1) Dada $A = \begin{pmatrix} 1/3 & k & 0 \\ 0 & 0 & 1 \\ k & 1/3 & 2 \end{pmatrix}$, se quiere resolver $Ax = b$

Se propone

$$\underbrace{\begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1/3 & 2 \end{pmatrix}}_{=M} * X_{N+1} = -\underbrace{\begin{pmatrix} 0 & k & 0 \\ 0 & 0 & 0 \\ k & 0 & 0 \end{pmatrix}}_{=N} x_N + b$$

a) Probar q' si $X_N \to X^*$ $\Rightarrow$ X^* es solucion de $Ax = b$

b) Hallar k para que haya convergencia $\forall x_0$

c) ¿Que condicion impondrias sobre k para generalizar que existe una norma $\|\cdot\|$ que verifique $\|X_M - X^*\| \leq \left(\frac{1}{4}\right)^N \|X_0 - X^*\|$

$$M x_{n+1} = -N \cdot x_n + b$$

$$\downarrow n\to\infty \qquad \downarrow n\to\infty$$

$$M x^* = -N \cdot x^* + b$$

$$M x^* + N x^* = b$$

$$(M + N)\, x^* = b$$

$$A\, x^* = b \quad ✓$$

b) $A = \begin{bmatrix} 1/3 & k & 0 \\ 0 & 0 & 1 \\ k & 1/3 & 2 \end{bmatrix}$ $M = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1/3 & 2 \end{bmatrix}$ $N = \begin{bmatrix} 0 & k & 0 \\ 0 & 0 & 0 \\ k & 0 & 0 \end{bmatrix}$

Matriz de iteración : $-M^{-1} . N$

$$x_{n+1} = -M^{-1} . N . x_n + M^{-1} . b$$

Quiero $\rho\left(-M^{-1}.N\right) = \max|\lambda_i|$

Pero tambien puedo calcular las raíces de

$$\det(\lambda M + N) = 0$$

$$\left|\lambda.\begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1/3 & 2 \end{bmatrix} + \begin{bmatrix} 0 & k & 0 \\ 0 & 0 & 0 \\ k & 0 & 0 \end{bmatrix}\right| = \left\|\begin{bmatrix} \lambda/3 & k & 0 \\ 0 & 0 & \lambda \\ k & \lambda/3 & 2\lambda \end{bmatrix}\right\|$$

$$= -\lambda.\begin{vmatrix} \lambda/3 & k \\ k & \lambda/3 \end{vmatrix} = -\lambda\left(\frac{\lambda^2}{9} - k^2\right) = 0$$

$$\lambda = 0$$

ó

$$\frac{\lambda^2}{3^2} = k^2 \Rightarrow k = \pm\frac{\lambda}{3}$$

$$\Updownarrow$$

$$\lambda = 3|k| < 1$$

$$|k| < \frac{1}{3}$$

c) $\|x_n - x^*\| \leq \left(\frac{1}{4}\right)^n \|x_0 - x^*\|$

$$e_n = \left(-M^{-1}.N\right)^n . e_0$$

$$\|e_n\| = \|(-M^{-1}\cdot N)^n \cdot e_0\|$$

$$\|e_n\| \leq \|(-M^{-1}\cdot N)^n\| \cdot \|e_0\|$$

$$\|$$

$$\|(-M^{-1}\cdot N)\|^n$$

$$\rho(-M^{-1}\cdot N) \leq \|(-M^{-1}\cdot N)\| < \rho(-M^{-1}\cdot N) + \varepsilon$$

$$\rho(-M^{-1}\cdot N) = |3k|$$