_j(p_2)\rangle_A = (2\pi)^3 \delta^3(p_1 - p_2) \delta_{ij}$ and $\langle X|\phi\rangle = 0$, since there is no X - ϕ mass mixing.⁷

We now compute the reduced density matrix by tracing out the ϕ in \mathcal{H}_B , $\hat{\rho}_A = \text{Tr}_B \hat{\rho}_{AB}$, as follows:

$$\hat{\rho}_A = \frac{1}{N} \sum_c \int_u \langle \phi_c(u)|_B (\hat{\mathbb{1}}_1 \hat{S} + \hat{\mathbb{1}}_2 \hat{S}) |\Omega\rangle \langle \Omega| (\hat{S} \hat{\mathbb{1}}_1^\dagger + \hat{S}^\dagger \hat{\mathbb{1}}_2) |\phi_c(u)\rangle_B. \quad (\text{S20})$$

The matrix elements $\langle u, c |_B \hat{\mathbb{1}}_2 \hat{S} |\Omega\rangle$ and its conjugate are essentially equivalent to those of the previous contact-term computation (of course with the amplitude itself now as given by the Lagrangian in this subsection). We note that the contribution from the internal X particle given by $\hat{\mathbb{1}}_1$ vanishes,

$$\langle \phi_c(u)|_B \hat{S} \hat{\mathbb{1}}_1 |\Omega\rangle = 0 = \langle \phi_c(u)|_B \hat{\mathbb{1}}_1 \hat{S} |\Omega\rangle, \quad (\text{S21})$$

since in this theory there is no mass mixing or other way to convert $\phi \leftrightarrow X$. This makes intuitive sense since because, to leading order, neither the final nor the initial state has overlap with \mathcal{H}_X . Thus we conclude that the density matrix is as we previously computed,

$$\Delta \mathcal{E}_{\text{UV}}[\Omega] = 4 \left(\frac{1}{2E_{k_1} 2E_{k_2}} \frac{T}{V} \right) \text{Im} \left[\sum_{abij} \Omega_{ij} \mathcal{A}_{ab}^{ij}(k_1 k_2 \rightarrow k_1 k_2) (\Omega^\dagger \cdot \Omega \cdot \Omega^\dagger)_{ab} \right] + O(g^4). \quad (\text{S22})$$

For a narrow X resonance, the imaginary part of the amplitude has support on the delta function $\delta(s - M_X^2)$.

The toy model considered above has a realization as collinear two-to-two photon scattering. A Lagrangian with a dilaton coupling $\mathcal{L} \supset \frac{g}{f} \Phi F_{\mu\nu} F^{\mu\nu}$, permits Φ -mediated light-by-light scattering, which at low energies generates the effective operator $\frac{C}{M_\Phi^2} (F_{\mu\nu} F^{\mu\nu})^2$, where $C = g^2/2f^2 > 0$; see, e.g., the model in Ref. [11]. In the forward limit, all polarizations are transverse to all momenta and thus behave like a complex internal quantum number [58], so the computations above for a scalar theory can be applied. For product initial states with polarizations ϵ_1 and ϵ_2 , the leading-order entanglement becomes

$$\Delta \mathcal{E}_{\gamma\gamma}[\Omega^{\text{prod}}] \propto C s^2 \text{Im} \left[\frac{(\epsilon_1 \cdot \epsilon_2)(\epsilon_1 \cdot \epsilon_2)^* + (\epsilon_1 \cdot \epsilon_2^*)(\epsilon_2 \cdot \epsilon_1^*)}{s - M_\Phi^2 + i\epsilon} \right] \propto C s^2 \delta(s - M_\Phi^2). \quad (\text{S23})$$

Wave Function Derivation

In the main text, we have encountered the factor of $\mathcal{N} = \left(\frac{1}{2E_{k_1} 2E_{k_2}} \frac{T}{V} \right)$, which we expect to be one. This factor is due to scattering plane wave states, which have infinite norm. Here, we try justify why we can ignore this factor in front with the cost of adding wave functions. We do so by scattering properly normalized states, which are obtained with the aid of wave packets. Hence, we do not scatter purely plane waves but states $|\psi\rangle$ with a given flavor index a ,

$$|\psi, a\rangle \equiv \int_p \psi(p) |p, a\rangle, \quad \int_p \equiv \int d^4 p \, \hat{\delta}(p^2 - m^2) \Theta(p^0), \quad \langle \psi, b | \psi, a \rangle = \delta_a^b \int_p \|\psi(p)\|^2 = \delta_a^b, \quad (\text{S24})$$

⁷ Note that if we had $\mathcal{L} = m_{ij} \phi_i \phi_j + g f_{ijk} \phi_i \phi_j \phi_k$, such vector products would not vanish and one could study the entanglement generated in mass mixing and more generically in particle-antiparticle oscillations. Studying such systems using our framework could, e.g., shed new insights on entanglement generated in neutrino oscillations [56, 57].

