Solving a coin-flip l2 mimimax estimation problem

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Abstract

We set up and solve a minimax under l2 loss problem derived from a coin flipping problem. The solution is interesting as it involves use of cancellation to solve the mimimax problem.

1 Introduction

Wald [Wald, 1949] set up statistical estimation as a game played against nature where the researcher picks a (possibly probabilistic) decision function and nature picks an adversarial distribution. Nature's distribution plays the role of Bayesian priors, but is not considered to be the an objective true distribution or a subjective estimate. It is instead a worst-possible distribution so that any inference bounds proven in this formalism hold in general. This game theoretic form of probability is fascinating and leads quickly to interesting questions and procedures.

2 The Problem

Take as our problem the task of estimating the unknown win-rate (or heads-rate) p of a random process or coin. We assume the process is memory-less and stationary (p is not changing and does not depend on earlier flips). We observe a sequence of n flips showing h wins/heads, and then are asked to return an estimate $\phi_n(h)$ for p. This problem was discussed and given context in [Mount, 2014a], [Mount, 2014c], [Mount, 2014b], and [Bauer et al., 2014].

Fix $n \in \mathbb{N}$, $n \geq 1$. Let $p \in [0, 1]$ and $\phi = (\phi(0), \ldots, \phi(n))$ be a $(n + 1)$-dimensional real vector in $[0, 1]^{n+1}$, and define:

$$L_n(p, \phi) := \sum_{h=0}^{n} \binom{n}{h} p^h (1 - p)^{n-h} (\phi(h) - p)^2. \tag{1}$$

$L_n(p, \phi)$ represents the expected square-error of encountered when using ϕ to estimate the win-rate of a coin with (unknown) true win-rate p by observing n flips/outcomes. The estimate is: use $\phi(h)$ when you see h wins/heads. This is related to Wald's game-theoretic formalism, but we are insisting on pure strategies for both the estimate (a single deterministic ϕ) and a single unknown true probability p. We are going to assume that nature picks p in an adversarial manner with full knowledge of ϕ.

Define:

$$f_n(\phi) = \max_{p \in [0,1]} L_n(p, \phi). \tag{2}$$

What we are looking for is $\mathrm{argmin}_{\phi \in [0,1]^{n+1}} f_n(\phi)$. The issue is: the definition of $f_n()$ has two quantifiers so it seems like it will be difficult to derive or even check solutions.

