ALMA MATER STUDIORUM
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Department of Statistical Sciences
Master of Science in Quantitative Finance

# Economics of Financial Markets - Project Work

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# Contents

**Abstract**

This paper aims to discuss and provide a computational application of the most known methodologies used in the field of asset allocation. We will deepen the Mean Variance optimization, the Black Litterman approach and the Bayesian approach. For each of this model, we will compute the relative statistics, i.e., mean, variance, standard deviation, skew, kurtosis, and Sharpe Ratio, in order to, at the end, make a comparison of the final results. Finally, we will analyse the structure and the statistics of a combination of all such portfolio.

# 1 Introduction

The data set used in this paper can be found in the attached file "data for exam 2024.xlsx", that includes a selection of stocks from the Italian Stock Market starting from January 1st, 2015 until January 16th, 2024. From the given data set we removed all the companies that in the last years were delisted, reducing the sample to 81 equities.
The results reported in the following sections have been computed using MATLAB® R2019b for Windows, and the related *Financial toolboox*.

# 2 Portfolio selection

## 2.1 Stock returns and statistics

By using the function `tick2ret`, we computed the daily and monthly returns of the 81 securities, then by considering 252 days and 12 months as the annual average, respectively, of days and months available for trading on Piazza Affari, we calculated the annualized returns (%) and volatility (%) both for daily and monthly frequencies. These statistics has been reported, respectively, as `Ann.ret` and `Ann.vol` in the following tables, along with the classical ones.

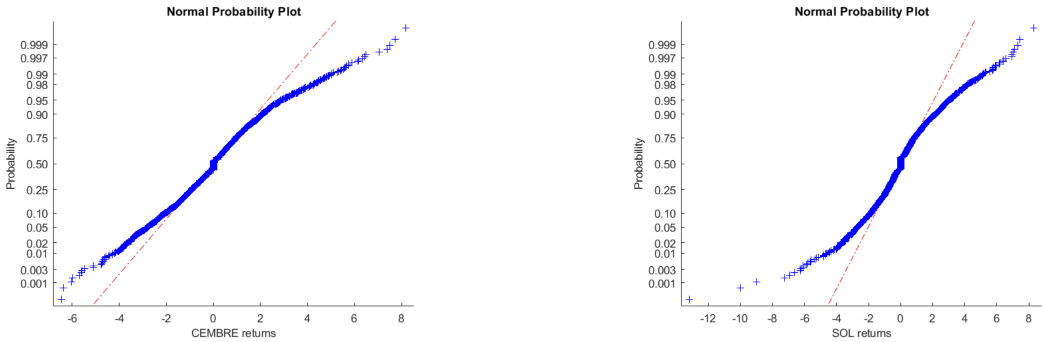
| Statistics for returns in daily terms | | | | | | | |
|---|---|---|---|---|---|---|---|
| | **Ann. ret** | **Ann. vol** | **Mean** | **STD** | **Variance** | **Skew** | **Kurtosis** |
| **I:LDO** | 8,475 | 36,741 | 0,034 | 2,314 | 5,357 | -0,862 | 18,168 |
| **I:ECK** | 4,795 | 48,478 | 0,019 | 3,054 | 9,326 | 1,869 | 17,458 |
| **I:LRZ** | -8,548 | 47,346 | -0,034 | 2,983 | 8,896 | 0,448 | 15,160 |
| **I:PIRL** | -3,988 | 34,479 | -0,016 | 2,172 | 4,718 | -0,323 | 10,189 |
| **I:STL** | 14,825 | 38,211 | 0,059 | 2,407 | 5,794 | -0,731 | 9,844 |
| **I:PINF** | -10,952 | 61,141 | -0,043 | 3,852 | 14,834 | -10,644 | 368,945 |
| **I:BRE** | 7,475 | 30,336 | 0,030 | 1,911 | 3,652 | -0,024 | 6,822 |
| **I:ISP** | 1,224 | 33,374 | 0,005 | 2,102 | 4,420 | -1,287 | 19,716 |
| **I:ILLB** | -6,923 | 30,099 | -0,027 | 1,896 | 3,595 | -0,174 | 9,445 |
| **I:UCG** | 0,290 | 42,561 | 0,001 | 2,681 | 7,188 | -0,452 | 12,072 |
| **I:BANC** | 4,366 | 30,370 | 0,017 | 1,913 | 3,660 | -0,295 | 11,603 |
| **I:BPE** | -1,684 | 45,252 | -0,007 | 2,851 | 8,126 | -0,292 | 11,730 |
| **I:FCBK** | 11,765 | 32,450 | 0,047 | 2,044 | 4,179 | -0,318 | 6,781 |
| **I:CPR** | 13,646 | 25,350 | 0,054 | 1,597 | 2,550 | -0,558 | 12,775 |
| **I:SPAC** | -15,266 | 39,949 | -0,061 | 2,517 | 6,333 | 0,110 | 9,546 |
| **I:CALT** | 8,337 | 27,408 | 0,033 | 1,727 | 2,981 | -0,040 | 7,562 |
| **I:ENEL** | 6,437 | 24,847 | 0,026 | 1,565 | 2,450 | -1,525 | 22,778 |
| **I:ARN** | 23,938 | 39,263 | 0,095 | 2,473 | 6,117 | 1,237 | 13,150 |
| **I:A2A** | 8,365 | 25,970 | 0,033 | 1,636 | 2,676 | -1,197 | 17,385 |
| **I:TRN** | 7,864 | 21,881 | 0,031 | 1,378 | 1,900 | -0,859 | 13,361 |
| **I:ACE** | 4,892 | 25,000 | 0,019 | 1,575 | 2,480 | -0,458 | 10,256 |
| **I:BMED** | 5,886 | 30,938 | 0,023 | 1,949 | 3,798 | -0,683 | 9,510 |
| **I:TIPS** | 13,219 | 25,367 | 0,052 | 1,598 | 2,553 | -0,069 | 12,863 |
| **I:MB** | 5,700 | 32,937 | 0,023 | 2,075 | 4,305 | -1,348 | 19,229 |
| **I:EQUI** | 3,455 | 24,574 | 0,014 | 1,548 | 2,396 | 0,618 | 13,818 |
| **I:ANI** | 0,295 | 37,138 | 0,001 | 2,339 | 5,473 | -0,133 | 8,312 |
| **I:TITR** | -9,088 | 37,828 | -0,036 | 2,383 | 5,678 | -0,484 | 17,932 |
| **I:TIT** | -12,133 | 38,144 | -0,048 | 2,403 | 5,774 | -0,047 | 16,195 |
| **I:ENV** | -0,660 | 29,110 | -0,003 | 1,834 | 3,363 | 0,368 | 16,829 |
| **I:VAL** | -4,995 | 27,362 | -0,020 | 1,724 | 2,971 | 0,474 | 9,541 |
| **I:CLT** | 1,046 | 30,527 | 0,004 | 1,923 | 3,698 | 2,248 | 24,609 |
| **I:HER** | 4,723 | 24,834 | 0,019 | 1,564 | 2,447 | -0,844 | 17,076 |
| **I:IRE** | 8,372 | 25,757 | 0,033 | 1,623 | 2,633 | -0,696 | 10,214 |
| **I:IG** | 3,932 | 23,720 | 0,016 | 1,494 | 2,233 | -0,719 | 10,595 |
| **I:ELN** | 18,531 | 37,130 | 0,074 | 2,339 | 5,471 | 0,073 | 6,562 |
| **I:AMP** | 19,532 | 31,896 | 0,078 | 2,009 | 4,037 | -0,622 | 10,609 |
| **I:DLG** | 7,307 | 33,164 | 0,029 | 2,089 | 4,364 | 0,063 | 6,996 |
| **I:FD** | -19,772 | 48,757 | -0,078 | 3,071 | 9,433 | 1,781 | 18,738 |
| **I:IP** | 14,553 | 30,416 | 0,058 | 1,916 | 3,671 | -0,249 | 5,761 |
| **I:IKG** | 11,578 | 34,123 | 0,046 | 2,150 | 4,621 | 0,708 | 11,891 |
| **I:ENAV** | -1,147 | 23,965 | -0,005 | 1,510 | 2,279 | 0,704 | 16,631 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| **I:PST** | 4,839 | 27,779 | 0,019 | 1,750 | 3,062 | -1,800 | 26,995 |
| **I:RCS** | -2,308 | 38,807 | -0,009 | 2,445 | 5,976 | 0,647 | 12,250 |
| **I:CAI** | -10,402 | 32,403 | -0,041 | 2,041 | 4,166 | 0,232 | 8,169 |
| **I:MON** | -18,849 | 38,665 | -0,075 | 2,436 | 5,932 | 1,680 | 19,983 |
| **I:GAMB** | -12,022 | 49,986 | -0,048 | 3,149 | 9,915 | 2,195 | 21,098 |
| **I:UNI** | 2,832 | 33,191 | 0,011 | 2,091 | 4,372 | -0,662 | 13,874 |
| **I:G** | 1,769 | 24,561 | 0,007 | 1,547 | 2,394 | -1,093 | 18,199 |
| **I:SRG** | 3,890 | 23,126 | 0,015 | 1,457 | 2,122 | -1,613 | 25,889 |
| **I:ENI** | 0,306 | 28,456 | 0,001 | 1,793 | 3,213 | -1,523 | 25,532 |
| **I:TOD** | -9,006 | 35,331 | -0,036 | 2,226 | 4,953 | 0,322 | 16,037 |
| **I:REC** | 14,576 | 27,182 | 0,058 | 1,712 | 2,932 | -0,331 | 15,545 |
| **I:RN** | -12,272 | 53,200 | -0,049 | 3,351 | 11,231 | 0,877 | 11,651 |
| **I:BRI** | -4,194 | 36,489 | -0,017 | 2,299 | 5,284 | 0,246 | 9,715 |
| **I:FUL** | -13,602 | 50,180 | -0,054 | 3,161 | 9,992 | 1,542 | 18,070 |
| **I:AISW** | 9,060 | 44,814 | 0,036 | 2,823 | 7,969 | 1,362 | 15,061 |
| **I:JUVE** | 4,576 | 40,998 | 0,018 | 2,583 | 6,670 | -0,142 | 11,458 |
| **I:SSL** | 4,942 | 38,672 | 0,020 | 2,436 | 5,934 | -0,178 | 14,686 |
| **I:CLE** | -27,721 | 47,043 | -0,110 | 2,963 | 8,782 | 0,926 | 14,561 |
| **I:B** | -15,914 | 35,696 | -0,063 | 2,249 | 5,056 | 0,767 | 12,165 |
| **I:CEM** | 6,895 | 32,489 | 0,027 | 2,047 | 4,189 | 0,033 | 5,749 |
| **I:US** | 0,468 | 24,580 | 0,002 | 1,548 | 2,397 | -0,345 | 8,362 |
| **I:BZU** | 11,059 | 31,178 | 0,044 | 1,964 | 3,857 | -0,199 | 7,569 |
| **I:CE** | 2,923 | 28,624 | 0,012 | 1,803 | 3,251 | -0,227 | 6,279 |
| **I:DAN** | 4,028 | 32,284 | 0,016 | 2,034 | 4,136 | 0,257 | 11,178 |
| **I:ITM** | 11,530 | 26,587 | 0,046 | 1,675 | 2,805 | 1,556 | 26,169 |
| **I:ZUC** | -9,156 | 54,557 | -0,036 | 3,437 | 11,812 | 0,716 | 47,844 |
| **I:IPG** | -5,154 | 38,589 | -0,020 | 2,431 | 5,909 | -0,714 | 17,856 |
| **I:VIN** | -2,872 | 28,964 | -0,011 | 1,825 | 3,329 | 0,110 | 6,918 |
| **I:EDNR** | 6,190 | 22,147 | 0,025 | 1,395 | 1,946 | -0,708 | 18,738 |
| **I:RAT** | 0,606 | 31,645 | 0,002 | 1,993 | 3,974 | 0,515 | 13,297 |
| **I:GAB** | -0,634 | 43,779 | -0,003 | 2,758 | 7,605 | 0,903 | 10,191 |
| **I:MS** | -11,411 | 37,336 | -0,045 | 2,352 | 5,532 | 0,467 | 18,727 |
| **I:ERG** | 11,450 | 27,370 | 0,045 | 1,724 | 2,973 | -0,535 | 17,527 |
| **I:CMB** | 14,017 | 28,043 | 0,056 | 1,767 | 3,121 | -0,010 | 6,852 |
| **I:SAB** | 5,450 | 29,207 | 0,022 | 1,840 | 3,385 | 0,256 | 7,765 |
| **I:BE** | -4,935 | 35,908 | -0,020 | 2,262 | 5,117 | 1,550 | 14,863 |
| **I:SOL** | 14,706 | 27,541 | 0,058 | 1,735 | 3,010 | 0,254 | 4,675 |
| **I:DAL** | -3,881 | 38,226 | -0,015 | 2,408 | 5,798 | -0,035 | 10,469 |
| **I:BSS** | 2,512 | 42,857 | 0,010 | 2,700 | 7,289 | -0,493 | 10,231 |
| **I:SAFI** | -19,583 | 45,063 | -0,078 | 2,839 | 8,058 | -0,083 | 14,408 |

