

MASD Final Exam

MASD 2018
Department of Computer Science
University of Copenhagen

Eksamensnummer: 35

Version 1
Due: November 4th, 2018, 16:00

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1. Exercise

(a)

i)

Here we use the chain rule $\left(\frac{d}{dx}f(x) = \frac{d}{dx}h(g(x)) * \frac{d}{dx}g(x)\right)$ and the rule for exponential functions: $\left(\frac{d}{dx}e^x = e^x\right)$

$$\frac{d}{dx}e^{2x^2+5} = 4xe^{2x^2+5}$$

ii)

$$\frac{\partial}{\partial y} \ln(x+y)e^{xy}$$

First we use the product rule: $\left(\frac{d}{dx}f(x)g(x) = \frac{d}{dx}(f(x))g(x) + f(x)\frac{d}{dx}g(x)\right)$

$$\frac{\partial}{\partial y}(\ln(x+y))e^{xy} + \ln(x+y)\frac{\partial}{\partial y}e^{xy}$$

We can then use the chain rule $\left(\frac{d}{dx}f(x) = \frac{d}{dx}h(g(x)) * \frac{d}{dx}g(x)\right)$ and the rule for exponential functions: $\left(\frac{d}{dx}e^x = e^x\right)$

$$\frac{\partial}{\partial y}(\ln(x+y))e^{xy} + \ln(x+y)e^{xy}\frac{\partial}{\partial y}xy$$

We can now use the constant multiple rule: $\left(\frac{\partial}{\partial x}(cf(x)) = c\frac{\partial}{\partial x}f(x)\right)$

$$\frac{\partial}{\partial y}(\ln(x+y))e^{xy} + x\ln(x+y)e^{xy}\frac{\partial}{\partial y}y$$

We can now rewrite this to help us, and do the trivial derivative: $\left(\frac{\partial}{\partial x}(x^n) = nx^{n-1}\right)$

$$e^{xy}\left(\frac{\partial}{\partial y}\ln(x+y) + x\ln(x+y)\right)$$

Now we have to use the chain rule again $\left(\frac{d}{dx}f(x) = \frac{d}{dx}h(g(x)) * \frac{d}{dx}g(x)\right)$ and the rule for the natural logarithm: $\left(\frac{\partial}{\partial x}\ln(x) = \frac{1}{x}\right)$

$$e^{xy}\left(\frac{\frac{\partial}{\partial y}(x+y)}{(x+y)} + x\ln(x+y)\right)$$

We can now use the sum rule: $\left(\frac{\partial}{\partial x}(f(x) + g(x)) = \frac{\partial}{\partial x}f(x) + \frac{\partial}{\partial x}g(x)\right)$

$$e^{xy} \left(\frac{\frac{\partial}{\partial y}(x) + \frac{\partial}{\partial y}(y)}{(x+y)} + x \ln(x+y) \right)$$

We will now rewrite it so that it's easier to work with:

$$\frac{e^{xy}}{(x+y)} \left(\frac{\partial}{\partial y}(x) + \frac{\partial}{\partial y}(y) + (x+y)x \ln(x+y) \right)$$

We can now do the trivial derivatives $\left(\frac{\partial}{\partial x}c = 0\right)$ and $\left(\frac{\partial}{\partial x}(x^n) = nx^{n-1}\right)$:

$$\frac{e^{xy}}{(x+y)} (1 + (x+y)x \ln(x+y))$$

We can now see that:

$$\frac{\partial}{\partial y} \ln(x+y) e^{xy} = \frac{e^{xy}}{(x+y)} (1 + (x+y)x \ln(x+y))$$

iii)

$$\frac{\partial}{\partial x} \frac{x^2 y + y^2}{x^3 - y}$$

First, we will rewrite the function to make it easier to work with:

$$\frac{\partial}{\partial x} \frac{y(x^2 + y)}{x^3 - y}$$

We can use the constant multiple rule: $\left(\frac{\partial}{\partial x}(cf(x)) = c \frac{\partial}{\partial x}f(x)\right)$

$$y \frac{\partial}{\partial x} \frac{x^2 + y}{x^3 - y}$$

We then use the Quotient rule: $\left(\frac{\partial}{\partial x} \frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}\right)$

$$y \left(\frac{1}{(x^3 - y)^2} (-x^2 + y) \frac{\partial}{\partial x}(x^3 - y) + \frac{\partial}{\partial x}(x^2 + y)(x^3 - y) \right)$$

