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### Random-Matrix Description of Chaotic Scattering: Semiclassical Approach

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We present a semiclassical theory for the two-point correlation function of the eigenphases of the  $S$  matrix for chaotic scattering. It is expressed as a sum of contributions from unstable periodic orbits of the classical scattering mapping. Backed by numerical results and for correlation ranges  $r^*$  ( $\sim 1/\hbar$ ) we obtain a universal function which is consistent with the result of Dyson's circular ensemble. This result adds to the conjecture that universal fluctuations governed by random matrix theory are the quantum manifestation of classical chaos.

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The study of chaotic scattering from both the classical and the quantum-mechanical points of view has recently appeared at the forefront of chaos research. This is due to the large variety of fields where chaotic scattering is encountered and to the new and interesting theoretical problems which arise in the attempts to elucidate this phenomenon.<sup>1-14</sup> In this Letter we shall concentrate on the quantum description of chaotic scattering. To set the stage, we shall briefly describe the present state of the quantum theory.

The quantum description of scattering is given in terms of the scattering matrix  $S_{i,i'}(E)$ . It is labeled by the (discrete or continuous) quantum numbers  $i$  ( $i'$ ) which specify the state of the system before (after) the collision, and by the total energy  $E$ . Chaotic scattering is characterized by fluctuations in the dependence of the  $S$  matrix on  $i$  or  $i'$  at a fixed energy (i.e., "speckle" pattern in angular distributions<sup>5</sup>), or in the dependence on the energy for a fixed transition ("Ericson fluctuations"<sup>10,11,15,16</sup>). The unpredictability and chaoticity of scattering from simple systems can be appreciated by noting that the spectrum of Ericson fluctuations corresponds to colored noise, and approaches a white-noise spectrum in the limit  $\hbar \rightarrow 0$ .

In our previous work on quantum chaotic scattering<sup>10,11</sup> we showed semiclassically that the above-mentioned fluctuations and their statistics can be traced

back to the chaotic nature of the underlying classical dynamics. The correlation lengths can be expressed in terms of classical quantities which characterize the (fractal) set of unstable bounded orbits. On the basis of these results it was proposed that the  $S$  matrix is a typical representative of Dyson's orthogonal ensemble of unitary symmetric random matrices (DOE).<sup>17,18</sup> This conjecture was tested by a statistical analysis of the distribution of the  $S$ -matrix eigenphases for two models which were solved numerically.<sup>10,11</sup> The nearest-neighbor-spacing (NNS) distribution indeed follows the predictions of random-matrix theory<sup>18</sup> (RMT) as soon as the classical dynamics becomes chaotic. Together with the statistics of the transition probabilities  $|S_{i,i'}|^2$ , this provides convincing evidence in favor of the proposed hypothesis.

The main purpose of this Letter is to provide a theoretical foundation for the proposed link between RMT and quantum chaotic scattering and to study systematically its range of validity. This will be done by investigating the two-point correlations in the spectrum of  $S$ -matrix eigenphases. We shall present a semiclassical expression for the normalized cluster function,<sup>17</sup>  $Y_2(r)$ . This function yields the  $\Delta_3$  statistics by a simple integration, and gives relevant information on the NNS distribution for small spacings.<sup>19</sup> We shall show below that the derived function reproduces Dyson's expression for

the RMT cluster function in an interval whose length scales inversely with  $\hbar$ . The analytical results will be corroborated by a detailed numerical investigation.

