Week 3 Hand-in

NumIntro 2019
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Version 1
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1. Exercise

To find the general solution for the recurrence relation, of the form $p(E)x$, we first rewrite it:

$$\begin{aligned} x_n &= 2(x_{n-1} + x_{n-2}) \\ 0 &= -x_n + 2(x_{n-1} + x_{n-2}) \\ p(\lambda) &= -\lambda^2 + 2\lambda + 2 \end{aligned}$$

We then use Wolfram Alpha to solve for $p(\lambda) = 0$:

$$\lambda_1 = 1 - \sqrt{3}$$

$$\lambda_2 = 1 + \sqrt{3}$$

Thus, the general solution is:

$$x_n = \alpha(1 + \sqrt{3})^n + \beta(1 - \sqrt{3})^n$$

We can determine that the difference equation is unstable since all roots do not satisfy the requirement of being ≤ 1.

2. Exercise

In this exercise we have to show that for $n = 1$ and $n = 2$ with given solutions $x_1 = 1$ and $x_2 = (1 - \sqrt{3})$, the values $\alpha = 0$ and $\beta = (1 - \sqrt{3})^{-1}$ hold true. We do this by substituting the values in the equation:

$$\begin{aligned} x_n &= \alpha(1 + \sqrt{3})^n + \beta(1 - \sqrt{3})^n \\ x_1 &= 0(1 + \sqrt{3})^1 + (1 - \sqrt{3})^{-1}(1 - \sqrt{3})^1 = \frac{(1 - \sqrt{3})}{(1 - \sqrt{3})} = 1 \\ x_2 &= 0(1 + \sqrt{3})^2 + (1 - \sqrt{3})^{-1}(1 - \sqrt{3})^2 = \frac{(1 - \sqrt{3})^2}{1 - \sqrt{3}} = (1 - \sqrt{3}) \end{aligned}$$

3. Exercise

We can see that there are two parts to this function, $\alpha(1 + \sqrt{3})^n$ and $\beta(1 - \sqrt{3})^n$, and start from there.
We can then notice that $(1 + \sqrt{3}) > 1$. This means that $\alpha(1 + \sqrt{3})^n > 1, n \in \mathbb{R}$, in fact it diverges to infinity as n increases, as long as $\alpha > 0$.
We then look at $(1 - \sqrt{3}) < 1$. We know that any number $0 < x^n < 1 \to 0$ as n approaches infinity. We can see that this holds for this part of the function as well, as we can substitute x with $\beta(1 - \sqrt{3})$. This means that $\beta(1 - \sqrt{3})^n \to 0, \beta \neq 0$. If $\beta = 0$ then the function is trivially $= 0$.

We can combine this and realise that for any $\alpha \neq 0$, the series will diverge. Conversely, if $\alpha = 0$, $\beta \in \mathbb{R}$ and $\beta \neq 0$, the general solution will converge on 0.

For these values we can rewrite x_n:

$$x_n = (1 - \sqrt{3})^{n-1}$$

We then try to find a $c < 1$ to prove the order of convergence:

$$\begin{aligned} |x_{n+1} - x^*| &\leq c|x_n - x^*| \\ \frac{|x_{n+1}|}{|x_n|} &\leq c \\ \frac{(1-\sqrt{3})^n}{(1-\sqrt{3})^{n-1}} &\leq c \end{aligned}$$

We use Wolfram Alpha to find that:

$$\frac{(1-\sqrt{3})^n}{(1-\sqrt{3})^{n-1}} \to 1 - \sqrt{3}$$

This means that we have a solution for all $1 - \sqrt{3} \leq c < 1$. We have thus proven that the order of convergence is at least linear.

4. Exercise

This exercise has been completed in the Q0042.py file which has been handed in on Absalon.

5. Exercise

The following plots have been generated using the Q0042.py file: