CHENNAI MATHEMATICAL INSTITUTE

MASTERS THESIS

THE NAME
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Author:

RITAM RAHA

CHENNAI MATHEMATICAL INSTITUTE

Supervised by:

NICOLAS MARKEY & LOÏC HÉLOUËT

INRIA RENNES

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Abstract

Quantitative reachability games are finite two player turn-based games played on weighted graphs. The objective of the game combines reachability objective(qualitative) with the (quantitative) requirement that the weights along a path must satisfy certain constraints (bounds). Beside having direct applications in reactive system synthesis with resource constraints, it is one of the simplest models that combine quantitative and qualitative objectives. In this thesis, we ask the question whether Player 1 can reach a target state along a path for which the accumulated weight for any finite prefix satisfies some bound. We look both at single bound and dual bound problems. We also consider the weighted game where we weaken one of the bounds (e.g. we allow violations of a given bound but up to a bounded number of times).

Dedication

To

Declaration

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Acknowledgements

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Introduction

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Preliminaries

2.1 Definitions

Game graphs. A *game graph* $G = \langle Q, E, w\rangle$ consists of a finite set Q of states partitioned in to player-1 states Q_1 and player-2 states Q_2 (i.e., $Q = Q_1 \cup Q_2$), and a set $E \subseteq Q \times Q$ of edges such that for all $q \in Q$, there exists (at least one) $q' \in Q$ such that $(q, q') \in E$. A player-1 game is a game graph where $Q_1 = Q$ and $Q_2 = \phi$. A special subset $T \subseteq Q$ of vertices is known as the *Target states* or the *Goal states*. Now, $w : E \to \mathbb{R}$ is the weight function such that $w(q_j, q_{j+1})$ is the weight of the edge between the vertices q_j and q_{j+1}

Plays and strategies. A game on G starting from a state $q_0 \in Q$ is played in rounds as follows. If the game is in a player-1 state, then player 1 chooses the successor state from the set of outgoing edges; otherwise the game is in a player-2 state, and player 2 chooses the successor state. We always consider the reachability game, i.e. this ends as soon as any vertex from the set T has been reached. The game results in a play from q_0 , i.e., a finite path $\rho = q_0 q_1 \ldots q_n$ such that $q_n \in T$ & $(q_i, q_{i+1}) \in E$, $\forall i \geq 0$.
A strategy for player 1 is a function $\sigma : Q^* Q_1 \to Q$ such that $(q, \sigma(\rho.q)) \in E$, $\forall q \in Q_1$ and all $\rho \in Q^*$. An outcome of σ from q_0 is a play $q_0 q_1 \ldots$ such that $\sigma(q_0 \ldots q_i) = q_{i+1}$ for all $i \geq 0$ such that $q_i \in Q_1$. Strategy and outcome for player 2 are defined analogously.

Payoff functions. We consider two different payoff functions:

- *Finite Total payoff.* For a finite path $p = q_0 q_1 \ldots q_l$, the total payoff of the path p is defined as, $TP(p) = \sum_{i=0}^{l-1} w(q_i, q_{i+1})$.
- *Finite Mean payoff.* For a finite path $p = q_0 q_1 \ldots q_l$, the mean payoff of the path p is defined as, $MP(p) = (1/l) \cdot \sum_{i=0}^{l-1} w(q_i, q_{i+1})$.

Bounds on weights. We consider two kinds of bounds, strong and week bounds:

- *Strong bounds.* This is the usual notion of bounds used in literature. Weight of a path p is strongly bounded(upper or lower) by B means, for every finite prefix π of the path p, $w(\pi) \gtreqless B$.
- *Weak bounds.* This is a new notion of boundedness. Weight of a path $p = s_1 \to s_2 \to \ldots \to s_n$, starting from s_1 with $c \in \mathbb{R} \cup \{-\infty\}$ as a lower weak bound is defined inductively by $r_1 = max(0, c)$, $r_{i+1} = max(r_i + w(s_i, s_{i+1}), c)$. The notion of weak upper bound is analogously defined.
  So for computing $w \downarrow_c (p)$, costs are accumulated along the transitions of p, but if at some point it goes down below c. it is reset to c i.e. all possible decreases below c are simply discarded.

Finite Memory& Memoryless strategies. A strategy for $P1$, $\sigma : Q^* Q_1 \to Q$ is called a *finite-memory* strategy if every move depends on finite amount of history. The strategy is called a *memoryless* one, if it does not depend on the whole history and only depends on the current state he is in. Hence, a memoryless strategy can be seen as a function $\sigma : Q_1 \to Q$. The definitions are analogous for $P2$.