| Statistics for returns in monthly terms | | | | | | | |
|---|---|---|---|---|---|---|---|
| | **Ann. ret** | **Ann. vol** | **Mean** | **STD** | **Variance** | **Skew** | **Kurtosis** |
| **I:LDO** | 7,311 | 38,276 | 0,609 | 11,049 | 122,090 | -0,624 | 6,360 |
| **I:ECK** | 4,505 | 61,152 | 0,375 | 17,653 | 311,630 | 3,683 | 31,060 |
| **I:LRZ** | -7,944 | 49,159 | -0,662 | 14,191 | 201,381 | 1,547 | 11,915 |
| **I:PIRL** | -5,672 | 33,184 | -0,473 | 9,579 | 91,766 | -0,827 | 4,487 |
| **I:STL** | 16,082 | 41,698 | 1,340 | 12,037 | 144,896 | -0,866 | 5,930 |
| **I:PINF** | -11,443 | 39,215 | -0,954 | 11,320 | 128,152 | 0,310 | 5,858 |
| **I:BRE** | 7,722 | 32,649 | 0,643 | 9,425 | 88,829 | -0,222 | 2,780 |
| **I:ISP** | 0,972 | 33,696 | 0,081 | 9,727 | 94,621 | -0,970 | 5,879 |
| **I:ILLB** | -10,329 | 34,227 | -0,861 | 9,881 | 97,624 | -1,449 | 7,380 |
| **I:UCG** | -0,245 | 42,557 | -0,020 | 12,285 | 150,922 | -0,937 | 5,916 |
| **I:BANC** | 4,201 | 31,857 | 0,350 | 9,196 | 84,574 | -1,025 | 4,984 |
| **I:BPE** | -2,232 | 43,615 | -0,186 | 12,591 | 158,524 | -0,029 | 3,038 |
| **I:FCBK** | 11,869 | 30,414 | 0,989 | 8,780 | 77,084 | -0,331 | 2,972 |
| **I:CPR** | 15,290 | 23,509 | 1,274 | 6,786 | 46,055 | -0,513 | 3,609 |
| **I:SPAC** | -15,846 | 36,958 | -1,321 | 10,669 | 113,826 | -0,320 | 4,064 |
| **I:CALT** | 8,982 | 25,677 | 0,748 | 7,412 | 54,942 | -0,477 | 4,150 |
| **I:ENEL** | 6,659 | 21,978 | 0,555 | 6,344 | 40,252 | -0,344 | 4,053 |
| **I:ARN** | 24,951 | 38,660 | 2,079 | 11,160 | 124,552 | 1,412 | 5,340 |
| **I:A2A** | 8,860 | 25,713 | 0,738 | 7,423 | 55,097 | -1,279 | 6,334 |
| **I:TRN** | 7,752 | 16,358 | 0,646 | 4,722 | 22,300 | -0,203 | 2,561 |
| **I:ACE** | 4,848 | 27,114 | 0,404 | 7,827 | 61,265 | -0,482 | 3,293 |
| **I:BMED** | 5,314 | 31,405 | 0,443 | 9,066 | 82,191 | -1,119 | 7,910 |
| **I:TIPS** | 14,059 | 22,251 | 1,172 | 6,423 | 41,260 | -0,262 | 2,902 |
| **I:MB** | 5,598 | 33,112 | 0,467 | 9,559 | 91,368 | -1,303 | 7,468 |
| **I:EQUI** | 3,011 | 22,437 | 0,251 | 6,477 | 41,951 | -0,332 | 5,128 |
| **I:ANI** | 0,169 | 37,653 | 0,014 | 10,869 | 118,146 | -0,457 | 4,402 |
| **I:TITR** | -9,204 | 34,991 | -0,767 | 10,101 | 102,030 | 0,211 | 3,498 |
| **I:TIT** | -12,199 | 35,972 | -1,017 | 10,384 | 107,834 | 0,546 | 4,606 |
| **I:ENV** | -0,476 | 21,482 | -0,040 | 6,201 | 38,456 | -0,210 | 4,000 |
| **I:VAL** | -5,429 | 25,502 | -0,452 | 7,362 | 54,197 | 0,560 | 4,937 |
| **I:CLT** | 1,529 | 22,840 | 0,127 | 6,593 | 43,473 | 1,425 | 7,406 |
| **I:HER** | 4,711 | 22,741 | 0,393 | 6,565 | 43,096 | -0,827 | 4,246 |
| **I:IRE** | 8,654 | 26,688 | 0,721 | 7,704 | 59,353 | -0,797 | 3,996 |
| **I:IG** | 6,195 | 20,904 | 0,516 | 6,035 | 36,416 | -0,408 | 3,571 |
| **I:ELN** | 19,684 | 44,665 | 1,640 | 12,894 | 166,244 | -0,306 | 3,804 |
| **I:AMP** | 20,609 | 29,738 | 1,717 | 8,585 | 73,695 | -0,995 | 5,020 |
| **I:DLG** | 7,915 | 34,112 | 0,660 | 9,847 | 96,971 | -0,205 | 3,242 |
| **I:FD** | -13,102 | 52,553 | -1,092 | 15,171 | 230,150 | 2,631 | 17,149 |
| **I:IP** | 15,467 | 32,396 | 1,289 | 9,352 | 87,460 | -0,741 | 3,349 |
| **I:IKG** | 12,306 | 32,081 | 1,026 | 9,261 | 85,767 | 0,513 | 4,577 |
| **I:ENAV** | -0,837 | 22,929 | -0,070 | 6,619 | 43,810 | -0,503 | 5,304 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| **I:PST** | 5,420 | 23,831 | 0,452 | 6,880 | 47,328 | -0,605 | 4,162 |
| **I:RCS** | -2,578 | 39,210 | -0,215 | 11,319 | 128,116 | 0,443 | 5,015 |
| **I:CAI** | -10,973 | 33,341 | -0,914 | 9,625 | 92,634 | -0,034 | 3,450 |
| **I:MON** | -18,934 | 30,683 | -1,578 | 8,857 | 78,453 | 0,905 | 8,732 |
| **I:GAMB** | -13,026 | 55,521 | -1,086 | 16,028 | 256,883 | 2,623 | 18,673 |
| **I:UNI** | 2,505 | 33,885 | 0,209 | 9,782 | 95,686 | -1,109 | 6,470 |
| **I:G** | 1,297 | 24,993 | 0,108 | 7,215 | 52,056 | -0,636 | 5,627 |
| **I:SRG** | 3,543 | 17,942 | 0,295 | 5,179 | 26,826 | -0,337 | 2,681 |
| **I:ENI** | 0,624 | 25,368 | 0,052 | 7,323 | 53,628 | 0,136 | 5,059 |
| **I:TOD** | -8,291 | 36,347 | -0,691 | 10,493 | 110,095 | 0,498 | 4,829 |
| **I:REC** | 14,833 | 23,552 | 1,236 | 6,799 | 46,226 | -0,238 | 3,952 |
| **I:RN** | -11,240 | 54,606 | -0,937 | 15,763 | 248,481 | 0,227 | 5,527 |
| **I:BRI** | -4,469 | 31,248 | -0,372 | 9,021 | 81,371 | -0,539 | 5,366 |
| **I:FUL** | -13,210 | 48,561 | -1,101 | 14,018 | 196,517 | 3,177 | 21,222 |
| **I:AISW** | 9,473 | 48,913 | 0,789 | 14,120 | 199,374 | 0,686 | 8,786 |
| **I:JUVE** | 5,109 | 48,428 | 0,426 | 13,980 | 195,440 | 0,262 | 5,843 |
| **I:SSL** | 5,138 | 43,655 | 0,428 | 12,602 | 158,816 | -0,243 | 8,679 |
| **I:CLE** | -30,122 | 39,581 | -2,510 | 11,426 | 130,558 | 0,078 | 5,164 |
| **I:B** | -16,502 | 19,932 | -1,375 | 5,754 | 33,107 | 0,750 | 5,969 |
| **I:CEM** | 7,145 | 32,778 | 0,595 | 9,462 | 89,535 | 0,298 | 3,218 |
| **I:US** | 0,129 | 24,408 | 0,011 | 7,046 | 49,647 | -0,406 | 4,182 |
| **I:BZU** | 10,703 | 26,582 | 0,892 | 7,673 | 58,882 | -0,312 | 2,709 |
| **I:CE** | 2,834 | 26,881 | 0,236 | 7,760 | 60,217 | -0,017 | 4,855 |
| **I:DAN** | 3,944 | 29,931 | 0,329 | 8,640 | 74,654 | -0,460 | 4,101 |
| **I:ITM** | 12,028 | 23,417 | 1,002 | 6,760 | 45,698 | 1,380 | 8,626 |
| **I:ZUC** | -9,619 | 32,241 | -0,802 | 9,307 | 86,622 | 0,926 | 6,796 |
| **I:IPG** | -5,570 | 36,086 | -0,464 | 10,417 | 108,517 | 0,304 | 3,793 |
| **I:VIN** | -2,333 | 20,451 | -0,194 | 5,904 | 34,853 | -0,003 | 4,023 |
| **I:EDNR** | 6,537 | 19,122 | 0,545 | 5,520 | 30,471 | -0,185 | 5,983 |
| **I:RAT** | 1,309 | 20,548 | 0,109 | 5,932 | 35,184 | -0,090 | 4,838 |
| **I:GAB** | -0,602 | 54,104 | -0,050 | 15,619 | 243,938 | 1,193 | 6,137 |
| **I:MS** | -11,974 | 37,610 | -0,998 | 10,857 | 117,874 | 1,644 | 10,858 |
| **I:ERG** | 12,637 | 24,112 | 1,053 | 6,961 | 48,451 | -0,220 | 4,617 |
| **I:CMB** | 14,366 | 28,119 | 1,197 | 8,117 | 65,890 | -0,851 | 8,767 |
| **I:SAB** | 4,899 | 33,856 | 0,408 | 9,773 | 95,520 | -0,048 | 3,194 |
| **I:BE** | -4,064 | 30,324 | -0,339 | 8,754 | 76,626 | 0,862 | 5,542 |
| **I:SOL** | 15,885 | 21,124 | 1,324 | 6,098 | 37,184 | 0,177 | 2,490 |
| **I:DAL** | -3,056 | 37,721 | -0,255 | 10,889 | 118,575 | 0,007 | 2,645 |
| **I:BSS** | 3,391 | 45,001 | 0,283 | 12,991 | 168,754 | -0,605 | 3,533 |
| **I:SAFI** | -20,929 | 46,417 | -1,744 | 13,399 | 179,545 | 0,146 | 5,579 |

