J. Appl. Prob. **33**, 400–410 (1996)
Printed in Israel

ON THE PREDICTION OF FRACTIONAL BROWNIAN MOTION

GUSTAF GRIPENBERG,* *University of Helsinki*
ILKKA NORROS,** *VTT Information Technology*

Abstract

Integration with respect to the fractional Brownian motion Z with Hurst parameter $H \in (1/2, 1)$ is discussed. The predictor $\boldsymbol{E}[Z_a \mid Z_s, s \in (-T, 0)]$ is represented as an integral with respect to Z, solving a weakly singular integral equation for the prediction weight function.

FRACTIONAL BROWNIAN MOTION; PREDICTION; STOCHASTIC INTEGRATION

AMS 1991 SUBJECT CLASSIFICATION: PRIMARY 60G25; 60M20
SECONDARY 60G15; 60G18

1. Introduction

In this paper, we look for more or less explicit expressions for the predictors $\boldsymbol{E}[Z_a \mid Z_s, s \in (-T, 0]]$, $a>0$, $T \in (0, \infty]$, where Z is a positively correlated fractional Brownian motion (FBM). Practical motivation from recent developments in teletraffic theory, and some basic facts about the FBM, are given in this introductory section. It is natural to represent the predictors as integrals over the observed part of the process. This requires, however, some general discussion on integration with respect to Z, since it is not a semimartingale. Section 2 is devoted to this. In Section 3, the predictors are represented as integrals with respect to Z over $(-T, 0)$. The prediction weight function is found by solving a weakly singular integral equation. The variance of the predictor is calculated, and a closed form expression is found in the case $T=\infty$. We close with a characterization of the conditional distribution of the whole future given the past.

The impulse for this study came from the second author's work in teletraffic theory, where self-similar processes such as the FBM have recently drawn much attention because of the empirically observed positively dependent self-similarity of local area network traffic which seriously challenges the traditional Poissonian and Markovian traffic modelling — see, e.g., [5, 7]. The logically simplest traffic model which fits well to many of those empirical traces is the FBM-based model studied in [10], where the amount of traffic $A(s, t)$ (say, bits) arriving during an interval $(s, t]$ is presented by $A(s, t)=$

Received 9 June 1994; revision received 13 January 1995.
* Postal address: University of Helsinki, Department of Mathematics, P.O. Box 4, 00014 University of Helsinki, Finland. e-mail:gustaf.gripenberg@helsinki.fi
** Postal address: VTT Information Technology, Telecommunications, P.O. Box 1202, 02044 VTT, Finland. e-mail:ilkka.norros@vtt.fi

$m(t-s)+\sqrt{ma}(Z_t-Z_s)$, where m and a are model parameters. Assuming this type of traffic behaviour, it is natural to ask: how should the near future (say, the next five minutes) be predicted on the basis of observed fluctuations of the traffic intensity, and how accurate can such a prediction be? The answers are important for the development of real-time traffic management functions, e.g. dynamic preventive congestion control mechanisms.

Explicit predictor formulas are, of course, also useful additions to basic knowledge in other applications where the FBM is used to model long-range dependent time evolution (hydrology, economics etc.).

Let us now turn to the mathematical framework. Throughout this paper, we denote by $(Z_t : t \in (-\infty, \infty))$ the *normalized fractional Brownian motion* (FBM) with self-similarity parameter $H \in (\frac{1}{2}, 1)$. Z is the stochastic process characterized by the following properties:

(i) Z has stationary increments;

(ii) $Z_0 = 0$, and $\boldsymbol{E}Z_t = 0$ for all t;

(iii) $\boldsymbol{E}Z_t^2 = |t|^{2H}$ for all t;

(iv) Z has continuous paths;

(v) Z is a Gaussian process, i.e. all its finite-dimensional distributions are multivariate Gaussian distributions.

The parameter H is also called the Hurst parameter. In the limiting case $H = \frac{1}{2}$, Z would be the standard Brownian motion. However, the results presented in this paper hold only for $H > \frac{1}{2}$.

