Homework 1

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Problem 1

Let $F : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ and $F(X) = X^{-1}$ for any invertible matrix $X \in \mathbb{R}^{n\times n}$. F is differentiable at X indicates that given any perturbation matrix $\Delta X \in \mathbb{R}^{n\times n}$ such that $X+\Delta X$ is also invertible, it holds that $F(X + \Delta X) = F(X) + A(\Delta X) + o\left(\|\Delta X\|\right)$. Please show the form of $A(\Delta X)$.

Solution:

Firstly, let us to get a formula for $(I + \Delta X)^{-1}$, in which $\|\Delta X\|_F < 1$. We have:

$$f(X, n) = I - \Delta X + \Delta X^2 + \cdots + (-\Delta X)^n \tag{1}$$

And we can get:

$$f(X, n)(I + \Delta X) = I - \Delta X + \Delta X + \cdots - (-\Delta X)^{n+1} = I - (-\Delta X)^{n+1} \tag{2}$$

when $\lim_n = +\infty$, the addional term ΔX^{n+1} will be negligible. Because that $\|(-\Delta X)^{n+1}\|_F \leq \|\Delta X\|_F^{n+1}$, when $\lim_n = \infty$, obviously, it is zero. Therefore, we can get(known as the neumann series):

$$(I + \Delta X)^{-1} = \lim_{n\to\infty} \sum_{i=0}^{n} (-\Delta X)^i \tag{3}$$

So we can get:

$$F(X + \Delta X) = (X(I + X^{-1}\Delta X))^{-1} = (I + X^{-1}\Delta X)^{-1}X^{-1} = (\lim_{n\to\infty} \sum_{i=0}^{n} (-X^{-1}\Delta X)^i)X^{-1} \tag{4}$$

For that the first n term's degree is lower than n, as for the problem request, we get the first two terms,that are:

$$F(X + \Delta X) = X^{-1} - X^{-1}\Delta X X^{-1} + o(\|\Delta X\|^{-1}) = F(X) - X^{-1}\Delta X X^{-1} + o(\|\Delta X\|) \tag{5}$$

That is the answer form for the problem. Which means that the $A(\Delta X) = -X^{-1}\Delta X X^{-1}$.