Objectives. In this thesis, we will focus on *quantitative-reachability* objectives. We consider two kinds of quantitative functions $f : \varrho \to \mathbb{N}$, where ϱ denotes the set of all finite paths in the corresponding game graph G

- *Total Payoff.* Total payoff function of a path $\rho = s_1 \rightarrow s_2 \rightarrow \ldots \rightarrow s_n$ is defined as $TP(\rho) = \sum_{i=1}^{n-1} w(s_i, s_i + 1)$.
- *Mean Payoff.* Mean payoff function of a path $\rho = s_1 \rightarrow s_2 \rightarrow \ldots \rightarrow s_n$ is defined as $MP(\rho) = (1/n)\sum_{i=1}^{n-1} w(s_i, s_i + 1)$.

We will look at the two kind of objectives:

- *Single Bound Objectives.* Given a game graph G, a starting vertex q_0 and a target vertex t, a quantitative function f and a bound $b \in \mathbb{N}$, single bound objective of $P1$ asks that, if starting from q_0, $P1$ can reach t in a path ρ, such that $f(\rho) \lesseqgtr b$.
- *Dual Bound Objectives.* Given a game graph G, a starting vertex q_0 and a target vertex t, a quantitative function f and two bounds U & $L \in \mathbb{N}$, single bound objective of $P1$ asks that, if starting from q_0, $P1$ can reach t in a path ρ, such that $L \leq f(\rho) \leq U$.

  Note that, these bounds are kind of strong on the both upper and lower side. Hence, we are going to call these *Strong Dual Bound Objectives*. Consequently, it is obvious that, we are going to define the notion of *Weak Dual Bound Objectives* as follows:

  Weak Dual Bound Objectives. Let A be a weighted graph. $c \in \mathbb{R} \cup \{-\infty\}$ and let $\gamma = s_1 \rightarrow s_2 \rightarrow \cdots \rightarrow s_n \in Runs(A)$, starting from s_1. The accumulated weight with initial credit 0 under weak lower bound c is $w{\downarrow_c}(\gamma) = r_n$, where $r_1, \ldots, r_n \in \mathbb{R}$ are defined inductively by $r1 = max(0, c), r_{i+1} = max(r_i + w(s_i, s_{i+1}), c)$. So for computing $w \downarrow_c (\gamma)$, costs are accumulated along the transitions of γ, but if at some point it goes down below c. it is reset to c i.e. all possible decreases below c are simply discarded.

2.2 Thesis Description

In this thesis, we will look at the following games:
In **Chapter Three**, we will see *Quantitative-Reachability Games with Single Bound*, in **Chapter Four**, we will see *Quantitative-Reachability Games with Dual Bounds*, both strong and weak objectives. In **Chapter Five**, we will see a new kind of game, *Quantitative-reachability game with bounded number of violations* which we call as *Apna Game*.

Chapter 3

Quantitative Reachability Games with Single Bounds

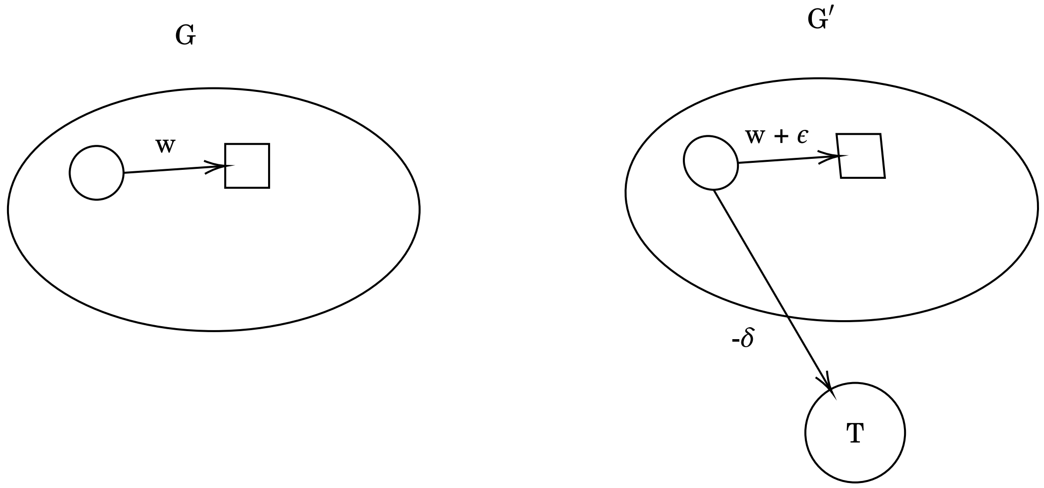
In this chapter, we will see different quantitative games with single bound.

3.1 Energy-Reachability Game

Given a game graph $G = \langle Q_1, Q_2, E, w, q_0, T \rangle$, with all usual notations, the energy-reachability objective is to reach the target states T starting from q_0 in a path ρ, such that for all finite prefixes $\pi(\rho)$, $w(\pi) \geq 0$.
Now, given a game graph, the decision problem for this game is that if Player 1 can win maintaining energy-reachability objective? We will prove the following theorem:

Theorem 3.1.1. The decision problem for Energy-Reachability game is in $NP \cap coNP$. For the lower bound, the problem is mean-payoff hard.