From the stationarity of the increments and the Gaussian character (properties (i) and (v)) it follows that the finite-dimensional distributions of Z are determined by the mean and variance functions (properties (ii) and (iii)). The continuity assumption then completes the characterization of Z. It is now obvious that Z is a *self-similar* process, i.e. $(Z_{\alpha t} : t \in \mathbb{R})$ is identical in distribution to $(\alpha^H Z_t : t \in \mathbb{R})$ for every $\alpha > 0$. As a general reference to self-similar processes see, e.g., articles in the collection edited by Eberlein and Taqqu [4].

The increments of Z are positively correlated. For $t_1 < t_2 \leqq t_3 < t_4$ one has

$$(1.1) \qquad \mathrm{Cov}(Z_{t_2} - Z_{t_1}, Z_{t_4} - Z_{t_3}) = \tfrac{1}{2}[(t_4 - t_1)^{2H} - (t_3 - t_1)^{2H} + (t_3 - t_2)^{2H} - (t_4 - t_2)^{2H}].$$

It follows that $r(n) \stackrel{\text{def}}{=} \mathrm{Cov}(Z_1, Z_{n+1} - Z_n) = H(2H-1)n^{-2(1-H)} + O(n^{-(3-2H)})$, which shows that Z possesses *long-range dependence* in the sense that $\Sigma_0^\infty r(n) = \infty$.

The stationary sequence $Z_{n+1} - Z_n$ (often called *fractional Gaussian noise*) is ergodic by the general result that any stationary Gaussian sequence with continuous spectral measure is ergodic and weakly mixing — see, e.g., [2], Theorem 14.2.1.

The FBM can also be obtained as a stochastic integral with respect to the standard Brownian motion, as was already observed in the pioneering paper [9], where the FBM was given its name. Suppose that $(B_t : t \in \mathbb{R})$ is a standard Brownian motion. Let $H \in (\frac{1}{2}, 1)$ and define a process $(Z_t : t \in \mathbb{R})$ by $Z_0 = 0$ and

$$Z_t - Z_s = \lim_{a \to -\infty} \left(c_H \int_a^t (t-\tau)^{H-\frac{1}{2}} dB_\tau - c_H \int_a^s (s-\tau)^{H-\frac{1}{2}} dB_\tau \right)$$

(1.2)

$$= c_H \int_s^t (t-\tau)^{H-\frac{1}{2}} dB_\tau + c_H \int_{-\infty}^s ((t-\tau)^{H-\frac{1}{2}} - (s-\tau)^{H-\frac{1}{2}}) dB_\tau$$

for $t > s$, where $c_H = \sqrt{2H\Gamma(\frac{3}{2}-H)/\Gamma(\frac{1}{2}+H)\Gamma(2-2H)}$ and Γ is the gamma function. Note that it is the dependence of the integrand on the upper end of the integration interval that brings us outside the world of semimartingales.

In order to show that Z is in fact the fractional Brownian motion, it is sufficient to show that it has the correct variance. This can be done directly, but it is also a consequence of Proposition 2.2 below (which also gives 1.1 as a special case).

2. Integration with respect to FBM

It is a well-known result of the theory of stochastic integration that 'reasonable' stochastic integration is possible only with respect to semimartingales (see, e.g., [3], Theorem VIII.80), and semimartingales can even be characterized by this property. The only non-trivial feature of this notion of integral is a very weak continuity condition with respect to the integrand. Since the FBM is not a semimartingale, it can be expected that stochastic integrals with respect to it are not continuous with respect to the integrand. However, the special case of a deterministic integrand turns out to be both easy to handle and sufficient for the present purposes.

Integration with respect to general Gaussian processes has been studied for decades. An extensive presentation of the theory was provided by Huang and Cambanis [6], Subsection 1.1 of which suffices as a reference for our needs. However, we give in this section a short and elementary FBM-specific, calculation-oriented introduction to the subject, in the hope that it is illuminating for a reader familiar only with classical stochastic integration.