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3 A Solution

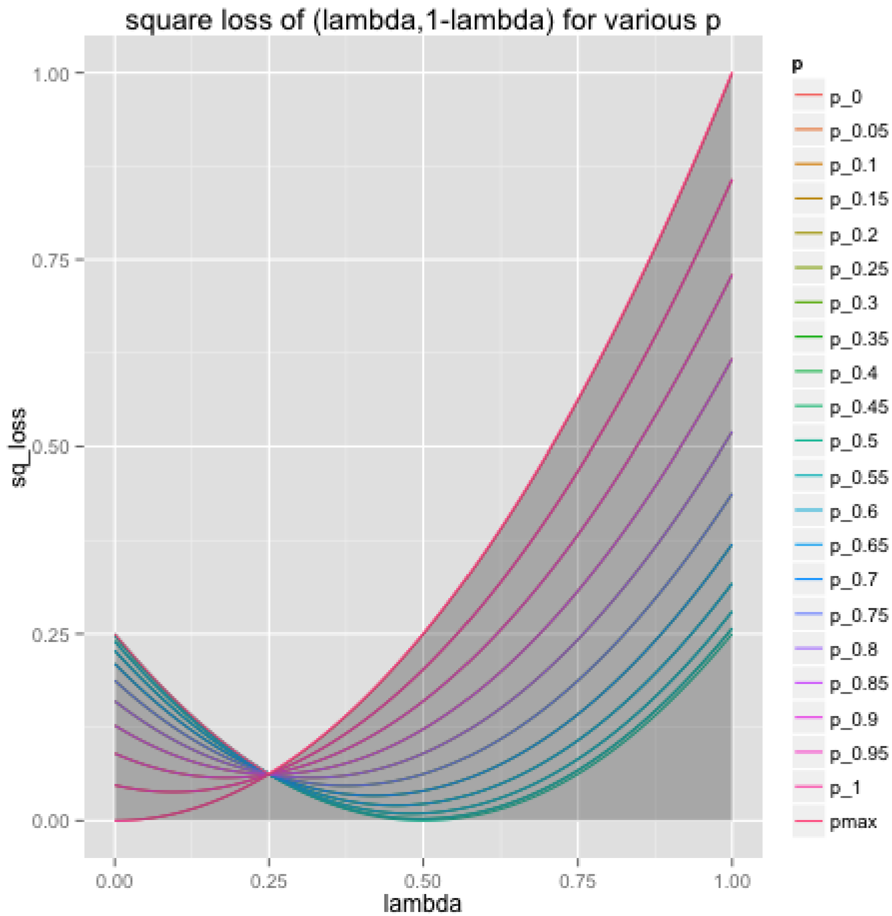

Figure 1: $L_1(p,(\lambda, 1-\lambda))$

Lemma 1. *Suppose ϕ is in the interior of $[0,1]^{n+1}$ and is such that $L_n(p,\phi)-\phi(0)^2 = 0$ simultaneously for all p. Then: ϕ is the unique global minimizer of $f_n()$.*

Proof. Suppose ϕ is as stated. We will confirm ϕ is an isolated local minimum by checking partial derivatives. Look at $\frac{\partial}{\partial \phi(h)} f_n(\phi)$ and $\frac{\partial}{\partial -\phi(h)} f_n(\phi)$. If we can show these are always both positive for all h we are done.

Because $L_n(p,\phi)$ is a constant independent of p we know $\frac{\partial}{\partial \phi(h)} f_n(\phi) \geq \frac{\partial}{\partial \phi(h)} L_n(p,\phi)$ for any $p \in [0,1]$ (i.e. all p curves are active or on the boundary boundary, figure 1 shows an example). So

$$\begin{aligned}
\frac{\partial}{\partial \phi(h)} f_n(\phi) &\geq \max_p \frac{\partial}{\partial \phi(h)} L_n(p,\phi) \\
&= \max_p \binom{n}{h} p^h (1-p)^{n-h} 2(\phi(h)-p) \\
&\geq \binom{n}{h} p^h (1-p)^{n-h} 2(\phi(h)-p)\Big|_{p=\phi(h)/2} \\
&> 0
\end{aligned}$$

Similarly we know $\frac{\partial}{\partial - \phi(h)} f_n(\phi) \geq \frac{\partial}{\partial - \phi(h)} L_n(p, \phi)$ for any $p \in [0,1]$. So

$$
\begin{aligned}
\frac{\partial}{\partial - \phi(h)} f_n(\phi) &\geq \max_p \frac{\partial}{\partial - \phi(h)} L_n(p, \phi) \\
&= \max_p \binom{n}{h} p^h (1-p)^{n-h} 2(p - \phi(h)) \\
&\geq \binom{n}{h} p^h (1-p)^{n-h} 2(p - \phi(h)) \Big|_{p=(1+\phi(h))/2} \\
&> 0
\end{aligned}
$$

So we know ϕ is an isolated local minimum of $f_n()$ But $L_n(p, \phi)$ is convex in ϕ for any fixed n, p ($n \geq 1, p \in [0,1]$), so $f_n(\phi)$ is also convex in ϕ. So an isolated local minimum ϕ is also the unique global minimum. □

Lemma 2. *If $L_n(p, \phi) - \phi(0)^2 = 0$ then*

$$
\phi(1)^2 = \phi(0)^2 + \frac{2}{k}\phi(0) \tag{3}
$$

and for all $h \geq 2$:

$$
\begin{aligned}
\phi(h)^2 = {} & \frac{(n+2)(n+1)}{(n+2-h)(n+1-h)} \phi(0)^2 \\
& + 2\frac{h}{n+1-h} \phi(h-1)(1 - \phi(h-1)) \\
& - \frac{h(h-1)}{(n+2-h)(n+1-h)} (\phi(h-2) - 1)^2.
\end{aligned} \tag{4}
$$

Proof. Perform a change of variables $z = p/(1-p)$ on $L_n(p, \phi) - \phi(0)^2$ and collect terms in powers of z. This yields the following equivalent equation:

$$
\sum_{h=0}^{n} \binom{n}{h} z^h ((1+z)\phi(h) - z)^2 = \phi(0)^2 \sum_{h=0}^{n+2} \binom{n+2}{h} z^h. \tag{5}
$$

Which yields the claimed equations organzied by powers of z. □

Define: ϕ_n as the vector in $\mathbb{R}^{n+1}$ such that

$$
\phi_n(h) := \frac{\frac{1}{2}\sqrt{n} + h}{\sqrt{n} + n}. \tag{6}
$$

Lemma 3. *ϕ_n from equation 6 is in the interior of $[0,1]^{n+1}$ has $L_n(p, \phi) - \phi(0)^2 = 0$ simultaneously for all p.*