For problems with time-reversal symmetry we consider unitary and symmetric  $S$  matrices of dimension  $L$ .  $L$  depends on the number of quantum numbers,  $f$ , needed to specify an initial or a final state,  $L \sim \hbar^{-f}$ . For the sake of simplicity we consider here systems with  $f=1$ . The  $S$  matrix is diagonal in the eigenchannel representa-

$$P_2(\eta) = \frac{1}{L(L-1)} \left\{ \int_{|\eta/2|}^{2\pi - |\eta/2|} d\left(x + \frac{\eta}{2}\right) d\left(x - \frac{\eta}{2}\right) dx - L\delta(\eta) \right\}.$$

It is convenient to discuss the two-point correlations in terms of the cluster function  $Y_2(r)$ ,<sup>17</sup> which is obtained from the density  $P_2(\eta)$  by subtracting the probability density in the absence of correlations. The argument of the cluster function,  $r$ , gives the phase difference  $\eta$  in units of the mean spacing  $2\pi/L$ . Substituting (1) into (2) and using the same notations and normalization as in Ref. 17, we get

$$Y_2(r) = \frac{1}{L} \left\{ 1 - 2 \sum_{N=1}^{\infty} s_N^{(L)} \cos \left( r \frac{2\pi}{L} N \right) \right\}. \quad (3)$$

The Fourier coefficients  $s_N^{(L)}$  are given by

$$s_N^{(L)} \equiv \frac{1}{L} |\text{Tr}S^N|^2 - 1 \\ = (L-1) \int_0^{2\pi} d\eta P_2(\eta) e^{i\eta N}. \quad (4)$$

The behavior of  $s_N^{(L)}$  in the limits  $N \rightarrow 0$  ( $\infty$ ) determines the behavior of  $Y_2(r)$  for large (small)  $r$ . For  $N=0$  we have  $s_0^{(L)} = L-1$  which is a large number. For  $N/L \gg 1$  we have  $(\omega_a - \omega_b)N \gg 2\pi$  for  $a \neq b$ , and hence  $|\text{Tr}S^N|^2 \rightarrow L$ , which implies  $s_N^{(L)} \rightarrow 0$ . In the sequel we shall provide a semiclassical expression for  $s_N^{(L)}$  which interpolates between these extreme  $N$  values.

$$S_{ii'} = \left( \frac{\hbar}{2\pi} \right)^{1/2} \sum_i \left( \left| \frac{\partial I'}{\partial \Theta} \right|^{(i)} \right)^{-1/2} \exp \left( \frac{i}{\hbar} \Phi(I, I')^{(i)} - i \frac{\pi}{2} v^{(i)} \right),$$

where the summation is over all scattering trajectories  $(i)$  which contribute to the transition  $I \rightarrow I'$ .

To calculate powers of  $S$  we use the semiclassical expression (7) and perform the intermediate sums by the saddle-point approximation. Because of relation (6), the saddle-point condition picks multiple-scattering trajectories which correspond to the multiple iterates of  $\hat{M}_S$ . Thus,  $\text{Tr}S^N$  is expressed semiclassically in terms of the *periodic points* of period  $N$  of  $\hat{M}_S$ ,

$$\text{Tr}S^N \sim \sum_{pp} \frac{1}{2|R_{pp}|^{1/2}} e^{i\Phi_{pp}/\hbar}, \quad R_{pp} = \frac{1}{4} \det \left( \frac{\partial(\Theta' I')}{\partial(\Theta I)} - 1 \right), \quad (8)$$

where the index  $pp$  is used to label the periodic points of length  $N$  (not necessarily primitive).  $R_{pp}$  is the residue of the periodic orbit and  $\Phi$  is the corresponding action (which includes also the Maslov-index contribution).

Considering  $N$  as a fictitious “time” we see that the expression for  $|\text{Tr}S^N|^2$  is formally equivalent to the expression derived by Berry<sup>21</sup> for the two-point correlation function in the spectra of bounded Hamiltonian systems. Proceeding now in his footprints we decompose the semiclassical expression for  $|\text{Tr}S^N|^2$  into two sums. A diagonal sum which is the sum of the squares of the contributions from individual periodic points and a nondiagonal sum which contains all the interference terms. As long as  $N < L$ , the difference between actions which correspond to different orbits exceeds  $\hbar$ .