For a simple process $Y_t = \Sigma_{j=1}^k X_j \, 1_{(T_{j-1}, T_j]}(t)$, the integral

$$\int_{-\infty}^{\infty} Y_t dZ_t \stackrel{\text{def}}{=} \sum_{j=1}^k X_j (Z_{T_j} - Z_{T_{j-1}}) \tag{2.1}$$

is of course problem-free as long as no limit operations are needed.

For a process Y with locally bounded variation it is always possible to use the definition by the integration by parts formula:

$$\int_a^b Y_t dZ_t \stackrel{\text{def}}{=} Y_b Z_b - Y_a Z_a - \int_a^b Z_t dY_t. \tag{2.2}$$

If Y is deterministic, it is easy to show that the integral defined in (2.2) is obtained as a limit in L^2 of Riemann sums of the type (2.1) with deterministic T_j.

The definition can be extended to a broader class of deterministic functions using L^2 limits. However, simple examples of non-continuity can be found when Y is allowed to be random.

Example 2.1. Let us denote

$$p_\beta(x) = \operatorname{sign}(x)\,|x|^\beta, \qquad x \in \mathbb{R},$$

and

$$U_{n,k} = p_\beta(\boldsymbol{E}[Z_{k/n} - Z_{(k-1)/n} \mid \mathscr{F}_{(k-1)/n}]),$$

where $\mathscr{F}_t = \sigma\{Z_s : s \leqq t\}$ and $\beta > 0$. Define

$$Y_n(t) = \sum_{k=1}^{n} U_{n,k}\, 1_{\{((k-1)/n,\, k/n]\}}(t).$$

The Y_n are predictable processes, and it is easy to see that $Y_n(t) \to 0$ for each t when $n \to \infty$. By the self-similarity of Z we now have

$$\begin{aligned}\int_0^1 Y_n(t)dZ_t &= \sum_{k=1}^{n} U_{n,k}(Z_{k/n} - Z_{(k-1)/n}) \\ &\underset{\mathscr{D}}{=} \sum_{k=1}^{n} n^{-\beta H} p_\beta(\boldsymbol{E}[Z_k - Z_{k-1} \mid \mathscr{F}_{k-1}]) n^{-H}(Z_k - Z_{k-1}) \\ &= n^{1-H-\beta H}\, \frac{1}{n} \sum_{k=1}^{n} (p_\beta(\boldsymbol{E}[Z_1 \mid \mathscr{F}_0]) Z_1) \circ T_1^{k-1},\end{aligned}$$

where $\underset{\mathscr{D}}{=}$ denotes equality in distribution and T_1 is the shift operator $Z_t(\omega) = Z_{t-1}(T_1\omega)$. As noted in Section 1, T_1 is ergodic, so that Birkhoff's ergodic theorem gives

$$\lim_{n\to\infty} \int_0^1 Y_n(t)dZ_t = \left(\lim_{n\to\infty} n^{1-H-\beta H}\right) \cdot \boldsymbol{E}(p_\beta(\boldsymbol{E}[Z_1 \mid \mathscr{F}_0]) Z_1) \quad \text{a.s.}$$

The expectation on the right is positive since $p_\beta(\cdot)$ is an increasing odd function. Thus the limit is infinite a.s. when $\beta < (1-H)/H$. Note that $\beta = 1$ is not sufficient.

The non-continuity with respect to the integrand, illustrated by Example 2.1, has the consequence that the Riemann sum definition (2.1) cannot be extended to a general definition with L^2 limits like in classical Itô integration. However, this is possible for deterministic integrands, using formula (2.4) below.

A general, although not very transparent, definition of a stochastic integral with respect to the FBM can be given by using the integral representation of the FBM given in (1.2). For simplicity, we restrict here to the case of a deterministic integrand. For $f \in L^2(\mathbb{R}; \mathbb{R}) \cap L^1(\mathbb{R}; \mathbb{R})$, define

$$\int_{\mathbb{R}} f(t)dZ_t = c_H(H-\tfrac{1}{2}) \int_{\mathbb{R}} \left(\int_\tau^\infty (t-\tau)^{H-\frac{3}{2}} f(t)dt \right) dB_\tau. \tag{2.3}$$

It is clear that for this definition to make sense, f must be such that the function $\tau \mapsto \int_\tau^\infty (t-\tau)^{H-\frac{3}{2}} f(t)dt$ is square integrable, and $f \in L^2(\mathbb{R}; \mathbb{R}) \cap L^1(\mathbb{R}; \mathbb{R})$ is sufficient for this. (Note that one could also formulate necessary and sufficient, but not very informative, conditions.)