We can then rewrite this again:

$$\frac{-y((x^2 + y) \frac{\partial}{\partial x}(x^3 - y) - (x^3 - y) \frac{\partial}{\partial x}(x^2 + y))}{(x^3 - y)^2}$$

We use the sum rule: $\left(\frac{\partial}{\partial x}(f(x) + g(x)) = \frac{\partial}{\partial x}f(x) + \frac{\partial}{\partial x}g(x)\right)$

$$\frac{-y((x^2 + y)(\frac{\partial}{\partial x}(x^3) - \frac{\partial}{\partial x}y) - (x^3 - y)(\frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial x}y))}{(x^3 - y)^2}$$

Now we can do the trivial derivatives of $\left(\frac{\partial}{\partial x}c = 0\right)$ and $\left(\frac{\partial}{\partial x}(x^n) = nx^{n-1}\right)$:

$$\frac{-y(3x^2(x^2 + y) - 2x(x^3 - y))}{(x^3 - y)^2}$$

We will rewrite this to something more legible:

$$\frac{-xy(3x^3 + 3xy - 2x^3 + 2y)}{(x^3 - y)^2}$$

We can now say that:

$$\frac{\partial}{\partial x} \frac{x^2y + y^2}{x^3 - y} = \frac{-xy(3x^3 + 3xy - 2x^3 + 2y)}{(x^3 - y)^2}$$

iv)

$$\nabla(x^T x) \text{ for } x = [x_1, \dots, x_n]^T$$

First we will write this into something that's usable, using vector multiplication rules:

$$\nabla \sum_{i=1}^n x_i^T * x_i$$

This can be rewritten as the following, using the sum rule: $\left(\frac{\partial}{\partial x}(f(x) + g(x)) = \frac{\partial}{\partial x}f(x) + \frac{\partial}{\partial x}g(x)\right)$

$$\sum_{i=1}^n \frac{\partial}{\partial x_i}(x_i^T * x_i)$$

This cannot be reduced further, so we have the following:

$$\nabla(x^T x) \text{ for } x = [x_1, \dots, x_n]^T = \sum_{i=1}^n \frac{\partial}{\partial x_i}(x_i^T * x_i)$$

(b)

First we will explain the function $E(f) = \sum_{n=1}^N (y_n - f(x_n))^2$, then $f^* = \operatorname{argmin}_{f \in \mathcal{F}} E(f)$

The function $E(f)$ goes through each coordinate of the dataset through the x-axis and compares the y-value of the given approximated function f to the y-value of the dataset. The difference of these is squared. The sum of this is taken for all coordinates. The result is then a measure of how close the function is to the dataset, where values closer to 0 mean it is more accurate

The way this is set up means that any difference between the given function and the dataset are exaggerated, meaning that even small deviations from the dataset result in large values of $E(f)$.

The second part $f^* = \operatorname{argmin}_{f \in \mathcal{F}} E(f)$ chooses the function that matches the best, aka which function that returns the smallest value $E(f)$, in a given family of functions. The choice of \mathcal{F} is a very important one in this function, since choosing one that does not fit the general curve of the dataset results in a function that is simply wrong. For example if we were to choose the family of linear functions, it is easy to see in the dataset given that past x-values of -3 and 3 , it would be wildly inaccurate. At the same time, choosing an n th degree function would not work either, since there is too much noise.

