Detection of financial bubbles using a log-periodic power law singularity (LPPLS) model

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Abstract

This paper provides a systematic review of the theoretical and empirical academic literature on the development and extension of the log-periodic power law singularity (LPPLS) model, which is also known as the Johansen-Ledoit-Sornette (JLS) model or log-periodic power law (LPPL) model. Developed at the interface of financial economics, behavioral finance and statistical physics, the LPPLS model provides a flexible and quantitative framework for detecting financial bubbles and crashes by capturing two salient empirical characteristics of price trajectories in speculative bubble regimes: the faster-than-exponential growth of price leading to unsustainable growth ending with a finite crash-time and the accelerating log-periodic oscillations. We also demonstrate the LPPLS model by detecting the recent bubble status of the S&P 500 index between April 2020 and December 2022, during which the S&P 500 index reaches its all-time peak at the end of 2021. We find that the strong corrections of the S&P 500 index starting from January 2022 stem from the increasingly systemic instability of the stock market itself, while the well-known external shocks, such as the decades-high inflation, aggressive monetary policy tightening by the Federal Reserve, and the impact of the Russia/Ukraine war, only serve as sparks.

Keywords: Log-periodic power law singularity (LPPLS) model, Johansen-Ledoit-Sornette (JLS) model, financial bubble and crash, bubble indicator, S&P 500.

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1. INTRODUCTION

Financial bubbles and crashes are not rare phenomena in modern financial markets. In the US financial markets over the past 30 years, there have been seven distinct crashes, including the 1987 stock market crash, the collapse of Long-Term Capital Management following the Russian debt crisis and the subsequent market crash of 1998, the bursting of the dot-com bubble during the period 1999–2001, financial meltdown subsequent to the subprime mortgage crisis during the period 2007–2009, and the Coronavirus crash of the 2020 US stock market. Each crash of bubbles can result in a permanent impairment of wealth, at least for some investors. Especially when a bubble originates in a commonly held asset such as stock market, a bursting of bubble can be catastrophic as it can have a significant impact on the lives and livelihoods of most people all over the country or world. For example, the 2007-2009 financial crisis leads to the cumulative economic toll in terms of lost output as much as $13 trillion, over six million job losses, and total household losses of about $9.1 trillion in the US, resulting in the worst economic contraction in US history since the Great Depression of the 1930s (GAO, 2013). However, due to the complexity of local and global economies and the lack of clearly defined fundamental value of assets, it is difficult to characterize, estimate, and more importantly, forecast and possibly avoid bubbles in advance.

In order to effectively detect the presence of a bubble, the log-periodic power law singularity (LPPLS) model (Drozdz et al., 1999; Feigenbaum & Freund, 1996; Johansen et al., 1999, 2000; Sornette et al., 1996; Sornette & Johansen, 1998, 2001), also known as the Johansen-Ledoit-Sornette (JLS) model or Log-Periodic Power Law (LPPL) model, has been developed by combining the economic theory of rational expectation bubbles, the behavioral finance on imitation and herding of traders, and the mathematical and statistical physics of bifurcations and phase transitions. The LPPLS model contains the ingredients: (1) a system of traders who are influenced by their neighbors, (2) local imitation propagating spontaneously into global cooperation, (3) global cooperation between traders leading to crash, and (4) prices related to the properties of this system (Johansen & Sornette, 2001a). The LPPLS model provides a flexible and quantitative framework for detecting financial bubbles by analyzing the temporal price series of an asset.

The LPPLS model captures two distinct characteristics of price trajectory of an asset normally observed in the regime of speculative bubbles: the transient super-exponential growth and the accelerating log-periodic oscillations, to model financial bubble as a process of faster-than-exponential power law growth punctuated by short-lived corrections. The faster-than-exponential growth of price results from positive feedback mechanisms, including (1) from a technical perspective: option hedging, insurance portfolio strategies, market makers bid-ask spreads for past volatility, business networks and human capital accumulation, bank-to-business procyclical financing, trend-following investment strategies, asymmetric information for hedging strategies, and the interplay of mark-to-market accounting and regulatory capital requirements, and (2) from a behavioral perspective: imitation, the social gregariousness, and the herding of humans (Yan, 2011). The log-periodic oscillations of price growth of an asset in a bubble regime results from the competition between different types of traders. The interplay between technical trading and herding (positive feedback) and fundamental valuation investments (negative mean-reverting feedback) is the main source of price dynamic. The second-order oscillations of price dynamics results from the presence of inertia between information gathering and analysis on the one hand and investment implementation on the other hand, and the coexistence of trend followers and value investing (Farmer, 2002). In recent years, many scholars have studied the LPPLS model for detecting speculative bubbles by analyzing the log-periodic signatures embedded in the asset price trajectories.

This paper provides a systematic review of the theoretical and empirical academic literature on the development and extension of the LPPLS model to detect financial bubbles and crashes. We also demonstrate this methodology by detecting the recent bubble status of the S&P 500 index between April 2020 and December 2022. The paper is organized as the following: Section 2 presents a brief description of the LPPLS methodology, Section 3 surveys both theoretical and empirical literature on the LPPLS model, Section 4 shows an application of LPPLS model to the S&P 500 index, and, finally, Section 5 concludes the study.