where the hatted notation means $\hat{dx} = dx/2\pi$ and $\hat{\delta}(x) = 2\pi\delta(x)$, so that we do not need to carry around factors of 2π [59]. It is also clear that from the definitions that the momentum-space representation of the wave function is $\langle k, b | \psi, a \rangle = \psi(k)\delta_a^b$. The most general pure initial state is given by

$$|\text{in}\rangle = \sum_{ab} \int_{k_1, k_2} \psi_{12}(k_1, k_2) \Omega_{ab} |k_1, a; k_2, b\rangle, \quad (\text{S25})$$

which in principle could be entangled in momenta and flavor space. We want a product state in momentum space $\psi_{12}(k_1, k_2) = \psi_1(k_1)\psi_2(k_2)$ while still permitting entanglement in flavor space. Our initial state becomes

$$|\text{in}\rangle \Rightarrow |\Omega\rangle \equiv \sum_{ab} \int_{k_1, k_2} \psi_1(k_1)\psi_2(k_2) \Omega_{ab} |k_1, a; k_2, b\rangle = \sum_{ab} \Omega_{ab} |\psi_1, a; \psi_2, b\rangle. \quad (\text{S26})$$

The normalization of these states $\langle \Omega | \Omega \rangle = 1$ implies that $\sum_{ab} \Omega_{ab}^\dagger \Omega_{ab} = 1$. We then evolve this initial state via the S-matrix $|\text{out}\rangle = \hat{S}|\text{in}\rangle = (1 + i\hat{T})|\text{in}\rangle$ and keep terms up to linear order in \hat{T} . Performing the partial trace over \mathcal{H}_B , we obtain the following reduced density matrix

$$\begin{aligned} \rho_A &= \sum_{crs} \Omega_{rc} \Omega_{sc}^\dagger |\psi_1, r\rangle \langle \psi_1, s| \\ &+ \left[i \sum_{abi} \sum_{nm} \Omega_{ab} \Omega_{nm}^\dagger \int_{q_1, q_2, k_1, k_2} \hat{\delta}^{(4)}(k_1 + k_2 - q_1 - q_2) \psi_1(k_1) \psi_2(k_2) \psi_2^\dagger(q_2) \mathcal{A}_{im}^{ab}(k_1 + k_2 \rightarrow q_1 + q_2) |q_1, i\rangle \langle \psi_1, n| + \text{h.c.} \right]. \end{aligned} \quad (\text{S27})$$

For the trace of ρ_A^2 , it is also useful to define the smeared amplitude, which will be an object acting purely on the flavor space,

$$\mathcal{A}_{cd}^{ab} = \int_{p_1, p_2, p_3, p_4} \hat{\delta}^{(4)}(p_1 + p_2 - p_3 - p_4) \psi_1(p_1) \psi_2(p_2) \psi_1^\dagger(p_3) \psi_2^\dagger(p_4) \mathcal{A}_{cd}^{ab}(p_1 + p_2 \rightarrow p_3 + p_4). \quad (\text{S28})$$

Squaring the reduced density matrix and finally tracing over \mathcal{H}_A leads to

$$\text{tr}[\rho_A^2] = \sum_{crs} \Omega_{rc} \Omega_{sc}^\dagger \left[\sum_{c' r' s'} \Omega_{sc'} \Omega_{r' c'}^\dagger + 2i \sum_{abi} \sum_{nm} \Omega_{ab} \Omega_{nm}^\dagger \left(\mathcal{A}_{im}^{ab} \delta_i^s \delta_r^n - \mathcal{A}_{nm}^{ib} \delta_a^s \delta_r^n \right) \right], \quad (\text{S29})$$

which can be put in simplified notation, $\Omega_{ab} \equiv |\Omega\rangle$ and $\Omega_{nm}^\dagger \equiv \langle \Omega|$, to obtain the entanglement

$$\mathcal{E}[\Omega] = 1 - \langle \Omega | \Omega \Omega^\dagger | \Omega \rangle + 4 \text{Im}[\langle \Omega | \Omega \Omega^\dagger \mathcal{A} | \Omega \rangle]. \quad (\text{S30})$$

In the case where the initial state is a product state, $(\Omega \cdot \Omega^\dagger \cdot \Omega)_{rc'} = \Omega_{rc'}$, i.e., $\Omega \Omega^\dagger | \Omega \rangle = | \Omega \rangle$. Then

$$\mathcal{E}[\Omega] = 4 \text{Im}[\langle \Omega | \mathcal{A} | \Omega \rangle]. \quad (\text{S31})$$

Coming back to $\text{tr}[\rho_A^2]$, if we were to calculate the quantum Tsallis entropy, we would need the n th power instead,

$$\text{tr}[\rho_A^n] = \langle \Omega | (\Omega \Omega^\dagger)^{n-1} | \Omega \rangle + 2n \text{Im}[\langle \Omega | (\Omega \Omega^\dagger)^{n-1} \mathcal{A} | \Omega \rangle], \quad (\text{S32})$$

which in the case of a product initial state reduces to $\mathcal{E}_n[\Omega] = \frac{2n}{n-1} \text{Im}[\langle \Omega | \mathcal{A} | \Omega \rangle]$.

The details of the wave function in the smeared amplitude should not be relevant to our discussion here. However, it is enlightening to show how one recovers the usual story when scattering infinite-norm states by choosing a Dirac delta for the squared wave function, cf. the final expression in Eq. (S24). For example, choosing

$$\|\psi_1(p)\|^2 \propto \delta^{(3)}(p - P_a) \quad \|\psi_2(p)\|^2 \propto \delta^{(3)}(p - P_b), \quad (\text{S33})$$

since the smeared amplitude is given by a combination of wave functions $\psi_1(p_1)\psi_2(p_2)\psi_1^\dagger(p_3)\psi_2^\dagger(p_4)$, this will only have support when $p_1 = p_3 = P_a$ and $p_2 = p_4 = P_b$, automatically selecting forward kinematics.