ä½ å¥¹ï¼Œæˆ˜æˆ˜æœ±ç»´ã˜ã€‚æ¬ç¿Žä½ ç»§ç»è·Ÿæˆ˜å¦ä¹ ç°¿æ§ä»£æ•°ï¼Œä»Šå¤©æˆ`è(è²çš,å†…å®ãˆ—â€œæ•°å€¼ç°¿æ§ä»£æ•°çš,è¿ä»£æ³•ï¼Œä»¥åŠšå¦¡,ä½•åœˆå®žè·µä¸è¿¨"è¿ä»£æ³•æ±,è§£ç°¿æ§æ—¹ç°¿»,â€ã€,

å¨§å¯†å°'ç°¿æ§æ—¹ç°¿»,çš,è®®¡ç®—å·²æˆ˜ã˜ä¸ºã¯ä¹ ç•Œä Šæœ€å¿«è®®¡ç®—æœºçš,æµ¶¯•æ ‡å‡†ã€,2008å¹ï¼ŒIBMä¸ºç¼Žå›½èƒ½æºéƒ"Los Alamoså›½å¶å®žéªŒå®®å»ºéᴱâ€œRoadrunnerâ€®¡ç®—æœºç³»»Ÿï¼Œï®ƒçš,è¿ç®—é€Ÿå°è³¾á°ä°†1.026 petaflop/sï¼âƒ ä¸æ¯æ—¡ç§ ï¼Œpetaflopæ¯ è¡¡é‡è®®¡ç®—æœºæ¶è¡¿çš,ä¸€ä¸é‡è§å•ä½ï¼Œ1 petaflopç‰ãŽe ç§ é Ÿè¿è¡Œ1åƒä¸‡ä°¿æ—¡çš,æ•°å¦è¿ç®—ï¼‰ã€,æŒ%æˆ°å°"㮚ä³«è®¡ç®—ï¼ŒçŽ°åœˆä ¸ç•Œä Šæœ€å¿«çš,è®®¡ç®—æœºå°²è³¾á°ä°†200 petaflopï¼Œæˆ`å»½ä'Ÿæ—©å°±è¿å…¥ä°†ä ¸ç•Œå%å—ï¼Œ¶æœ%æœœâ®žçŽ°1 exaflop/sï¼ç™¾ä°¿ä°¿æ—¡ç§ï¼‰ï¼Œæˆä¸ä¸ç•Œç—ä€ã€,

â¯è ƒ½ï½ ä½šæœ‰%ä°¿ç¯æ ƒ"ï¼Œ°ä¸€ä¬æ¯è ¯%çˆåŽæœŸæ¥ƒ®²æ•°å€¹ç°¿æ§ä»£æ•°å°¢ï¼Ÿ

é,£æ¯â¬ ä¸æˆ°å€¹ç°¿æ§ä»£æ•°æ˜ä¬é€—°ç‰ï°'æ®Šçš,å¦ç§ï¼Œæ¯ç‰ä°«ä¸°è®¡ç®—æœºä Šè¿¡è¨Œç°¿æ§ä»£æ•°è®¡ç®—æœæŠ§çš,ï¼Œä¸¥è €å®ƒ¯ ¯'ç©¯ç¯©é¬é¿ç®—ç®—æ³•çš,å¦ç§ï¼Œåå'ç®—æ•€è³è·µä¸Žä¥ç¬è®¾ï½è®¡®žæ‰¶¯†ä'å‰‰Ÿ°ç¡é¿çYŸè ¯†çš,é°ãŽáŽï¼Œä'åˆ±ä'æ•°å€¹ç°¿æ§ä»£æ•°ä¼šæ¯æœ%æœ¯ï¼Œè®¡ä'â¯ä»¥ç'æŽ¥¿ç'ˆæˆ°è®¡®—æœºç§ï°å¡ä¸çš,ï¼Œæ¯ ¯äˆï¼šåœ²á°¾ƒåfåŽç¼½©ä¸ï¼Œï¿½ç¯¨ä¥‡ä¹,å€¹á°†è§£ï¼"SVDï¼%æ¥åŠ,çœå†…å¯ï½åœ¨æ¬±å°¡å¦ä' ä¸ï¼Œï¿½¨å…±ï¼æ¢¯ä°¶æ¥åŠ é€Ÿç¥žç¼½°ç»œçš,æ"¶æ•»ã€,

## è¿ä»£æ—¹æ³•è´˜æ˜Ž

è ¾¿°å†…å®ï¼çš,å%œœŸä¸ç¿´éf½åœˆç"ᴾ¡æŽ¥¥á¥¤¥€§£ç°¿æ§æ—¡ç»»,ï¼Œä¸¬é‹æ—¬¶…f润³•ã€,ä½±†åœˆá®žè·µï¼Œæˆ°ä»»â¬æ¬é¬ é‹æ¬æ™—æ—¶¶ï¼Œæ¶âˆºãˆ†çš,ä¸šäï¿½ç¯è¿ä»£æ³•æ¥€§£¿ç°¿»,ï¼Œä¸é‹æ—€‡á°±æ¢ º%é°çš,é—'æ˜·ï¡°ä®'%ðÉâ ä ¸â¾ˆá°šâœ'æ™ ᴹ£ç"â°â˜°á¸§åœ¯€ç–ç'Y©é°µã€,æ%é³ï¼Œ"æ'œ'ç®—âœ'è¿†éˆÈ®²è®²æœ°áᵐãⅈᵃ ä çš,è¿ä»£æ³•á°'ç'ã€,è¿™é‡É𬀀ç'±æ¬,ï¼Œ¬â ¯'ç'æŽ¥¥äé‡è|ï¼Œéᴄ'æŽ¥¥è§£³†ᵃ°è®®å¤šç»,â¯ç‰€å•çš,é—®éç¯ï¼Œä¹Yâ¯â¦â¦¬â»—æ—¹æ³•çš,åŸ°ç¦ã€,

çŽ°åœˆæ¬°å°±æ¥€ä¸€ä¸æ¯â ¯ä»£æ³ᴑï¼Ÿ

æˆˆä»»è¿æ—é€Šè¿¿ç¿æ§æ—¹ç°¿»,$Ax=b$æ¥¢œ¿çœã€,å€è¿™é‡¶Œæ¨¨ä»»â¯†è§£$A$ï¼Œ䬽â³â—$A=S-T$ï¼Œä»£å…¥ç%å¼åŽŽ³¾—å‡ᵗï¼š$Sx=Tx+b$ï¼ç%å¼á¹ ï¼%ã€,

æŒ%è¿™æ¶·çš,æ—¹æ—æ¨©çⅈ¸åŽⅈï¼Œé鶀è¿‡é¬‡¿ä»£çš,æ—¹å¼æ¥¢£$Sx$ã€,è¿™å°÷ç±»ä¼¼á°ŽæŠŠâ¾æ,é—®é¢²ä±,ä±,å‡†è§£áᶜç®€å®Œ—ï¼Œæœ€ç»»ᵃä¿¦á¾—è¿™ä°¨è¿ä»£ç%å¼æᴄç«¨ï¼š$Sx_{k+1}=Tx_{k}+b$ï¼ç%å¼á¹ ᴹ¼%ã€,

æ¯'å…ᵈá½ᴑ¯á €ç¡ᴑ¥â ¯ï¼Œæ¯â¸â…¶â®žæ¯â»Ž$Sx_{0}$å¼åœ¨åᵍ¼ᴑï¼Œè§£$Sx_{1}=Tx_{0}+b$ã€,ç,¬âŽï¼Œç»¾»»è§£$Sx_{2}=Tx_{1}+b$ã€,ä¸ç¡áˆ°$x_{k+1}$éᴑ,æŽ¥¥ç¿'$x_{k}$æ—¶ï¼Œåˆâˆˆe…â€ㆂᴄ'æ¥ä½ᴹ⺈¼$r_{k}=b-Ax_{k}$åᴑŽ¥¿'$0$æ—¶ï¼Œå¿°ᵗá°ᵃ³Žç°,æ¥€§£'¬Æⅈ¸è¿™ᵃᵃᵅ†¿¿ç¬ᵃᴞ†áᵗ ç™¾æ—¡çš,è¿ä»£éf½æ̃ æœ¶%å è·f½çš,ã€,æ%é³ä¥ï¼Œï¿½ä»£æ³•çš,ç»®æ ‡á°±æ ᵐæᵐ¶á…fæ³•æ¬'á¿«é€Ÿæ¥é¬€¿è¿ᴄçœŸå®žè§£ã€,

é,£ä'ᴑ'ç©ᴙç«¨Ÿá°ᵗè ¥á¦,ä½¿å¿«é€Ÿæœ°é¬€¿è¿ᴄçœŸå®žè§£á¯¢ï¼Ÿ

è™€é‡Éⅈ¼Œ$A=S-T$ï¼Œä€$A$çš,âᵗ†è§£æ̃áᵗá…³é¬°Œï¼Œ$A$çš,âᵗ†è§£ç»®æ ‡æᵗᵒ¶æᵄ%çš,„è¿ç®—é€Ÿ㺰â'Œæ̃"¶æ•»é€Ÿᵃᵃᵒᵈf½è¶á¿«ã€,æ̃ æ¥çš,è¿ç®—é€Ÿᵃᵒá̃—á†³ᵃŽ$$Sï¼ŒèᵒᵒŒæ̃"¶æ•»é€Ÿᵃᵒᵃ—á†³ᵃŽâ¾œéᴹ™è—ㆀ(error)ï¼Œ$e_{k}$ï¼Œè¿™é‡ᴑŒçš,é̄ᵗ™è—$e_{k}$æᵗ$x-x_{k}$ï¼Œᵃᴞ†á°±æ̃—è—$x$å¸ᵒŒ$x_{k}$çš,âᵍ®ᵃᵐᴹᵃ¥á¿«é€Ÿé€¿è¿ᴑᵒᵗᴑ¼Œᵃᵃä»̄ æŠŠé̄ᵗ™è—ㆀç°ᵃᵃᴄᴛᴹæ̃·ï¼š$e_{k+1}=S^{-1}Te_{k}$ï¼ᴄç%åᵍ¼áᶜᴑᵗᴑ¼%ã€,

å®ƒæᶜᴄç%åᵍ¼áᶜᴑᵃᴄå·å®žŽᴑ³¾—å‡ᴑçš,ç»̄'æŽœᵗï¼Œé²ᵈᵗᵐᵃ™è—éf½ᴑᵗᵃᵒᴄ<$S^{-1}T$ᴑᵗᴑ¼Œᴑ¦,æŽœ$S^{-1}T$è̄¶Šᴑᵗᵒᴑᵗᵃᴄ£é€ᵗᵃᴑᵗᵃᴑ¼è¿ᴑ0çš,é€Ÿᵃᵒᵃᴑᵃᴑᵗᴄᴄㆀᴛᴛᴄᵗ†è§£æf…á†µäᴄï¼Œ$S=A$ï¼Œ$T=0$ï¼Œé¿£$Ax=b$åᶜáᵗ»æᵃ¶ᵗ†ï¼Œç—ᴑǟᵃᴄ—è¿ä»£åᵗᵒ±éfᵒᴑá®Œæᶜᴑᴑᵐᴄ•ㆂᴑᴄᴑᴑ…ᵓᴄ$S^{-1}T$ç%åᵍᴈᴄᴄ,

ä½†æᴑᴑ¼Œè¿™ä¸æᵗᴑᴄᴄ,æᶜᴑᵐᵒᴑᵃᵒᴑᵗ¬é«ᵗᴑᴑ¼Œᴑᴑ»̄âᵓ³ᵃᵒᵗ†éᵓᴑᴄᴑᵒᴑ—ᴑ¼çš,åᵍᵗᵗ'¸ᴑᴄᴄᴄ%ᴑᴄ¥ᴑᴑ¼Œᴑᵗᴄᴄᵗᵒᵗᴄᵗᵗᴑ¼Œᴑᴄᵗᴑᵗ'áᶜᴑᵗᵒᴑᵗ᭄ᴄ͎åᴄᴄ£ᴑᵗᴄᴑ⑊ᴑᵗᴄᴄᴄᴄᴄᴈᴄᵗ¼š $$ᴑᴄ%æœ©çšᴑᴄᵗᵒᵗᵒᴄᴑ,è§åᴄ,èᵒᴑ—ᴈᴈᴛᴑᵗ¼š

1. é›…å̄ᵃ æᵗᴑᵃᵑᵗᴑ¼Œᴑᴄᵗᴑᵃᴄ$Sᴑᴄᴄᴑᵒᴑᵗᴄᴑᵒᴄᴄᴄᴈᴈᴄᴄᴛᴄᴄᴄᴄᴈᴄᴄᴄᴄ
2. é«̄ᴄᵗ—è̲ᵗ¾âᴄᴄᴄ
3. ILUᴄᵗᴑᵃᴑᵑᴑᴑᵑᴑᴄᴄ

## é›…å̄ᵃᴑ"æᵗᴑᵃ•åᴄᴄᴄ

æ€ä½½ᴑᴑᴄᴄᴄ

é¦â€¦ᴑ̄ᴑᴑᴄ

$$
Ax=b
$$

$$
\left\{\begin{array}{c}
2 u-v=4 \\\\
-u+2 v=-2
\end{array}\right.
$$

æ̂ᴑ»̄ᴈᴛᴈᴄᴄ

$$

$$
\left[\begin{array}{l}
u \\
v
\end{array}\right]=\left[\begin{array}{l}
2 \\
0
\end{array}\right]
$$

çŽ°åœ¨æ ˆ¨ä»¬å°±ç”é›…å¯ æ¯ æ˜¯æ³•æ¥çœ‹çœ‹æ€Žä¹ˆè§£è¿™ä ªæ–'ç¨‹ç»„ï¼š

é¦–å…ˆï¼Œæ ˆ¨ä»¬æŠŠç¿¿æ€§æ–'ç¨‹ç»»è½¬æ¢æ çŸ©é¿å½½¢å¼ã€‚

$$
\left[\begin{array}{cc}
2 & -1 \\
-1 & 2
\end{array}\right]=\left[\begin{array}{c}
4 \\
-2
\end{array}\right]
$$

æŽ¥ç€ï¼ŒæŠŠAçš,å¯¹è§'çº°æ¾åœ¨ç‰ªå¼å··è¾¼ï¼Œå¾—å‡ºS$S$çŸ©é¿µã€‚

$$
S=\left[\begin{array}{ll}
2 & 0 \\
0 & 2
\end{array}\right]
$$

å…¶ä½™éƒ¨ˆ†ç§§å°°ç‰åå¾³è¾¼ï¼Œå‡ºT$ç Ÿ©é¿µã€‚

$$
T=\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]
$$

äºŽæ¯ï¼Œé¦–å…ˆæ¯ è¿ä»£å°±ä»¬å¥¡ç¼ºˆæˆä¸‹é¢è¿™æ·çš,å½½¢å¼ã€‚

$$
\mathrm{S} x_{k+1}=T x_{k}+\mathrm{b}
$$

$$
\left\{\begin{array}{l}
2 u_{k+1}=v_{k}+4 \\
2 v_{k+1}=u_{k}-2
\end{array}\right.
$$

çŽ°åœ¨æ¯ æ¶™â€™è¿¿è¡ä»££äº†ï¼Œæˆ¨ä»¬ä»Ž$u_{0}=v_{0}=0$å¼€å§‹ã€‚

$$
\left[\begin{array}{l}
u_{0} \\
v_{0}
\end{array}\right]=\left[\begin{array}{l}
0 \\
0
\end{array}\right]
$$

ç¬¬ä¸€æ¬¡è¿ä»£Ž¿æˆ¨ä¾—å¾—ï¼š

$$
\left[\begin{array}{l}
u_{1} \\
v_{1}
\end{array}\right]=\left[\begin{array}{c}
2 \\
-1
\end{array}\right]
$$

ç¬¬äºŒæ¬¡è¿ä»¬å¾å¾—ï¼š

$$
\left[\begin{array}{l}
u_{2} \\
v_{2}
\end{array}\right]=\left[\begin{array}{l}
\frac{3}{2} \\
0
\end{array}\right]
$$

ç¬¬ä¸‰æ¬¡è¿ä»¬å¾å¾—ï¼š

$$
\left[\begin{array}{l}
u_{3} \\
v_{3}
\end{array}\right]=\left[\begin{array}{c}
2 \\
-\frac{1}{4}
\end{array}\right]
$$

ç¬¬å››æ¬¡è¿ä»¬å¾å¾—ï¼š

$$
\left[\begin{array}{l}
u_{4} \\
v_{4}
\end{array}\right]
$$

$$
\end{array}\right]=\left[\begin{array} {l}
\frac{15} {8} \\\\
0
\end{array}\right]
$$

ç⼀ä°"æ¬¡è¿ä»£åŽï¼Œæˆ"ä»¬å¾—å°°ï¼š

$$
\left[\begin{array} {l}
u_{5} \\\\
v_{5}
\end{array}\right]=\left[\begin{array} {c}
2 \\\\
-\frac{1} {16}
\end{array}\right]
$$