(c)

(d)

i)

To show that $E(f_w) = (Xw - y)^T(Xw - y)$, we will first write this out according to the error function given, part by part. First of all, we will define D as the highest value of d , to avoid confusion. Now, $f_w(x)$ can be written in a more useful way:

$$f_w(x_n) = w_1x_n + w_2x_n^2 + \dots + w_Dx_n^D = \sum_{d=1}^D w_dx_n^d$$

This can be set into the error function:

$$E(f_w) = \sum_{n=1}^N (y_n - f(x_n))^2 = \sum_{n=1}^N \left(y_n - \left(\sum_{d=1}^D w_dx_n^d \right) \right)^2$$

Now, we will show this backwards, by rewriting $(Xw - y)^T(Xw - y)$ to $E(f_w)$. First we will write the mathematical expression for each element in the resulting vector Xw :

$$Xw_n = \begin{bmatrix} x_1 & x_1^2 & \dots & x_1^D \\ x_2 & x_2^2 & \dots & x_2^D \\ \vdots & \vdots & \ddots & \vdots \\ x_N & x_N^2 & \dots & x_N^D \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_D \end{bmatrix} = w_1x_n + w_2x_n^2 + \dots + w_Dx_n^D = \sum_{d=1}^D w_dx_n^d$$

We can see that this is the same as $f_w(x_n)$, and will substitute that following.

Since $\sum_{d=1}^D w_dx_n^d$ is only the expression for a single element in Xw , we can write the vector as thus:

$$Xw = \begin{pmatrix} f_w(x_1) \\ f_w(x_2) \\ \vdots \\ f_w(x_n) \end{pmatrix}$$

Then, $Xw - y$ must look like this:

$$(Xw - y) = \begin{pmatrix} f_w(x_1) - y_1 \\ f_w(x_2) - y_2 \\ \vdots \\ f_w(x_n) - y_n \end{pmatrix}$$

Now, to write $(Xw - y)^T(Xw - y)$, we can see the following:

$$\begin{pmatrix} f_w(x_1) - y_1 \\ f_w(x_2) - y_2 \\ \vdots \\ f_w(x_n) - y_n \end{pmatrix}^T \begin{pmatrix} f_w(x_1) - y_1 \\ f_w(x_2) - y_2 \\ \vdots \\ f_w(x_n) - y_n \end{pmatrix} = \sum_{n=1}^N (f_w(x_n) - y_n)^2$$

We know that $a^n = (-a)^n$, so we can rewrite it as thus:

$$\sum_{n=1}^N (-f_w(x_n) + y_n)^2 = \sum_{n=1}^N (y_n - f_w(x_n))^2$$

Which is exactly what we wanted to show:

$$E(f_w) = (Xw - y)^T(Xw - y) = \sum_{n=1}^N (y_n - f_w(x_n))^2$$

2. Exercise

(a)

i)

$$\int_1^4 x^3 - 4x^2 + \sqrt{x} dx$$

First we use the sum rule:

$$\int x^3 dx - \int 4x^2 dx + \int \sqrt{x} dx$$

These are trivial, so we calculate them separately and get the following:

$$\frac{x^4}{4} - \frac{4x^3}{3} + \frac{2}{3}x^{\frac{3}{2}}$$

Now we calculate these at the boundaries and subtract the result of the lower from the upper:

$$\left(\frac{4^4}{4} - \frac{4 * 4^3}{3} + \frac{2}{3} * 4^{\frac{3}{2}} \right) - \left(\frac{1}{4} - \frac{4}{3} + \frac{2}{3} * 1^{\frac{3}{2}} \right) = -16 - \left(\frac{-5}{12} \right)$$

Thus, we have found that the definite integral equals:

$$-16 - \left(\frac{-5}{12} \right)$$

ii)

$$\int \ln(\sqrt{x}) dx$$

First we rewrite this to the following:

$$\int \frac{\ln(x)}{2} dx$$

We now take the constant out:

$$\frac{1}{2} \int \ln(x)$$

We now do integration by parts:

$$\frac{1}{2} \left(x \ln(x) - \int 1 dx \right)$$

The integral of a constant is $\int c = cx$:

$$\frac{1}{2} \left(x \ln(x) - x \right)$$

Now we have to add a constant C , and we're done:

$$\frac{1}{2} \left(x \ln(x) - x \right) + C$$

iii)

$$\int x^2 e^{x^3} dx$$

Here, we have to do u-substitution:

$$\int x^2 e^u dx$$

We now have:

