Mar. 3, 2025 (Due: 08:00 Mar. 10, 2025)

1. Suppose that a, b, c, d satisfy

$$\begin{aligned} a + b + c + d &= 5, \\ 2a + 4b + 8c + 16d &= 7, \\ 3a + 9b + 27c + 81d &= 11, \\ 4a + 16b + 64c + 256d &= 1. \end{aligned}$$

Calculate $5a + 25b + 125c + 625d$ by hand.

Note: Try to simplify the calculation as much as you can.

2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a sufficiently smooth function. Show that

$$\lim_{(x_1,\dots,x_k)\to(x_*,\dots,x_*)} f[x_1, x_2, \dots, x_k] = \frac{f^{(k-1)}(x_*)}{(k-1)!}$$

for any $x_* \in \mathbb{R}$.

3. Let $p(x)$ be a polynomial of degree n with roots μ_1, μ_2, …, μ_n. Suppose that $\gcd\bigl(p(x), p'(x)\bigr) = 1$, i.e., $p(x)$ has no repeated roots. Show that

$$\sum_{i=1}^{n} \frac{1}{p'(\mu_i)} = 0.$$

4. Polynomial interpolation provides one way to approximate a given function. For instance, let $0 = x_1 < x_2 < \cdots < x_n = 2\pi$ be equally spaced interpolation nodes. The interpolation polynomial passing through all $(x_i, \sin x_i)$'s can be used to approximate the sine function $f(x) = \sin x$. Try to visualize the difference between the interpolation polynomial and the sine function for a few different choices of n.

5. This exercise is about an atypical approach for two-dimensional interpolation.

Interpolating a data set $\{(x_i, y_i, z_i)\}_{i=1}^{n} \subset \mathbb{R}^3$ can be understood as interpolating $\{(x_i + \mathrm{i}y_i, z_i)\}_{i=1}^{n} \subset \mathbb{C} \times \mathbb{R}$, where the interpolation nodes $x_i + \mathrm{i}y_i$'s are complex numbers. The polynomial interpolation techniques we have learned from this course theoretically carry over to complex inputs, while the resulting interpolation polynomial is in general complex-valued. Nevertheless, we can take the real part of the output.

Use this approach to interpolate the following data set over the unit disk and visualize the result.

x_i	y_i	z_i
1.00000	0.00000	−1.0000
0.80902	0.58779	−2.6807
0.30902	0.95106	5.6161
−0.30902	0.95106	5.6161
−0.80902	0.58779	−2.6807
−1.00000	0.00000	−1.0000
−0.80902	−0.58779	−2.6807
−0.30902	−0.95106	5.6161
0.30902	−0.95106	5.6161
0.80902	−0.58779	−2.6807

(If you use MATLAB/Octave, the functions `imagesc` and `colorbar` are useful for visualizing a bivariate function.)

6. (optional) Let

$$a_1, a_2, \ldots, a_n, a_1, a_2, \ldots, a_n, \ldots, a_1, a_2, \ldots, a_n, \ldots$$

be a periodic sequence. Can you find a smooth function that interpolates this infinite sequence?