ç»è¿‡ä°"æ¬¡è¿ä»£åŽå'çŽ°æ¶¶æ•›ï¼Œå› ä¸°å®ƒçš¸ç»"æžœæŽ¥è¿'çœŸå®žè§£ã€,

$$
çœŸå®žè§£
\left[\begin{array} {l}
2 \\\\
0
\end{array}\right]
$$

çŽ°âœ¨æˆ'ï¼Œå†æ¥çœä¸€ä¸‹é'™è¯ç‰å¼ï¼Œ$\mathrm{Se}_{k+1}=T e_{k}$ï¼Œæˆ"ä»¬æŠŠ$$å'Œ$T$ä»£å…¥ç‰å¼ï¼Œâ¾—å‡°ï¼š

$$
\left[\begin{array} {ll}
2 & 0 \\\\
0 & 2
\end{array}\right] e_{k+1}=\left[\begin{array} {ll}
0 & 1 \\\\
1 & 0
\end{array}\right] e_{k}
$$

è®¡ç®—$$$çš,é€†çŸ©é˜µå'Œ$T$ç›,ä¹˜$S^{-1}$å¾—å‡°ï¼š

$$
e_{k+1}=\left[\begin{array} {cc}
0 & \frac{1} {2} \\\\
\frac{1} {2} & 0
\end{array}\right] e_{k}
$$

è¿™é‡Œï¼ŒŠŠ$çš,é€†çŸ©é˜µå'Œ$T$ç›,ä¹˜$S^{-1}$T$æœ‰%‰¹å¾'å€½$\frac{1} {2}$å'Œ$-\frac{1} {2}$ï¼Œæ‰€è¥¼å'Œ$ƒçš,è°±åŠå¾,æ¯$\rho(B)=\frac{1} {2}$ã€,è¿™é‡Œçš,è°±åŠå¾,æ¯¯ç"æ¥æŽ§å¶"¶æ•›çš,ï¼Œæ‰€å€¥ï¼Œå› é‡è¦ã€,è°±åŠå¾,ä»žŽ•°å¦å®šä'‰ä·Šæ¯¯ï¼šçŸ©é˜µµ'¼'æˆ—è€…æœ‰ç•Œ°¿æ€§ç®—âçš,è°±åŠå¾,ï¼‰æœ¯æŒ‡â…¶ç‰°¹å¾'å€¼ç»å'ä€¼ä¾,çš,ä¸Šç¡®ç•Œã€,è¿™ä,æ¦¦,å¿µœ¯ä¸¾—å¾'é§¾ç†è§£ï¼Ÿâ…ä½'è°±åŠå¾,çš,æ¦,å¿µ½'½ä ä»¥¥Ž•å"è'¿½'æ¥Ž·â—ï¼Œä¸°†æ¬'ä¾,ä½½ ç†è§£ï¼Œè¿™Œè¿™¯è¿¯æ¯¯°å¦æ—'æ³•æ¥ç€å•è¡¨è¿'¼'ã€,

$$
B=S^{-1} T=\left[\begin{array} {ll}
0 & \frac{1} {2} \\\\
\frac{1} {2} & 0
\end{array}\right]
$$

é€šè¿‡$$$çš,é€†çŸ©é˜µå'Œ$T$ç›,ä¹˜$S^{-1}$T$ï¼Œæˆ"ä»¬å¾—å‡°ï¼š$|\lambda|_{\max}=\frac{1} {2}$ï¼Œå»¥Šï¼š

$$
\left[\begin{array} {cc}
0 & \frac{1} {2} \\\\
\frac{1} {2} & 0
\end{array}\right]^{2}=\left[\begin{array} {cc}
\frac{1} {4} & 0 \\\\
0 & \frac{1} {4}
\end{array}\right]
$$

è¿™é‡Œçš,ç‰°¹å¾'å€¼$\frac{1} {2}$éžå,â°å'Œæ‰€ä»¥10æ¬¡è¿ä»£åŽï¼Œé€™™è¯—â°±å¾'ä½'Žä°†ï¼Œâ³$\frac{1^{10}} {2}=\frac{1} {1024}$ã€,è€Œå¦¦,æžœç‰'å¿'å€¼æˆ¯0.99æˆˆè€……0.999ï¼Œé,£å¾'â~³½ç,¶è¿-ä»¬æ¬¡æ•°å°±è¦å£¤Šå¾—â¾—šï¼Œä¸Ÿå'æ•—¶è€ ´éœ€è¦æ›'â¤šæ—¶é—´æ¥¥åŠè¿ç®—ã€,

# é«˜æ¯¯-èµ¶å¾·å°°æ¬'æ³å®žè·µ

çŽ°âœ¨æˆ'ä»¬å†æ¥çœä¸€ä¸ªé«˜æ¯¯-èµ¶å¾'å°°æ¬'æ³•ï¼Œé«˜æ¯¯-èµ¶å¾'å°°è¿ä»£â ä»¥¥š,ç"å~ä,"ã'Œåš é€Ÿè¿¿£ï¼Œæ¯¯ è¿ä»£â ä»¥¥ã'¶é€œ€ä €ç»»,å'â,"â•å…fï¼Œè€ŒŒé….â æ¯¯è¿ä»£é€œ€è'¦ä¤»,å•å…fã€,

$$$å–$A$çš,ä¸‰‰§'éƒˆ†ï¼Œè¿'æ¯¯ä½²ç›"â¹'å‰'â¹é¿…â æ¯¯æ¬'æ³•â•ä¿çš,ä¸‹äÆ¼Œæˆ"ä»¬å¾—å‡°æ¬'ç¯ç»,ï¼š

$$
\left\{\begin{array} {c}
u_{k+1}=\frac{1} {2} v_{k}+2 \\\\
v_{k+1}=\frac{1} {2} u_{k+1}-1
\end{array}\right.
$$

