2.0 Mathematical Modeling and LP Formulation

Ramon Miguel C. Panis
Instructor
Department Industrial Engineering and Operations Research
University of the Philippines – Diliman

Module 4 of the Business Intelligence and Analytics Track of UP NEC and the UP Center of Business Intelligence

Today's Topics:

- More LP Formulation Problems
- Graphical Interpretation

Prepared by: Ramon Miguel C. Panis
Instructor, UP Diliman

Decision Variables

$$x_1, \ x_2, \ x_3$$

Objective Function

$$\text{Max } z = 2x_1 + 3x_2 - x_3$$

$$x_1 + x_2 + x_3 \leq 10$$
$$3x_1 - x_2 \leq 8$$
$$2x_2 + 3x_3 \leq 12$$

Technological Constraints

$$x_1, \ x_2, \ x_3 \geq 0$$

Non-negativity constraint

A company that operates 10 hours a day manufactures two products on three sequential processes. The following table summarizes the data of the problem

	Minutes per unit		
	Process 1	Process 2	Process 3
A	10	6	8
B	5	20	10

The profit margin for Product A is \$2 while that of Product B is \$5. Determine the optimal product mix for the company

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Instructor, UP Diliman
Source: Taha

An industrial center uses two scrap aluminum metals, A and B, to produce at least 6 tons of special alloy. The contents of the aluminum metals are shown below

	Percent contribution		
	Aluminum	Silicon	Carbon
A	6%	3%	4%
B	3%	6%	3%

The cost per ton of scraps are $100 for A and $80 for B. The special alloy must contain at least 3% Aluminum, 5% Silicon and 4% Carbon. Determine the optimal product mix that should be used in producing the special alloy.

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Source: Taha

Pepito wishes to invest $5000 over the next year in two types of investment. Investment A yields 5% and investment B yields 8%. Market research recommends an allocation of at least 25% in and A and at most 50% in B. Moreover, the investment in A should be at least half the investment in B. What should Pepito do with his money?

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Source: Taha

A trucking company hires workers who are contracted to work for 5 consecutive days and has a daily wage of $100. The minimum number of workers needed for each day (starting from Monday) is 20, 22, 17, 20, 18, 22, 24. Determine the optimal hiring plan for the company

Source: Taha

Juice Co. produces three types of Juices, A, B and C, using fresh strawberries, apples and grapes. The daily supply of each fruit is 2000 lbs, 1750 lbs and 2100 lbs respectively. The ratio of ingredients for each juice and other relevant information is shown in the table

	Ratio of Ingredients			Minimum Demand
	Strawberry	Apple	Grape	
A	1	1	0	1100 lbs
B	1	1	2	1300 lbs
C	0	2	3	1250 lbs

A pound of fresh fruit costs $12, $8, $7 for strawberry, apples and grape respectively. A pound of juice A sells for $20, juice B for $22 and juice C for $26. Determine the optimal product mix for the three drinks.

Source: Taha

A shoe company sells men's shoes and women's shoes. They may also hire a full-time worker or a part-time worker. A full time worker's salary is at $100 for an 8-hour working day while that of a part-time worker is $60 for a 4-hour working day. The time it takes to produce a pair of shoe is 2 hours and 4 hours for men's and women's shoes respectively. As a company policy a worker may only work on the men's shoe department or women's shoe department. The minimum demand for men's shoes is 280 pairs per day while women's shoes has a demand of 350 pairs per day. Additionally labor union rules state that there must be at least 2 full-time employees for every part-time employee regardless of shoe department. How many of each employee must the company hire?

A company produces A, B, and C and can sell these products in unlimited quantities at the following unit prices: A, \$10; B, \$56; C, \$100. Producing a unit of A requires 1 unit hour of labor; a unit of B, 2 hours of labor plus two units of A; and a unit of C, 3 hours of labor plus 1 unit of B. Any A that is used to produce B cannot be sold. Similarly, any B that is used to produce C cannot be sold. A total of 40 hours of labor are available. Formulate an LP to maximize the company's revenues.

