

# Solving a coin-flip l2 minimax estimation problem

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## Abstract

We set up and solve a minimax under l2 loss problem derived from a coin flipping problem. The solution is interesting as it involves use of cancellation to solve the minimax problem.

## 1 Introduction

Wald [Wald, 1949] set up statistical estimation as a game played against nature where the researcher picks a (possibly probabilistic) decision function and nature picks an adversarial distribution. Nature's distribution plays the role of Bayesian priors, but is not considered to be the an objective true distribution or a subjective estimate. It is instead a worst-possible distribution so that any inference bounds proven in this formalism hold in general. This game theoretic form of probability is fascinating and leads quickly to interesting questions and procedures.

## 2 The Problem

Take as our problem the task of estimating the unknown win-rate (or heads-rate)  $p$  of a random process or coin. We assume the process is memory-less and stationary ( $p$  is not changing and does not depend on earlier flips). We observe a sequence of  $n$  flips showing  $h$  wins/heads, and then are asked to return an estimate  $\phi_n(h)$  for  $p$ . This problem was discussed and given context in [Mount, 2014a], [Mount, 2014c], [Mount, 2014b], and [Bauer et al., 2014].

Fix  $n \in \mathbb{N}$ ,  $n \geq 1$ . Let  $p \in [0, 1]$  and  $\phi = (\phi(0), \dots, \phi(n))$  be a  $(n + 1)$ -dimensional real vector in  $[0, 1]^{n+1}$ , and define:

$$L_n(p, \phi) := \sum_{h=0}^n \binom{n}{h} p^h (1-p)^{n-h} (\phi(h) - p)^2. \quad (1)$$

$L_n(p, \phi)$  represents the expected square-error of encountered when using  $\phi$  to estimate the win-rate of a coin with (unknown) true win-rate  $p$  by observing  $n$  flips/outcomes. The estimate is: use  $\phi(h)$  when you see  $h$  wins/heads. This is related to Wald's game-theoretic formalism, but we are insisting on pure strategies for both the estimate (a single deterministic  $\phi$ ) and a single unknown true probability  $p$ . We are going to assume that nature picks  $p$  in an adversarial manner with full knowledge of  $\phi$ .

Define:

$$f_n(\phi) = \max_{p \in [0, 1]} L_n(p, \phi). \quad (2)$$

What we are looking for is  $\operatorname{argmin}_{\phi \in [0, 1]^{n+1}} f_n(\phi)$ . The issue is: the definition of  $f_n()$  has two quantifiers so it seems like it will be difficult to derive or even check solutions.

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### 3 A Solution

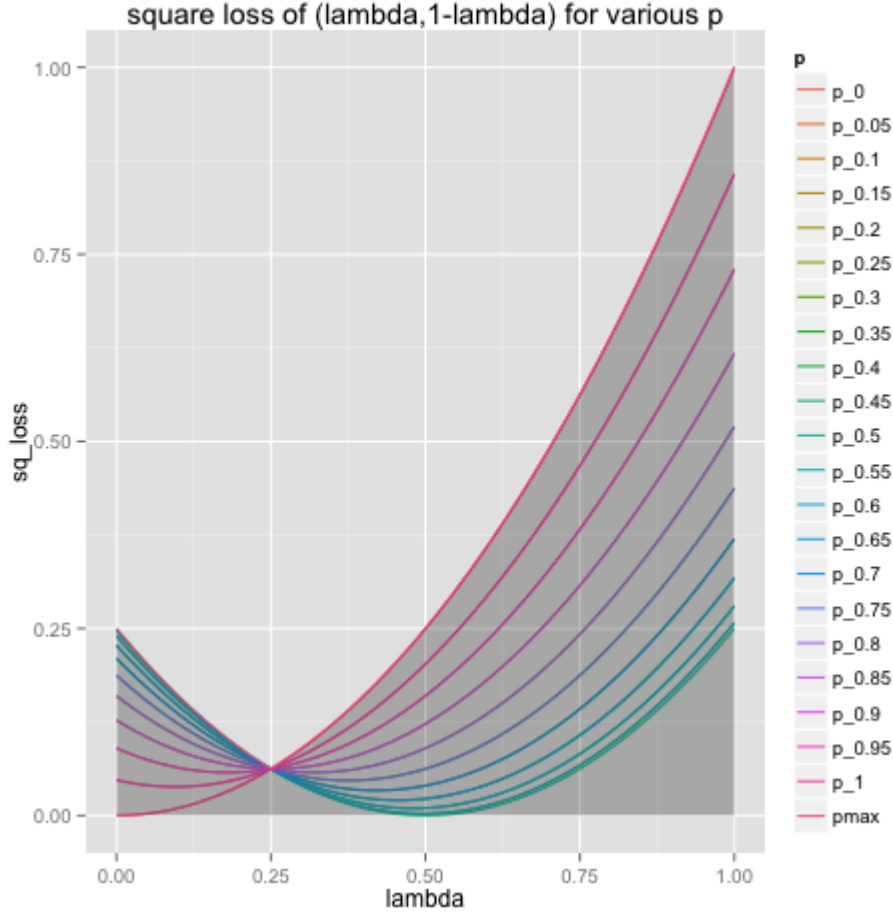


Figure 1:  $L_1(p, (\lambda, 1 - \lambda))$

**Lemma 1.** Suppose  $\phi$  is in the interior of  $[0, 1]^{n+1}$  and is such that  $L_n(p, \phi) - \phi(0)^2 = 0$  simultaneously for all  $p$ . Then:  $\phi$  is the unique global minimizer of  $f_n()$ .