è¿™é‡Œæœ‰¾ä¸,æ""è³¼ƒâ¤§çš,åˆ'å¶-ï¼Œé…å¾'±æ¯$u_{k}$æ¶'â¤±äºæˆ"ï¼Œé¿é€è¿‡$v_{k}$ï¼Œæˆ"ä»¬‾â¤™ä¼ç€'æŽŽ¾— â°$u_{k+1}$å'Œ$v_{k+1}$ï¼Œè¿¯™æ æœ‰%‰‰¦ä'â¥½ä°â¿,å¼'éï¼Ÿä¤ä°š¼%½â¿,æ¯¯æ¯¾ç€ŒŒ"è§çš,ï¼Œå¦±æ¯¯ èŠ,çª"å~ã',"ã'ŒåŠ é€Ÿè¿ä»£ã€,

æŽŽ¥ä¸,æ¥¥ï¼Œæ¯¯ä»¬žŽ$u_{0}=0$ï¼Œ$v_{0}=-1$æ¥æµ¤è¯•ä€ä¸,è¿ä»£ã€,

ç⼀ä€æ¬¡è¿ä»£åŽï¼Œæˆˆ"ä»¬å¾—â°°ï¼š

$$
\left[\begin{array} {l}
u_{1} \\\\
v_{1}
\end{array}\right]=\left[\begin{array} {c}
\frac{3} {2} \\\\
\frac{-1} {4}
\end{array}\right]
$$

ç⼀ä°Œæ¬¡è¿ä»£åŽå¾—å°°ï¼š

$$
\left[\begin{array} {l}
u_{2} \\\\
v_{2}
\end{array}\right]=\left[\begin{array} {l}
\frac{15} {8} \\\\
\frac{-1} {16}
\end{array}\right]
$$

ç¬¬ä¸‰æ¬¡è¿å£åŽå¾—å°ºï¼š

$$
\left[\begin{array} {l}
u_{3} \\\\
v_{3}
\end{array}\right]=\left[\begin{array} {r}
\frac{63} {32} \\\\
-\frac{1} {64}
\end{array}\right]
$$

ç»è¿‡ä¸‰æ¬¡è¿å£åŽå'çŽ°æ"¶æ•›ï¼Œå ä¸ºç¬¬ä¸‰æ¬¡è¿å£åŽçš„ç»"æžœæŽ¥è¿'çœŸå®žè§£ã€,

é'™™è¯¯ç»è¿‡è®¡ç®—â†'å°†æ¯\$-1, \frac{-1} {4}, \frac{-1} {16}, \frac{-1} {64}\$ï¼Œå'Œâ€šæ%ä½¿ç"é¯…â€¯æ¯'æ³•å¾—å‡'çš„é"™è¯¯\$2, \frac{1} {2}, \frac{1} {8}, \frac{1} {32}\$ã€,æ¯'è½fåŽæ"ä»¯â€¯ä¸¥å'çŽ°ï¼Œæ—è®®ªæ¯'è¿å»¡æ•°è¿¯æ¯'æ"¶æ•›ï¼Œç¯'é¢ï¼Œé<ã€¯-è¡på¾·å°"æ¯'æ³•æ¯'ï¼…ä¸æ¯'æ³•é¯'Yâº¡¿«ã€ç²³¼ç¡®å°¡ä'Yé‹ã¾—â¤šã€,

## é€æ¬¡è¶…æ¾¾å¼›æ³•

æœ€åŽ¾ï¼Œæ¯'ä»¯â€œé‹æ—-è¡på¾·å°"æ¯'æ³•ä Šåšä ªå°°ƒæ•'ï¼Œâœè¿±ä»£å å½•å…¥ä ¥å‚ªä¸æ€œomegaâ€ï¼ŒŠï‰ï¼Œâ³è¶…æ¾¾å¼›å› ââ€ç,¶åŽé€%œ©ä¸€ä ªâˆé€çš'\$Ï‰\$ï¼Œï½¿å¾—\$S^{-1}T\$çš'å°±åŠâ¾'å°½â è ƒ¹½ºï¼Œè¿'™ä ªæ—'æ³•å°±å«âšé€æ¬¡è¶…æ¾¾å¼›æ³•ï¼Šuccessive over-relaxation methodï¼Œç®€ç§°SORï¼‰ã€,

SORæ—'æ³•çš'æ—'ç¯'æ¯'ï¼š\$Ï‰Ax=Ï‰b\$ï¼Œç"©é"µ\$S\$æœ‰\$A\$çš'â¯è§'ç°ï¼Œâ¯è§'ç°¿ä'æ¯'\$Ï‰A\$ï¼Œç‰å/å³½ï¼'\$T\$æ¯'\$S-Ï‰A\$ï¼Œäª'Žæ¯'ï¼Œæ¯'ä»'è¯'ä¸«ä½¿ç¯'ä'ä'å%é…ä¸æ¯'æ³•ä¸çš'ä¾"ä'åª"ï¼Œå'¾—â°SORæ—'ç¯'ç»'å'ä¸ã€,

$$\left\[\begin{array} {c}
2 u_{k+1}=(2-2 \omega) u_{k}+\omega v_{k}+4 \omega \\\\
-\omega u_{k+1}+2 v_{k+1}=(2-2 \omega) v_{k}-2 \omega
\end{array}\right.$$

æ¯'ä¸æ¯'çœèµ·æ¥æ¿'å¤æ¡ä°†ï¼Ÿ

æ²¡â€¦³ç³»ï¼Œå…¶å®žå®ƒfâ°æ¯'âœ'æ¯'ä»'çœ½ä¸çœèµ·æ¥â°æ¿'ï¼Œå¯'è®¡ç®—âœ®æ¥è¯'æ¯'æ²¡åŒ¹å«'çš,ã€,ã¯'SORæ¥è¯'ï¼Œæ¯'æ—'â¤šä»†ä¸ã°\$Ï‰\$ï¼Œè€€Œ\$Ï‰\$å¾'å°±å¶·Ša¾'ã€,å…·ä½'\$Ï‰\$çš,é€%œ©ï¼Œä¾"åŠŠè¿»å¿çš'â¯'å°±ä¸µ¡è¿'ä°¯ï¼Œæ¯'ç»™ä¸ã¸ä ¥âœ'â€œ\$Ï‰\$â°§âºŽ1â€â'Œâ€œ\$Ï‰\$â°âºŽ1â€â¤¡ç§æƒ…å†µä¸æ¥â¶ºç‰œ©ã‡ä¸\$Ï‰\$è¿'è¿'è°è¯•ï¼Œæœ€åŽ¿½ â°"è ¥ä½'å³½—â°ç»"è®°ï¼š