tion where it takes the form  $S = \sum_a |\alpha\rangle e^{i\omega_a \langle \alpha |}$ . Here,  $|\alpha\rangle$  and  $\omega_a$  denote the eigenstates and the eigenphases of the  $S$  matrix, respectively. The *spectral density* of the eigenphases then reads as follows:

$$d(\omega) = \sum_a \delta(\omega - \omega_a) = \frac{1}{2\pi} \sum_{-\infty}^{\infty} e^{-i\omega N} \text{Tr}S^N. \quad (1)$$

The two-point correlations are expressed in terms of the probability density  $P_2(\eta)$  of finding two levels at a distance  $\eta$ ,

$$P_2(\eta) = \frac{1}{L(L-1)} \left\{ \int_{|\eta/2|}^{2\pi - |\eta/2|} d\left(x + \frac{\eta}{2}\right) d\left(x - \frac{\eta}{2}\right) dx - L\delta(\eta) \right\}. \quad (2)$$

The operator  $S^N$  ( $N > 1$ ) has no direct physical meaning. However, since  $S$  is a unitary mapping of the initial (and final) space of states onto itself, we may consider  $S^N$  formally as the  $N$ th iteration of a unitary quantum mapping. The classical analog of  $S$  is the Poincaré scattering map,  $\hat{M}_S$ , introduced by Jung and Scholz in their pioneering work on chaotic classical scattering.<sup>13</sup> It is constructed by following classical scattering trajectories from the initial conditions  $(I, \Theta)$  to the final conditions  $(I', \Theta')$ .  $I$  is the action variable corresponding to the quantum number  $i$ , and  $\Theta$  is the conjugate angle. Of prime importance is the action  $\Phi(I, I')$  along the classical trajectory

$$\Phi(I, I') = - \int_{-\infty}^{+\infty} (\Theta i + r \dot{p}) dt, \quad (5)$$

where  $r$  and  $p$  are the scattering coordinate and momentum. (As  $|r| \rightarrow \infty$ , the action  $I$  becomes an integral of the motion.) The Poincaré mapping is generated by  $\Phi(I, I')$ ,

$$\Theta = \partial \Phi / \partial I, \quad \Theta' = - \partial \Phi / \partial I', \quad (6)$$

thus ensuring that the mapping is area preserving.

The semiclassical formula for the  $S$  matrix<sup>20</sup> is given by

$$S_{ii'} = \left( \frac{\hbar}{2\pi} \right)^{1/2} \sum_i \left( \left| \frac{\partial I'}{\partial \Theta} \right|^{(i)} \right)^{-1/2} \exp \left( \frac{i}{\hbar} \Phi(I, I')^{(i)} - i \frac{\pi}{2} v^{(i)} \right), \quad (7)$$

Hence, the contributions from the nondiagonal part will average to zero. The contribution from the diagonal part can be estimated by using the Hannay and Ozorio De Almeida sum rule,<sup>22</sup> adapted for the present situation (see discussion below). The sum rule holds only if  $N > N^*$ , where  $N^*$  denotes the time after which the periodic points are uniformly distributed over the entire available phase space. Thus, for  $N^* < N < L$ ,  $s_N^{(L)} \sim N/L - 1$ .

For  $N \gg L$ , the classical periodic orbits proliferate and action differences may become arbitrarily small. The interference terms in this regime cannot be neglected. We have given above a simple argument to show that in this regime  $s_N^{(L)} \rightarrow 0$ . To summarize, we have

$$s_N^{(L)} = \begin{cases} L-1, & \text{for } N=0, \\ \text{nongeneric,} & \text{for } N < N^* \ll L, \\ \sim N/L - 1, & \text{for } N^* < N < L, \\ \sim 0, & \text{for } N \gg L. \end{cases} \quad (9)$$

Apart from the nongeneric region  $N < N^*$ , this is the proper behavior expected for the Fourier transform of  $Y_2$  of the DOE, which displays the same characteristic transition between the regimes  $N < L$  and  $N > L$ .