2. METHODOLOGY

2.1 The log-periodic power law singularity model

The LPPLS model (Johansen et al., 1999, 2000) assumes that the asset price $p(t)$ follows a standard diffusive dynamics with drift $\mu(t)$ and discrete discontinuous jumps:

$$\frac{dp}{p} = \mu(t)dt + \sigma(t)dW - kdj \tag{1}$$

where $\sigma(t)$ is the volatility, dW is the infinitesimal increment of a standard Wiener process, κ is the crash amplitude when it occurs, and dj represents a discontinuous jump with the value of zero before the crash and one afterwards. The dynamics of the jumps is governed by a crash hazard rate $h(t)$, which is the probability per unit of time that the crash will happen in the next instant given that it has not happened yet. The crash risk rate $h(t)$ quantifies the probability that a large group of agents place sell orders at the same time to create enough imbalance in the order book that market makers cannot absorb the other side without significantly lowering the price. Thus, we have the expectation of jump $E_t[dj] = 1 \times h(t)\mathrm{d}t + 0 \times (1 - h(t)\mathrm{d}t) = h(t)\mathrm{d}t$. Under the assumption of no arbitrage condition and rational expectations, the price process is a martingale by ignoring the interest rate, information asymmetry and the market-clearing condition, leading to the conditional expectation of the price dynamics $E_t[\mathrm{d}p] = 0$ and $\mu(t) = \kappa h(t)$ which indicates that the drift $\mu(t)$ is controlled by the crash hazard rate $h(t)$, and the $\mu(t)$ increases as $h(t)$ increases to compensate the traders for the increased risk.

The network structure underlying the LPPLS model assumes two types of agents presented in the market: a group of traders with rational expectations and a group of noise traders who exhibit herding behavior to destabilize the asset price and make irrational and erratic decisions to buy, sell, or hold. The LPPLS model assumes that the crash hazard rate aggregated by the noise traders has the dynamics:

$$h(t) = \alpha\left(t_c - t\right)^{m-1}\left(1 + \beta cos(\omega\, ln(t_c - t) - \phi')\right) \tag{2}$$

where $\alpha, \beta, m, \omega,$ and ϕ' are the parameters and t_c is the critical time for the end of a bubble, i.e., the theoretical termination time of a bubble or the most probable time of a crash. Bubbles can be terminated in the form of large crashes or changes of the average growth rate (regime change), so super-exponential growth rates of price change to an exponential or lower growth rates as accelerating oscillations end.

The power law singularity $(t_c - t)^{m-1}$ embodies the mechanism of positive feedback related to the herding behavior of noise traders, resulting in the formation of bubbles, and it reaches the singularity when time t equals to the critical time t_c. The log-periodic function $\cos(\omega \ln(t_c - t) - \phi')$ considers the existence of a possible hierarchical cascade of panic acceleration punctuating the bubble growth, which may be due to a pre-existing hierarchies in noise trader sizes (Sornette & Johansen, 1997) and/or the interplay between market price impact inertia and nonlinear fundamental value investing (Ide & Sornette, 2002).

Using $\mu(t) = \kappa h(t)$, the dynamics of the logarithmic price expectation in the form of the original LPPLS model can be derived as:

$$\mathrm{LPPLS}(t) \equiv E_t[\ln p(t)] = A + B(t_c - t)^m + C(t_c - t)^m \cos[\omega\, ln(t_c - t) - \phi] \tag{3}$$

where the constant $A = \ln p(t)$ gives the expected terminal log-price at the critical time t_c. $B = -\frac{k\alpha}{m}$ controls the amplitude of the power law acceleration and $B < 0$ ($B > 0$) when the price is indeed growing (decreasing) super-exponentially as time goes towards t_c. $C = -\frac{k\alpha\beta}{\sqrt{m^2+\omega^2}}$ controls the proportional magnitude of the log-periodic oscillations around the power law singular growth. The exponent $m \in (0, 1)$ quantifies the degree of faster-than-exponential and $m > 0$ ensures that the price remains finite at t_c, while $m < 1$ ensures that a singularity exist at t_c, i.e., the expected log-price diverges at t_c. ω is the log-periodic angular frequency of the oscillation which is related to the fundamental scaling ratio $\lambda = \exp\left(\frac{2\pi}{\omega}\right)$ of the temporal hierarchy of accelerating oscillations converging to t_c. The local maxima of the price function are separated by time intervals that tend to zero on the critical date and do so in a geometric progression such that the ratio of consecutive time intervals is a constant λ of the underlying discrete scale invariance, which is very useful from an empirical perspective, because such oscillations are more evident in real data than simple power law and the fitted oscillations contain information about t_c (Johansen et al., 1999). $\phi \in (0, 2\pi)$ is a phase parameter that embodies the characteristic time scale of the oscillations. Equation (3) is the first-order log-periodic correction to a pure power law exhibiting a singularity at t_c and can be thought of as just the first term in a general log-periodic Fourier series expansion of the general form.

The cosine term of Equation (3) can be expanded with $C_1 = C\cos\phi$ and $C_2 = C\sin\phi$, leading to rewrite Equation (3) as (Filimonov & Sornette, 2013):

$$\begin{aligned} \text{LPPLS}(t) \equiv E_t[\ln p(t)] = A + B(t_c - t)^m + C_1(t_c - t)^m \cos[\omega\, ln(t_c - t)] \\ + C_2(t_c - t)^m \sin[\omega\, ln(t_c - t)] \end{aligned} \tag{4}$$

The reformed LPPLS model reduces the 4 nonlinear parameters $\{t_c, m, \omega, \phi\}$to 3 nonlinear parameters $\{t_c, m, \omega\}$, which greatly improve the efficiency and stability of the model calibration.

It should be noted that the LPPLS model can only diagnose endogenous bubbles and crashes where the changes in price trajectories are caused by the self-reinforcing synergistic herding and imitative behaviors through interactions of market participants in the long-memory process of endogenous organizations, such as the Wall Street Crash of October 1929 and the dot.com crash of April 2000. In exogenous bubbles and crashes, the exogenous news or shocks can change an asset's fundamental value underlying a financial market and leave observable signatures in the price trajectories; however, two distinct characteristics, i.e., faster-than-exponential growth and the accelerating log-periodic oscillations, are absent from the corresponding price trajectories, such as the Nazi invasion of the western Europe on May 10, 1940, and 2022 Russian Rubles Crash.