1. âœ'\$Ï‰\$å°§âºŽ1æ—'ï¼Œ\$Ï‰\$è¶¶Šâ¤§ï¼Œè¿»å£çš,æ—'æ•°å°±è¶Šâˆ±ï¼Œæ"¶æ•›é€Ÿå°'å±·è¶¶Šæ…¢ï¼Œ\$Ï‰\$æŽ¥è¿'1æ—'ï¼Œè¿»å£çš,æ—'æ•°è¶Šï°ï¼Œæ"¶æ•›é€Ÿå°'è¶Šå¿«ã€,
2. âœ'\$Ï‰\$å°âºŽ1æ—'ï¼Œ\$Ï‰\$è¶'Šå°'ï¼Œè¿»å£çš,æ—'æ•°å°±è¶Šâˆ±ï¼Œæ"¶æ•›é€Ÿå°'å±·è¶¶Šæ…¢ï¼Œ\$Ï‰\$æŽ¥è¿'1æ—'ï¼Œè¿»å£çš,æ—'æ•°è¶Šï°ï¼Œæ"¶æ•›é€Ÿå°'è¶Šå¿«ã€,

æ%ä»¥ï¼ŒSORè¿»å£æ³•çš,å…³é"®å°±æ¯'\$Ï‰\$çš,é€%œ©ï¼Œå®'fâ¸ä»¥è¢'çœäï½œæ¯'é‹æ—-è¡på¾·å°"æ³•çš,æ%¶â€¦…ã€,

é›… å¯'æ¯'æ³•â€é€‹æ—-è¡på¾·å°"æ³'ï¼Œä¥åŠSORè¿»å£æ³•éf½æ¯"âˆ'å³Ÿšâ,è¿»å£â¸'ã€,æŽ¥ä¸æ¥'è®'ã'ªä¸å'ŒåˆŠ,è¿»å£æ³•ä,åŒçš,å¦ä €ç±»æ—'æ³•ï¼Œä¹Ÿæ¯'å®žè·µä¸ç''çš,æ¯'è½fåˆ½çš,æ—'æ³•â€"â€"å…±è½½-æ¢¯åº'æ³•ï¼Œ(Conjugate gradientï¼‰%ï¼Œå®'fâ±zä°ŽKrylovå¿©é—'æ—'æ³•ã€,ç®€å•æ¥è¯'ï¼ŒKrylovå¿©é—'æ—'æ³•â ä€§ â€œé"™ç»'æ%¶"äŠ»â€ ç‰%æ‰«è½½ï½'ç''çš,æœ'â¤'ï¼Œ£å½'à¹'æ¯'å…±è½½â…³ç³»ã€,

## å…±è½½æ¢'å°¦æ³•

è¦è®²å…±è½½æ¢'å°¦æ³•ï¼Œæ¯'ä»'å…¶è§£é‡Šã,€ã,â€œå…±è½½æ—'å°ï¼Œå…±è½½å°±æ¯'æŒ%ä,ã®šçš,è§,å¾«ç›,é…çš,ä,€ä¸'ï¼Œé€šå¿—ç,'è¯'â°±æ¯'â°ç'Ÿã€,â€œè½½â€œ ç%%æ«â%è½½ï½'ç''çš,æœ'â¤'ï¼Œé,£å½¹ä°'æ¯'å…±è½½â…³ç³»ï¼Œæ—'æ%¶â €¾'è½½ï½,çš,ä,â¤'ç%%ï¼Œå°±æ¯'â…±è½½å…³ç³»ã€,



æˆ'ä»»æ'ä'æ®è¿'™ä ªåˆšä'%å†æ¥è§£é‡Šä,€ä¸,å…±è½½æ—'å°ï¼Œåˆ'é‡'\$p, q \in R\$ï¼Œ è¥æ»¡è¶'æ¡ä»¶\$pAq=0\$ï¼Œ å'™ç§°\$p\$å'\$q\$å…³äº'Ž\$A\$æ¯'â…±è½½æ—'ï¼Œæˆ-è€…\$p\$å'\$q\$å…³äºŽ\$A\$å…±è½½ã€,æœ%ä°†â…±è½½æ—'å'çš,æ¦,å¿µåŽï¼Œåˆæ¥çœå…±è½½æ¢'å°¦æ³•ã€,å…±è½½æ¢'å°¦æ³'çš,å°ƒçŽ'ã)â¸â€¦æ"ä°ªâºè£å†³æ¢'å°¦ä,é€'™Œæ³•çš,æ'¶æ•›é€Ÿå°'ï¼Œé‹ï¼ŒèŒŽä'ä'Ÿé¿å…ä°ç%%é¿¡æ³•ä'éœ'è¦å'â,'ã,Œè®¡ç®—â€—é›'â¡žç'Ÿ©é'µ(ï¼"Hessian Matrixï¼%å'¶æ±,é€†çš,ç¼'ç,'ã€,

çŽ°åœ'æ¥çœçœå…±è½½æ¢'å°¦æ³•â³ï¼Œè¯'\$Ax=b\$ï¼Œâ…'¡ä,\$A\$æ¯'ä,€ä ªâˆžä'¡ç§°æ£å®šçŸ©é'µã€,

