$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ 非定常热传导方程

$$\begin{cases} u(x,0) = f(x) \\ u(0,t) = a(t) = 0 \\ u(1,t) = b(t) = 0 \end{cases} \qquad \begin{cases} a(t) = 0 \\ b(t) = 0 \end{cases}$$

$$f(x) = \begin{cases} 0 & 0 \le x < 0.3 \\ 1 & 0.3 \le x \le 0.7 \\ -\frac{10}{3}x + \frac{10}{3} & 0.7 \le x \le 1 \end{cases}$$ 分段

分离变量法解题的思路：

将PDE变为多个ODE

$u(x,t) = V(x) \cdot W(t)$

先求出微分方程的特解.

由线性无关的特解叠加出通解

再用定解条件 去确定叠加系数

Step1: 分离变量

Step2: 求解特征值

Step3: 求特解并叠加出一般解

Step4: 利用本征函数的正交性叠加系数

$\lambda \le 0$ 时不符合边界条件.

不讨论

$u(x,t) = V(x) W(t)$

~~u(t) = k~~

$u(x,t)_t = k\, u(x,t)_{xx}$

$V(x) W'(t) = k \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$

$= k \frac{\partial}{\partial x}\left[W(t) \cdot V'(x)\right]$

$= k\, W(t) \frac{\partial V'(x)}{\partial x}$

$= k\, W(t)\, V''(x)$

$VW' = kWV''$ 引入待定常数

$\frac{W'}{kW} = \frac{V''}{V} = -\lambda^2$ 令 $\lambda > 0$

$$★\begin{cases} W' + \lambda^2 k W = 0 & ① \\ V'' + \lambda^2 V = 0 & ② \end{cases}$$

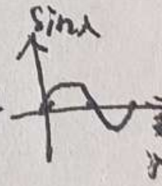
$$\begin{cases} V(x) W(0) = f(x) \\ V(0) W(t) = a(t) = 0 \\ V(1) W(t) = b(t) = 0 \end{cases}$$

齐次边界条件. $\begin{cases} V(0) W(t) = 0 \\ V(1) W(t) = 0 \end{cases}$

$X(0) = X(1) = 0$

得到②式通解：

$V(x) = A\sin(\lambda x) + B\cos(\lambda x)$



引入边界条件 得到 $\sin(\lambda) = 0$

$\lambda = n\pi \quad n = 1, 2, 3, \dots n.$

①

~~置~~ ~~[illegible]~~

$$\cancel{X_n(x)}\ \ u_n(x) = A_n \sin(n\pi x) \quad n=1,2,3\cdots n$$

对②式：设通解为 $W(t) = C_n \exp[-(n\pi)^2 k t]$

得到 $u_n(x,t) = a_n \sin(n\pi x) \exp[-n^2\pi^2 k t] \quad n=1,2,3\cdots n.$

由叠加原理：

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) \exp(-n^2\pi^2 k t)$$

代入初值条件 $u(x,0) = f(x)$ 得到

$$\sum_{n=1}^{\infty} a_n \sin(n\pi x) = f(x)$$

傅立叶级数定义：

$$a_n = 2\int_0^1 [f(x)]\underbrace{\sin(n\pi x)}_{\text{分段 2种}} dx \quad \text{积分}$$

原问题的解：

$$u(x,t) = \cancel{?}\sum_{n=1}^{\infty} a_n \sin(n\pi x) \exp(-n^2\pi^2 k t)$$

数值法：均匀网格数 $M = 16.32.64.128.256$, $\alpha = 0.1$ $t = 0.01$
(有限差分)