Proof. We will prove that, energy-reachability game is equivalent to infinite-energy game by both side reduction. The theorem will be the consequence of the proof from [BFL+08].
Given a game graph $G = \langle Q_1, Q_2, E, w, q_0 \rangle$, with all usual notations, the decision problem for infinite-energy game asks, if player 1 has a strategy such that with zero initial weight for any infinite play γ from initial state q_0, if $w(\gamma') \geq 0$ for all finite prefixes γ' of γ?
First we reduce the infinite-energy game to energy-reachability game. Consider a game graph $G = \langle Q_1, Q_2, E, w, q_0 \rangle$. We produce another game graph $G' = \langle Q_1, Q_2, E', w', q_0, T \rangle$, where we add a target vertex T. For every player 1 vertex $v \in Q_1$, we add an edge $(v, T) \in E'$ with $w'(v, T) = -\delta$, where δ is larger than the sum of all the positive weights of the graph G. For, all other edges $e \in E$, we have the same corresponding edges $e' \in E'$ with $w'(e') = w(e) + \epsilon$, where $\epsilon << 1/2n$, where n is the total number of vertices. The overview of the construction is shown in Figure 3.1.
If player 1 has a strategy to win infinite-energy in G, then all the cycles player 1 forces in the infinite winning path ρ is either a positive or a zero cycle. The same path in G' will produce positive cycles for ϵ. Hence, after



inf-reach.tex

Figure 3.1: Infinite-energy to Energy-reachability

a large number of iterations, player 1, will have weight $\geq \delta$ in some of its vertex and then it can reach T in G' and win energy-reachability. For the other side, let player 1 has a winning strategy in energy-reachability in G'. That means, in every winning path, at some player 1 vertex it has weight $\geq \delta$, which is impossible without forcing a positive cycle as δ is large enough. Hence, iterating the same cycle player 1 wins infinite-energy game in G.
Now, we will show the opposite reduction i.e. from energy-reachability to infinite-energy. Again consider, a game graph $G = \langle Q_1, Q_2, E, w, q_0, T\rangle$. We produce another game graph $G' = \langle Q_1', Q_2', E', w', q_0\rangle$, where we add a self loop of weight 0 on T and we keep only those vertices and corresponding edges in G' which are in the attractor set of T in G. Clearly, if player 1 wins energy-reachability in G, he wins infinite-energy in G' by taking the zero loop in T after reaching there. The other direction is also true as if player 1 wins infinite energy in G', either he takes the loop in T or he goes into a positive loop inside. If he reaches T, he wins energy-reachability in G trivially. If he can loop inside, as the graph is restricted to attractor set of T, he can force to reach T and win energy-reachability in G. Hence, it is proved that, both the games are equivalent complexity -wise. □

3.2 Finite Mean-Payoff Reachability

In this section, we will take the mean-payoff function as our quantitative function. In general mean-payoff is used in theory mostly for the case of infinite games, but in this thesis we restrict ourselves to reachability, hence finite games. For this reason, we will define a notion of mean payoff for the finite case:

Given a game graph $G = \langle Q_1, Q_2, E, w, q_0, T\rangle$ and a bound $b \in \mathbb{N}$, the decision problem for finite mean-payoff reachability game asks, starting from q_0 with zero initial weight, if player 1 has a strategy to reach T in some path γ such that, for all finite prefixes γ' of γ, if $MP(\gamma') \lesseqgtr b$? We will only consider the case of $MP(\gamma') \leq b$ here. The other case is just the dual. Remember that, for a prefix $\gamma' = q_0, q_1, \ldots q_l$, $MP(\gamma') = (1/l) \cdot \Sigma_{i=0}^{l-1} w(q_i, q_i + 1)$.
Now, we will prove the following result about the finite mean-payoff reachability game:

Theorem 3.2.1. The decision problem for finite mean-payoff reachability game is in $NP \cap coNP$ and also mean-payoff hard.