$$\begin{aligned} du &= 3x^2 dx \\ du * \frac{1}{3x^2} &= dx \end{aligned}$$

With this we can substitute dx :

$$\int e^u x^2 \frac{1}{3x^2} du$$

This can be rewritten;

$$\int \frac{1}{3} e^u du$$

We take the constant out, and know that $\int e^x = e^x$, so we now have the result:

$$\frac{1}{3} \int e^{x^3} dx$$

We end up with:

$$\frac{1}{3} e^{x^3}$$

iv)

$$\int e^{-5r} dr$$

Here, we have to do substitution again:

$$\int e^u dx$$

We know that:

$$du = -5dx$$

We can then replace dx :

$$\int \frac{e^u}{-5} du$$

Now we have to take the constant out:

$$\frac{-1}{5} \int e^u du$$

And following the same rule as before $\int e^x = e^x$:

$$\frac{-1}{5} e^{-5r}$$

3. Exercise

(a)

The expression *pmf* stands for the "probability mass function".

The *pmf* of $X \sim \text{Poiss}(\lambda)$ can be written as follows, as per table 1.1 in the "MASD_essentials_ch1-updated.pdf" on Absalon:

$$p(x) = e^{-\lambda} \frac{\lambda^x}{x!}$$

This expression is used to calculate the probability of x events happening, when there is an average of λ events happening.

(b)

The expectation of Y , $\mathbb{E}Y$ can be written as follows, as per definition 1.8 in the "MASD_essentials_ch1-updated.pdf" on Absalon:

$$\mathbb{E}Y = \sum_y yp(y)$$

The variance of Y , $\text{var}(Y)$, can be written as follows, as per definition 1.10 in the "MASD_essentials_ch1-updated.pdf" on Absalon:

$$\text{var}(Y) = \mathbb{E}(Y - \mathbb{E}Y)^2 = \sum_y y^2 p(y) - (\sum_y yp(y))^2$$

To find $\mathbb{P}(a < V \leq b)$ we first have to make the observation that the cdf $F_V(x)$ is defined as being the probability that the variable V will be *less than or equal to* x , thus we can say that:

$$F_V(x) = \mathbb{P}(V \leq x)$$

This means that to calculate the probability of it being inside the interval $a < V \leq b$, we calculate $\mathbb{P}(V \leq b)$, and subtract $\mathbb{P}(V \leq a)$.

$$\mathbb{P}(a < V \leq b) = F_V(b) - F_V(a)$$

4. Exercise

(a)

For this, I assume that the given values for x and y are all the values possible. If that assumption is true, then the final joint pmf will look as follows:

$p(x, y)$	$y=-1$	$y=1$	$y=3$
$x=-1$	0.1	0.1	0.4
$x=1$	0.3	0.05	0.05

The reason for this is because the joint pmf must be equal to 1. So we simply calculate:

$$1 - (0.1 + 0.1 + 0.3 + 2(0.05)) = 0.4$$

(b)

To compute $\mathbb{E}(X + Y)$ we have to make the following observation:

$$\mathbb{E}(X + Y) = \mathbb{E}X + \mathbb{E}Y$$

This means that we simply have to compute:

$$\mathbb{E}X + \mathbb{E}Y = \sum_x x p_X(x) + \sum_y y p_Y(y)$$

To find the marginal pmfs $p_X(x)$ and $p_Y(y)$ we use the following calculations:

$$p_X(x) = \sum_y p_{X,Y}(x, y)$$

$$p_Y(y) = \sum_x p_{X,Y}(x, y)$$

First we will find $\mathbb{E}X$, then $\mathbb{E}Y$:

$$\mathbb{E}X = -1(0.1 + 0.1 + 0.4) + 1(0.3 + 0.05 + 0.05) = -0.2$$

$$\mathbb{E}Y = -1(0.1 + 0.3) + 1(0.1 + 0.05) + 3(0.4 + 0.05) = 1.1$$

Then we simply write:

$$\mathbb{E}(X + Y) = -0.2 + 1.1 = 0.9$$

This should be interpreted as expecting to make back 90% of the money put in.