The following result is basic in the integration with respect to Z. It provides a Hilbert space isomorphism which is central in the integration theory of Gaussian processes as presented in [6]. We give a proof which proceeds with direct calculation from formula (2.3).

Proposition 2.2. For $f, g \in L^2(\mathbb{R}; \mathbb{R}) \cap L^1(\mathbb{R}; \mathbb{R})$ we have

$$\boldsymbol{E}\left(\int_{\mathbb{R}} f(s)dZ_s \int_{\mathbb{R}} g(t)dZ_t \right) = H(2H-1) \iint_{\mathbb{R}^2} f(s)g(t)\,|s-t|^{2H-2} dtds. \tag{2.4}$$

Proof. Since by [1], 6.2.1, 6.2.2,

$$\begin{aligned}\int_{-\infty}^{\min(s,t)} (s-\tau)^{H-\frac{3}{2}}(t-\tau)^{H-\frac{3}{2}} d\tau &= \int_0^\infty (|t-s|+\tau)^{H-\frac{3}{2}} \tau^{H-\frac{3}{2}} d\tau \\ &= |t-s|^{2H-2} \int_0^\infty (1+\tau)^{H-\frac{3}{2}} \tau^{H-\frac{3}{2}} d\tau \\ &= |t-s|^{2H-2} \frac{\Gamma(H-\frac{1}{2})\Gamma(2-2H)}{\Gamma(\frac{3}{2}-H)},\end{aligned}$$

the result is obtained by direct calculation:

$$\begin{aligned}&\boldsymbol{E}\left(\int_{\mathbb{R}} f(s)dZ_s \int_{\mathbb{R}} g(t)dZ_t \right) \\ &\quad = c_H^2 (H-\tfrac{1}{2})^2 \iint_{\mathbb{R}^2} dtds f(s)g(t) \int_{-\infty}^{\min(s,t)} d\tau (s-\tau)^{H-\frac{3}{2}}(t-\tau)^{H-\frac{3}{2}} \\ &\quad = H(2H-1) \iint_{\mathbb{R}^2} f(s)g(t)\,|s-t|^{2H-2} dtds.\end{aligned}$$

Equation (2.4) can also be interpreted as a precise formulation of the infinitesimal rule

$$\operatorname{Cov}(dZ_s, dZ_t) = H(2H-1)\,|s-t|^{2H-2} dsdt.$$

3. Predictor formulae

Let us now return to our main theme, the prediction of $Z_a (a>0)$ on the basis of the values obtained by Z_t in an interval $(-T, 0)$.

Remark. By the stationarity of the increments of Z, this is equivalent to the problem of predicting, for any t, the difference $Z_{t+a}-Z_t$ on the basis of the differences Z_s-Z_t, $s\in(t-T, t)$. However, the prediction of Z_{t+a} on the basis of Z_s, $s\in(t-T, t)$ is more complicated, unless we have $0\in(t-T, t)$.

The main result of this paper is the following:

Theorem 3.1. Let Z be a fractional Brownian motion with $H\in(1/2, 1)$. For each $a>0$ and $T\in(0,\infty]$, the predictor $\hat{Z}_{a,T}=\boldsymbol{E}[Z_a \mid Z_s, s\in(-T, 0)]$, can be represented as an integral $\int_{-T}^0 g_T(a, t)dZ_t$, where for $T<\infty$, $t\in(0, T)$

$$g_T(a,-t)=\frac{\sin(\pi(H-\frac{1}{2}))}{\pi}\, t^{-H+\frac{1}{2}}(T-t)^{-H+\frac{1}{2}}\int_0^a \frac{\sigma^{H-\frac{1}{2}}(\sigma+T)^{H-\frac{1}{2}}}{\sigma+t}\,d\sigma, \tag{3.1}$$

and for $T=\infty$, $t>0$

$$\begin{aligned} g_\infty(a,-t)&=\frac{\sin(\pi(H-\frac{1}{2}))}{\pi}\, t^{-H+\frac{1}{2}}\int_0^a \frac{\sigma^{H-\frac{1}{2}}}{\sigma+t}\,d\sigma \\ &=\frac{\sin(\pi(H-\frac{1}{2}))}{\pi}\left(\frac{1}{H-\frac{1}{2}}\left(\frac{t}{a}\right)^{-H+\frac{1}{2}}-B_{a/(t+a)}(H-\tfrac{1}{2}, \tfrac{3}{2}-H)\right), \end{aligned} \tag{3.2}$$

where $B.(\cdot,\cdot)$ is the incomplete beta function. The function $g_T(a,\cdot)$ is a solution of the integral equation

$$(2H-1)\int_0^T g_T(a,-t)\,|t-s|^{2H-2}dt=(a+s)^{2H-1}-s^{2H-1}, \qquad s\in(0, T), \tag{3.3}$$

and has the scaling property

$$g_T(a, t)=g_{T/a}(1, t/a). \tag{3.4}$$

Proof. Since Z is Gaussian, $\hat{Z}_{a,T}$ is a linear functional of $(Z_s; s\in(-T, 0))$. Thus we try to find a smooth function $g_T(a,\cdot):(-T, 0)\to\mathbb{R}$ satisfying the orthogonality condition

$$\boldsymbol{E}\left(\left(Z_a-\int_{-T}^0 g_T(a, t)dZ_t\right)(Z_u-Z_v)\right)=0, \qquad -T<v<u\leqq 0. \tag{3.5}$$

Using Proposition 2.2, we arrive at the integral equation (3.3), from which (3.4) follows easily by self-similarity. Let us now solve (3.3).

Denote $\alpha=2H-1$ and let $h_{T,a}$ be the function $h_{T,a}(t)\stackrel{\text{def}}{=}g_T(a,-tT)$, $t\in[0, 1]$. With this definition, (3.3) becomes equivalent to $\int_0^1 h_{T,a}(s)\,|t-s|^{\alpha-1}ds=f_{T,a}(t)$, $t\in[0, 1]$, where

$f_{T,a}(t)=(1/\alpha)(t+a/T)^\alpha-(1/\alpha)t^\alpha$, $t\in[0,1]$. A formula for the solution of this equation can be found in [8], (31), p. 310 (but note that the number α there corresponds to $1-\alpha$ here). Thus we get

$$h_{T,a}(t)=-c(\alpha)t^{-\alpha/2}\frac{d}{dt}\int_t^1 ds\, s^\alpha(s-t)^{-\alpha/2}\frac{d}{ds}\int_0^s du\, u^{-\alpha/2}(s-u)^{-\alpha/2}f_{T,a}(u), \tag{3.6}$$

where $c(\alpha)=(\Gamma(1-\frac{1}{2}\alpha)^2\Gamma(\alpha)2\cos(\frac{1}{2}\pi\alpha))^{-1}$. The solution (3.6) can be further elaborated by writing $f_{T,a}$ as $f_{T,a}(u)=\int_0^{a/T}(\sigma+u)^{\alpha-1}d\sigma$ so that

$$h_{T,a}(t)=-c(\alpha)\int_0^{a/T}d\sigma\, t^{-\alpha/2}\frac{d}{dt}\int_t^1 ds\, s^\alpha(s-t)^{-\alpha/2}\frac{d}{ds}\int_0^s du\, u^{-\alpha/2}(s-u)^{-\alpha/2}(\sigma+u)^{\alpha-1}.$$

Now, by a change of variable and [1], 15.3.1, we have

$$\int_0^s u^{-\alpha/2}(s-u)^{-\alpha/2}(\sigma+u)^{\alpha-1}du=\frac{\Gamma(1-\alpha/2)^2}{\Gamma(2-\alpha)}\left(\frac{s}{\sigma}\right)^{1-\alpha}F\left(1-\alpha,1-\frac{\alpha}{2},2-\alpha,-\frac{s}{\sigma}\right),$$

where F is the hypergeometric function. Because by [1], 15.2.3, 15.3.1, and 15.3.3,

$$\frac{d}{dz}(z^{1-\alpha}F(1-\alpha,1-\tfrac{1}{2}\alpha,2-\alpha,z))=(1-\alpha)z^{-\alpha}(1-z)^{\alpha/2-1},$$

it follows that

$$\begin{aligned}h_{T,a}(t)&=-\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)2\cos(\frac{1}{2}\pi\alpha)}t^{-\alpha/2}\int_0^{a/T}\sigma^{\alpha-1}\frac{d}{dt}\int_t^1(s-t)^{-\alpha/2}\left(1+\frac{s}{\sigma}\right)^{\alpha/2-1}dsd\sigma\\&=-\frac{\sin(\frac{1}{2}\pi\alpha)}{\pi}t^{-\alpha/2}\int_0^{a/T}\sigma^{\alpha/2}\frac{d}{dt}\int_0^{(1-t)/(\sigma+t)}\tau^{-\alpha/2}(1+\tau)^{\alpha/2-1}d\tau d\sigma\\&=\frac{\sin(\frac{1}{2}\pi\alpha)}{\pi}t^{-\alpha/2}(1-t)^{-\alpha/2}\int_0^{a/T}\frac{\sigma^{\alpha/2}(1+\sigma)^{\alpha/2}}{\sigma+t}d\sigma,\end{aligned}$$

which yields (3.1). Finally, (3.2) is obtained by noting that

$$\begin{aligned}\lim_{T\to\infty}g_T(a,-t)&=\frac{\sin(\pi(H-\frac{1}{2}))}{\pi}t^{-H+\frac{1}{2}}\int_0^a\frac{\sigma^{H-\frac{1}{2}}}{\sigma+t}d\sigma\\&=\frac{\sin(\pi(H-\frac{1}{2}))}{\pi}\left(\frac{1}{H-\frac{1}{2}}\left(\frac{t}{a}\right)^{-H+\frac{1}{2}}-\int_{t/a}^\infty\frac{\sigma^{-H+\frac{1}{2}}}{1+\sigma}d\sigma\right)\\&=\frac{\sin(\pi(H-\frac{1}{2}))}{\pi}\left(\frac{1}{H-\frac{1}{2}}\left(\frac{t}{a}\right)^{-H+\frac{1}{2}}-B_{a/(t+a)}(H-\tfrac{1}{2},\tfrac{3}{2}-H)\right).\end{aligned}$$

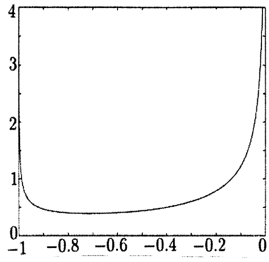

Figure 3.1. The prediction weight function $g_1(1, t)$ for $H=0.9$. The function approaches infinity at both ends of the interval

It is easy to verify that since $H \in (1/2, 1)$, we have $g_T(a, \cdot) \in L^1 \cap L^2$ for both finite and infinite T. For small T note also that

$$\lim_{T \searrow 0} \min_{-T<t<0} g_T(a, t) = \infty, \qquad \lim_{T \searrow 0} \int_{-T}^{0} g_T(a, t)dt = 0.$$

It is interesting that the weight function goes to infinity at the origin and also at $-T$ when T is finite — see Figure 3.1. In particular, the weight function is not monotone in the latter case. Intuitively, this can be understood so that the 'closest witnesses' to the unobserved past have special weight.