②

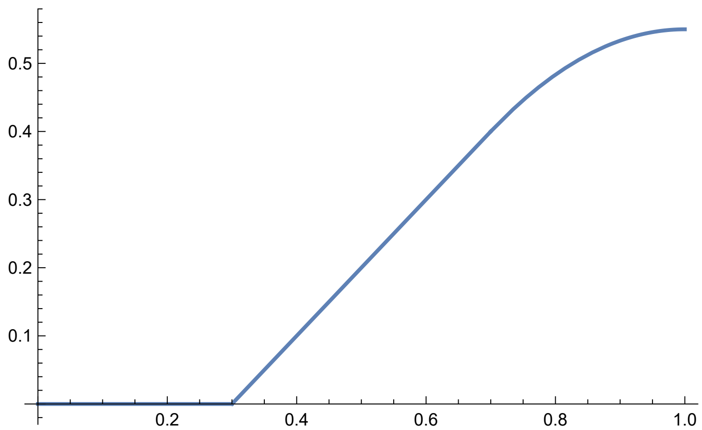
分离变量法求解
非稳态导热微分方程 - 精确解

```
f[x_] := Piecewise[{{0, 0 < x < 0.3}, {1, 0.3 ≤ x ≤ 0.7}, {-10/3 x + 10/3, 0.7 ≤ x ≤ 1}}];
```
分段函数

```
g = Integrate[f[x], x]
```
积分

In[∘]:=
```
Plot[g, {x, 0, 1}]
```
绘图

Out[∘]=

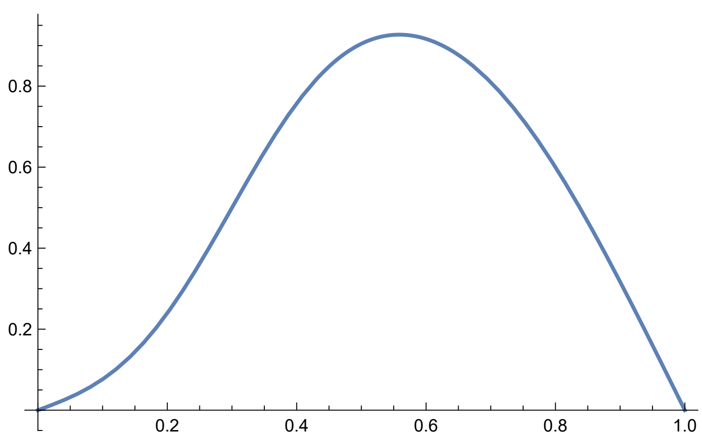


$$u = \sum_{n=1}^{10} \left(\left(2 \int_0^1 f[x] \times \mathrm{Sin}[n\,\mathrm{Pi}\,x] \, dx \right) \mathrm{Sin}[n\,\mathrm{Pi}\,x] \, e^{-n^2 \mathrm{Pi}^2 t} \right);$$

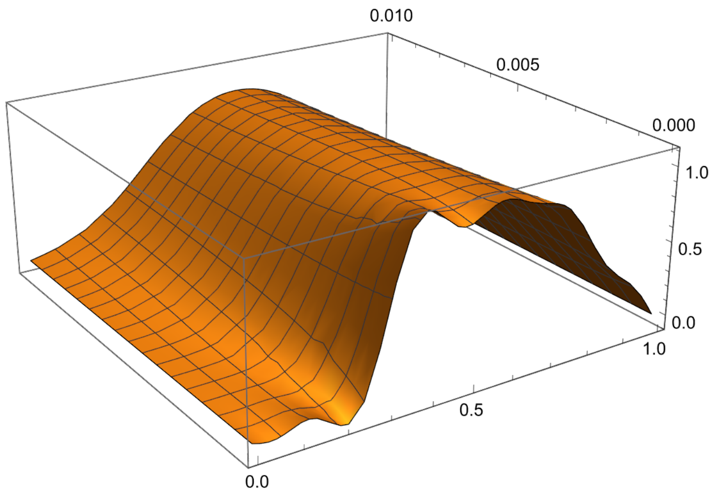
正弦 圆周率 正弦 圆周率

In[∘]:=
```
t = 0.01;
Plot[u, {x, 0, 1}]
```
绘图

Out[∘]=



In[ ]:=
```
Plot3D[u, {x, 0, 1}, {t, 0, 0.01}]
```
绘制三维图形



In[ ]:=