- *Proof.* We will reduce this game to and from energy-reachability game. As usual, we consider a game graph $G = \langle Q_1, Q_2, E, w, q_0, T\rangle$. We produce another game graph, $G' = \langle Q_1, Q_2, E, w' = b - w(\text{or } w - b, \text{accordingly}), q_0, T\rangle$. We claim that, player 1 wins energy-reachability in G iff he wins finite mean-payoff reachability in G' and also vice-versa. This will prove that, energy-reachability is equivalent to mean-payoff reachability complexity-wise. The result of the theorem will be then the consequence of Theorem 3.1.1
First we prove the reduction from energy reachability to mean-payoff reachability. The opposite reduction will be exactly similar. Let player 1 has a winning strategy λ in G for energy reachability. That means, for all the path ρ conforming λ, reaches T and for all finite prefixes ρ' of ρ, $w(\rho) \geq 0$. Consider such a prefix $q_0, q_1, \ldots q_l$. We have, $\Sigma_{i=0}^{l-1} w(q_i, q_{i+1}) \geq 0$. Now, let in G', also player 1 plays with same strategy λ. Now, for the same prefix, mean-payoff will be:
$$\begin{aligned} MP(\rho') &= (1/l)\Sigma_{i=0}^{l-1} w'(q_i, q_{i+1}) \\ &= (1/l)\Sigma_{i=0}^{l-1}(b - w(q_i, q_{i+1})) \\ &= (1/l)(lb - \Sigma_{i=0}^{l-1} w(q_i, q_{i+1}) \\ &= b - (1/l)\Sigma_{i=0}^{l-1} w(q_i, q_{i+1}) \\ &\leq b \end{aligned}$$

Hence, player 1 wins in finite mean-payoff reachability objective in G'. The other proofs can be obtained similarly.
Hence, we prove that, finite mean-payoff reachability is equivalent to energy-reachability.

□

3.3 Finite Total Payoff Reachability

Now, it comes down to the final quantitative function we are concerned with in this thesis, which is total payoff. Given a game graph $G = \langle Q_1, Q_2, E, w, q_0, T\rangle$ and a bound $b \in \mathbb{N}$, the decision problem for finite total payoff reachability game asks, starting from q_0 with zero initial weight, if player 1 has a strategy to reach T in some path γ such that, for all finite prefixes γ' of γ, if $TP(\gamma') \lesseqgtr b$? We will only consider the case of $TP(\gamma') \geq b$ here.

The other case is just the dual. Remember that, for a prefix $\gamma' = q_0, q_1, \ldots q_l$, $TP(\gamma') = \Sigma_{i=0}^{l-1} w(q_i, q_i + 1)$. Note that, energy-reachability is just a special case of total payoff reachability, where $b = 0$. We will show that, it is equivalent to energy-reachability game. From that, the following theorem is evident:

Theorem 3.3.1. The decision problem for finite total payoff reachability game is in $NP \cap coNP$ and also mean-payoff hard.

Proof. Incomplete $\square$

3.4 Conclusion

This brings us to the end of the Quantitative Reachability Games with Single Bounds. We showed that, all the reachability games with single bounds (for the quantitative functions we are concerned with in this thesis) are equivalent to energy-reachability game which is complexity-wise equivalent to infinite energy games. So, we have $NP \cap coNP$ complexity for all of them. For the lower bound also, all of them are at least mean-payoff hard. Next we move to the chapters for the similar kind of games with two bounds.

Chapter 4

Quantitative Reachability Games with Strong Dual Bounds

In this chapter, we will consider the quantitative games, where player 1 has to reach his goal, always maintaining his weights inside two bounds. This chapter we will consider both the bounds to be strong.

4.1 Finite Total Payoff Reachability

In this section, we will use total payoff as our quantitative function. Given a game graph $G = \langle Q_1, Q_2, E, w, q_0, T \rangle$, an upper bound $U \in \mathbb{N}$ and a lower bound $b \in \mathbb{N}$, finite total payoff reachability objective with dual bound says that, with zero initial energy starting from q_0, player 1 has to reach T in a path ρ such that, for all finite prefixes ρ' of ρ, $l \leq TP(\rho') \leq U$, where TP is simply the sum of the weight of the edges as defined earlier. We call this **LU -Reachability Game**. We will consider, both one player and two player version of the game.

4.1.1 One Player LU-Reachability Game

We will consider the case where $Q_2 = \phi$. Hence, all the vertices are player 1 vertices. We will prove the following theorem:

Theorem 4.1.1. One player LU-reachability game is PSPACE-complete.