As an application of Theorem 3.1, we obtain an expression for the variance of the predictor $\boldsymbol{E}[Z_a \mid Z_s, s \in (-T, 0)]$:

Corollary 3.2. For $a>0$, $T \in (0, \infty]$ we have

$$D^2(\boldsymbol{E}[Z_a \mid Z_s, s \in (-T, 0)]) = D^2(Z_a)H \int_0^{T/a} g_{T/a}(1, -s)((1+s)^{2H-1} - s^{2H-1})ds. \tag{3.7}$$

For $T=\infty$, the result can be stated in terms of the gamma function:

$$D^2(\boldsymbol{E}[Z_a \mid Z_s, s \leqq 0]) = D^2(Z_a)\left(1 - \frac{\sin(\pi(H-\frac{1}{2}))}{\pi(H-\frac{1}{2})} \frac{\Gamma(\frac{3}{2}-H)^2}{\Gamma(2-2H)}\right). \tag{3.8}$$

Proof. Using Proposition 2.2 and the integral equation (3.3), we get

$$D^2\left(\int_{-T}^{0} g_T(a, t)dZ_t\right) = H(2H-1)\int_0^T \int_0^T g_T(a, -s)g_T(a, -t)|s-t|^{2H-2}dsdt$$
$$= D^2(Z_a)H \int_0^{T/a} g_{T/a}(1, -t)((1+t)^{2H-1} - t^{2H-1})ds.$$

In the case $T=\infty$, we can integrate by parts and obtain

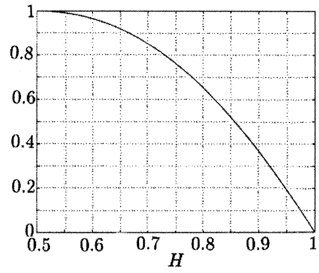

Figure 3.2. The relative variance of error $D^2(Z_a-\hat{Z}_{a,\infty})/D^2(Z_a)$ as a function of the self-similarity parameter H

$$H\int_0^\infty g_\infty(1,-t)((1+t)^{2H-1}-t^{2H-1})dt=\frac{\sin(\pi(H-\frac{1}{2}))}{2\pi}\int_0^\infty\frac{(1+t)^{2H}-t^{2H}-1}{(1+t)t^{H+\frac{1}{2}}}\,dt.$$

For $-\frac{1}{2}<H<0$, the last integral can be evaluated using formulas 6.2.1 and 6.2.2 of [1], which gives

$$\begin{aligned}\int_0^\infty\frac{(1+t)^{2H}-t^{2H}-1}{(1+t)t^{H+\frac{1}{2}}}\,dt&=\frac{\Gamma(\frac{1}{2}-H)\Gamma(\frac{1}{2}-H)}{\Gamma(1-2H)}-2\Gamma(\tfrac{1}{2}+H)\Gamma(\tfrac{1}{2}-H)\\&=-\frac{2\Gamma(\frac{3}{2}-H)^2}{(H-\frac{1}{2})\Gamma(2-2H)}+\frac{2\pi}{\sin(\pi(H-\frac{1}{2}))}.\end{aligned}$$

Now, by analytic continuation this result can be extended to the case $\frac{1}{2}<H<1$ as well, and (3.8) follows.

Figure 3.2 shows the relative variance of error

$$\frac{D^2(Z_a-\hat{Z}_{a,\infty})}{D^2(Z_a)}=\frac{\sin(\pi(H-\frac{1}{2}))}{\pi(H-\frac{1}{2})}\,\frac{\Gamma(\frac{3}{2}-H)^2}{\Gamma(2-2H)}$$

as a function of H. Note that, as a result of self-similarity, this quantity is independent of a. It is seen that the predictive force of the past is not very high unless H is rather large. The past before 0 explains half of the variance of Z_a when H is about 0.85. According to [7], this a rather typical Hurst parameter value in local area network traffic, so that in this application we obtain a convenient rule of thumb for the possibilities of prediction.

Figure 3.3 depicts the relative variance of error $D^2(Z_1-\hat{Z}_{1,T})/D^2(Z_1)$ as a function of T with $H=0.9$. It is seen that for the prediction of Z_a, it makes relatively little difference whether we know Z on $(-a,0)$ or $(-\infty,0)$. (Of course, this little difference becomes essential if, for example, one tries to simulate the FBM step-by-step forward, which is not a good idea!) Thus, we obtain another simple rule: use only the latest second to predict the next second, latest minute for next minute, etc.

