Assignment 1

MASD 2018

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Contents

Exercise 1

(a)

For this proof we can assume the following:

$$\{a_n\} \text{ with } lim_{n\to\infty} a_n = a$$

And we need to prove that the following is true:

$$\{-a_n\} \text{ with } lim_{n\to\infty} - a_n = -a$$

To do this, we need to prove the following:

$$|-a_n + a| \leq |a_n - a| < \epsilon$$

From the definition of limits, we know that the following is true for a:
For every $\epsilon > 0$ there exists an integer $N \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ for all $n > N$.
We know, using the rule that $|a| = |-a|$, that the following must be true:

$$|a_n - a| = |-(a_n - a)| = |-a_n + a| < \epsilon \implies |-(a_n - a)| < \epsilon$$

Now that we know this is true, our claim must also be true.

(b)

First we want to prove this one way; thus, we want to show that:

$$|(a_n)_i - a_i| \leq ||a_n - a|| < \epsilon$$

We assume that: $a_n \to a$ is true, and thus, the definition of convergence must also be true for it, hence $|a_n - a| \leq \epsilon$ must be true. Since we know that the square root of the sum of all squared numbers in the vector is equal to the length of the vector, we can say the following:

$$\sqrt{\sum_{i=1}^{d} (a_n)_i^2} \leq ||a_n||$$

and

$$\sqrt{\sum_{i=1}^{d} a_i^2} \leq ||a||$$

Thus, we can show that:

$$\sqrt{\sum_{i=1}^{d} (a_n)_i^2} - \sqrt{\sum_{i=1}^{d} a_i^2} \leq ||a_n - a|| < \epsilon$$

With this, we have shown that:

$$a_n \to a \implies (a_n)_i \to a_i$$

Now we have to show the opposite:

$$(a_n)_i \to a_i \implies a_n \to a$$

This means that

$$|(a_n)_i \to a_i| < \epsilon$$

Following the same rules as before, we can see that we have to prove the following:

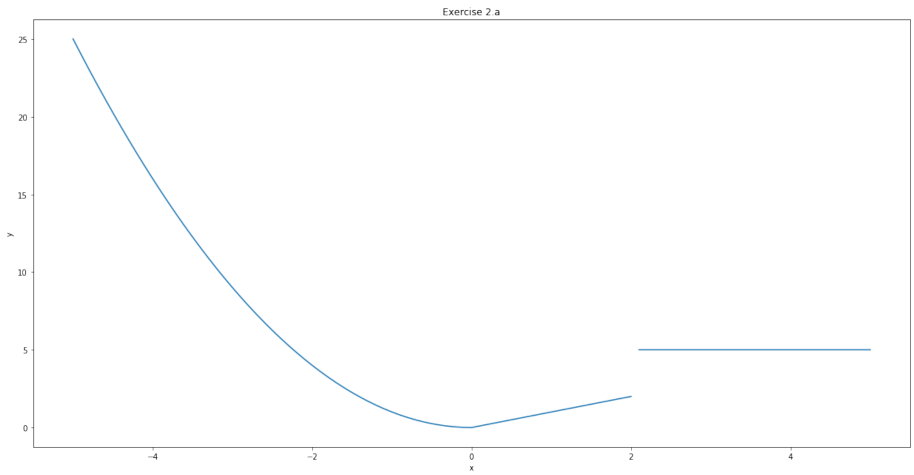
$$|a_n - a| \leq |(a_n)_i - a_i| < \epsilon$$

Exercise 2

(a)

Python code to plot the graph:

```
xs = np.linspace(-5, 5, 101)
ys = np.zeros(xs.shape)

# Drawing the function at all values of x
for i in range(101):
    if xs[i] < 0:
        ys[i] = xs[i]**2
    elif xs[i] >= 0 and xs[i] <= 2:
        ys[i] = xs[i]
    else:
        ys[i] = 5

# Removing lines drawn where the function is discontinuous
pos = np.where(np.abs(np.diff(ys)) >= 1)[0]+1
xs = np.insert(xs, pos, np.nan)
ys = np.insert(ys, pos, np.nan)

# Plotting
plt.plot(xs, ys)
plt.title('Exercise 2.a')
plt.xlabel('x')
plt.ylabel('y')
```



There is a single point in which the function is not continuous, in the point $x = 2$, where the function jumps from 2 to 5.

(b)

We know trivially that the functions $f(x) = x^2$, $f(x) = x$ and $f(x) = c$ are continuous. Thus, we have to prove that the function is continuous in $x = 0$, which is the place the two different functions meet, and discontinuous in $x = 2$.
First we want to prove continuity in $x = 0$. We do this by proving the following:
$lim_{x \to a} f(x)$ exists for all $a \in \mathbb{R}$.
To do this, we will show the following:
If $a < 0$, then $lim_{x \to a} f(x) = lim_{x \to a} x^2 = a^2$
If $a \in [0, 2]$, then $lim_{x \to a} f(x) = lim_{x \to a} x = a$
If $a > 2$, then $lim_{x \to a} f(x) = lim_{x \to a} 5 = 5$
If $a = 0$, then $lim_{x \to a} f(x) = lim_{x \to a} x = a$ and $lim_{x \to a} f(x) = lim_{x \to a} x^2 = a^2$
If $a = 2$, then $lim_{x \to a} f(x) = lim_{x \to a} x = a$ and $lim_{x \to a} f(x) = lim_{x \to a} 5 \neq a = 2$

