May. 12, 2025 (Due: 08:00 May. 26, 2025)

1. Use the finite difference method and the finite element method (with linear elements), both on $n+1$ equispaced nodes, to solve the boundary value problem

$$\begin{cases} -u''(x) + u(x) = x^2, & (0 < x < 1) \\ u(0) = 0,\ u(1) = 1. \end{cases}$$

Try a few different values of n and compare your solutions with the exact one.

2. Use the finite difference method and the finite element method (with linear elements), both on $n+1$ equispaced nodes, to solve the eigenvalue problem

$$\begin{cases} -u''(x) = \lambda u(x), & (0 < x < \pi) \\ u(0) = 0,\ u(\pi) = 0. \end{cases}$$

Compute a few smallest eigenvalues and the corresponding eigenfunctions. Compare your solutions with the exact ones.

3. Solve the partial differential equation

$$\begin{cases} \dfrac{\partial^2 u(x,y)}{\partial x^2} + \dfrac{\partial^2 u(x,y)}{\partial y^2} = 0, & (-1 < x < 1,\ -1 < y < 1) \\ u(x,-1) = u(x,1) = x + 1, & (-1 < x < 1) \\ u(-1,y) = y^2 - 1,\ u(1,y) = y^2 + 1, & (-1 < y < 1). \end{cases}$$

Visualize your solution.

4. Solve the partial differential equation

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \dfrac{\partial^2 u(x,t)}{\partial x^2}, & (0 < x < 1) \\ u(x,0) = 1 - x^2, & (0 < x < 1) \\ u(0,t) = 1 - 2t,\ u(1,t) = -2t, & (t \geq 0) \end{cases}$$

with different finite difference schemes. Observe the convergence and error propagation using a few different step sizes.

5. (optional) Use the Galerkin method to solve the Fredholm integral equation

$$u(t) - \int_0^1 (2t+3)s^t u(s)\,\mathrm{d}s = t^{1/2} - 2.$$

Compare your solution with the exact one ($u(t) = t^{1/2}$).