Since  $N^*$  is a classical quantity, independent of  $\hbar$ , the effects of the region  $N < N^*$  on the function  $Y_2$  will show up for  $r > r^* \sim L/N^* \sim 1/\hbar$ .

To justify the application of the classical sum rule<sup>22</sup> in the present context, we observe that it was originally derived for chaotic area-preserving mappings on a compact volume of phase space. The only ingredients of the chaoticity that are used are that (a) the mapping is hyperbolic, and (b) the periodic points cover it uniformly and densely as the period becomes large. The scattering mapping is area preserving [see (6)] and its domain is finite since we always assume conservation of energy and a scattering interaction of a finite range. A new feature which characterizes the mapping  $\tilde{M}_S$  is that as a function of its variables it is a fractal function<sup>23</sup> which is piecewise monotonic on the complement of a Cantor set and is thus distinguished from other types of mappings (e.g., the standard map) which can be expressed in terms of simple analytical functions. This property does not pose any difficulty as far as the requirements (a) and (b) mentioned above are concerned.

Our analytical results will now be illustrated by very recent numerical results obtained from the exact solution of a two-dimensional model described in detail in Refs. 11 and 12. The model consists of a periodic array of strong nonoverlapping spherically symmetric scattering potentials whose centers are located on equidistant points along the  $y$  axis in the plane. Because of the discrete translational symmetry of the problem in the  $y$  direction, scattering is allowed only into a finite set of Bragg directions which results in a  $S$  matrix of dimension  $L \times L$ ,

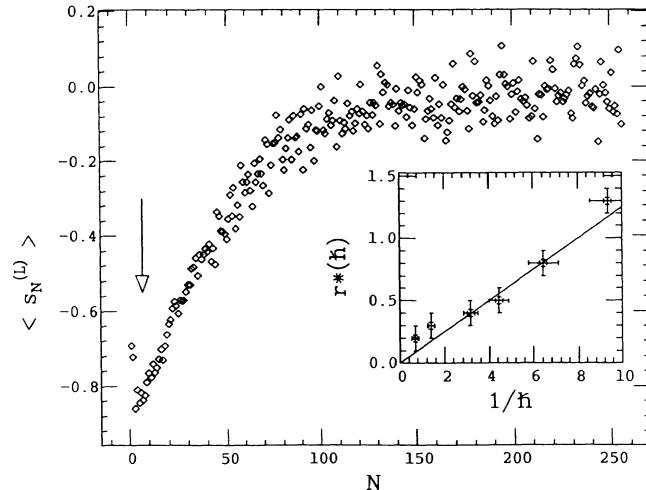


FIG. 1. The Fourier coefficients of the cluster function  $s_N^{(L)}$  as a function of  $N$  for  $L \sim 100$ . The arrow points to the value of  $N^*$  where departure from the universal  $N$  dependence is observed [see (9)]. Inset: The dependence of  $r^*(\hbar)$  on  $1/\hbar$  (in arbitrary units) as extracted from Fig. 2.

where  $L$  is (half) the number of allowed directions. ( $L \sim \sqrt{ED}/\hbar$ , where  $D$  is the spacing between scatterers.)

In Fig. 1 we show  $s_N^{(L)}$  as a function of  $N$  for  $L \sim 100$ . It was obtained as an ensemble average over 400 matrices corresponding to energies within a range of  $\pm 10\%$  around the nominal energy  $E=1$ . The chosen energy step ( $\Delta E=0.0005$ ) is about twice the width of the corresponding  $S$ -matrix-energy autocorrelation function (see Figs. 7 and 8 in Ref. 11). This guarantees that the matrices in the ensemble are in fact statistically independent. Figure 1 clearly exhibits the three regimes summarized in (9).