Source: Winston

Recall:

When does a solution become a feasible solution?

When the solution satisfies all constraints

When does a solution become an optimal solution?

When the solution is feasible and there is no other solution that will yield a better objective function value

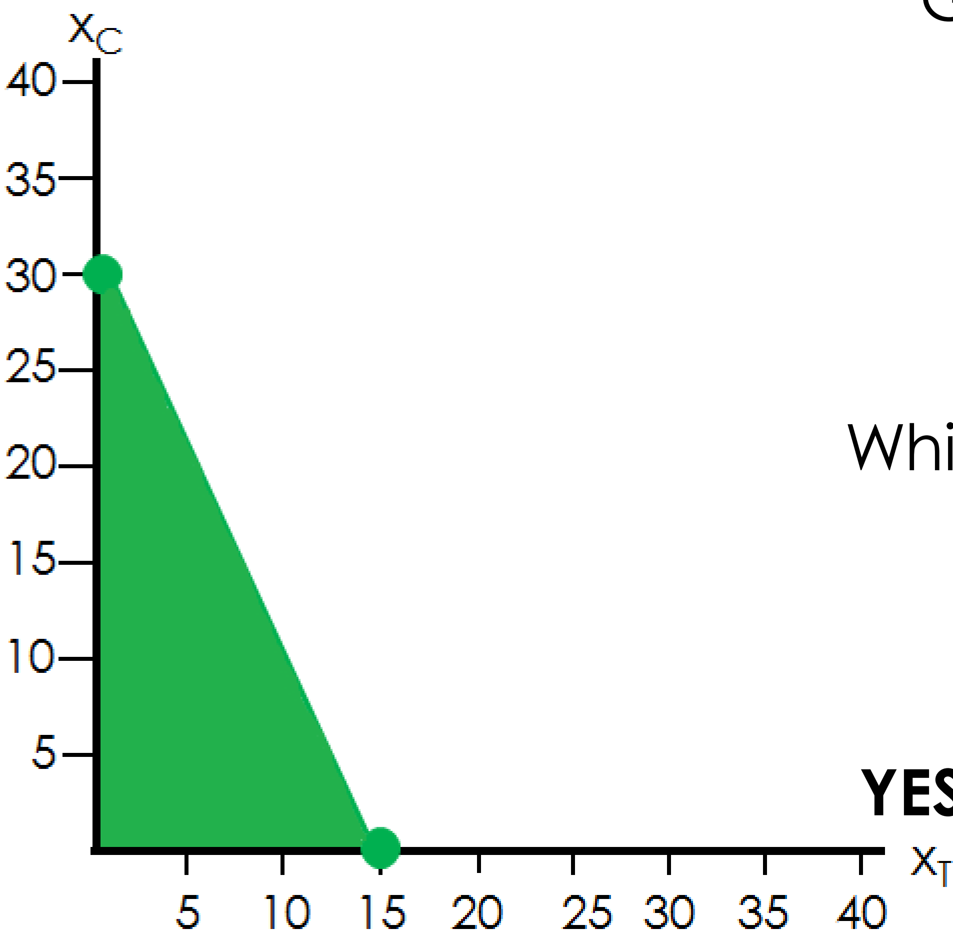
A furniture company produces tables and chairs which are both processed through assembly and finishing departments. Assembly has 60 hours available; finishing can handle up to 48 hours of work. Producing a table requires 4 hours in assembly and 2 hours in finishing, each chair requires 2 hours in assembly and 4 hours in finishing. If profit is $8 per table and $6 per chair, determine the best possible combination of tables and chairs to produce and sell in order to realize maximum profit.