2.2 Model calibration

The LPPLS model in Equation (4) is composed of the three nonlinear parameters (t_c, m, ω) and the four linear parameters (A, B, C_1, C_2). The LPPLS model can be calibrated using the Least-Squares method of minimizing the sum of squared residuals:

$$\begin{aligned} F(t_c, m, \omega, A, B, C_1, C_2) = \sum_{i=1}^{N} [ln\, p\,(\tau_i) - A - B\big(t_c - \tau_i\big)^m - C_1(t_c - \tau_i)^m \cos(\omega\, ln(t_c - \tau_i)\big) \\ - C_2(t_c - \tau_i)^m \sin(\omega\, ln(t_c - \tau_i))]^2 \end{aligned} \tag{5}$$

where $\tau_1 = t_1$ and $\tau_N = t_2$. Slaving the four linear parameters (A, B, C_1, C_2) to the three nonlinear parameters (t_c, m, ω), we can obtain a nonlinear optimization problem:

$$\{\hat{t}_c, \hat{m}, \hat{\omega}\} = \arg \min_{\{t_c, m, \omega\}} F_1(t_c, m, \omega) \tag{6}$$

where the cost function $F_1(t_c, m, \omega) = \min_{(A,B,\, C_1, C_2)} F(t_c, m, \omega, A, B,\, C_1, C_2)$, and this optimization problem can be rewritten as:

$$\begin{aligned} (\hat{A}, \hat{B},\, \hat{C}_1, \hat{C}_2) &= \arg \min_{(A,B,\, C_1,C_2)} F(t_c, m, \omega, A, B,\, C_1, C_2) \\ &= \arg \min_{(A,B,\, C_1,C_2)} \sum_{i=1}^{N} [\ln p\,(\tau_i) - A - Bf_i - \, C_1 g_i - C_2 h_i]^2 \end{aligned} \tag{7}$$

where $f_i = (t_c - t_i)^m$, $g_i = (t_c - t_i)^m \cos(\omega \ln(t_c - t_i))$, and $h_i = (t_c - t_i)^m \sin(\omega \ln(t_c - t_i))$. Being linear in terms of the variable $A, B,\, C_1, C_2$ for $m \neq 0$, $\omega \neq 0$, and $t_c > t_2$, one unique explicit analytical solution can be obtained from the matrix equation:

$$\begin{pmatrix} N & \sum f_i & \sum g_i & \sum h_i \\ \sum f_i & \sum {f_i}^2 & \sum f_i g_i & \sum f_i h_i \\ \sum g_i & \sum f_i g_i & \sum {g_i}^2 & \sum h_i g_i \\ \sum h_i & \sum f_i h_i & \sum g_i h_i & \sum {h_i}^2 \end{pmatrix} \begin{pmatrix} \hat{A} \\ \hat{B} \\ \hat{C}_1 \\ \hat{C}_2 \end{pmatrix} = \begin{pmatrix} \sum \ln p_i \\ \sum f_i \ln p_i \\ \sum g_i \ln p_i \\ \sum h_i \ln p_i \end{pmatrix} \tag{8}$$

which can be solved by using the LU decomposition algorithm. Therefore, the global fitting procedure is reduced to solve the nonlinear optimization problem in Equation (6).

Due to complex multiple extrema structure of the cost function, metaheuristics algorithms are usually used to calibrate the LPPLS model, such as the taboo search algorithm (Cvijovic & Klinowski, 1995), the genetic algorithm (Gulsen et al., 1995), and the covariance matrix adaptation evolution strategy (CMA-ES) (Hansen et al., 1995). Metaheuristics algorithms are universal rather than problem-specific, typically make fewer or no assumptions about cost function, do not need to calculate derivatives and can search a very large multidimensional spaces of candidate solutions, but are no guaranteed to find the optimal or even a satisfactory near-optimal solution. Based on the preliminary conditions of metaheuristics explorative search, a local search algorithm may be applied to find the optimal solution with the smallest sum of squares between the fitted model and the observations, such as, the Levenberg-Marquardt nonlinear least squares algorithm (Levenberg, 1944; Marquardt, 1963) and the Nelder-Mead simplex method (Nelder & Mead, 1965).

Although any fitting procedures with Equation (4) can produce calibration results, this does not mean that the results should be trusted. To ensure the meaningfulness of results to describe developing bubbles from the spurious ones, a range of search space and filter conditions which are derived from the empirical evidence gathered in investigations of previous bubbles, can be used during and after fitting process. The search space and filter conditions used in different literatures may be slightly different. We recommend using the search space (Shu & Zhu, 2020b): $m \in [0,1]$, $\omega \in [1, 50]$, $t_c \in \left[t_2, t_2 + \frac{t_2 - t_1}{3}\right]$, and $m|B|/\left(\omega\sqrt{{C_1}^2 + {C_2}^2}\right) \geq 1$ ensuring the non-negative crash hazard rate $h(t)$ (Bothmer & Meister, 2003); and the filter conditions (Shu & Zhu, 2020b): $m \in [0.01, 0.99]$, $\omega \in [2, 25]$, $t_c \in \left[t_2, t_2 + \frac{t_2 - t_1}{5}\right]$, $(\omega/2\pi)\ln[(t_c - t_1)/(t_c - t_2)] \geq 2.5$ for distinguishing a genuine log-periodic signal from noise-generated signal if $|C/B| \geq 0.05$ (Gerlach et al., 2019), $\max\left(\frac{|\hat{p}_t - p_t|}{p_t}\right) \leq 0.15$ ensuring that the fitted price of an asset $\hat{p}_t$

should be not too far from the observations, $p_{lomb} \leq \alpha_{\text{sign}}$ for the existence the logarithm-periodic oscillations in the fitting LPPLS model (Johansen & Sornette, 1999). and $\ln(\widehat{p_t}) - \ln(p_t) \sim \text{AR}(1)$ ensuring that the LPPLS fitting residuals can be modeled by a mean-reversal Ornstein–Uhlenbeck process when the logarithmic price in the bubble regime is attributed to a deterministic LPPLS component (Lin et al., 2014). Only calibration results that meet the filter conditions are considered valid and have the stylized features of LPPLS model.