Since we are dealing with optimal allocation models, it is crucial to test whether the data exhibits a normal distribution. As can be seen by the tables above, the skewness and kurtosis values for the daily data deviate significantly from those expected in a normal distribution (0 for the skewness and 3 for the kurtosis). In order to better visualized this deviation, we selected Cembre S.p.A. and Sol S.p.A. as the securities with, respectively, the skewness closest to 0 and the kurtosis closest to 3, and we used the Matlab function `normplot` that compares the distribution of the chosen securities returns to the normal distribution (Figure 1 and Figure 2).




(a) Cembre returns.

(b) Sol returns.

Figure 1: Comparison to the normal distribution (daily).

It is evident from both figures that the time series do not closely resemble a Gaussian distribution. This is aligned with the general fact that stocks returns in daily frequency are often characterized by high skewness and kurtosis that deviate from the normal distribution.

On the other hand, for monthly data the skewness and kurtosis values are closer to those expected for a normal distribution. Indeed, even taking into account the stock with the skewness and kurtosis values further away from the ideal ones, say Ecosuntek S.p.A., its returns distribution apparently better approximates the normal distribution, as shown in Figure 3.

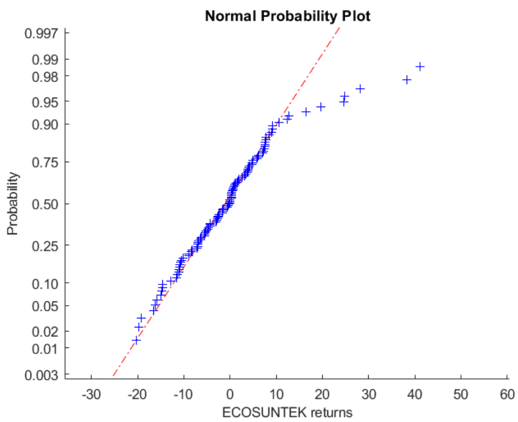



Figure 2: Comparison with the Ecosuntek returns (monthly).

Hence, we can reject the null hyphotesis of a normal variable for all the daily data, while more comprehensive tests would be necessary for the monthly data. However, we decided not to delve deeper into that analysis, as it is widely recognized that financial returns typically do not follow a normal distribution.
In conclusion, it's worthwhile to say that we must treat the data and the results carefully.

## 2.2 Variance, covariance and correlation matrices

Figure 4 and Figure 5 display the heatmaps, respectively, of the variance-covariance and correlation matrices for the whole sample. The heatmaps use colour gradients, with darker shades indicating lower values. These matrices are computed using the Matlab function `heatmap` (see the details in the code attached).

(a) Variance-covariance matrix. (b) Correlation matrix.

Figure 3: Daily frequency.

(a) Variance-covariance matrix. (b) Correlation matrix.

Figure 4: Monthly frequency.

## 2.3 Securities selection

Our sample selecting process consists in 3 steps, both for daily and monthly data.
Since the stocks returns shall be treated as random variables and the portfolio theoretically represents a sample of random variables, we decided to focused on the independence of the securities. In this way, even though we might not build the less risky portfolio, since we were not primarily choosing the less correlated assets (i.e. correlation less than or equal to zero), we tried to maximize the diversification, as it can be deduced by the first two steps.

1. *Correlation matrix:* we computed the $norm_1$ $||\cdot||_1$ for every row of the correlation matrix and we sort the stocks in ascending order with respect to this quantity by the function `sort` (see the function `best_choice` in the code for details). We utilized the $norm_1$ rather than, i.e., the average since the negative values of the correlation for some stock pairs would have contributed to a final outcome of the average correlation of both stocks closer to 0, inappropriately for our purpose. Indeed, the mean correlation would have led us to choose some stocks that were only apparently the less dependent from the others, since in each row the positive entries could have been balanced by the negative ones.
   We recall the definition of $norm_1$:
$$||x||_1 = \sum_{i=1}^{n} |x_i| \qquad \forall x = (x_1, ..., x_n) \in \mathbb{R}^n.$$
   At the end of this step we select the first 25 securities of the sorted sample.

2. *Differentiation:* among the 25 securities, firstly we pick one for each business sector, then, among them, we concentrated on the stocks with a negative annualized return.

3. *Volatility:* among the latter group, we select the ones with the lowest annualized standard deviation (annualized volatility).

**Remark 1.** Due the observations discussed in section 2.1, the correlation matrix used at step one is the one generated according to the monthly data.

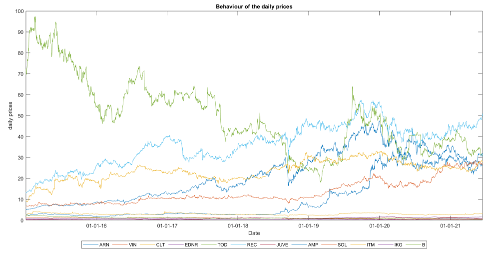
Below, we report our final sample of 12 securities.

| **Asset Names** | **Code** | **Sector** |
|---|---|---|
| ALERION CLEAN POWER | ARN | Utilities |
| VIANINI INDR. | VIN | Real Estate |
| CENTRALE DEL LATTE D'ITALIA | CLT | Consumer Defensive |
| EDISON RSP | EDNR | Utilities |
| TOD'S | TOD | Consumer Cyclical |
| RECORDATI INDUA.CHIMICA | REC | Health Care |
| JUVENTUS FOOTBALL CLUB | JUVE | Communication Services |
| AMPLIFON | AMP | Health Care |
| SOL | SOL | Chemical Industry |
| ITALMOBILIARE | ITM | Holding |
| INTEK GROUP | IKG | Personal Safety |
| BASTOGI | B | Holding |

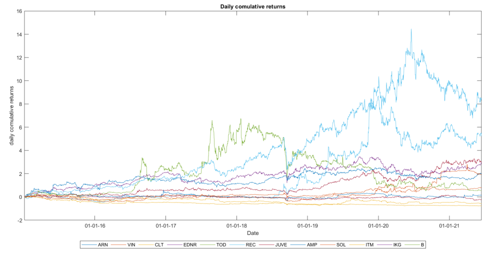
Table 3: Selected securities.

## 2.4 Behaviour of the selected securities prices

Figures 5 and Figure 6 illustrate the stocks prices and cumulative returns graphs of the 12 selected securities, using both daily and monthly data.

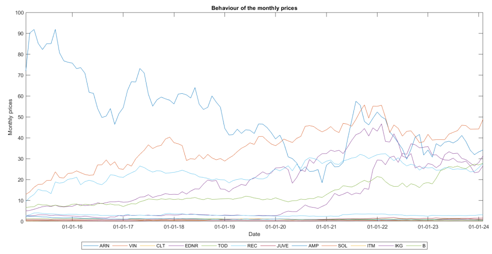



(a) Stock prices.

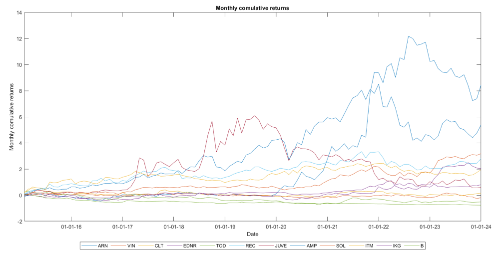



(b) Cumulative returns.

Figure 5: Daily frequency.




(a) Stock prices.




(b) Cumulative returns.

Figure 6: Monthly frequency.

