# Feb. 17, 2025 (Due: 08:00 Feb. 24, 2025)

**1.** In the lecture we briefly discussed the convergence of the Gregory–Leibniz series

$$\frac{\pi}{4} = \arctan 1 = \sum_{k=1}^{\infty} \frac{(-1)^k}{2k-1}.$$

If we compute $\pi/4$ through the partial sum

$$S_{2n} = \sum_{k=1}^{2n} \frac{(-1)^k}{2k-1},$$

the convergence is slow. Estimate the truncation error, and propose a simple correction to $S_{2n}$ based on your error estimate.

(optional) Estimate the truncation error again for the corrected scheme.

**2.** In principle, the sine function can be evaluated through the Taylor series expansion

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Let us consider two computational schemes to evaluate the sine function.
(a) Directly truncate the Taylor series. Make sure that the truncation error is less than the rounding error bound for any input $x$.
(b) First shift $x$ to the interval $(-\pi/2, \pi/2]$, and then apply scheme (a).
Scheme (b) is in general more accurate. Can you provide a theoretical analysis?

Sample at least 1000 points in $[-50, 50]$ (e.g., using the MATLAB/Octave statement `linspace(-50, 50, 1000)`) and plot the errors (e.g., using the MATLAB/Octave function `semilogy`). Can you explain the result?

**3.** Using Newton's method to find the root of $\arctan x = 0$ is an overkill, since the unique solution, $x_* = 0$, is trivial. However, this is a good example to see that the convergence of Newton's method relies on the initial guess. The set of real initial guesses such that Newton's method converges to $x_*$ is of the form $(-\alpha, \alpha)$, where $\alpha > 0$. Try to calculate $\alpha$ with at least 10 significant decimal digits. What happens if $\alpha$ is used as the initial guess?

**4.** Write a program to plot the Newton fractal for the cubic equation $x^3 - 1 = 0$ over the region $\{z \in \mathbb{C} \colon -2 \le \operatorname{Re} z \le 2 \text{ and } -2 \le \operatorname{Im} z \le 2\}$.

(optional) You are encouraged to try other polynomial equations and other regions in the complex plane.

**5.** (optional) Solve the following problem at `pintia.cn` and explain your algorithm.

“ *Produce a table of the values of the series*

$$\psi(x) = \sum_{k=1}^{\infty} \frac{1}{k(k+x)}$$

*for the* 2001 *values of* $x$, $x = 0.000, 0.001, 0.002, \ldots, 2.000$. *All entries of the table must have an absolute error less than* $0.5 \times 10^{-12}$. *This problem is based on a problem from Hamming (1962), when mainframes were very slow by today's microcomputer standards.*”

**6.** (optional) Design an algorithm to compute the reciprocal of a nonzero real number (i.e., $\alpha \mapsto \alpha^{-1}$) without any division operation.