Apr. 7, 2025 (Due: 08:00 Apr. 14, 2025)

1. Show that a quadratic Bézier curve is part of a parabola.

2. Write a program to plot the Bézier curve with arbitrary set of control points. Visualize a few examples.

3. (a) Determine the degree of exactness of the quadrature rule

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{b-a}{2}\left(f\Big(\frac{2a+b}{3}\Big)+f\Big(\frac{a+2b}{3}\Big)\right).$$

(b) By partitioning $[a,b]$ into n subintervals of equal length and applying the quadrature rule from part (a) on each subinterval, we obtain a composite quadrature rule. Implement such a composite quadrature rule and determine the asymptotic behavior of the error ($O(h)$, $O(h^2)$, $O(h^3)$, or ...) when integrating

$$\int_0^\pi \sin x\,\mathrm{d}x = 2,$$

where $h=(b-a)/n$.

4. Suppose that $f\colon [a,b]\to\mathbb{R}$ is twice continuously differentiable.
(a) Show that the remainders (i.e., truncation errors) of the composite midpoint rule and the composite trapezoidal rule are given by

$$K(b-a)h^2|f''(\xi)|$$

for some $\xi\in(a,b)$, where $K=1/24$ for the midpoint rule, and $K=1/12$ for the trapezoidal rule.
(b) Suppose that we need to use the composite midpoint rule to approximate $\int_0^1 \mathrm{e}^x\,\mathrm{d}x$. Determine how many sampling points do we need in order to obtain six correct digits after the decimal point based on the a priori error estimate.

5. Use Romberg's method to compute $\int_0^1 \mathrm{e}^x\,\mathrm{d}x$ and $\int_0^1 x^{3/2}\,\mathrm{d}x$. Keep a record for intermediate results. What can you say about the convergence?

6. (optional) Linear and quadratic Bézier curves can exactly fit line segments and parabolas, respectively. However, many computer programs only support cubic Bézier curves. Suppose you want to plot a segment of parabola of the form $y=ax^2+bx+c$ for $x\in[x_1,x_2]$. How to construct four control points for this purpose?