é¦-â…'ï¼Œæ¯'ä»'è®'å°å'å§å€¡\$x_{0}\$ä,º\$0\$æˆ-è€…ä,€ä ªä¼'è®¡ä'€'ä½ï¼Œæ¥è®¡ç®—\$r_{0}\:=b-A x_{0}\$ã€,å¦,æžœ\$r_{0}\$éžå,å°'ï¼Œé,£\$x_{0}\$å°±æ¯'ç»"æžœï¼Œå'¦,æžœä¸æ¯'å°±ç»Šç»¿ã€,

æŽ¥ä¸,æ¥è®'å¾'\$p_{0}\:=r_{0}\$ï¼Œ\$k=0\$ã€,çŽ°åœ'æ¯'ä»'å½€å§'è¿»å£å¾'¿çŽ ã€,

a.è®®¡ç®—\$\alpha_{k}\$ ã€,

$$\alpha_{k}\:=\frac{r_{k}^{T} r_{k}} {p_{k}^{T} A p_{k}}$$

b.è®¡ç®—$x_{k+1}$ ã€‚

$$x_{k+1}:=x_{k}+\alpha_{k} p_{k}$$

c.è®¡ç®—$r_{k+1}$ã€‚

$$r_{k+1}:=r_{k}-\alpha_{k} A p_{k}$$

d.å¦‚æžœ$r_{k+1}$éžå¸¸å°ï¼Œåˆ™ç»“æŸï¼Œå¦åˆ™æ¯ç»§ç»ã€‚

e.è®¡ç®—$Î²_{k}$ ã€‚

$$\beta_{k}:=\frac{r_{k+1}^{T} r_{k+1}}{r_{k}^{T} r_{k}}$$

f.è®¡ç®—$p_{k+1}$ ã€‚

$$p_{k+1}:=r_{k+1}+\beta_{k} p_{k}$$

g.$k:=k+1$ã€‚

4. è¿"å›žæœ$x_{k+1}$ã€‚

äžŽç®—æ³•æˆ"ä»¬ä½¿ç"¨å¼ï¼Œ... ±è½æ¢ å°œ³•çš,ä¼ç,¹æ ̄â~â¸'é‡å°ªŒå…æœ‰æ¥æ"¶æ›æ€§ã€‚å¦æžœä½ ç†Ÿ‰‰MATLABï¼Œå°ä½šâ'çŽ°â... ±è½æ¢ å°œ³•çš‚å®žç°Ž'è¶…ç°§ç®€å•ï¼Œå°éœ€è¦çŸçŸ-âå‡è¡Œå£ ï¼ˆ ̄æ—¹ä'æ£ç ̈âŽMATLAB/GNU Octaveçš,ä¾ å›¼½‰Œ,
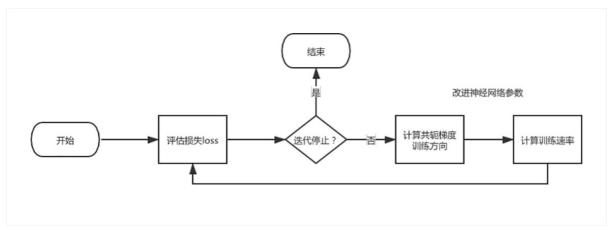
```
function x = conjgrad(A, b, x)
    r = b - A * x;
    p = r;
    rsold = r' * r;

    for i = 1:length(b)
        Ap = A * p;
        alpha = rsold / (p' * Ap);
        x = x + alpha * p;
        r = r - alpha * Ap;
        rsnew = r' * r;
        if sqrt(rsnew) < 1e-10
              break;
        end
        p = r + (rsnew / rsold) * p;
        rsold = rsnew;
    end
end
```

## æœºå™¨å¦ä¹ ä¸çš„å…±è½¢¢ å°¦

å…±è½æ¢ å°œ³•çš,è¢¢ç‰ 'âœè®ç®›fç¥žç»ç½'çœä ̈ï¼Œåœ'ä®žè·µä‚²ç»è ̄æŽ̄ï¼Œå®ffæ œ'æ' æ¢ å°'ä‚é™'æ´æœ‰æ•°çš,æ—æ³•ï¼Œ ä‚°å̄±fš‰æ‰š®²çš,ï¼Œå®fä¾éœ€è¦è®¡ç®—é¾'å¡žç'Ÿ©ï¼Œé,£æˆ̄çŽ°âœ±æ¥è®²ä è¢̄®²ï¼Œ½¹'ç̄â... ±è½æ¢ å°œ³•çš,ç¥žç»ç½'çœœ®ç»fè¿‡ç̄ã€‚



âœ•â•'ä'ªè®ç»fè¿‡ç ̈ä ï¼Œæ¥•æ"'è¿¿æ ̄é‡ç,'ï¼Œå½'ç,¶è¿‰™ä'Ÿæ ̄æ‰æ€¢‰æ•°çš,é€ç,'ã€,è¿‰™ä'ªè¿ç̄æ' é€'Žè®'ç‡è®¡ç®—å... ±è½æ¢ å°¦çš,è®ç»fæ—'å'ï¼Œ¶åŽž®è®ç®—è®ç»fé€ŸçŽ'Žå¥¥žçŽ°åˆ"ãçŽ,ãæ ̄â... ±è½çš,æ,æ‰%‰®ï¼Œ¶'Ÿæ¥̄é€Ÿâ°|æ¯̄ ǣå°|ä ̄ã€,

çŽ°âœ'æ'̄»æ¥¥œæ± è®ç»fè¿ç̄â—æ—æ³•ã€,é¦—å... ï'Œæ'̄»æ›'è¢¢½½®è®ç̄»fæ—'å'é‡ $d$ï¼Œ¶åŽž®ä®°â ̈æ ä°å§å,æ»°å'é‡$w^{0}$ï¼Œ¶åˆŠä€ã'åºå§å,è®ç̄»fæ—'å'é‡$d^{0}=-g^{0}$ï¼Œ¶ä'Žæˆ̄Œï'Œå... ±è½æ¢ å°'•æž,é€'å†‡çš,è®ç»fæ—'å̄ ä»¥è¦'ç'ƒæ ̄ï¼š$d^{i+1}=g^{i+1}+d^{i} \cdot \gamma_{i}$ã€,