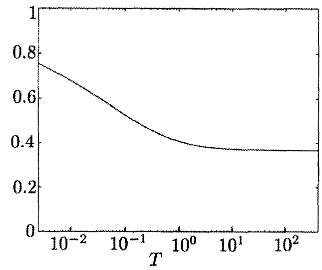

Figure 3.3. The relative variance of error $D^2(Z_1-\hat{Z}_{1,T})/D^2(Z_1)$ as a function of T, $H=0.9$

In fact, we have now sufficient information for a complete characterization of the distribution of the process $(Z_t : t \geqq 0)$ given $(Z_t : t \leqq 0)$. Because of its Gaussian character, it is sufficient to know $\boldsymbol{E}[Z_t \mid \mathscr{F}_0]$ and $D^2[Z_t - Z_s \mid \mathscr{F}_0]$, from which the first was given in Theorem 3.1. By a general property of multivariate Gaussian distributions, the second is non-random, so that

$$D^2[Z_t - Z_s \mid \mathscr{F}_0] = D^2(Z_t - Z_s) - D^2(\hat{Z}_t - \hat{Z}_s),$$

where the last term can be calculated in a similar way as the result of Corollary 3.2.

It is also interesting to note that the process $(\hat{Z}_t, t \geqq 0)$ is self-similar with parameter H, although with non-stationary increments: for any $\alpha > 0$ we have

$$\begin{aligned}(\hat{Z}_{\alpha t} : t > 0) &= \left(\int_{-\infty}^{0} g_\infty(\alpha t, u)dZ_u : t > 0\right) \\ &= \left(\int_{-\infty}^{0} g_\infty(t, u/\alpha)dZ_u : t > 0\right) = \left(\int_{-\infty}^{0} g_\infty(t, u)dZ_{\alpha u} : t > 0\right) \\ &\underset{\mathscr{D}}{=} \left(\alpha^H \int_{-\infty}^{0} g_\infty(t, u)dZ_u : t > 0\right) = (\alpha^H \hat{Z}_t : t > 0).\end{aligned}$$

Acknowledgement

We thank Professor Murad Taqqu for his valuable comments on the original manuscript.

References

[1] Abramowitz, M. A. and Stegun, I. A. (1970) *Handbook of Mathematical Functions.* Dover, New York.

[2] Cornfeld, I. P., Sinai, Ya. G. and Fomin, S. V. (1982) *Ergodic Theory.* Springer, Berlin.

[3] Dellacherie, C. and Meyer, P.-A. (1982) *Probabilities and Potential.* North Holland, Amsterdam.

[4] Eberlein, E. and Taqqu, M. S. (1986) *Dependence in Probability and Statistics.* (*Prog. Prob. Stat.* Vol. 11). Birkhäuser, Boston.

[5] Fowler, H. J. and Leland, W. E. (1991) Local area network traffic characteristics, with implications for broadband network congestion management. *IEEE J. Sel. Areas Commun.* **9**, 1139–1149.

[6] Huang, S. T. and Cambanis, S. (1978) Stochastic and multiple Wiener integrals for Gaussian processes. *Ann. Prob.* **6**, 585–614.

[7] Leland, W. E., Taqqu, M. S., Willinger, W. and Wilson, D. V. (1994) On the self-similar nature of Ethernet traffic (extended version). *IEEE/ACM Trans. Networking* **2**, 1–15.

[8] Lundgren, T. and Chiang, D. (1967) Solution of a class of singular integral equations. *Quart. J. Appl. Math.* **24**, 303–313.

[9] Mandelbrot, B. B. and Van Ness, J. W. (1968) Fractional Brownian motions, fractional noises and applications. *SIAM Rev.* **10**, 422–437.

[10] Norros, I. (1994) A storage model with self-similar input. *Queueing Systems* **16**, 387–396.