# 3 Mean Variance Optimization

We implemented four portfolios, one without any constraint on short positions (i.e. weights may assume negative values) and the other one with the possibility to take only long positions (i.e. all weights must be positive), both with daily and monthly data.
Since it is of direct derivation and on asset allocation issues it does not add much more than the mean-variance model of Markowitz (1952), we used the CAPM for what concerns the expected returns of our securities. However, it is important to remind the potential sensitivity of this model and that it is not the optimal strategy. Indeed, despite CAPM is widely used in financial theory, yet, it requires several strong assumptions:

- each investor has an amount of wealth that is infinitely small compared to the market. Therefore, purchases and sales made by each individual investor cannot change the price of securities;

- investors express their preferences on each security only in relation to the mean and variance of those securities;
- investors have homogeneous expectations about securities;
- markets are competitive and efficient: all investors have costless access to market information, and the prices instantly respond to new public information;
- there are no taxes on dividends or capital gains, no transaction costs in buying or selling securities and no restrictions on short selling;
- for any given asset, expected returns are normally distributed. However, we have previously discussed that is often not the case in financial markets.

The CAPM is also referred to as a single index model, since it identifies the market portfolio as the only non-diversifiable risk factor.

## 3.1 Portfolios statistics

In Table 4 and in Table 6 we reported the weights we should allocate to each stock, based on the maximization of the Sharpe index ("Sharpe" column) and on the minimization of the portfolio volatility ("GMV" column), for both daily and monthly data and under both assumptions of negative-allowed and positive-restricted weights.
We recall the definition of Sharpe index:

$$S = \frac{\mu_P - R_f}{\sigma_P}, \tag{3.1}$$

where $\mu_P$ and $\sigma_P$ are respectively the mean and the volatility of the portfolio P, and $R_f$ is the risk-free rate.

We computed the weights for the previous model by the Matlab function `EstimateMaxSharpeRatio`, while for the latter model by `EstimateFrontierLimits`.

### 3.1.1 Daily data

Firstly, we do not report the unconstrained-case for the GMV portfolio since as we can see from the third column of Table 4, and as expected, the model is not suggesting any negative value even if we allow for short selling. Therefore, for the GMV portfolio, the two cases coincide, at least for daily frequency.
On the other hand, in the case without constraints the MaxSharpe portfolio allows for both long and short positions, resulting in non-zero weights for all securities. Short positions are taken in certain stocks to finance assets that are expected to generate higher returns. Conversely, when imposing the constraint of non-negative positions, only half of the assets have non-zero weights. This outcome is driven by the objective of maximizing the Sharpe ratio, which aims to minimize volatility

| Code | Unconstr. | | Constr. |
|---|---|---|---|
| | **Max Sharpe** | **GMV** | **Max Sharpe** |
| **ARN** | 0,38151 | 0,03733 | 0,21855 |
| **VIN** | -0,16395 | 0,17394 | 0,00000 |
| **CLT** | -0,08809 | 0,09624 | 0,00000 |
| **EDNR** | 0,15202 | 0,20985 | 0,00000 |
| **TOD** | -0,55176 | 0,00591 | 0,00000 |
| **REC** | 0,34217 | 0,06734 | 0,14193 |
| **JUVE** | -0,04337 | 0,01727 | 0,00000 |
| **AMP** | 0,42296 | 0,02382 | 0,19266 |
| **SOL** | 0,42133 | 0,11792 | 0,21057 |
| **ITM** | 0,34363 | 0,11237 | 0,12477 |
| **IKG** | 0,26680 | 0,06935 | 0,11152 |
| **B** | -0,48323 | 0,06866 | 0,00000 |

Table 4: Weights according to daily data.

and exclude securities that are considered less favourable or dominated in terms of risk-return trade-off.

| Statistics | Unconstr. | | Constr. |
|---|---|---|---|
| | **Max Sharpe** | **GMV** | **Max Sharpe** |
| **Mean** | 0,00196 | 0,00024 | 0,00067 |
| **STD** | 0,02444 | 0,00852 | 0,01151 |
| **Variance** | 0,00060 | 0,00007 | 0,00013 |
| **Skewness** | -0,14359 | -0,88176 | -0,48204 |
| **Kurtosis** | 5,88902 | 13,85726 | 12,45987 |
| **Sharpe Ratio** | 0,08021 | 0,05822 | 0,05822 |

Table 5: Portfolios statistics (daily).

Table 5 above contains the main statistics for the two portfolios, under both assumptions of negative-allowed and positive-restricted weights. It is evident that both MaxSharpe portfolios significantly outperform the GMV portfolio and present skew and kurtosis values much closer to the ones expected for a normal distribution. However, as expected and due to the asset allocation criteria of two approach, the MaxSharpe portfolios volatility is widely higher than the GMV one.
Finally, the MS portfolio without constraints exhibits higher return compared to the long-only MS portfolio. However, the long-only portfolio demonstrated lower volatility, in accordance with the theoretical predictions.

### 3.1.2 Monthly data

Due to the same reason discussed for daily frequency, in Table 6 we did not report the unconstrained-case for the GMV portfolio. As well as for the daily frequency,

| Code | Unconstr. | | Constr. |
|---|---|---|---|
| | **Max Sharpe** | **GMV** | **Max Sharpe** |
| **ARN** | 0,18028 | 0,01122 | 0,14517 |
| **VIN** | -0,36585 | 0,16579 | 0,00000 |
| **CLT** | -0,09292 | 0,10598 | 0,00000 |
| **EDNR** | 0,43961 | 0,25947 | 0,07924 |
| **TOD** | -0,14302 | 0,00000 | 0,00000 |
| **REC** | 0,34435 | 0,05872 | 0,14584 |
| **JUVE** | 0,00473 | 0,01905 | 0,00000 |
| **AMP** | 0,25434 | 0,03398 | 0,20871 |
| **SOL** | 0,72271 | 0,13052 | 0,34748 |
| **ITM** | 0,44051 | 0,05072 | 0,06909 |
| **IKG** | 0,21527 | 0,00000 | 0,00448 |
| **B** | -1,00000 | 0,16455 | 0,00000 |

Table 6: Weights according to monthly data.

in the case without constraints the MaxSharpe portfolio allows for both long and short positions. In addition, dealing with monthly data allocated a weight equal to -1 to the stock I:B when there were no constraints (second column) and a non-zero weight to the stock I:EDNR in the long-only MaxSharpe portfolio, unlike for the correspondent daily frequency portfolio.

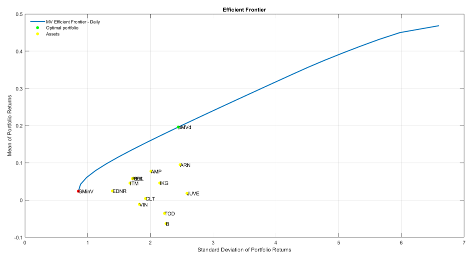
| Statistics | Unconstr. | | Constr. |
|---|---|---|---|
| | **Max Sharpe** | **GMV** | **Max Sharpe** |
| **Mean** | 0,04631 | 0,00282 | 0,01417 |
| **STD** | 0,09920 | 0,03255 | 0,04494 |
| **Variance** | 0,00984 | 0,00105 | 0,00202 |
| **Skewness** | -0,10701 | -0,14697 | -0,06569 |
| **Kurtosis** | 2,91764 | 2,92582 | 2,93114 |
| **Sharpe Ratio** | 0,46682 | 0,31537 | 0,31537 |

Table 7: Portfolios statistics (monthly).

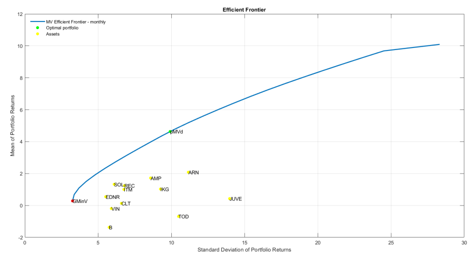
As well as for the daily frequency, MaxSharpe portoflios significantly outperform the GMV portfolio while the MaxSharpe portfolios volatility is widely higher than the GMV one. However, for monthly data the three skew and kurtosis values result to be very close to each other. In particular, for both daily and monthly data, all three negative skew values indicate that the distributions have a longer left tail or

is "skewed to the left." This means that the distributions are characterized by a concentration of values towards the right side of the distribution, with fewer values towards the left side.
As well as for the daily frequency, the MS portfolio without constraints exhibits higher return compared to the long-only MS portfolio. However, the long-only portfolio demonstrated lower volatility, in accordance with the theoretical predictions.

## 3.2 Efficient frontier

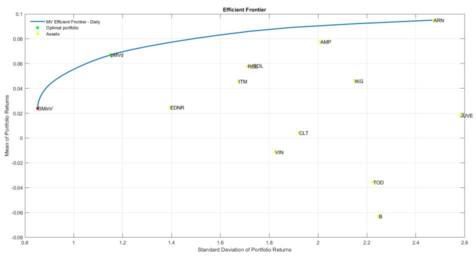



(a) Efficient Frontier with daily data.

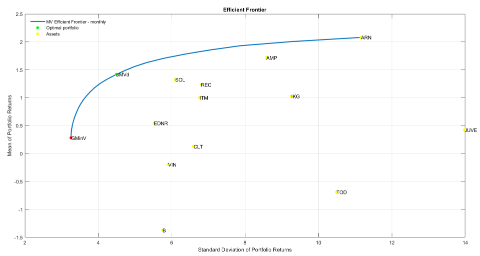



(b) Efficient Frontier with monthly data.

Figure 7: Mean-Variance Efficient Frontier (Unconstrained).




(a) Efficient Frontier with daily data.




(b) Efficient Frontier with monthly data.

Figure 8: Mean-Variance Efficient Frontier (Constrained).

In accordance to what discussed in Section 2.1, by observing Figure 7 we have clear evidence of a non-gaussian distribution of the daily returns even in our restricted sample. Indeed, Max Sharpe portfolio returns do not follow a Gaussian distribution as shown by the presence of "fatter" tails in the EF graphs, highlighting the presence of outliers and that the probability to meet extreme values is high.

Furthermore, by comparing Figure 7 and Figure 8 we can verify that portfolios that allow short positions tend to have higher returns but also exhibit higher risk compared to the long-only portfolios, as already inferred by analysing both Table 5 and Table 7.