Now we have shown that the limit exists everywhere except in $x = 2$. Thus we only have to prove the continuity in $x \neq 2$. We do this by proving the following:
$lim_{x \to a} f(x) = f(a)$ for all $a \in \mathbb{R}$.
We do this for all cases except $x = 2$, since we already know it is discontinuous at that point.
If $a < 0$, then $lim_{x \to a} f(x) = a^2 = f(a)$
If $a \in [0, 2]$, then $lim_{x \to a} f(x) = a = f(a)$
If $a > 2$, then $lim_{x \to a} f(x) = 5 = f(a)$
If $a = 0$, then $lim_{x \to a} f(x) = a^2 = f(a)$ and $lim_{x \to a} f(x) = a = f(a)$

Exercise 3

(a)

The sequence defined as thus will break the test, and be defined as convergent, when in reality it is divergent at $x \geq 50$. It will be considered convergent when $t = 1$, but may pass the test at smaller t values. The first part for $x < 50$ is the convergent series taken from the assignment. The second series is the divergent series $\sum_{n=50}^{\infty} \frac{1}{n}$. I have found this series here: `https://en.wikipedia.org/wiki/Harmonic_series_(mathematics)`

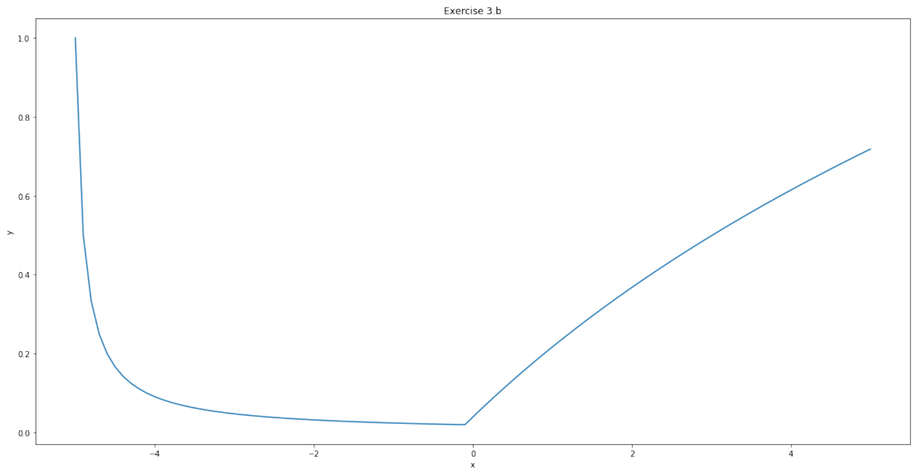
$$f(x) = \begin{cases} \frac{1}{n} & when \quad x \leq 50 \\ \sum_{n=1}^{\infty} \frac{1}{n} & when \quad x > 50 \end{cases}$$

(b)

The code used to check whether the test will determine the sequence as convergent or divergent. At the bottom is a plotted graph of the sequence made in 3a.

```
xs = np.linspace(-5, 5, 101)
ys = np.zeros(xs.shape)

epsilon = 1

for i in range(101):
    #put the function to be tested here
    if i < 50:
        ys[i] = 1/(i+1) #1/n
    else:
        ys[i] = ys[i-1] + 1/(i+1)




isconvergent = True

for i in range(ys.size):
    if i > 1:
        if abs(ys[i-1]-ys[i]) > epsilon and isconvergent:
            isconvergent = False
            print ("is not convergent")

if isconvergent:
    print ("is convergent")
```

```
# Removing lines drawn where the function is discontinuous
pos = np.where(np.abs(np.diff(ys)) >= 1)[0]+1
xs = np.insert(xs, pos, np.nan)
ys = np.insert(ys, pos, np.nan)

plt.plot(xs, ys) # Plotting
plt.title('Exercise 3.b')
plt.xlabel('x')
plt.ylabel('y')
```



(c)

As seen in the test of the code, the criterion holds only when t (named epsilon in the code) is small enough. The only reason the test didn't pass, was because the epsilon value was too big to be accurate. This means that the test is only good when the user knows which scale the epsilon value should be at. It is obviously not a proper test of convergence since it can be made to fail when given a too large input from the user.

Exercise 4

In this proof, we have to show that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

To do this, we have to make the following assumption:

$$lim_{n \to \infty} \frac{1}{2^n} = 0$$

First, I'll define what this sequence looks like:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + ... + \frac{1}{2^n}$$

Then, we have to look at the following, different sequence:

$$\sum_{n=1}^{\infty} 2(\frac{1}{2^n}) = 1 + \frac{1}{2} + \frac{1}{4} + ... + \frac{1}{2^n}$$

We can then observe that the following is true:

$$\sum_{n=1}^{\infty} 2(\frac{1}{2^n}) = 1 + \sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$$

Now we just reduce this to the following:

$$\sum_{n=1}^{\infty} 2(\frac{1}{2^n}) - \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 + \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{2^n}$$

$$\Longleftrightarrow$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

Now we use the assumption that $lim_{n\to\infty} \frac{1}{2^n} = 0$, and can thus write that the following is true as n approaches infinity.

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$