Proof. We will first show that the one player LU-reachability game is in PSPACE. $\boxed{\text{Prove that it is in PSPACE}}$.
Now, we will prove the hardness. We will prove it by reduction from the reachability of bounded one counter automaton, which is proven to be PSPACE-complete in [FJ13].
A bounded one-counter automaton has a single counter that can store values between 0 and some bound $b \in \mathbb{N}$. The automaton may add or subtract values from the counter as long as the bounds of 0 and b are not overstepped.
For two integers $a, b \in \mathbb{Z}$ we define $[a, b] = \{n \in \mathbb{Z} : a \leq n \leq b\}$ to be the subset of integers between a and b. A bounded one-counter automaton is defined by a tuple (L, b, Δ, l_0), where L is a finite set of locations, $b \in \mathbb{N}$ is a global counter bound, specifies the set of transitions, and $l_0 \in L$ is the initial location. Each transition in Δ has the form (l, p, g_1, g_2, l'), where l and l' are locations, $p \in [b, b]$ specifies how the counter should be modified, and $g_1, g_2 \in [0, b]$ give lower and upper guards for the transition. All numbers used in the specification of a bounded one-counter automaton are encoded in binary.
Each state of the automaton consists of a location $l \in L$ along with a counter value c. Thus, we define the set of states S to be $L \times [0, b]$. A transition exists between a state $(l, c) \in S$, and a state $(l', c') \in S$ if there is a transition $(l, p, g_1, g_2, l') \in \Delta$, where $g_1 \leq c \leq g_2$, and $c' = c + p$.
The reachability problem for bounded one-counter automata is specified as follows. An input to the problem is a pair (β, t), where β is a bounded one-counter automaton, and t is a target location. To solve the problem, we must decide whether there is a sequence of transitions between state $(l_0, 0)$ and the state $(t, 0)$.

Now, we will give reduce this problem to our game. Given the instance of a bounded one counter automaton and a target location (β, t), we construct the following graph G.
The states of the graph are exactly the locations of the counter automaton. For every transition in $(l, p, g_1, g_2, l') \in \Delta$, we create following transitions in our graph:

$$l \xrightarrow{-g_1} Sp_1^l \xrightarrow{g_1+b-g_2} Sp_2^l \xrightarrow{g_2-b+p} l'$$

We add a new target t' for G and add the edge $t \xrightarrow{b} t'$. Now, in this new graph G, with upper bound b and lower bound 0 we ask, if there exists a path from q_0 to t' such that the bounds are respected.
Notice that, if in a location l, the counter value c does not follow the constraint $g_1 \leq c \leq g_2$, then here we cannot reach from l to l' as that will violate the bound constraints in Sp_1^l or Sp_2^l vertices.
Now, player 1 wins the game in graph G iff it can reach t' maintaining the bound constraint which is possible if it reaches t with weight 0 i.e. it reaches $(t,0)$ configuration in the one counter automata. This completes the reduction and hence the hardness result is proved. □

4.1.2 Two player LU-Reachability Game

Now we will move to the case of two player LU-reachability Game. Here $Q_2 \neq \phi$ anymore. We will prove the following theorem:

Theorem 4.1.2. Two players LU-reachability game is EXPTIME-complete

Proof. We will first give a very obvious EXPTIME algorithm to solve two player LU game. The algorithm is simply to blow up the state space. In details, given the game graph $G = \langle Q_1, Q_2, E, w, q_0, T\rangle$ and two bounds l and U, we will create a new game graph, $G' = \langle {Q_1}', {Q_2}', E', {q_0}', T'\rangle$, where for every state $q \in Q_i$, $i \in \{1,2\}$ and for all $j \in [l,U]$, $(q,j) \in {Q_i}'$. Now, for all $(u,v) \in E$ with $w(u,v) = w$, we have $\langle (u,j),(v,j+w)\rangle \in E'$ for all $j \in [l,U]$ iff $j+w \in [l,U]$. Otherwise, it goes in to a dead state. This construction intuitively means, we encode the energy levels in the state space instead of edges. Now, the game is just reduced to check, if for some j, (T,j) is reachable from $(q_0,0)$ in G'. We know that, reachability can be solved in linear time. Hence, our game can be solved using linear time reachability algorithm on this new exponential size graph. Hence, we have an EXPTIME algorithm for two player LU-reachability game.
Now, we come to the hardness part. We will give a reduction from *countdown game* to our game. Countdown game has been proved to be EXPTIME-complete in [JLS07].
A countdown game C consists of a weighted graph (S,T), where S is the set of states and $T \subseteq S \times \mathbb{N} \setminus \{0\} \times S$ is the transition relation. If $t = (s,d,s') \in T$, then we say that the duration of the transition t is d. A configuration of a countdown game is a pair (s,c), where $s \in S$ is a state and $c \in \mathbb{N}$. A move of a countdown game from a configuration (s,c) is performed in the following way: first player 1 chooses a number d, such that $0 < d \leq c$ and $(s,d,s') \in T$, for some state $s' \in S$; then player 2 chooses a transition $(s,d,s') \in T$ of duration d. The resulting new configuration is (s',cd). There are two types of terminal configurations, i.e., configurations (s,c) in which no moves are available. If $c = 0$ then the configuration (s,c) is terminal and is a winning configuration for player 1. If for all transitions $(s,d,s') \in T$ from the state s, we have that $d > c$, then the configuration (s,c) is terminal and it is a winning configuration for player 2. The algorithmic problem of deciding the winner in countdown games is, given a weighted graph (S,T) and a configuration (s,c), where all the durations of transitions in C and the number c are given in binary, to determine whether player 1 has a winning strategy from the configuration (s,c).
Given a countdown game (S,E) with initial configuration (s_0,c_0), we construct a weighted game as follows: let $S_1 = S$, $S_2 = \{(s,d) | (s,d,s') \in E\}$, T a target state and

$$T = \{s \xrightarrow{d} (s,d) | (s,d) \in S_2\} \cup \{(s,d) \xrightarrow{0} s' | (s,d,s') \in E\} \cup \{s \xrightarrow{-c_0} T | s \in S\}$$

