Solving a coin-flip l2 mimimax estimation problem

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Abstract

We set up and solve a minimax under l2 loss problem derived from a coin flipping problem. The solution is interesting as it involves use of cancellation to solve the mimimax problem.

1 Introduction

Wald [Wald, 1949] set up statistical estimation as a game played against nature where the researcher picks a (possibly probabilistic) decision function and nature picks an adversarial distribution. Nature's distribution plays the role of Bayesian priors, but is not considered to be the an objective true distribution or a subjective estimate. It is instead a worst-possible distribution so that any inference bounds proven in this formalism hold in general. This game theoretic form of probability is fascinating and leads quickly to interesting questions and procedures.

2 The Problem

Take as our problem the task of estimating the unknown win-rate (or heads-rate) p of a random process or coin. We assume the process is memory-less and stationary (p is not changing and does not depend on earlier flips). We observe a sequence of n flips showing h wins/heads, and then are asked to return an estimate $\phi_n(h)$ for p. This problem was discussed and given context in [Mount, 2014a], [Mount, 2014c], [Mount, 2014b], and [Bauer et al., 2014].

Fix $n \in \mathbb{N}$, $n \geq 1$. Let $p \in [0,1]$ and $\phi = (\phi(0), \ldots, \phi(n))$ be a $(n+1)$-dimensional real vector in $[0,1]^{n+1}$, and define:

$$L_n(p, \phi) := \sum_{h=0}^{n} \binom{n}{h} p^h (1-p)^{n-h} (\phi(h) - p)^2. \tag{1}$$

$L_n(p, \phi)$ represents the expected square-error encountered when using ϕ to estimate the win-rate of a coin with (unknown) true win-rate p by observing n flips/outcomes. The estimate is: use $\phi(h)$ when you see h wins/heads. This is related to Wald's game-theoretic formalism, but we are insisting on pure strategies for both the estimate (a single deterministic ϕ) and a single unknown true probability p. We are going to assume that nature picks p in an adversarial manner with full knowledge of ϕ.

Define:

$$f_n(\phi) = \max_{p \in [0,1]} L_n(p, \phi). \tag{2}$$

What we are looking for is $\text{argmin}_{\phi \in [0,1]^{n+1}} f_n(\phi)$. The issue is: the definition of $f_n()$ has two quantifiers so it seems like it will be difficult to derive or even check solutions.