# 4 FTSE Italia All Share index (FTSE)

The FTSE Italia All Share Total Return index represents the italian stock market, including big, mid and small cap italian societies. We reported in Table 8 and Table 9 the main statistics of the FTSE Italia All Share index in its total return version, along with those of our MaxSharpe portfolio (indicated as "MS"), both for daily and monthly data.

| | **FTSE** | **MS (Constr.)** | **MS (Unconstr.)** |
|---|---|---|---|
| **Mean** | 0,00034 | 0,00067 | 0,00196 |
| **STD** | 0,01365 | 0,01151 | 0,02444 |
| **Variance** | 0,00019 | 0,00013 | 0,00060 |
| **Skewness** | -1,68797 | -0,48204 | -0,14359 |
| **Kurtosis** | 22,57938 | 12,45987 | 5,88902 |

Table 8: Statistics related to daily data.

| | **FTSE** | **MS (Constr.)** | **MS (Unconstr.)** |
|---|---|---|---|
| **Mean** | 0,00443 | 0,01418 | 0,04631 |
| **STD** | 0,05822 | 0,04495 | 0,09920 |
| **Variance** | 0,00339 | 0,00202 | 0,00984 |
| **Skewness** | -0,90975 | -0,06569 | -0,10701 |
| **Kurtosis** | 6,76204 | 2,93114 | 2,91764 |

Table 9: Statistics related to monthly data.

By observing the statistics in the tables, it arises that FTSE index deviates from a normal distribution, as evidenced by the non-zero skewness and non-three kurtosis values, both for daily and monthly data. In particular, similarly at the MaxSharpe and GMV portfolios, the FTSE index present a skewness negative value.

It is worth noting that our portfolio, both in the constrained and non-constrained versions and both for daily and monthly data, outperforms the FTSE index. This indicates that the portfolio selection process has resulted in superior performance compared to investing in the index itself. Specifically, the constrained portfolio yields a higher return compared to the FTSE index, while simultaneously exhibiting a lower standard deviation, both in the daily and monthly case. This implies that the constrained monthly portfolio dominates the index, providing both higher returns and lower volatility. On the other hand, the unconstrained daily portfolio exhibits significantly higher returns relative to the index but also carries a much higher level of volatility, with a standard deviation approximately twice that of the FTSE, both in the daily and monthly case.

# 5 Beta

The CAPM enables investors to determine the pricing of financial assets based on the risk premium, which represents the difference between the market return and the risk-free return, using the following formula.

$$r_i = r_f + \beta_{i,m}(r_m - r_f), \tag{5.1}$$

where $r_i$ is the return of the asset i, $r_f$ is the risk-free rate and $r_m$ is the return of the market, which in our case is proxied by FTSE, and $\beta_{i,m}$ is the factor loading of the model which indicates the marginal contribution to the market risk and delivers the intensity.
The concept of beta is central to the CAPM and the Security Market Line (SML), and it can be calculated as:

$$\beta_{i,m} = \frac{Cov(r_i, r_m)}{Var(r_m)}. \tag{5.2}$$

Beta may assume different values:

1 $\underline{\beta_{i,m} > 1}$: so-called *aggressive beta*, it represents that the security/portfolio is theoretically more volatile than the market;

2 $\underline{\beta_{i,m} < 1}$: so-called *defensive beta*, it represents that the security/portfolio is theoretically less volatile than the market;

3 $\underline{\beta_{i,m} = 1}$: it represents that the security/portfolio volatility is exactly equal to the market volatility;

4 $\underline{\beta_{i,m} < 0}$: the security/portfolio shows a negative correlation coefficient with respect to the market return.

**Remark 2.** As required, we used as benchmark the FTSE Italia All Share Total Return index. However, since it represents the whole Italian stock market, it would have been the best choice anyway.

As represented in Table 10, all our assets have a defensive beta, which means that the assets are not prone to market fluctuations and then the volatility is reduced with respect to the market. We might say that this result partially derive from our portfolio selection process, since in the step 3 we chose among the remaining securities looking at their volatility.

The beta of the portfolio is a weighted average of the betas of each asset composing the portfolio, so it should be, and indeed it is, defensive as well.

| | Daily Beta | Monthly beta |
|---|---|---|
| **ARN** | 0,4462 | 0,4760 |
| **VIN** | 0,1290 | 0,1662 |
| **CLT** | 0,3962 | 0,4523 |
| **EDNR** | 0,4428 | 0,3684 |
| **TOD** | 0,7447 | 0,6820 |
| **REC** | 0,6233 | 0,4130 |
| **JUVE** | 0,7423 | 0,7428 |
| **AMP** | 0,6547 | 0,6207 |
| **SOL** | 0,4208 | 0,3794 |
| **ITM** | 0,4994 | 0,5201 |
| **IKG** | 0,4393 | 0,6590 |
| **B** | 0,3776 | 0,4290 |

Table 10: Securities betas.

| Daily | | Monthly | |
|---|---|---|---|
| **Beta (Unconstr.)** | **beta (Constr.)** | **Beta (Unconstr.)** | **beta (Constr.)** |
| 0,5122 | 0,5120 | 0,5672 | 0,4588 |

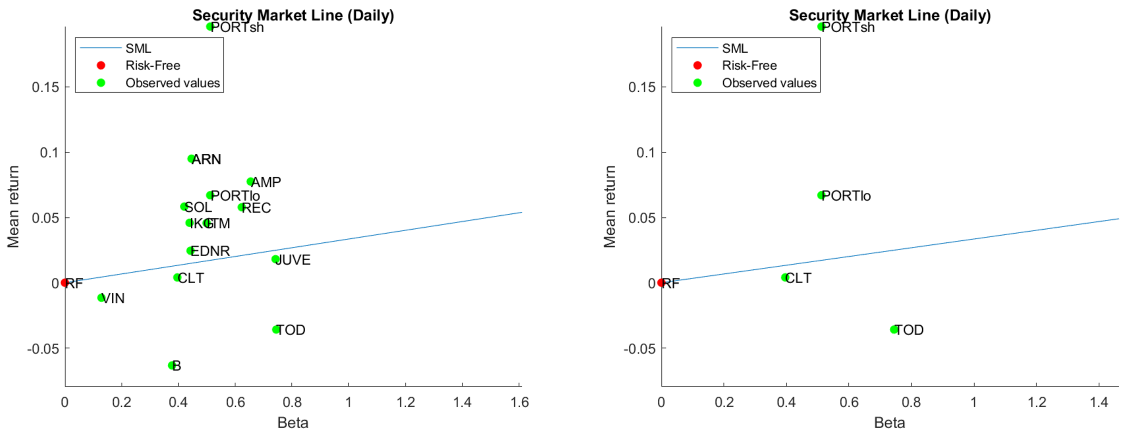
Table 11: MaxSharpe Portfolios beta.

# 6 Security Market Line (SML)

The Security Market Line (SML) is closely related to the Capital Asset Pricing Model (CAPM). The fundamental difference between the SML and the Capital Market Line (CML), where the Sharpe index is the slope of the latter, is that the latter represents the risk premium of the efficient portfolio as a function of the portfolio's standard deviation. The SML, on the other hand, represents the relationship between expected returns and systematic risk of a single security or portfolio. The SML states that the expected return of a security or portfolio depends on its system beta, which measures the sensitivity of a security or portfolio to market movements. The SML is graphically a line plotted on a Cartesian plane, with the x-axis representing risk (measured by beta) and the y-axis representing expected return. It starts from the risk-free rate (typically represented by the risk-free interest rate) and extending upward with a slope equal to the market risk premium.

A security is considered fairly priced when its mean return aligns with the required return indicated by the SML, which lies on the line. If the expected return of the security is above the SML line but not directly on it, it suggests that the return is higher than the equilibrium point, therefore the security is considered underpriced, making it advantageous to buy. Conversely, if the expected return falls below the SML, it implies that the security is overpriced, hence in such situations, it is generally recommended to sell the security.

As required, we set a risk free rate of 3% and we choose arbitrarily two securities from our portfolio to verify the SML: Centrale del Latte d'Italia (I:CLT) and Tod's (I:TOD). In Figure 9 we plotted the SML both for daily and monthly data.



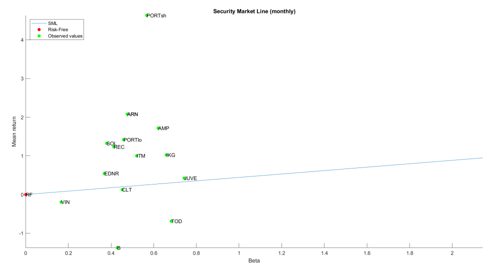

(a) SML for all the securities.

(b) SML for only two securities.

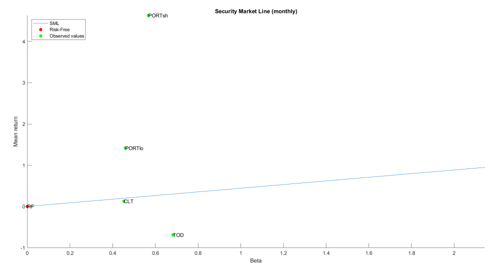
Figure 9: Security Market Line (daily).

Based on Figure 9(a), it is evident that CLT eand TOD lie under the Security Market Line (SML), therefore both of them are offering lower expected returns than what is required by their levels of systematic risk. Consequently, they are considered overpriced according to the SML. As a result, it would be prudent to consider selling them. On the other hand, both the portfolios (short-positions allowed and long-only) are positioned above the SML in the plot, indicating that it offers a higher expected return than what is warranted by its level of systematic risk. As a result, it would be advantageous to hold onto the portfolios.
If we consider Figure 9(b), where only TOD and CLT appear, the result still holds, since the SML plotted with only two securities is equivalent to the SML with all the stocks.
Back to Figure 9(a), as can be seen, half of the stocks considered for our portfolio when represented by the SML turn out to be undervalued.

The conclusions reached for the monthly portfolios and the two selected securities align with the observations for the daily case. Both portfolios exhibit the same pattern, with CLT and TOD positioned under the SML, indicating their overpriced nature, while the portfolios lie above the SML, suggesting to hold onto them. And even in this case, when considering only two stocks, the result holds.
The relevant difference between Figure 9 and Figure 10 lies in the increasing number of securities that are considered underpriced for monthly data.




(a) SML for all the securities.




(b) SML for only two securities.

Figure 10: Security Market Line (monthly).

# 7 Black-Littermann approach

The Black-Litterman asset allocation model is a sophisticated portfolio construction method based on Bayesian analysis that overcomes the problem of highly-concentrated portfolios, input-sensitivity, and estimation error maximization. These three related and well-documented problems with mean-variance optimization are the most likely reasons that more practitioners do not use the Markowitz paradigm (1952), in which return is maximized for a given level of risk. In Markowitz's mean-variance model the asset allocation is computed once the expected returns and co-variances of the assets are known. However, its application in the financial markets has always faced the concrete difficulty to come up with reasonable estimates of expected returns, even though the covariances of a few assets has been adequately estimated.
The Black-Litterman model overcomes the problem of input-sensitivity using a Bayesian approach to combine the subjective views of an investor regarding the expected returns of one or more assets with the market equilibrium vector of expected returns (the prior distribution) to form a new, mixed estimate of expected returns. Subjective views can be of two types: absolute views, which are related exclusively to one asset, or relative views, which relate to securities and measure the intensity one outperforms the other.

