Mar. 17, 2025 (Due: 08:00 Mar. 24, 2025)

1. Show that 2D Lagrange interpolation over a rectangular mesh is always well-posed, as long as there are no repeated interpolation nodes.

2. Let

$$L[y] = \int_0^1 f(t, y, y')\,\mathrm{d}t$$

be a sufficiently smooth functional defined for all sufficiently smooth functions $y(t)$ satisfying $y(0) = y(1) = 0$. Show that any stationary point of $L[y]$ satisfies the *Euler–Lagrange equation*

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}t}\Bigl(\frac{\partial f}{\partial y'}\Bigr) = 0.$$

3. Use Gaussian functions (i.e., density of the normal distribution) to interpolate the Runge function $f(x) = (1 + 25x^2)^{-1}$ over $[-1, 1]$ with a few equispaced nodes. Visualize the result.

4. Interpolate the following data set and visualize your solution on $[-1, 1] \times [-1, 1]$.

x_i	y_i	z_i
−1.0000	−1.0000	1.6389
−1.0000	1.0000	0.5403
1.0000	−1.0000	−0.9900
1.0000	1.0000	0.1086
−0.7313	0.6949	0.9573
0.5275	−0.4899	0.8270
−0.0091	−0.1010	1.6936
0.3031	0.5774	1.3670

Note that different interpolation strategies will lead to different results. You are encouraged to try different 2D interpolation strategies learned from the lecture. (If Delaunay triangularization is used, you can make use of the MATLAB/Octave function `delaunay`.)

5. (optional) In the lecture we have proved the minimum energy property for cubic splines. Can you derive the minimum energy property for certain types of quintic splines?

6. (optional) Let us consider Exercise 4 from last week again. Can you give a theoretical analysis on how much error Bob will make on each subinterval?