3. LITERATURE REVIEW

A wealth of literatures have contributed to the development and extension of the LPPLS method, in particular various alternative forms of the crash hazard rate $h(t)$. Anifrani et al. (1995) proposed the universal log-periodic correction to renormalization group scaling for predicting rupture stress from acoustic emissions. Feigenbaum and Freund (1996) and Sornette et al., (1996) independently introduced the log-periodic structure that has been found in a variety of nonlinear complex physical systems constituting the self-organization and complexity paradigms to identify precursory patterns of stock market crashes by using the first-order log-periodic correction to pure power laws based on the renormalization group theory of critical phenomena, and demonstrated that stock market can be viewed as a self-organizing cooperative system. Sornette and Johansen (1997) extended the first-order LPPLS model using the second-order Landau expansion of phase transition to include both the first-order and second-order log-periodic corrections to power laws. Johansen and Sornette (1999) generalized the LPPLS models by introducing the third-order Landau expansion of phase transitions to better capture behavior away from the critical point and including the qualitative ensemble symmetry with respect to the time direction for anti-bubble. It should be stressed that the first-order log-periodic formula is embedded as a special case of the second-order formula which is itself embedded as a special case of the third-order formula. Gluzman and Sornette (2002) developed the generalized Weierstrass-type LPPL model from discrete renormalization group equations and the Weierstrass-type LPPLS model can conveniently take into account the fundamental log-frequency and its higher-order harmonics.

Sornette and Johansen (1998) presented a simple hierarchical model of trader exhibiting imitative behavior, first modeling the dynamics of financial market at the microscopic level and showing how systemic instability emerges leading to crash. Johansen et al. (2000) showed that the log-periodic signatures before large stock market crashes are not purely accidental by fitting data sets generated from GARCH models and truncated real data. Johansen and Sornette (2001a) generated empirical distributions of the exponent m and the log-periodic angular frequency ω for the bubble-fits using the price and the logarithm of the price, respectively, by investigating the bubbles and anti-bubbles in major financial markets and a set of secondary stock markets. Bothmer and Meister (2003) derived a new restriction of the free variables in the LPPLS model to ensure that the crash hazard rate must remain positive by definition.

Sornette and Zhou (2002) extended the antibubble analysis of Johansen and Sornette (1999) by using the absolute value of the time interval between the analysis time and the critical time to provide significant better and more stable fits and make the critical time weakly sensitive to the selection of start time, estimated the critical time using the Shank transformation on a hierarchy of characteristic times, adopted both the parametric Lomb periodogram method and the non-parametric (H, q) analysis of fractal signals to confirm the present of log-periodic structures, and detected second-order harmonic using the statistical theory of nested hypotheses with the Wilk's statistics. Zhou and Sornette (2003b) generalized the antibubble analysis of Sornette and Zhou (2002), in terms of the renormalization group framework to model critical points. Zhou and Sornette (2005) proposed an algorithm of detecting regime change of antibubble phase by combining the likelihood-ratio test and bootstrap method to track the cross-over regime in log-frequency shift, i.e., change of regime from the first-order to the second-order log-periodic formula, and develop a battery tests for the possible end of antibubbles.

Zhou and Sornette (2002) generalized q analysis to detect log-periodic structure in complex systems by constructing a signature (H, q) derivative of discrete scale invariance and this nonparametric tool is a useful complement and confirmation to the parametric Lomb periodogram method. Zhou and Sornette (2003a) applied two nonparametric methods: (H, q) derivative and Hilbert transform to test the existence of log-periodicity in financial time series preceding crash or strong corrections. Sornette and Zhou (2006) proposed the alarm index based on the pattern recognition approach to predict the changes of regimes in stock markets and applied the error diagrams to evaluate the prediction performance. . Zhou and Sornette (2006a) presented a general methodology to incorporate fundamental economic factors into the LPPLS model, including interest rate, interest spread, historical volatility, implied volatility and exchange rates, and found that the second-order LPPL model without economic factors has a better performance.

Yan et al. (2010) adapted the LPPLS formula to model the negative bubbles, so that the market rebounds can be detected through pattern recognition. Yan et al. (2012) extended the LPPLS model by incorporating the Zipf factor measuring the diversification risk of the stock market portfolio, to provide information about the concentration of stock gains over time. Yan et al. (2014) generalized the LPPLS model to infer the fundamental value of an asset price and the crash nonlinearity from bubble calibration, and tested the performance of model using both the likelihood-ratio test under the assumption of Gaussian residuals and the bootstrap method under the assumption of non-Gaussian residuals.

Lin and Sornette (2013) developed two rational expectation models of financial bubbles with heterogeneous rational arbitrageurs and positive feedbacks, i.e., finite-time singularity in the price dynamics with stochastic critical time model and finite-time singularity in the momentum price dynamics with stochastic critical time, and demonstrated the feasibility of early warning of bubbles using stochastic models embodying positive feedback mechanisms. Lin et al. (2014) proposed the volatility-confined LPPLS model by combining a mean-reverting volatility process with a stochastic conditional return which provides a consistent universal description of financial bubbles as a super-exponential acceleration of price decorated with log-periodic oscillations with mean-reverting residuals, so the LPPLS fitting residuals can be modeled by a mean-reversal Ornstein-Uhlenbeck process if the logarithmic price in the bubble regime is attributed to a deterministic LPPLS component.

The LPPLS method has also been criticized by many researchers (Bothmer & Meister, 2003; Brée et al., 2013; Brée & Joseph, 2013; Chang & Feigenbaum, 2006, 2008; Feigenbaum, 2001a, 2001b; Ilinski, 1999; Laloux et al., 1999, 2002). By examining the first difference of the logarithms of the S&P 500 before the 1987 crash, Feigenbaum (2001a) criticized the statistical insignificance of the log-periodic component when removing the data for the last year before the crash. Sornette and Johansen (2001) responded to the criticism that it is naive to reliably analyze critical point phenomena in nonlinear complex dynamical systems by removing the most significant part of the data closest to the critical point. Sornette and Johansen (2001) also extended the LPPLS model for general and arbitrary risk-aversion within the general stochastic discount factor theory. Chang and Feigenbaum (2006) firstly applied Bayesian methods to test the log-periodic returns of LPPLS model using S&P 500 data prior to the 1987 crash and found no evidence that LPPLS model could explain the observed oscillations since a null hypothesis model without log-periodicity outperformed the LPPLS model in term of marginal likelihood if crash probabilities were not taken into account, and the fitted parameters have small posterior probability assuming the LPPLS model is correct. Chang and Feigenbaum (2008) considered a regime-switching model in which expected returns alternate between high-drift and low-drift regimes fitted by Bayesian methods and showed that the parameters did not satisfy restrictions imposed by the LPPLS model when fixing the frequency and critical time by the regime-switching model. In contrast, Lin et al. (2014) found a very strong statistical preference for the LPPLS model compared with a standard benchmark using Bayesian inference. Brée et al. (2013) showed that the LPPLS functions are intrinsically hard to fit by accounting for the sloppiness. Brée and Joseph (2013) demonstrated that the martingale conditions are rejected as the mechanism underlying the LPPLS model since it requires the expected price to be increasing throughout the bubble and the fitting parameters being confined within specific ranges is only partially supported. Sornette et al. (2013) summarized the common questions and criticisms on the LPPLS model and provided systematic clarifications to remove misconceptions on both theoretical and empirical aspects.

The formulation of the LPPLS formula was transformed by Filimonov and Sornette (2013) to reduce the number of nonlinear parameters in the function from four to three, which reduces the complexity and improves the stability of the calibration. Filimonov, Demos and Sornette (2017) developed the modified profile likelihood inference method to calibrate the LPPLS model for financial bubbles and obtained the interval estimation for the critical time.

Sornette et al. (2015) constructed the DS LPPLS confidence indicator (also known as the LPPLS confidence indicator) to measure the bubble status from the ensemble of qualified fits, which are widely used for real-time bubble detection later. The LPPLS confidence indicator is defined as the fraction of fitting windows where the LPPLS calibrations satisfy the specified filter conditions to measure the sensitivity of detected bubble pattern to the choice of start time in fitting window. The larger the LPPLS confidence indicator, the more reliable the LPPLS bubble pattern is detected, while a small value of the indicator signals a possible fragility, because the LPPLS bubble patterns only appears in few fitting windows. Sornette et al. (2015) also constructed the DS LPPLS trust indicator to quantify the sensitivity of the calibrations to different instance of the realized residuals.

Demos and Sornette (2017) carried out systematic tests of the precision and reliability in determining the beginning and end time of a bubble, and found that the beginning of bubbles is much better determined than their end. Demos and Sornette (2019) proposed the Lagrange regularization method to estimate the beginning time of bubbles. Shu, Song, and Zhu (2021a) developed the modified Lagrange regularization method to mitigate the impact of potential LPPLS model overfitting on estimating bubble start time.

Zhang, Zhang and Sornette (2016) adopted quantile regression for the LPPLS model calibration to provide a family of LPPLS fits indexed by probability levels and combined the quantile regression with a multi-scale analysis. Seyrich and Sornette (2016) presented a plausible microfoundational model for the finite-time singular form of the crash hazard rate in the LPPLS model based on a percolation picture of the network of traders and the concept of clusters of related traders sharing the same point of view. Hu and Li (2017) extended the LPPLS model by incorporating interest rate, deposit reserve rate and historical volatilities of targeted indices and US equity indices. Dai et al. (2018) calibrated the LPPLS model using a two-step algorithm by combining the price gyration method and the constrained genetic algorithm. Demirer et al. (2019) introduced the LPPLS confidence multi-scale indicators using different time-scale fitting window length. Shu and Zhu (2020a) developed an adaptive multilevel time series detection methodology for real-time bubble detection.

Since its introduction, a wealth of studies have widely adopted the LPPLS model for the ex-ante diagnosis and postmortem analysis of financial bubbles and crashes in various kinds of stock markets, including the US Stock Market: Standard & Poor (S&P) 500 index (Drożdż et al., 2003; Shu et al., 2021b; Sornette & Zhou, 2002, 2006; Zhang, Sornette, et al., 2016; W.-X. Zhou & Sornette, 2003b, 2006a), the Dow Jones Industrial Average index (Bolonek-Lason & Kosinski, 2011; Gustavsson et al., 2016; Vandewalle et al., 1998), and the Nasdaq Composite index (Johansen & Sornette, 2000b); the Global Stock Market: global stock market indexes (Drożdż et al., 2008; Giorgis et al., 2022; Johansen & Sornette, 2001a; Song et al., 2022), the Chinese stock market (Jiang et al., 2010; Li, 2017; Shu & Zhu, 2020a, 2022; Yan et al., 2012; W.-X. Zhou & Sornette, 2004), the Japanese stock market (Johansen & Sornette, 1999, 2000a; Lynch & Mestel, 2017), the Korean stock market (Ko et al., 2018), the German stock market (Bartolozzi et al., 2005; Kurz-Kim, 2012; Wosnitza & Leker, 2014a), the Brazilian stock market (Cajueiro et al., 2009), the Polish stock market (Gnaciński & Makowiec, 2004), the Greek Stock Market (Papastamatiou & Karakasidis, 2022), the Portuguese stock market (Gonçalves et al., 2022) and South African stock market (W.-X. Zhou & Sornette, 2009).

In addition to stock markets, the LPPLS model has been widely used in other financial markets to detect bubbles and crashes, including (1) the Foreign exchange markets (Johansen et al., 1999; Johansen & Sornette, 2001a), (2) the Commodity Market: precious metals (Akaev et al., 2010; Drożdż et al., 2008; Geraskin & Fantazzini, 2013; W. Zhou et al., 2018), and oil (Cheng et al., 2018; Fantazzini, 2016; Sornette et al., 2009; Wątorek et al., 2016); (3) the Property Market: Real estate in the United States (Ardila et al., 2018; Brauers et al., 2014; W.-X. Zhou & Sornette, 2006b, 2008), the China (Zhi et al., 2019; W.-X. Zhou & Sornette, 2004), the United Kingdom (Fry, 2014), the Hong Kong and Seoul (Xiao, 2010), and the Switzerland (Ardila et al., 2018, 2021); (4) the Bond Market: Corporate bond yield (Clark, 2004), government bond CDS spread (Wosnitza & Denz, 2013), and financial institutions' CDS spread (Wosnitza & Leker, 2014b; Wosnitza & Sornette, 2015); (5) the Cryptocurrency Market (Gerlach et al., 2019; Geuder et al., 2019; Shu et al., 2021a; Shu & Zhu, 2020b; Wheatley et al., 2019), and 6) others: Foreign capital inflow (Sornette & Zhou, 2004) and world population growth (Johansen & Sornette, 2001b).

4. AN EMPIRICAL APPLICATION

In order to demonstrate the LPPLS methodology, we have conducted the real-time detection for the recent positive and negative bubble status of the S&P 500 index from April 2020 to December 2022. After the "COVID" crash of the 2020 US Stock market, the S&P 500 index has increased by 114.5% just within 21 months, from 2237.4 on March 23, 2020, to its all-time high value of 4798.1 on December 30, 2021. Then the S&P 500 index has experienced a series of strong corrects and lost nearly a quarter of its value by October 2022, wiping more than $10 trillion off market value. In this study, the data of S&P 500 index comes from Yahoo Finance (https://finance.yahoo.com/). Positive bubbles are associated with accelerating growth trends that are susceptible to regime changes in the form of large crashes or volatile sideway plateaus, while negative bubbles are associated with accelerating downward trends that are vulnerable to state change in the form of rallies or volatile sideway plateaus.

In this study, the LPPLS confidence indicator is calculated by: (1) creating a series of fitting time windows through shrinking the fitting window length from 650 trading days to 30 trading days in steps of 5 trading days for a given fictitious "present" data point t_2, which is moving from April 1, 2020, to December 30, 2022, (2) calibrating the LPPLS model within the given search space mentioned in Section 2.2 for each fitting time window and counting the number of fitting time windows meeting the specified filter conditions in Section 2.2, and (3) dividing this number by 125 of the total fitting window number. Because only data prior to t_2 are used, the LPPLS confidence indicator is causal.

Figure 1 shows the daily LPPLS confidence indicator for positive bubbles shown in red and negative bubbles in green (right scale) along with the S&P 500 index in blue (left scale), from 4/1/2020 to 12/30/2022. We can perceive intuitively the confidence level of the detected LPPLS bubble pattern in Figure 1 as the indicator measures the sensitivity of the detected bubble pattern to the start time selection and the larger LPPLS confidence indicator indicates the more reliable LPPLS bubble pattern. It can be clearly observed that Figure 1 presents a series of positive LPPLS confidence indicator clusters from August 2020 to January 2022.

From Figure 1, the positive LPPLS confidence indicator reaches the global peak value of 26.4% on Nov 30, 2021, i.e., 33 out of 125 fitting windows can successfully pass the filters which is a rare situation for stock markets, indicating that the detected LPPLS bubble pattern is very reliable and the price trajectory of S&P 500 index can be confirmed in a positive speculative bubble regime. The accelerating growth trend of the S&P 500 index is unsustainable. As the positive bubble is maturing, the growth rate is very likely to change because the average state of the system becomes exquisitely sensitive to a small perturbation. Thus, the strong corrections of S&P 500 index starting from January 2022 stem from the increasingly systemic instability of the stock market itself, while the well-known external shocks only serve as sparks, such as the decades-high inflation, aggressive monetary policy tightening by the Federal Reserve, and the impact of the Russia/Ukraine war. Interestingly, the date of the all-time high value of 4798.1, i.e., December 30, 2021, falls within the time range of a positive bubble cluster formed from November 11, 2021 to January 14, 2022, during which period the positive LPPLS confidence indicator also reaches the global peak value of 26.4% a month earlier. Further, a cluster of negative bubble can be also observed between June 13 and July 14, 2022, with the peak value of 5.6% on June 21, 2022. In this period, the S&P 500 index reaches a local trough.