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3 A Solution

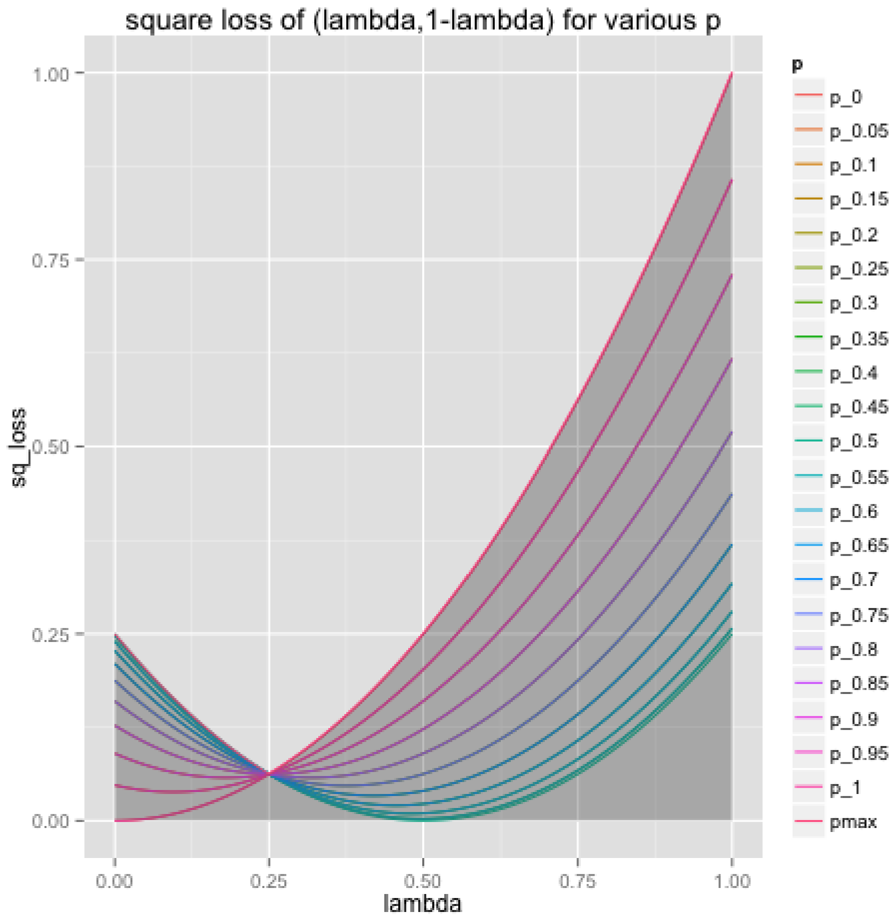

Figure 1: $L_1(p, (\lambda, 1 - \lambda))$

Lemma 1. *Suppose ϕ is in the interior of $[0,1]^{n+1}$ and is such that $L_n(p,\phi) - \phi(0)^2 = 0$ simultaneously for all p. Then: ϕ is the unique global minimizer of $f_n()$.*

Proof. Suppose ϕ is as stated. We will confirm ϕ is an isolated local minimum by checking partial derivatives. Look at $\frac{\partial}{\partial \phi(h)} f_n(\phi)$ and $\frac{\partial}{\partial - \phi(h)} f_n(\phi)$. If we can show these are always both positive for all h we are done.

We know $\frac{\partial}{\partial \phi(h)} f_n(\phi) \geq \frac{\partial}{\partial \phi(h)} L_n(p, \phi)$ for any $p \in [0,1]$ (i.e. all p curves are active or on the boundary boundary, figure 1 shows an example). So

$$\begin{aligned}
\frac{\partial}{\partial \phi(h)} f_n(\phi) &\geq \max_p \frac{\partial}{\partial \phi(h)} L_n(p, \phi) \\
&= \max_p \binom{n}{h} p^h (1-p)^{n-h} 2(\phi(h) - p) \\
&\geq \binom{n}{h} p^h (1-p)^{n-h} 2(\phi(h) - p) \Big|_{p = \phi(h)/2} \\
&> 0
\end{aligned}$$

Similarly we know $\frac{\partial}{\partial -\phi(h)} f_n(\phi) \geq \frac{\partial}{\partial -\phi(h)} L_n(p,\phi)$ for any $p \in [0,1]$. So

$$
\begin{aligned}
\frac{\partial}{\partial - \phi(h)} f_n(\phi) &\geq \max_p \frac{\partial}{\partial - \phi(h)} L_n(p,\phi) \\
&= \max_p \binom{n}{h} p^h (1-p)^{n-h} 2(p - \phi(h)) \\
&\geq \binom{n}{h} p^h (1-p)^{n-h} 2(p - \phi(h)) \bigg|_{p=(1+\phi(h))/2} \\
&> 0
\end{aligned}
$$

So we know ϕ is an isolated local minimum of $f_n()$ But $L_n(p,\phi)$ is convex in ϕ for any fixed n, p ($n \geq 1, p \in [0,1]$), so $f_n(\phi)$ is also convex in ϕ. So an isolated local minimum ϕ is also the unique global minimum. ☐

Lemma 2. *If $L_n(p,\phi) - \phi(0)^2 = 0$ then*

$$\phi(1)^2 = \phi(0)^2 + \frac{2}{k}\phi(0) \tag{3}$$

and for all $h \geq 2$:

$$
\begin{aligned}
\phi(h)^2 = &\frac{(n+2)(n+1)}{(n+2-h)(n+1-h)}\phi(0)^2 \\
&+ 2\frac{h}{n+1-h}\phi(h-1)(1-\phi(h-1)) \\
&- \frac{h(h-1)}{(n+2-h)(n+1-h)}(\phi(h-2)-1)^2.
\end{aligned} \tag{4}
$$

Proof. Perform a change of variables $z = p/(1-p)$ on $L_n(p,\phi) - \phi(0)^2$ and collect terms in powers of z. This yields the following equivalent equation:

$$\sum_{h=0}^{n} \binom{n}{h} z^h ((1+z)\phi(h) - z)^2 = \phi(0)^2 \sum_{h=0}^{n+2} \binom{n+2}{h} z^h. \tag{5}$$

Which yields the claimed equations organized by powers of z. ☐

Define: ϕ_n as the vector in $\mathbb{R}^{n+1}$ such that

$$\phi_n(h) := \frac{\frac{1}{2}\sqrt{n} + h}{\sqrt{n} + n}. \tag{6}$$

Lemma 3. *ϕ_n from equation 6 is in the interior of $[0,1]^{n+1}$ has $L_n(p,\phi) - \phi(0)^2 = 0$ simultaneously for all p.*