The upper bound is set to c_0 and the lower bound is 0 . Player 1 can now from a state $s \in S$ choose a particular number d and Player 2 from the temporary state (s,d) choose a transition to a state $s' \in S$. The number d is added to the accumulated weight and the same repeats. As the accumulated weight is bounded by c_0 , Player 1 has to eventually take some transition labeled c_0 and reach the target state T . In order not to drop below zero, this is only possible if the accumulated weight is exactly c_0 , hence the first player in the countdown game has a winning strategy if and only if Player 1 has a winning strategy in the two player LU-reachability game. □

4.2 Conclusion

In this chapter we have seen that for LU-reachability game, one player version is PSPACE-complete and two player version is EXPTIME-complete. Now, in the next chapter we will move to the cases, when one of the bound is weak and the other bound remains strict.

Chapter 5

Quantitative Reachability Games with Weak Dual Bounds

In this chapter, we will explore the dual bound quantitative reachability games where one bound is weak. Recall the notion of weak bound: a bound(w.l.o.g say, lower bound) l is weak means, if the weight hits l, it never goes below l, it stays at l until it goes higher. Also recall that, the weight of a path γ with weak lower bound l is denoted as $w{\downarrow_l}(\gamma)$. Here we will consider the case where the upper bound is strong and the lower bound is weak. Note that, the other case can be obtained just by reversing the sign of the weights. Also, we will take the lower weak bound as 0. Note that, the game with the lower weak bound as c can be simulated with the game with lower weak bound as 0 converting all weights w to $w-c$. Now, like the strong bound case, we will consider both the one player version and two player version of this game:

5.1 One Player QR Games with Weak Dual Bounds

We consider the one player version of this game, where $Q_2 = \phi$. We will prove the following theorem:

Theorem 5.1.1. Given a game graph G , a strong upper bound U and a lower weak bound 0, deciding if P_1 can win the one player QR games with weak dual bound game in G is is P.

Proof. Before proving it formally, let us look at some intuition: consider a winning strategy σ of P_1. Intuitively, any outcome of σ will not have any positive cycle, as P_1 can just ignore the cycle and still win. Hence, it will be either an acyclic path maintaining the objective, or he has to choose a negative cycle where he can rotate enough number of times maintaining the objective, lowers the energy to a certain stable value and then continues forward along the path.

Now, we examine a negative cycle in a graph from a vertex v. Starting, from initial energy 0 from v, we will reach v_1 in the cycle, where the energy is the highest, let's say a. Then, the energy level decreases along the cycle and reaches v_2, where the energy level is minimum in the cycle,let's say x. Then, it goes back to v, with energy, say y. Let $b = y - x$. Now, if player 1 can reach vertex v, with at most $U - a$ energy, he will be able to rotate through this negative cycle many times and as the lower weak bound is 0, after sufficient number of rotation, he can reach v_2 with energy level 0 and reach v with b amount of energy. Hence, we associate the pair (a, b) with the vertex v i.e. if P_1 can manage to reach v with at most $U - a$ energy, he can lower his energy level up to b in v. The phenomena has been depicted with an example in Figure 5.1.

In the example, $U = 15$. Hence, $a = 5$ and $b = 1$, i.e. if player 1 can reach $s1$ with at most 10 energy level, he can rotate the cycle as much as he wants, and can get the energy level down to energy 1.

Consider a winning path ρ in G. Suppose a negative cycle C exists in that path. That means, the path visits the vertex v_1, the point with the highest energy level in C at least once. As, the path is winning, it visits v_1 with energy level $\leq U$. Hence, if we can compute (U, b) for that vertex v_1, we can say that, after visiting the cycle C, P_1 can rotate around the cycle and get the energy level down to b at v_1. Using the idea, we can do the following:

- For every vertex v, we will check, if it is the highest point of any negative cycle i.e. will check if pair (U, b) exists for some $b \in \mathbb{N} \cup \{0\}$. If yes, then we will calculate the optimal b.
- We will fill the table $H_{vertex \times steps}$ such that, $H(v, i) =$ The minimum energy level we can have at v after at least i - steps starting from q_0.

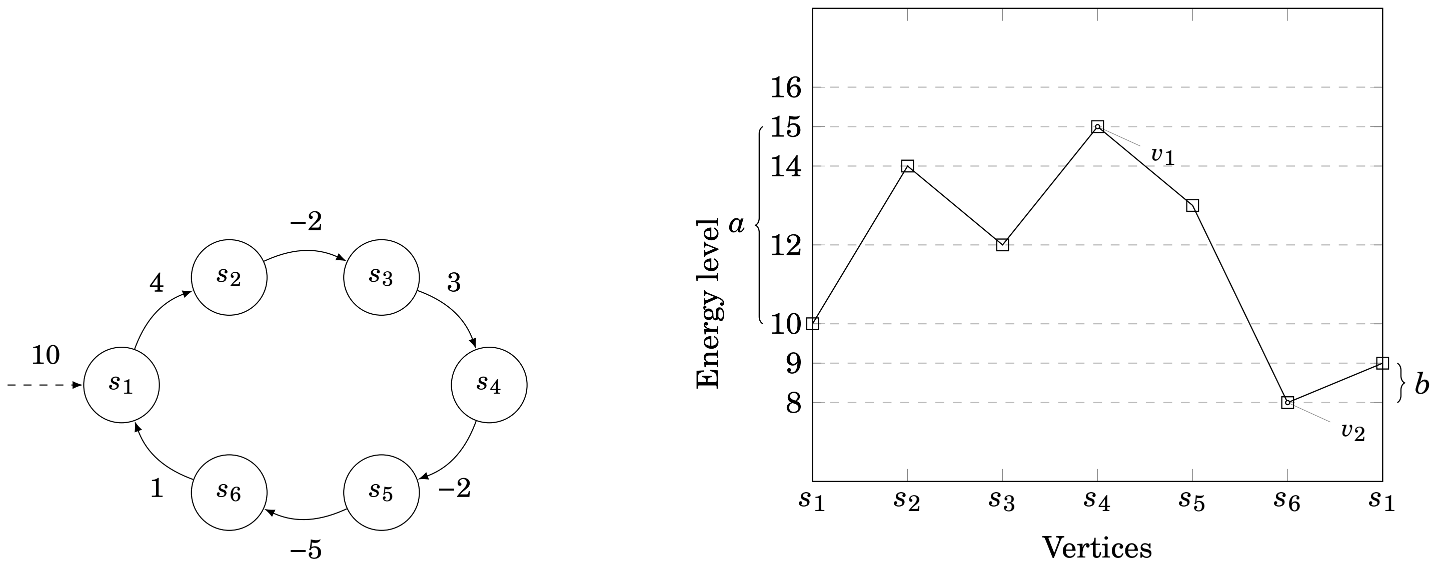


Figure 5.1: Energy level of a negative cycle

Notice that, if (U,b) pair exists for a vertex v, there must exist a negative edge out of it. From the above observation about the energy levels of a negative cycle, it is easy to see that, if (U,b) exists for v, from v, along the negative cycle, there will be a vertex u, where the energy level will be minimum, say x. Similarly, there must be a positive weight edge out of t. Then, starting from t, there will be a path back to v. Hence, for every vertex v, which has at least one negative edge out of it, we will start with energy level U, and for every other vertex t with at least one positive weight edge out of it, we will try to find, if there exists a path of at most length $|V|$, such that it reaches t maintaining the strict upper and the weak lower bound, with energy level x, and along the path the energy level never goes below x. If we can find such vertex t, with energy level x, similarly we will try to find, starting from t with x energy, do we have a path of length at most $|V|$ back to v, maintaining all the bounds and also such that x is the minimum energy level along the path. Let, we reach v with energy y. Then, $b = y - x$. If, we find several such b, we will take the minimum one as the optimal. Both the path-checking can be done by simple DFS maintaining the energy level with the vertices. We stop at each path in the DFS when the bound is violated or the length of the path becomes $|V|$. We can also, maintain a counter to remember the minimum energy level seen along the path. Hence, we can compute (U,b) pair for vertices, if exists, in polynomial time.

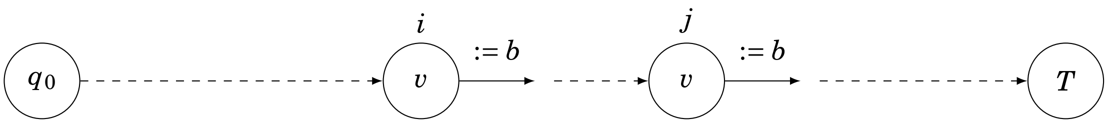
Once, we have computed the (U,b) pairs for vertices, we can add the following special edges in the graph: If (U,b) exists for v, we add a special loop $v \xrightarrow{:=\text{b}} v$ in the game graph. The idea is, if player 1 can reach v with at most U energy, he can take some negative cycle in the original graph and get the energy level to b after sufficient many steps. In this new graph, player 1 can take this special loop and in one step he can lower the energy level of v, down to b at one step. Clearly, player 1 can win in the original graph iff he can win in this new graph with the special transitions.