Figure 2 shows the function  $Y_2(r)(\pi r)^2$  for six different values of  $\hbar$ , with corresponding dimensions  $L \sim 20, \dots, 280$ . We plotted  $Y_2(r)(\pi r)^2$  since this function (for the DOE) approaches unity where  $r > 1$ , and it amplifies the large- $r$  region where we expect to observe the deviations from the DOE predictions. The break points  $r^*$  are indicated by arrows. Within the uncertainties of extraction of the break points, the scaling of  $r^*$  is consistent with  $r^* \sim 1/\hbar$ , as is evident from the inset in Fig. 1.

In summary, we would like to suggest that the results discussed here, together with the previous findings in Refs. 10 and 11, bring the study of fluctuations in quantum chaotic scattering to the same level of detail and depth as was already achieved for the study of fluctuations in the spectrum of *bounded* chaotic Hamiltonian systems. For a third class of systems, namely those driven by periodic time-dependent perturbations, there is accumulating evidence in support of the proposition that their fluctuation properties also follow the predictions of RMT.<sup>24-26</sup> The evidence from all the three classes of

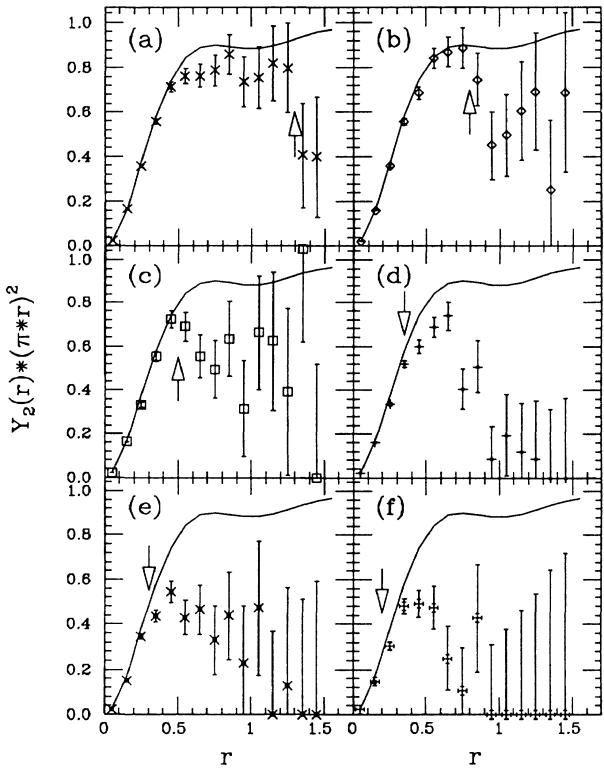


FIG. 2. The function  $Y_2(r) * (\pi r)^2$  (see text) for six different values of  $\hbar^{-1}$ . The actual values of  $\hbar^{-1}$  (in arbitrary units) for (a)–(f) are given by  $\hbar^{-1} = 9.35, 6.45, 4.46, 3.16, 1.41$ , and  $0.71$ , respectively. The points  $r^*$  where the numerical functions deviate from the RMT prediction are marked by an arrow. The continuous line is the RMT prediction. The error bars on the numerical values denote the statistical uncertainty.

systems, put together, suggests the universality of the connection between RMT and the quantum description of classically chaotic systems. Genuine quantum correlations set a limit to the applicability of RMT, but as  $\hbar$  decreases, these limitations are progressively relaxed.

To the best of our knowledge, the explicit use of the periodic orbits of the classical scattering mapping in calculating spectral properties of the quantal  $S$  matrix was presented here for the first time. Phase-space periodic orbits were introduced in the past (see, e.g., Refs. 9 and 27) to calculate the poles of the  $S$  matrix in the complex energy plane. It should be emphasized, however, that for this purpose one needs the classical orbits which correspond to *complex* energies. Such trajectories are very complicated, and little is known about them. Their relation to the periodic orbits of the scattering mapping, is an open problem which deserves further study.

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