In addition, we also study the performance of LPPLS confidence indicator under two-scale fitting window length, by dividing the 125 fitting windows into two subgroups: short-term and long-term. The fitting windows in the short-term time scale subgroup are generated by moving the t_1 towards the t_2 from 200 data points to 30 data points in a step of 5 data points to get 35 fitting windows. Similarly, the length of fitting window is shrunk by moving the t_1 towards the t_2 from 650 data points to 205 data points in a step of 5 data points to generate 90 fitting windows for the long-term time scale subgroup.

Figure 2 and Figure 3 show the LPPLS confidence indicator for positive bubbles in red and negative bubbles in green (right scale) along with the S&P 500 index in blue (left scale) from 4/1/2020 to 12/30/2022 in the long-term time scale and short-term time scale, respectively. In this study, the long-term LPPLS confidence indicator has better and more stable performance than the short-term LPPLS confidence indicator. Figure 2 presents the distinct cluster of long-term positive confidence indicator from November 11, 2021 to January 12, 2022 with the peak of 36.7% on November 30, 2021, which accurately predict the positive bubble regime change of the S&P 500 index one month later. While the noticeable cluster of the short-term negative confidence indicator in Figure 3 from June 13 to July 14, 2022, with the peak of 20% on June 21, 2022, successfully captures the negative bubble regime change in a short-term time scale.

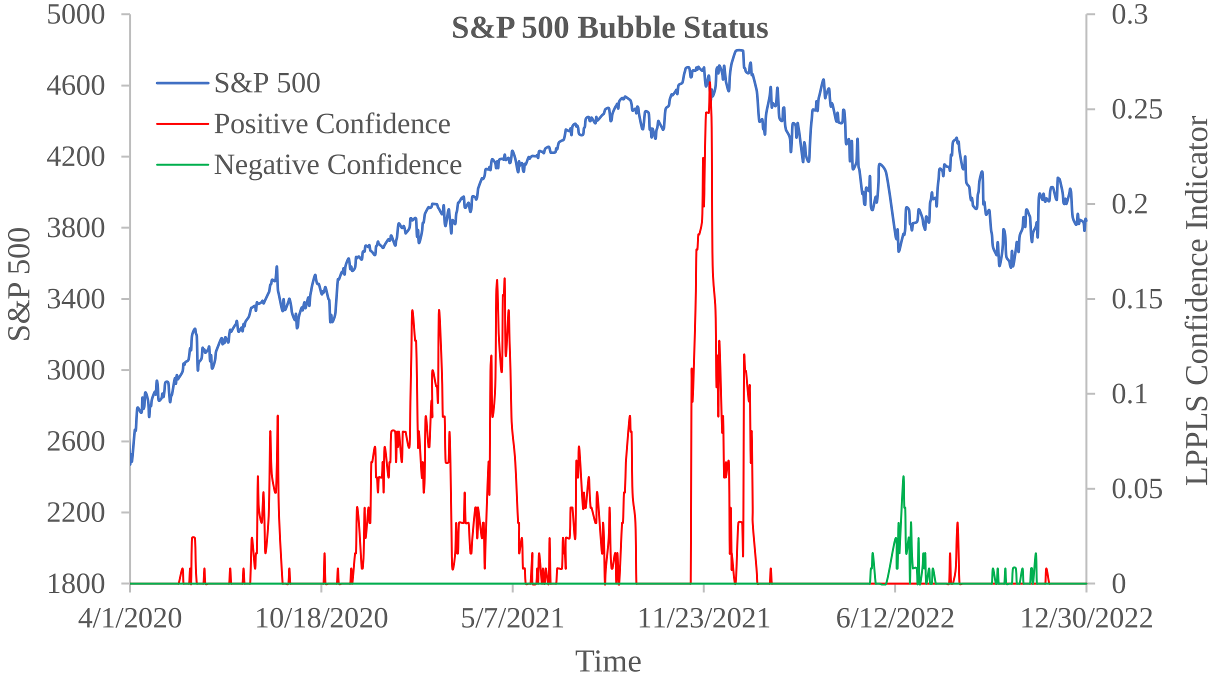

Figure 1. Daily LPPLS confidence indicator for positive bubbles is shown in red and negative bubbles in green (right scale) along with the S&P 500 index in blue (left scale), from 4/1/2020 to 12/30/2022.