Now, in this special graph, we will calculate the table $H_{vertex \times steps}$. For any vertex v, let $e_i = min\{H(u,i-1) + w(u,v) | (u,v) \in E \ \& \ H(u,i-1) \leq U\}$. Then,

$$H(v,i) = \begin{cases} min(e_i, b), & \text{if } v \xrightarrow{:=\text{b}} v \text{ exists and } H(v,i-1) \leq U \\ e_i, & \text{otherwise} \end{cases}$$

Now, if for some i, $H(T,i) \leq U$, player 1 wins, otherwise, he loses.

Consider a winning path ρ for player 1. We have already argued that, there is no positive cycle in this path. Now, the claim is, every special loop can appear at most once in ρ. This can be easily seen as the following: Let, same special transition appears twice in the winning path ρ at i^{th} and j^{th} positions.



Then, the configuration of the path ρ in both the positions are same. Hence, player 1 can just forget the middle

part between the positions, and continue similarly from the i^{th} position, as progressed from the j^{th} position. As, the original path is winning, the new truncated path will also be winning for player 1.
Now, to prove the correctness of the algorithm proposed earlier, it is enough to prove the following lemma:

Lemma 5.1.2. If there exists a winning path for player 1 in G, there exists a winning path of length at most $|Q|^2$ for player 1.

Proof of Lemma 5.1.2 Let the optimal shortest winning path for player 1 has length $> |V|^2$. By pigeon hole principle, $\exists u \in Q$, which appears in the path $|Q|+1$ times. Therefore the path has admitted at least $|Q|+1$ cycles through u. As, there can not be any positive cycle in the optimal winning path, all of them are negative cycles. For every negative cycles, the will admit a special transition of the form $v \xrightarrow{:=b} v$ as shown earlier. Hence, there exists at least $|Q|+1$ special transitions in the path. But the number of unique special transitions are bounded by $|Q|$ and we know, each special transition can appear at most once in the optimal path. Then the contradiction arises

From Lemma 5.1.2, it is evident that, calculating $H_{vertex \times steps}$ up to at most $|Q|^2$ many steps are enough. This completes the proof for our theorem. □

Now, we will move to the case for the two player version.

5.2 Two Player QR Games with Weak Dual Bounds

Now, $Q_2 \neq \phi$ anymore. We will first prove the following lemma about the memory requirements for both the players in the game.

Lemma 5.2.1. For two player QR games with weak dual bounds, exponential memory may be necessary for player 1. For player 2, memoryless strategies are sufficient.

Proof. As reachability is one of the objective, trivially finite memories are sufficient for both the players. Now, we will show, a class of game graphs, where exponential memory may be necessary for player 1.

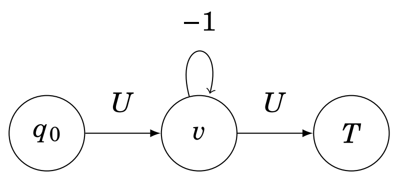


Figure 5.2: Exponential memory is necessary for player 1

In the game graph of Figure 5.2, all the vertices are player 1 vertices and U is the strict upper bound. It is easy to see, player 1 needs at least U- memory to win this game. Considering binary encoding. the value of U is exponential in the size of the input.
Now, we will prove that, memoryless strategies are enough for player 2. Let's fix a winning strategy $lambda$ for player 2 WLOG, every player 2 vertices have two outgoing ages. Fix one player 2 vertex, say q_2. Let the weight of the two outgoing edges from q_2 are e_l and e_r to q_l and q_r respectively. Now, if any path reaches q_l with at most w_l energy, player 1 can win from q_l and similarly if any path reaches q_r with at most w_r energy, player 1 can win from q_r. Clearly, if any path reaches q_2 with at most $min(w_l - e_l, w_r - e_r) = w'$,say, energy, no matter what player 2 chooses from q_2, player 1 wins. As, λ is winning for player 2, no outcome of λ reaches q_2 with $\leq w'$ energy. W.L.O.G., let $w' = w_l - e_l$. It is easy to see that, we can construct a memoryless strategy λ' for player 2, such that $\lambda'(q_2) = q_l$. Hence, memoryless strategy is enough for player 2. □

To be added the short certificate for the NP proof

5.3 Conclusion

In this chapter, we have showed that QR games with weak dual bounds are in P for the single player case and is in $coNP$ for the two player case. We also believe that the two player version of the game is in NP. We have shown the short certificate for the problem, but unfortunately could not prove the verification part.

This ends all the QR game versions with bounds on weight functions. In the next chapter, we will introduce a new QR game, with a notion of bound on number of violations.

Chapter 6

Apna Game

Chapter 7

Conclusion

Chapter 8

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Appendix A

Appendix Title