Proof. It is obvious is in the interior of $[0,1]^{n+1}$. So it is just a matter of checking $L_n(p,\phi_n) - \phi_n(0)^2 = 0$ using arguments from [Bauer et al., 2014] or by checking ϕ_n obeys the recurrences in lemma 2. ☐

Theorem 1. *ϕ_n from equation 6 is the unique minimizer of $f_n(\phi)$ and the only ϕ in the interior of $[0,1]^{n+1}$ such that $L_n(p,\phi) - \phi(0)^2 = 0$.*

Proof. By lemma 3 we know ϕ_n meets the conditions of lemma 1. Therefore ϕ_n is the unique global minimizer of $f_n()$. The uniqueness of the minimizer of $f_n()$ means there can be no other solutions of $L_n(p, \phi) - \phi(0)^2 = 0$ that meet the pre-conditions of lemma 1, so ϕ_n must be the only solution $L_n(p, \phi) - \phi(0)^2 = 0$ in the interior of $[0, 1]^{n+1}$.

Note: $L_n(p, \phi) - \phi(0)^2 = 0$ may well have solutions outside of $[0, 1]^{n+1}$ (it is known to have non-real solutions). □

It is kind of neat we get that these is no more than one solution of $L_n(p, \phi) - \phi(0)^2 = 0$ in the interior of $[0, 1]^{n+1}$ from the convexity of the related optimization problem.

4 Discussion

The proof of solution is similar to ideas found in the Majorize-Minimization algorithm[Wikipedia, 2014] where we are using information from functions coincident with $f()$ to get bounds on directional gradients.

The motivating problem (estimating the win-rate of a coin by observing n flips) is standard in probability theory. The derived solution corresponds to Bayesian inference using a $\beta(\frac{1}{2}\sqrt{n}, \frac{1}{2}\sqrt{n})$ prior (or pseudo-observations). This is not a common prior: more common being $\beta(1, 1)$ (Laplace additive smoothing, also minimizes the expected square error under an assumed uniform distribution of the unknown quantity p), and the Jeffreys prior $\beta(\frac{1}{2}, \frac{1}{2})$.

Here we check the claim about +1 smoothing minimizing expected square error under a uniform prior for p.

The expected square error under the uniform prior is given by $D_n()$:

$$D_n(\phi) := \int_{p=0}^{1} \sum_{h=0}^{n} \binom{n}{h} p^h (1-p)^{n-h} (\phi(h) - p)^2 \, \mathrm{d}p \tag{7}$$

Lemma 4. *$D_n(\phi)$ is minimized at $\phi = (\frac{1}{n+2}, \frac{2}{n+2}, \cdots \frac{n+1}{n+2})$.*

One way to look at this is remember $\beta(x, y) := \int_{p=0}^{1} t^{x-1}(1-t)^{y-1} \, \mathrm{d}p$ and then notice $\beta(1, 1)$ is the uniform density on p. So Laplace "add one" smoothing models the use of a uniform prior. This is a sign that naive classical probability (where all indifferent primitive events are assumed to have equal probability, an idea associated with Laplace) may have been an original justification for Laplace additive smoothing (though obviously not acceptable now).

Proof. Look at $\frac{\partial}{\partial \phi(h)} D_n(\phi)$.

$$\begin{aligned}
\frac{\partial}{\partial \phi(h)} D_n(\phi) &= \int_{p=0}^{1} \binom{n}{h} p^h (1-p)^{n-h} 2(\phi(h) - p) \, \mathrm{d}p \\
&= 2\binom{n}{h} \left(\int_{p=0}^{1} p^h (1-p)^{n-h} \phi(h) \, \mathrm{d}p - \int_{p=0}^{1} p^h (1-p)^{n-h} p \, \mathrm{d}p \right) \\
&= 2\binom{n}{h} (\phi(h)\beta(h+1, n-h+1) - \beta(h+2, n-h+1))
\end{aligned}$$

At the optimum we expect these derivatives to be zero. So: $\phi(h) = \beta(h+2, n-h+1)/\beta(h+1, n-h+1) = (h+1)/(n+2)$, which is the claim. Really all we are doing is re-deriving the use of $\beta(,)$ as a conjugate prior to Bernoulli/binomial distributions. □

References

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