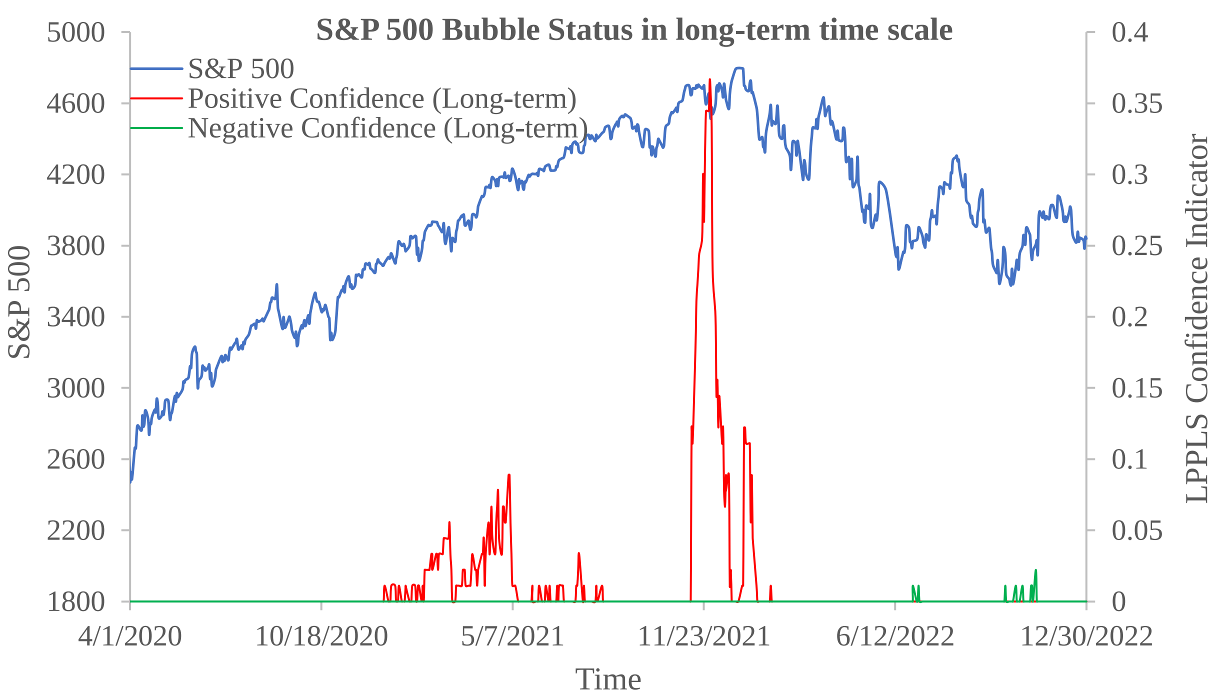

Figure 2. Long-term daily LPPLS confidence indicator for positive bubbles is shown in red and negative bubbles in green (right scale) along with the S&P 500 index in blue (left scale), from 4/1/2020 to 12/30/2022.

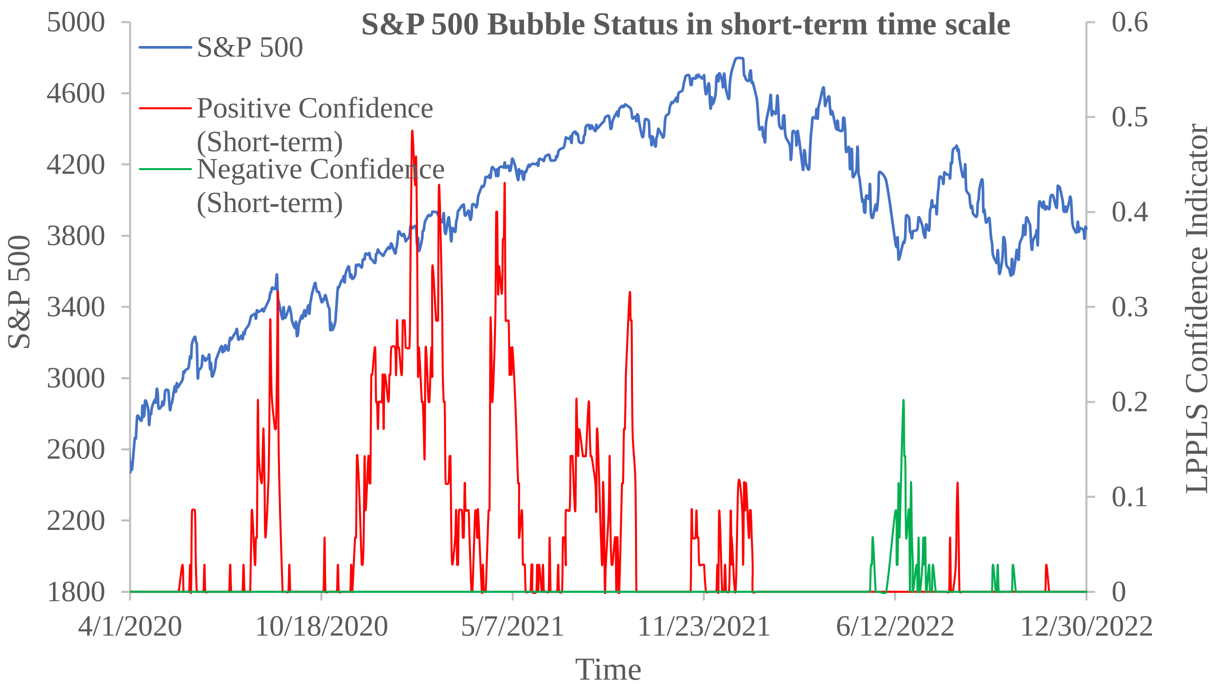

Figure 3. Short-term daily LPPLS confidence indicator for positive bubbles is shown in red and negative bubbles in green (right scale) along with the S&P 500 index in blue (left scale), from 4/1/2020 to 12/30/2022.

5. CONCLUSIONS

We conduct a systematic review of the theoretical and empirical academic literature on the development and extension of the LPPLS method as well as the related criticisms. We also summarize the applications of the LPPLS model to detect bubbles in a variety of financial markets. In addition, we apply the LPPLS model to carry out the real-time detection for the recent positive and negative bubble status of the S&P 500 index from April 2020 to December 2022. We find that a series of positive LPPLS confidence indicator clusters of the S&P 500 index have formed from August 2020 to January 2022. The positive LPPLS confidence indicator reaches the global peak value of 26.4% on Nov 30, 2021, which is a rare situation for stock markets, confirming that the price trajectory of S&P 500 index is indeed in a positive bubble regime and the system goes critical and becomes exquisitely sensitive to a small perturbation. Thus, the strong corrections of S&P 500 index from January 2022 stem from the increasingly systemic instability of the stock market itself, while the well-known external shocks, e.g., the decades-high inflation, aggressive monetary policy tightening by the Federal Reserve, and the impact of the Russia/Ukraine war, only serve as sparks. Further, the results show that the long-term LPPLS confidence indicator has better and more stable performance than the short-term LPPLS confidence indicator.

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