å...¶ä¸ï¼Œ$g$æ ̄æ¢ å°¦ï¼Œ¶é‡$Î³$æ ̄å...±è½å•è ̈Œ,å,æ»°å'é¢Šè¿‡ç̄‡è®¡ç™â'ç̄ï¼ç̄‡çš,è¿ç̄â—ï€ç'Ÿ©ï¼Œ¶é'Šâ,è¢¢½½®è®¡¢½½…®,è®ç»fé€ŸçŽ'$\eta$â̄ä½¿ç¨'â•ä'é‡å½ä¾ã°ä'âŒ—æ—ä'•æ±,å¾—ã€,

$$w^{i+1}=w^{i}+d^{i} \cdot \eta^{i}$$

## æœ¬èŠ,å°ç»"

å½'ä'†ï¼Œâˆ°è¿™é‡Œâ¢°â'çã°è¿ç»fä¢çš,è¢ä»£¢³•è¿‡ä ̄ä¿è¯§å... ç'ç¢¢²±æ'çš,å®¼€€åŽæ̂'â'†æ¥é¢'ç»̄ä,å'«é¼œ€é½2çš,å›£¢şå,å‡...â®³ã€,

é¦—å... ̄ï¼Œâ...è¢å£é‡ŠȘâ' †æˆ•å•€'¼çŽ°°æ€¢§å£æ•'çš,è¢ä»£¢³•è·¢"ç̄,¶‡¶Žä®ä'ä¾'ä€ã̄,¶'çç̄æ€¢§â¡'ç'ç»,ä€„è,å%¼é¢²ï¼Œå'é'éŸ'æ̂'Žå̂...±è½æ¢ å°'•ç'̄æ̂'â†ǣ ±'˘å'ǟ çš,ç¥žç½'çœ̄ç̄â—fã̄ã€,

å'Œæœ½ä½ f}/âœ¢è̄†è§‡£â''â†è£'ä'ïæ•â'¼çŽ'°æ€¢§â£æ•'çš,è¢ä»£ï¼Œå›¥åŠŠè̄ä'æ³•åŽ°ï¼Œ'â¤â'ç̄—æœ¢ç§§'å¼é¢‡â†åŸŸä'ï¼Œè¢¿ç̄‡'ç̄è¢ï»£'•å¤çŸ©é¿ç®—â€‹ã€,å¦,æžœæœ‰å... ́â¦¶ï¼Œ½ä'Ÿ'å'Ÿä'âŒ—ä'â¼å,¹ä'...¶å¹fâ'æ®èžè·'µä,½ç,è¢ä»£ï¼Œ

## ç°ï¼æ€§ä»£æ•°ç»̄fä'åœ

ç°fä'æ—¶å¨âˆ°'â'†ï¼Œè¢™â'¬ç»ä½½ç'̄̄çâ'ä¸€€ç̄‡çï¼æ€§æ€—ç'‹ç'ï¼Œé‡Œæçš,ä¾â̄âä'Œä½¼â̄æ¥Œ'é€'é‰%'‰å'»æ¥,â€ç'ä¸̄è¢¿â'è£'ä¯ç'ï¼Œ,

å‡è®¾ï¼Œâ€ã,'æ¥—...æ,â¢çç'±å°'âå'Œâ§°º'ç»,æ'̄ï¼ŒåŽžç̂'¬æ—¶—å'»ä€é,µâå‡'å̄'Žæ̂ ́â°å'âçš,ç¥¼»'3å...ƒï¼Œâǟ'°â§°ç¥̄»'3.2å...ƒï¼Œæ€¢å...±è¿è±è́'118.4å...ƒã€,å›žç̄'æ—¶—â'€é,µâå‡'Žæ̂ ́â°å'âçš,ç¥¼»'3.5å...ƒï¼Œâǟ'°â§°ç¥̄»'3.6å...ƒï¼Œæ€¢å...±è½è±è́'135.2å...ƒã€,è¯'-é—®ç'™æ—'æ—...æ,â¢ç¢ æœ‰å¢šå°'â€å'Œâ§°œï¼Ÿ

è®¾â'å̂'â€œæ•°ä'º$x_{1}$ï¼Œâ'â'Žæ̂'æ•°ä'º$x_{2}$ï¼ŒçŽ'ŽæˆʻŽæ̂ ́â»'ä»—â̂'° â'°ä€§â'ä,̄ ̈ª'ç»,ï¼š

$$\left\{\begin{array} {c}
3 x_{1}+3.2 x_{2}=118.4 \\\
3.5 x_{1}+3.6 x_{2}=135.2
\end{array}\right.$$

è¿™ä'æ—'ç»,çš,è§£æ ̄ï¼š

$$\left\{\begin{array} {l}
x_{1}=16 \\\
x_{2}=22
\end{array}$$

\end{array}\right.$$

ä½¯â¯ä»¥è®¡ç®—ä¸€ä¸ªå¤šå°°æ¬¡è¿°åŁåŽå®ƒèƒ½å½œè¯´â—¶ï¼Œä¹åº±æ¯é€¼ç¿ŸçœŸå®žè§£ï¼Ÿä»¥¥åŠå®ƒ©çš„é²™™è¯¯$eå¥å¯†å—æ¯â— šå°°ï¼Ÿ

æ¬¤è¿¿ŽâœçåŸè°•€åŒÞæ™å‡ºçš,ä½¯çš,è¿¿ç®—è¿¿‡çˆ¯å'Œç›â—æžœæ¯â€,å¦,æžœæœ‰æ¬¶èŽ·ï¼Œä¹¤äÔ¯æ¬¤è¿¿Žåº¾æŠŠè¿¿™ç¿¯‡æ–‡ç«â—â†ä°«ç»â„¢ä½¯çš,æœœâå‹¿â€,