Proof. It is obvious is in the interior of $[0,1]^{n+1}$. So it is just a matter of checking $L_n(p, \phi_n) - \phi_n(0)^2 = 0$ using arguments from [Bauer et al., 2014] or by checking ϕ_n obeys the recurrences in lemma 2. □

Theorem 1. *ϕ_n from equation 6 is the unique minimizer of $f_n(\phi)$ and the only ϕ in the interior of $[0,1]^{n+1}$ such that $L_n(p, \phi) - \phi(0)^2 = 0$.*

Proof. By lemma 3 we know ϕ_n meets the conditions of lemma 1. Therefore ϕ_n is the unique global minimizer of $f_n()$. This follows by combining lemma 1 and lemma 3. The uniqueness of the minimizer of $f_n()$ means there can be no other solutions of $L_n(p, \phi) - \phi(0)^2 = 0$ that meet the pre-conditions of lemma 1, so ϕ_n must be the only solution $L_n(p, \phi) - \phi(0)^2 = 0$ in the interior of $[0, 1]^{n+1}$. Note $L_n(p, \phi) - \phi(0)^2 = 0$ can have solutions outside of $[0, 1]^{n+1}$ (for example it is known to have non-real solutions). □

It is kind of neat we get that these is no more than one solution of $L_n(p, \phi) - \phi(0)^2 = 0$ in the interior of $[0, 1]^{n+1}$ from the convexity of the related optimization problem.

4 Discussion

The proof of solution is similar to ideas found in the Majorize-Minimization algorithm[Wikipedia, 2014] where we are using information from functions coincident with $f()$ to get bounds on directional gradients.

The motivating problem (estimating the win-rate of a coin by observing n flips) is standard in probability theory. The derived solution corresponds to Bayesian inference using a $\beta(\frac{1}{2}\sqrt{n}, \frac{1}{2}\sqrt{n})$ prior (or pseudo-observations). This is not a common prior: more common being $\beta(1, 1)$ (Laplace additive smoothing, also minimizes the expected square error under an assumed uniform distribution of the unknown quantity p), and the Jeffreys prior $\beta(\frac{1}{2}, \frac{1}{2})$.

Here we check the claim about $+1$ smoothing minimizing expected square error under a uniform prior for p.

The expected square error under the uniform prior is given by $D_n()$:

$$D_n(\phi) := \int_{p=0}^{1} \sum_{h=0}^{n} \binom{n}{h} p^h (1-p)^{n-h} (\phi(h) - p)^2 \, \mathrm{d}p \tag{7}$$

Lemma 4. *$D_n(\phi)$ is minimized at $\phi = (\frac{1}{n+2}, \frac{2}{n+2}, \cdots \frac{n+1}{n+2})$.*

One way to look at this is remember $\beta(x, y) := \int_{p=0}^{1} t^{x-1}(1-t)^{y-1} \, \mathrm{d}p$ and then notice $\beta(1, 1)$ is the uniform density on p. So Laplace "add one" smoothing models the use of a uniform prior. This is a sign that naive classical probability (also associated with Laplace) may have been an original justification for Laplace additive smoothing (though obviously not acceptable now).

Proof. Look at $\frac{\partial}{\partial \phi(h)} D_n(\phi)$.

$$\begin{aligned}
\frac{\partial}{\partial \phi(h)} D_n(\phi) &= \int_{p=0}^{1} \binom{n}{h} p^h (1-p)^{n-h} 2(\phi(h) - p) \, \mathrm{d}p \\
&= 2\binom{n}{h} \left(\int_{p=0}^{1} p^h (1-p)^{n-h} \phi(h) \, \mathrm{d}p - \int_{p=0}^{1} p^h (1-p)^{n-h} p \, \mathrm{d}p \right) \\
&= 2\binom{n}{h} (\phi(h)\beta(h+1, n-h+1) - \beta(h+2, n-h+1))
\end{aligned}$$

At the optimum we expect these derivatives to be zero. So: $\phi(h) = \beta(h+2, n-h+1)/\beta(h+1, n-h+1) = (h+1)/(n+2)$, which is the claim. Really all we are doing is re-deriving the use of $\beta(,)$ as a conjugate prior to Bernoulli/binomial distributions. □

References

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