We split our work in three main steps: Priors (prior equilibrium distribution), Likelihood (investor's view, both relative and absolute) and the Posterior (posterior distribution).

## 7.1 Priors

The original model assumes the returns are normally distributed, that means:

$$r \sim D_{prior} = N(\pi, \tau\Sigma), \tag{7.1}$$

where $r$ represents the vector of the expected returns, $D_{prior}$ indicates the prior distribution, $\Sigma$ is the variance-covariance matrix, $\tau$ is a constant of proportionality

which is usually close to zero, calculated in our case as $\frac{1}{\text{length of the available time series}}$ (Meucci, 2006), and $\pi$ represents the excess equilibrium returns vector.
Equilibrium returns are the set of returns that clear the market and are used as a neutral starting point in the Black-Litterman model. The latter are derived using a reverse optimization method in which the vector of implied excess equilibrium returns is extracted from known information using:

$$\pi = \lambda \Sigma \omega_{market}, \tag{7.2}$$

where $\lambda$ is the risk aversion coefficient and $\omega_{market}$ is the market capitalization weight. The risk-aversion coefficient $\lambda$ characterizes the expected risk-return trade-off. It is the rate at which an investor will forego expected return for less variance (Idzorek, 2004).

## 7.2 Likelihood

Assuming the investor has $k$ different views on a total of $n$ assets, the views are represented as linear combinations of expected returns:

$$P * \mu = Q + \epsilon, \tag{7.3}$$

where:

- $P$ is a $k \times n$ matrix containing the weight of each investor view for each asset in the portfolio. The rows represent the views while the columns represent the portfolio asset. With absolute views, the matrix receives the value $+1$ for the considered asset while all others are 0, while in the relative views the better-expected performance securities get a value of $+1$, the other assets subjected to the view get -1 and all the others get 0;

- $\mu$ is the $n \times 1$ vector of the average of the expected returns;

- $Q$ is the $k \times 1$ vector of the expected returns for each view;

- $\epsilon$ is the $k \times 1$ vector of the random errors committed in the views. The uncertainty of the views implies that $\epsilon$ results in a random, unknown, independent, normally-distributed error term vector with a mean of 0 and covariance matrix $\Omega$.

Since the model assumes that the views are independent of one another, $\Omega$ is a diagonal covariance matrix, while the diagonal entries represent the uncertainty of each view.

We started creating our views vector setting 4 views in total, 2 absolutes and 2 relatives, both in daily and monthly frequency, as the following:

1. **Vianini** will soon experience a +0,6% daily return and a +6% monthly return. The reason is linked to the monetary policy announced by the ECB for the next months and its implication on the real estate sector. The BCE has recently claimed that it is going to progressively cut the interest rates that significantly rose throughout the last year. This would reduce the pressure on real estate sector and, for us, lead to a period of financial confidence, after months of turmoil.

2. **Centrale del latte** will soon experience a -0.8% daily return and a -7% monthly return. The reason is linked to the Borsa Italiana statement of 01/12/2024 that announces the withdrawal of the status of STAR for CLT common shares, as the requirements about the minimum free float have not been fulfilled since January 2023. For us, this communicate will generate a period of lack of confidence in this security.

3. **Sol** will outperform **Recordati** by 0.25% in daily terms and by 2% in monthly terms. The estimation is based on the fact that in the last two years Sol has strengthened its foreign presence, like in South-America and India, while Recordati has invested primarily in its national scenario. However, due to the significant increasing of Recordati stock price, we have estimated just a slight difference between the performance of the two securities.

4 **Edison** will outperform **Tod**'s by 0.8% in daily terms and by 9% in monthly terms. The main reason is linked to the fact that the world is currently pursuing towards a shift from the use of fossil fuels, from which companies whose business is based on the renewables could widely benefit. Moreover, we retained that Tod's business sector is not going to face neither a similar expansion, neither stronger. Since this is a weaker view compared with the others, we expect to obtain a higher weight in position $\Omega(4, 4)$.

Therefore, in our case $k = 4$, $n = 12$,

$$Q_{daily} = \begin{pmatrix} 0.006 \\ -0.008 \\ 0.0025 \\ 0.008 \end{pmatrix} \quad \text{and} \quad Q_{monthly} = \begin{pmatrix} 0.06 \\ -0.07 \\ 0.02 \\ 0.09 \end{pmatrix},$$

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We calculated $\Omega$, in both daily ($\Omega_d$) and monthly ($\Omega_m$) frequency, with the method used in He-Litterman (He & Litterman, 2002):

$$\Omega_{daily} = diag(P\tau\Sigma_d P') = \begin{pmatrix} 0.0014 & 0 & 0 & 0 \\ 0 & 0.0016 & 0 & 0 \\ 0 & 0 & 0.0020 & 0 \\ 0 & 0 & 0 & 0.0023 \end{pmatrix}$$

and

$$\Omega_{monthly} = diag(P\tau\Sigma_m P') = \begin{pmatrix} 0.3227 & 0 & 0 & 0 \\ 0 & 0.4025 & 0 & 0 \\ 0 & 0 & 0.5888 & 0 \\ 0 & 0 & 0 & 1.1967 \end{pmatrix}.$$

Finally, since we did not have sufficient data to compute the market-implied returns, we set the coordinates of $\omega_{market}$ according to the capitalization of our portfolio securities, in such a way that the sum of the assigned weights were equal to 1.

Combining all terms computed above, we extract the likelihood distribution, in both daily and monthly frequency: $D_{likelihood} = N(Q, \Omega)$.
By definition, $\Omega = 0$ in case of extreme confidence of the views (100%), while $\Omega = \infty$ with no confidence (0%). Furthermore, we can already eye that the entry of $\Omega$ related to the view 4 is the highest among all the entries, both in daily and monthly frequency, as predicted. In particular, in $\Omega_{monthly}$ it is higher than one, confirming that our view 4 is a weak and risky prediction.

## 7.3 Posterior

Finally, according to the Black Litterman approach, we used the following distribution:

$$\begin{aligned} D_{BL} &= N(\mu_{BL}, \Sigma_{BL}), \qquad \text{where} \\ \mu_{BL} &= ((\tau\Sigma)^{-1} + P^T\Omega^{-1}P)^{-1}((\tau\Sigma)^{-1}\pi + P^T\Omega^{-1}Q), \\ \Sigma_{BL} &= ((\tau\Sigma)^{-1} + P^T\Omega^{-1}P)^{-1}. \end{aligned}$$

In Figure 10-11 we compared our portfolio composition based on the Black-Litterman approach and the one built with the mean-variance optimization (MaxSharpe portfolio), both in daily and monthly frequency and under both assumptions of negative-allowed and positive-restricted weights.
The figures report only the securities with positive weight allocated. It arises that with the BL approach we have a portfolio with more long-positions the MS approach, since we include views on assets that were not profitable before.
Furthermore, the BL unconstrained and constrained portfolios are very similar, both in daily and monthly frequency.

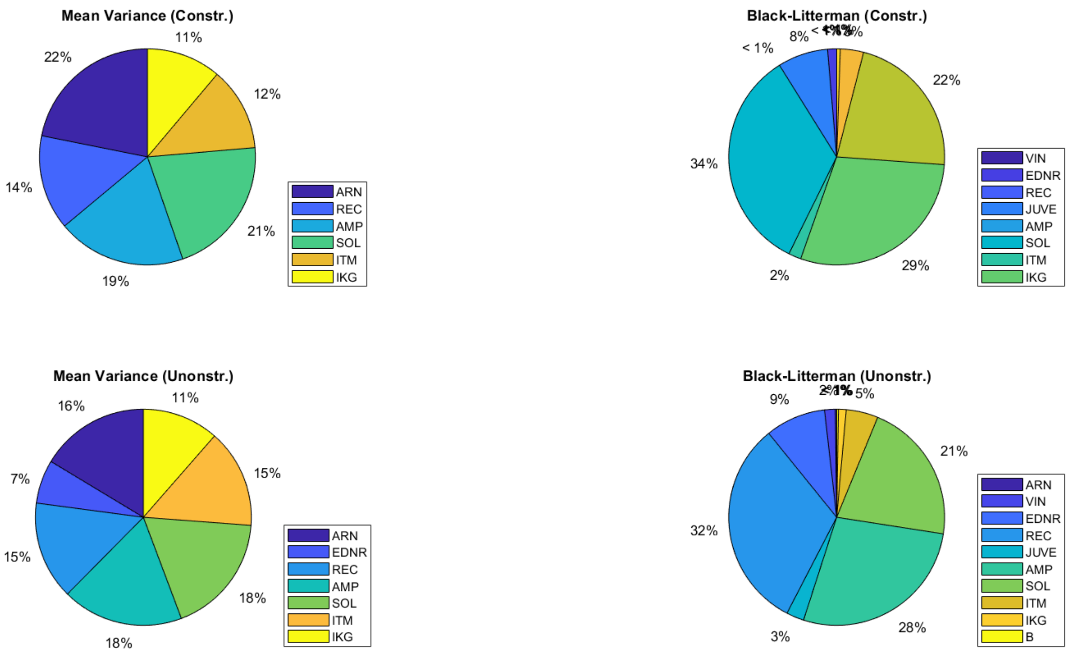



Figure 11: Daily data.

