Apr. 14, 2025 (Due: 08:00 Apr. 21, 2025)

1. Show that the n-point Gauss–Chebyshev quadrature rule reads

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,\mathrm{d}x \approx \frac{\pi}{n}\sum_{k=1}^{n} f\biggl(\cos\frac{(2k-1)\pi}{2n}\biggr).$$

2. Develop a quadrature rule for the integral $\int_a^b \cos(mx)f(x)\,\mathrm{d}x$ such that it provides exact results for polynomials of degree up to three.

3. Write a program to compute $\int_0^1 \mathrm{e}^x\,\mathrm{d}x$ and $\int_0^1 x^{3/2}\,\mathrm{d}x$ using the Gauss–Legendre quadrature rule. Plot the quadrature error with respect to the number of quadrature nodes.

4. Determine the degree of exactness of the following 2-D quadrature rule:

$$\int_0^1 \int_0^{1-y} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \approx \frac{1}{6}\biggl(f\Bigl(\frac{2}{3},\frac{1}{6}\Bigr) + f\Bigl(\frac{1}{6},\frac{2}{3}\Bigr) + f\Bigl(\frac{1}{6},\frac{1}{6}\Bigr)\biggr).$$

Hint: Check whether the quadrature rules provide exact results for bivariate polynomials 1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, $\dots$

5. Let $\mathcal{D} = \{(x,y)\in\mathbb{R}^2 \colon x+y\le 1,\ x\ge 0,\ y\ge 0\}$. Estimate

$$\iint_{\mathcal{D}} \mathrm{e}^x \sin y\,\mathrm{d}x\,\mathrm{d}y$$

by partitioning $\mathcal{D}$ with a triangular mesh and applying a composite quadrature rule. Compare your result with the exact one.

6. (optional) Use Peano's linear functional representation to analyze the truncation error of the composite trapezoidal rule.

7. (optional) Let us ignore all planets in the solar system and assume that the orbit of Halley's comet is a perfect ellipse. Try to estimate the total orbit length of Halley's comet.

Note that the circumference of an ellipse cannot be represented by an elementary function in terms of its major and minor axes. You will need to do the calculation using numerical integration.

Here are some useful data:

- The eccentricity of the orbit is 0.967.
- The shortest distance between the comet and the sun is 88 million kilometers.

You can also search online to find some other information about the comet.