*Proof.* Suppose  $\phi$  is as stated. We will confirm  $\phi$  is an isolated local minimum by checking partial derivatives. Look at  $\frac{\partial}{\partial \phi(h)} f_n(\phi)$  and  $\frac{\partial}{\partial -\phi(h)} f_n(\phi)$ . If we can show these are always both positive for all  $h$  we are done.

Because  $L_n(p, \phi)$  is a constant independent of  $p$  we know  $\frac{\partial}{\partial \phi(h)} f_n(\phi) \geq \frac{\partial}{\partial \phi(h)} L_n(p, \phi)$  for any  $p \in [0, 1]$  (i.e. all  $p$  curves are active or on the boundary boundary, figure 1 shows an example). So

$$\begin{aligned}
 \frac{\partial}{\partial \phi(h)} f_n(\phi) &\geq \max_p \frac{\partial}{\partial \phi(h)} L_n(p, \phi) \\
 &= \max_p \binom{n}{h} p^h (1-p)^{n-h} 2(\phi(h) - p) \\
 &\geq \binom{n}{h} p^h (1-p)^{n-h} 2(\phi(h) - p) \Big|_{p=\phi(h)/2} \\
 &> 0
 \end{aligned}$$

Similarly we know  $\frac{\partial}{\partial -\phi(h)} f_n(\phi) \geq \frac{\partial}{\partial -\phi(h)} L_n(p, \phi)$  for any  $p \in [0, 1]$ . So

$$\begin{aligned} \frac{\partial}{\partial -\phi(h)} f_n(\phi) &\geq \max_p \frac{\partial}{\partial -\phi(h)} L_n(p, \phi) \\ &= \max_p \binom{n}{h} p^h (1-p)^{n-h} 2(p - \phi(h)) \\ &\geq \binom{n}{h} p^h (1-p)^{n-h} 2(p - \phi(h)) \Big|_{p=(1+\phi(h))/2} \\ &> 0 \end{aligned}$$

So we know  $\phi$  is an isolated local minimum of  $f_n()$ . But  $L_n(p, \phi)$  is convex in  $\phi$  for any fixed  $n, p$  ( $n \geq 1, p \in \mathbb{R}$ ), so  $f_n(\phi)$  is also convex in  $\phi$ . So an isolated local minimum  $\phi$  is also the unique global minimum.  $\square$

**Lemma 2.** *If  $L_n(p, \phi) - \phi(0)^2 = 0$  then*

$$\phi(1)^2 = \phi(0)^2 + \frac{2}{k} \phi(0) \quad (3)$$

and for all  $h \geq 2$ :

$$\begin{aligned} \phi(h)^2 &= \frac{(k+2)(k+1)}{(k+2-h)(k+1-h)} \phi(0)^2 \\ &\quad + 2 \frac{h}{k+1-h} \phi(h-1)(1 - \phi(h-1)) \\ &\quad - \frac{h(h-1)}{(k+2-h)(k+1-h)} (\phi(h-2) - 1)^2. \end{aligned} \quad (4)$$

*Proof.* Perform a change of variables  $z = p/(1-p)$  on  $L_n(p, \phi) - \phi(0)^2$  and collect terms in powers of  $z$ . This yields a tri-diagonal system with the claimed equations as the general steps.  $\square$

Define:  $\phi_n$  as the vector in  $\mathbb{R}^{n+1}$  such that

$$\phi_n(h) := (\frac{1}{2}\sqrt{n} + h) / (\frac{1}{2}\sqrt{n} + n). \quad (5)$$

**Lemma 3.**  *$\phi_n$  from equation 5 is in the interior of  $[0, 1]^{n+1}$  has  $L_n(p, \phi) - \phi(0)^2 = 0$  simultaneously for all  $p$ .*

*Proof.* It is obvious is in the interior of  $[0, 1]^{n+1}$ . So it is just a matter of checking  $L_n(p, \phi_n) - \phi_n(0)^2 = 0$  using arguments from [Bauer et al., 2014] or by checking  $\phi_n$  obeys the recurrences in lemma 2.  $\square$

**Theorem 1.**  *$\phi_n$  from equation 5 is the unique minimizer of  $f_n(\phi)$  and the only  $\phi$  in the interior of  $[0, 1]^{n+1}$  such that  $L_n(p, \phi) - \phi(0)^2 = 0$ .*

*Proof.* By lemma 3 we know  $\phi_n$  meets the conditions of lemma 1. Therefore  $\phi_n$  is the unique global minimizer of  $f_n()$ . This follows by combining lemma 1 and lemma 3. The uniqueness of the minimizer of  $f_n()$  means there can be no other solutions of  $L_n(p, \phi) - \phi(0)^2 = 0$  that meet the pre-conditions of lemma 1, so  $\phi_n$  must be the only solution  $L_n(p, \phi) - \phi(0)^2 = 0$  in the interior of  $[0, 1]^{n+1}$ . Note  $L_n(p, \phi) - \phi(0)^2 = 0$  can have solutions outside of  $[0, 1]^{n+1}$  (for example it is known to have non-real solutions).  $\square$

It is kind of neat we get that there is no more than one solution of  $L_n(p, \phi) - \phi(0)^2 = 0$  in the interior of  $[0, 1]^{n+1}$  from the convexity of the related optimization problem.

# References

- [Bauer et al., 2014] Bauer, A., Mount, J., and Dotsenko, V. (2014). Mathoverflow: Existence of solutions of a polynomial system. <http://mathoverflow.net/questions/177574/existence-of-solutions-of-a-polynomial-system>. Accessed: 2014-08-08.
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