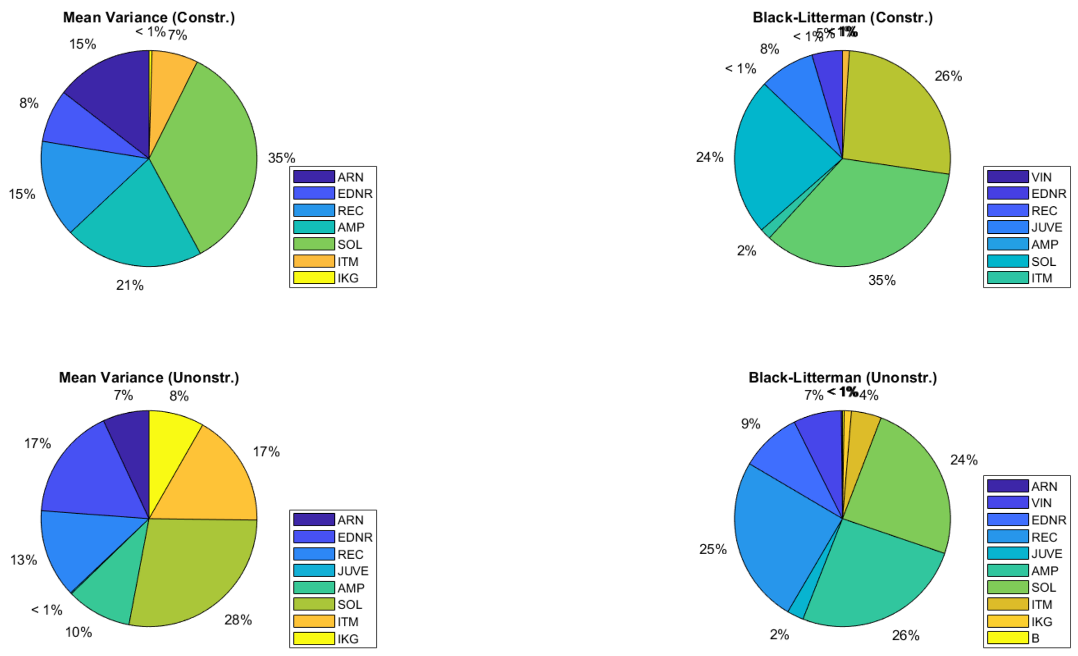



Figure 12: Monthly data.

In Table 12-13 we reported the main statistics of the portfolio based on the Black-Litterman approach and the one built with the mean-variance optimization (MaxSharpe portfolio), both for daily and monthly data and under both assumptions of negative-allowed and positive-restricted weights.
With this new approach, it is obvious to experience a drop in the mean return as

| Statistics | Daily | | Monthly | |
|---|---|---|---|---|
| | **MaxSharpe** | **Black-Litt.** | **MaxSharpe** | **Black-Litt.** |
| **Mean** | 0,00196 | 0,00037 | 0,04631 | 0,00869 |
| **STD** | 0,02444 | 0,00980 | 0,09920 | 0,04412 |
| **Variance** | 0,00060 | 0,00010 | 0,00984 | 0,00195 |
| **Skewness** | -0,14359 | -1,12878 | -0,10701 | -0,19838 |
| **Kurtosis** | 5,88902 | 18,23657 | 2,91764 | 3,13528 |
| **Sharpe Ratio** | 0,08021 | 0,01881 | 0,46682 | 0,01892 |

Table 12: MaxSharpe portfolio vs Black-Litterman portfolio (Unconstr.).

| Statistics | Daily | | Monthly | |
|---|---|---|---|---|
| | **Max Sharpe** | **Black-Litt** | **Max Sharpe** | **Black-Litt** |
| **Mean** | 0,00067 | 0,00035 | 0,01418 | 0,00726 |
| **STD** | 0,01151 | 0,00940 | 0,04495 | 0,03818 |
| **Variance** | 0,01325 | 0,00883 | 0,00202 | 0,00146 |
| **Skewness** | -0,48204 | -1,17606 | -0,06569 | -0,38114 |
| **Kurtosis** | 12,45987 | 19,51424 | 2,93114 | 2,95172 |
| **Sharpe Ratio** | 0,05822 | 0,01863 | 0,31537 | 0,01781 |

Table 13: MaxSharpe portfolio vs Black-Litterman portfolio (Constr.).

we deduce a higher allocation percentage in favour of less favourable investments.

# 8 Standard Bayesian Asset Allocation

From an holistic point of view the main differences existing between the classical and bayesian asset allocation that we analysed in this paper, arise from the choice of the density to be considered in performing the asset allocation. The density condenses whether the investor accounts for parameter uncertainty or focuses on predictability in asset returns. Therefore, true advantage of Bayesian approach lies in the possibility to specify a set of subjective views depicted by prior PDF and the data.
Our goal is to derive the posterior by the means of a specification of a prior PDF, so we follow the steps from the previous section to get our posterior distribution.
Firstly, we assume that vector of continuously compounded asset excess returns $r_t$ is normally distributed, that is:

$$f(r_t|\mu, \Sigma) \sim N(\mu, \Sigma),$$

where $\Sigma$ is the known variance-covariance matrix.
The conditional likelihood distribution is proportional to the multivariate normal density, using the same $\tau$ of the previous section:

$$f(Y|\mu, \Sigma) \sim N(\hat{\mu}, \tau\Sigma). \tag{8.1}$$

As required, our multivariate prior distribution is defined as:

$$f_{pr}(\mu) = N(\mu_0, \Lambda_0) f_{pr}(\mu) = N(\hat{\mu} + \hat{\sigma}, 2\Sigma).$$

The Bayes' rule allows us to write the posterior as:

$$f_{po}(\mu|Y, \Sigma) \sim f(Y|\mu, \Sigma) f_{pr}(\mu), \tag{8.2}$$

then using (8.1) and (8.2) we obtain the following posterior distribution:

$$f_{po}(\mu|Y, \Sigma) \sim N(\mu_{Bayes}, \Sigma_{Bayes}), \tag{8.3}$$

where

$$\begin{aligned} \mu_{Bayes} &= (T\Sigma^{-1} + \Lambda_0^{-1})^{-1}(T\Sigma^{-1}\hat{\mu} + \Lambda_0^{-1}\mu_0) \\ \Sigma_{Bayes} &= (T\Sigma^{-1} + \Lambda_0^{-1})^{-1}. \end{aligned}$$

Figure 12 and Figure 13 show a comparison among the Black-Litterman model, the Mean-Variance model and the Standard Bayesian model, both for daily and monthly data, considering only the constrained portfolio for the BL and MS portfolio.

We may observe that for every portfolio the number of securities remain unchanged comparing the daily and monthly data.
The Bayesian portfolio does not allocate any significant weight to the assets, it results to be more balanced in the asset allocation than the other two portfolios.

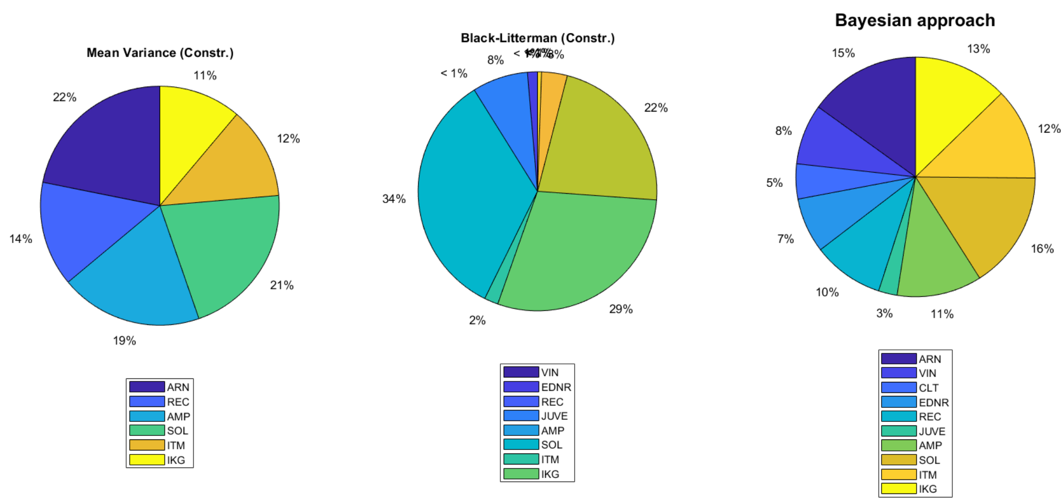



Figure 13: Daily data.

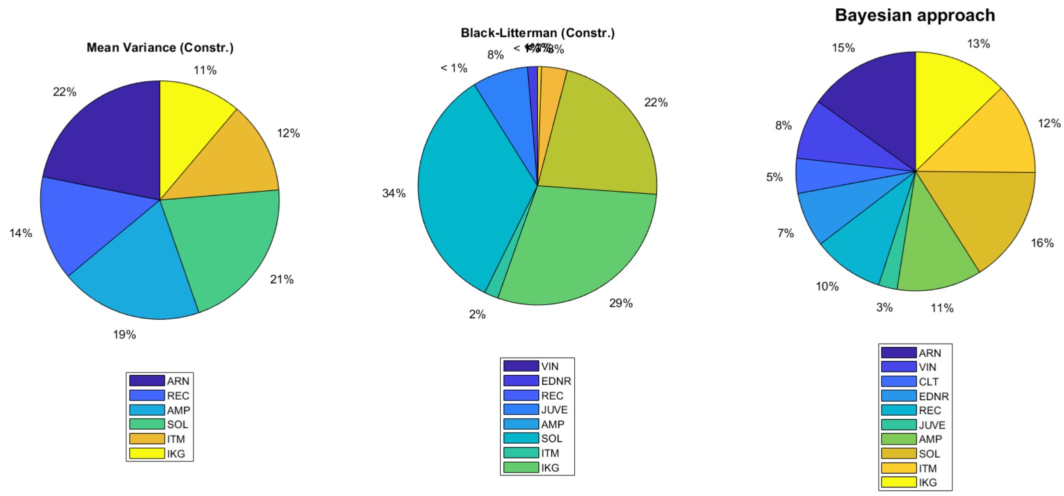



Figure 14: Monthly data.

Table 14 and Table 15 report the statistics of the Bayes portfolio, the Black-Litterman portfolio, the Mean-Variance model and the Standard Bayesian model, both for daily and monthly data, considering only the constrained portfolio for the BL and MS portfolio.
From the tables it arises that the Bayesian portfolio is the least volatile in monthly frequency, while it is a middle ground between the BL and mean-variance portofolio according to the return, both for daily and monthly data. Furthermore, it gets the highest Sharpe Ratio, both in daily and monthly cases, therefore the Bayesian model gives us the best portfolio since our purpose is to maximize the Sharpe ratio.

| | **MaxSharpe** | **Black-Litt.** | **Bayes** |
|---|---|---|---|
| **Mean** | 0,00067 | 0,00035 | 0,00051 |
| **STD** | 0,01151 | 0,00940 | 0,00973 |
| **Variance** | 0,01325 | 0,00883 | 0,00009 |
| **Skewness** | -0,48204 | -1,17606 | -0,69655 |
| **Kurtosis** | 12,45987 | 19,51424 | 13,98251 |
| **Sharpe Ratio** | 0,05822 | 0,01863 | 0,46658 |

Table 14: Daily statistics.