(c)

First we will compute the pmf p_Z of $Z = X * Y$.
To do this, first we will define what p_Z looks like:

$$p_Z(z) = \mathbb{P}(Z = z)$$

We can see from the table from Exercise 4a that there are six possible values for Z , since there are 2 values for X and 3 for Y . These values are the following:

$X * Y$	$y = -1$	$y = 1$	$y = 3$
$x = -1$	1	-1	-3
$x = 1$	-1	1	3

Now we simply compute $\mathbb{P}(X * Y)$ for each value of x and y , and get the following table:

$p_Z(z)$	$\mathbb{P}(X * Y = z)$
$z = -3$	0.4
$z = -1$	0.4
$z = 1$	0.15
$z = 3$	0.05

To figure out if X and Y are independent, we have to look at their covariance, which is defined as thus:

$$\text{cov}(X, Y) = \mathbb{E}XY - \mathbb{E}X\mathbb{E}Y$$

We start off by finding $\mathbb{E}XY$:

$$\mathbb{E}XY = \sum_{x,y} xyp(x,y)$$

I have done this calculation in the attached Jupyter Notebook named: "Calculations" since it would be tedious to do by hand. It has given me the following:

$$\mathbb{E}XY = -1.3$$

We have already calculated $\mathbb{E}X$ and $\mathbb{E}Y$ further up, in exercise 4b. So we can write the following:

$$\text{cov}(X, Y) = (-1.3) - (-0.2) * 1.1 = -1.08$$

Since we now know that the covariance $\text{cov}(X, Y) \neq 0$, we also know that X and Y are dependant on each other.

(d)

To show that $cov(V, W) = 0 \implies V$ and W are independent for Bernoulli distributions of V and W , we first have to look at the definition of independence, as per Definition 1.15 in "MASD_essentials_ch1-updated.pdf" on Absalon. I will rewrite it slightly so it fits the exercise at hand:

$$p(x, y) = p(x)p(y) \text{ for all } x, y \implies \text{mutual independence}$$

So now we can see that we need to show that:

$$\mathbb{E}VW - \mathbb{E}V\mathbb{E}W = 0 \implies p(v, w) - p(v)p(w) = 0 \implies p(v, w) = p(v)p(w)$$

We know that for Bernoulli distributions $\mathbb{E}V = \theta$ since:

$$\mathbb{E}V = \sum_v vp(v) = \mathbb{P}(V = 0) * 0 + \mathbb{P}(V = 1) * 1 = \theta * 1 + (1 - \theta) * 0 = \theta$$

The same of course holds for $\mathbb{E}W = \phi$

We can do the same for $\mathbb{E}VW$:

$$\mathbb{E}VW = \sum_{v,w} vwp(v, w)$$

Since this expression always will be zero except in $(V = 1, W = 1)$, we can simplify the equation:

$$\mathbb{E}VW = \theta\phi$$

Now we can insert these values into $p(v)$, $p(w)$ and $p(v, w)$:

$$p(v) = \begin{cases} 1 - \mathbb{E}V & \text{if } v = 0 \\ \mathbb{E}V & \text{if } v = 1 \end{cases}$$

$$p(w) = \begin{cases} 1 - \mathbb{E}W & \text{if } w = 0 \\ \mathbb{E}W & \text{if } w = 1 \end{cases}$$

$$p(v, w) = \begin{cases} (1 - \mathbb{E}V) * (1 - \mathbb{E}W) & \text{if } v = 0, w = 0 \\ \mathbb{E}V * (1 - \mathbb{E}W) & \text{if } v = 1, w = 0 \\ (1 - \mathbb{E}V) * \mathbb{E}W & \text{if } v = 0, w = 1 \\ \mathbb{E}V\mathbb{E}W & \text{if } v = 1, w = 1 \end{cases}$$

We can now see that for all values of V and W , the following must be true:

$$\mathbb{E}VW - \mathbb{E}V\mathbb{E}W = 0 \implies p(v, w) - p(v)p(w) = 0 \implies p(v, w) = p(v)p(w)$$

And since $cov(V, W) = \mathbb{E}VW - \mathbb{E}V\mathbb{E}W$, the proof is finished.