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| | MaxSharpe | Black-Litt. | Bayes |
|---|---|---|---|
| **Mean** | 0,01418 | 0,01287 | 0,01360 |
| **STD** | 0,04495 | 0,04756 | 0,04326 |
| **Variance** | 0,00202 | 0,00226 | 0,00187 |
| **Skewness** | -0,06569 | -0,49864 | -0,06631 |
| **Kurtosis** | 2,93114 | 3,07640 | 2,92914 |
| **Sharpe Ratio** | 0,31537 | 0,02393 | 1,35442 |

Table 15: Monthly statistics.

# 9 Global Minimum Variance Portfolio and Comparisons

Since we had already mentioned the GMV portfolio in section 3.1, along with its statistics, we combined below the statistics the corresponding allocated weights of all portfolios discussed in the previous sections, both for daily and monthly data, without allowing for short selling.

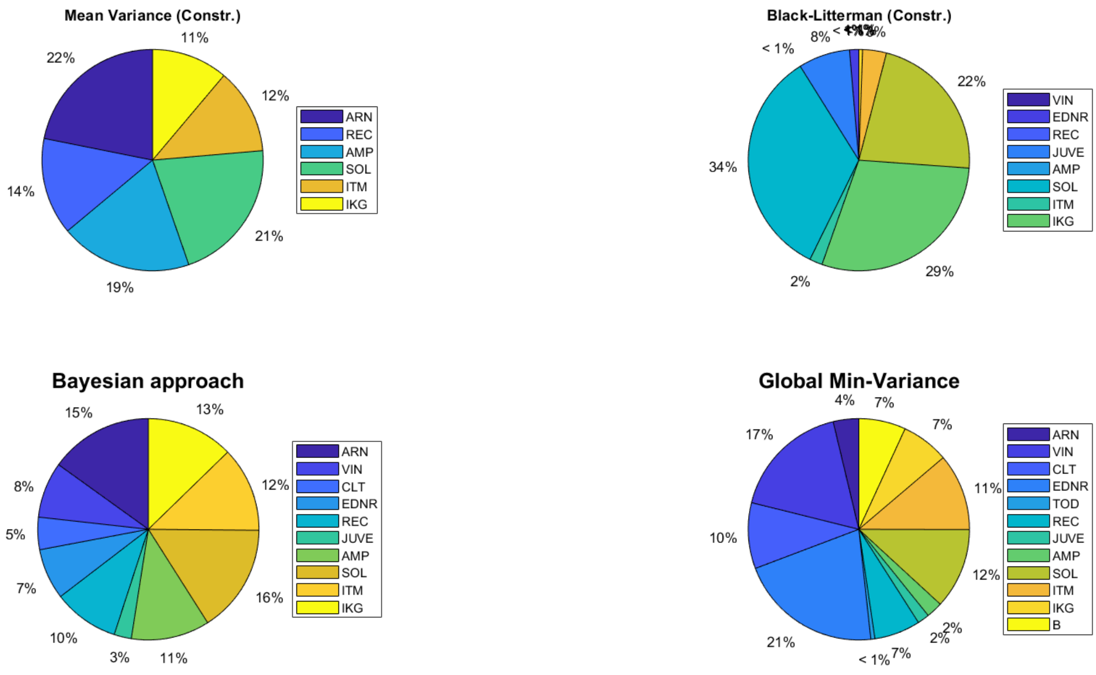


Figure 15: Daily data.

The disparities in asset allocation observed among the portfolios derived from different assumptions can be attributed to various factors. In terms of assumptions, each portfolio strategy relies on distinct suppositions and methodologies. For instance, the Mean Variance Portfolio employs historical return and covariance data and is based on the strong assumption that returns are normally distributed, which is something we have shown is almost every time not true and that asset alloca-

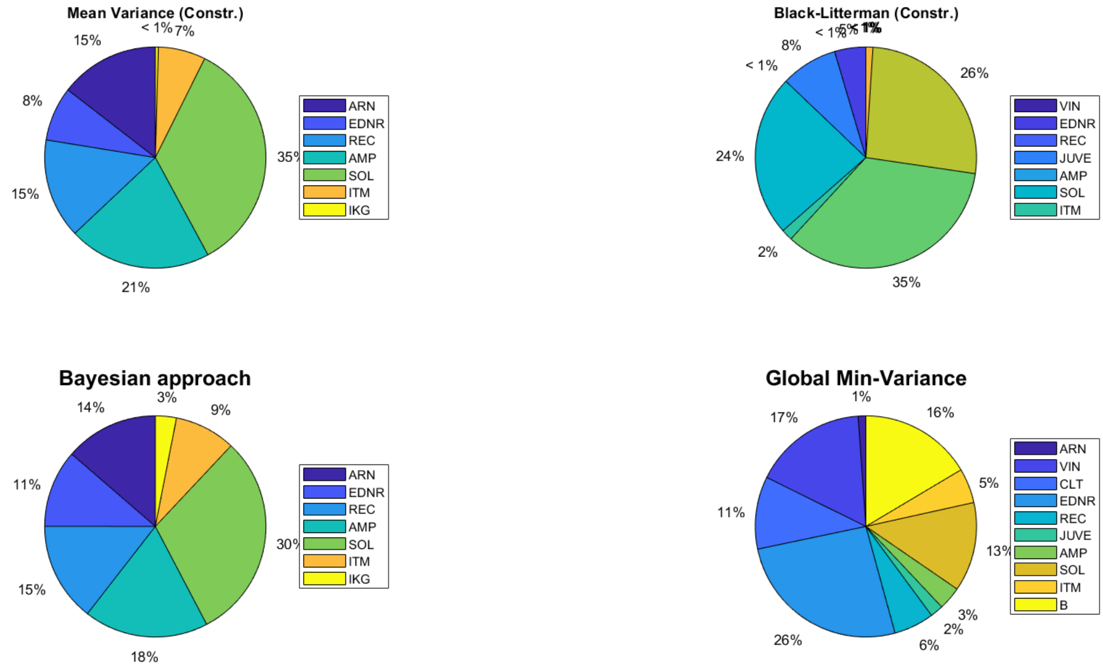


Figure 16: Monthly data.

tions are determined based on a single period of investment. The Black-Litterman approach extends the Mean-Variance Optimization approach and is reliant on the subjective views of investors about expected returns on different assets or asset classes, which are combined with a prior estimate of expected returns based on a benchmark equilibrium distribution. However, the latter should be computed using the implied market capitalization-weight of the asset, but in our case this was replaced by the sample capitalization-weight of the asset. The GMV, computed starting from the application of the already cited models, relies on the fact that the investor choices the portfolio with the minimum variance, regardless the expected return. The bayesian approach is mainly based on the posterior function, which is the respective density function of the estimate that produces a random variable that gets value within a given range of the risk, that allows the blend between the prior density, which represents the investor's experience, and the likelihood.

These divergent assumptions give rise to discrepancies in asset allocation decisions, which can be influenced also by the risk preferences of each strategy. The GMV, for instance, as stated before, places a premium on risk reduction, while others may seek higher returns. Differences can be found also in the inputs and constraints, where the availability and quality of data, investment constraints (such as minimum/maximum allocations), and investor preferences influence asset allocation decisions (example: our absolute negative view on CLT). Differences in these inputs and constraints across portfolio strategies lead to divergent asset allocations.
Another relevant distinction that must be underlined is the significant changes in

the weights allocation led by the Black-Littrman approach.

| | **MaxSharpe** | **Black-Litt.** | **Bayes** | **MGV** |
|---|---|---|---|---|
| **Mean** | 0,00067 | 0,00035 | 0,00051 | 0,00024 |
| **STD** | 0,01151 | 0,00940 | 0,00973 | 0,00852 |
| **Variance** | 0,01325 | 0,00883 | 0,00009 | 0,00007 |
| **Skewness** | -0,48204 | -1,17606 | -0,69655 | -0,88176 |
| **Kurtosis** | 12,45987 | 19,51424 | 13,98251 | 13,85726 |
| **Sharpe Ratio** | 0,05822 | 0,01863 | 0,46658 | 0,05822 |

Table 16: Daily statistics.

| | **MaxSharpe** | **Black-Litt.** | **Bayes** | **MGV** |
|---|---|---|---|---|
| **Mean** | 0,01418 | 0,01287 | 0,01360 | 0,00282 |
| **STD** | 0,04495 | 0,04756 | 0,04326 | 0,03255 |
| **Variance** | 0,00202 | 0,00226 | 0,00187 | 0,00106 |
| **Skewness** | -0,06569 | -0,49864 | -0,06631 | -0,14697 |
| **Kurtosis** | 2,93114 | 3,07640 | 2,92914 | 2,92582 |
| **Sharpe Ratio** | 0,31537 | 0,02393 | 1,35442 | 0,31537 |

Table 17: Monthly statistics.

The statistics in Table 16-17 reveal, as already noticed, that the Bayesian approach seems to be the best choice in terms of return compared to the volatility, both in daily and monthly data.

# 10 Mixed portfolio

A possible way to improve upon all such results is to create a mixed portfolio. Due to the high volatility of both MaxSharpe and Black-Litterman portfolio, we decided to set the weights of the mixed portfolio as it follows: 18 % Black-Litterman, 20% MaxSharpe, 30% GMV in order to reduce the overall volatility, 32% Bayes for what just mentioned looking at Table 16 and Table 17.

| | **Daily** | **Monthly** |
|---|---|---|
| **Mean** | 0,00043 | 0,01035 |
| **STD** | 0,00924 | 0,03847 |
| **Variance** | 0,00009 | 0,00148 |
| **Skewness** | -0,87753 | -0,21666 |
| **Kurtosis** | 16,03343 | 2,75050 |
| **Sharpe Ratio** | 0,04685 | 0,27026 |

Table 18: Mixed portfolio.

The mixed portfolio has performance comparable to the Bayesian portfolio, so it's a compromise between all the portfolios regarding the return. Furthermore, it has the lowest standard deviation among all the portfolios seen previously, a part for GMV. Problems come up with the Sharpe ratio, which is very low compared to the highest value detected among all the portfolios.

# 11 Conclusions

Throughout the paper, various portfolio optimization techniques were employed, and each approach yielded different results depending on the assumptions and parameter calculations involved, as previously discussed in section 9.
To conclude, it is important to note that using only 12 securities out of the 81 available in the sample may lead to limited diversification. A possible way to improve diversification would be to include bonds and enlarge the number of securities in the portfolio allocation. Furthermore, it is worth considering more advanced models beyond the simple Mean-Variance approach. Autoregressive models, for example, offer a different perspective by considering the dynamics and time-varying nature of the market. Exploring these models would provide a richer understanding of portfolio optimization and the evolving nature of financial markets.

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