Graphical Interpretation of a Linear Program

Let x_i be number of product i to be produced
(i = T for tables and C for chairs)

Maximize $z =$	$8x_T + 6x_C$	Maximize Total Profit
subject to:	$4x_T + 2x_C \leq 60$	Assembly Dept.
	$2x_T + 4x_C \leq 48$	Finishing Dept.
	$x_T \geq 0$ $x_C \geq 0$	Non-negativity Constraints

Let us analyze the graphical interpretation of this Linear Programming Problem



Graphing the constraint:

$$4x_T + 2x_C \leq 60$$

If $x_T = 0$; $x_C = 30$

If $x_C = 0$; $x_T = 15$

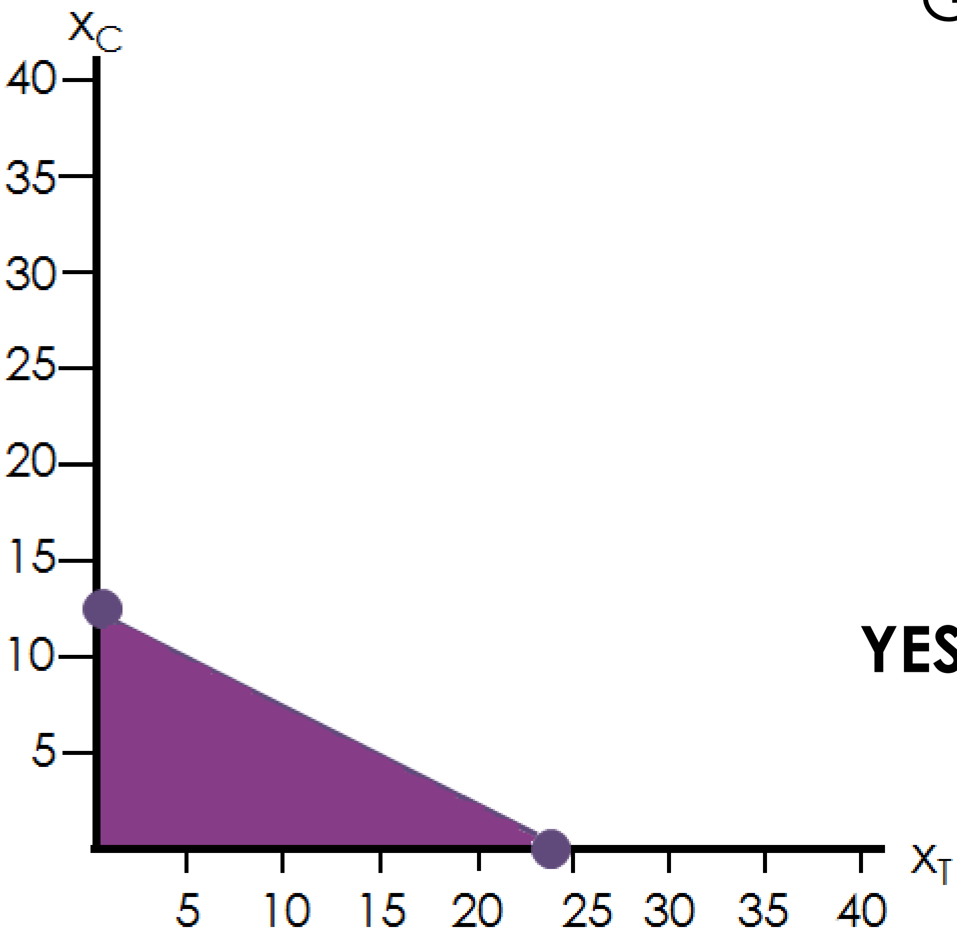
Which part of the graph satisfies the constraint?

Test (0,0)

$$4(0) + 2(0) \leq 60$$

YES! (0,0) is part of the graph

Graphical Interpretation of a Linear Program



Graphing the constraint:

$$2x_T + 4x_C \leq 48$$

If $x_T = 0$; $x_C = 12$

If $x_C = 0$; $x_T = 24$

Test $(0,0)$

$$2(0) + 4(0) \leq 48$$

YES! $(0,0)$ is part of the graph

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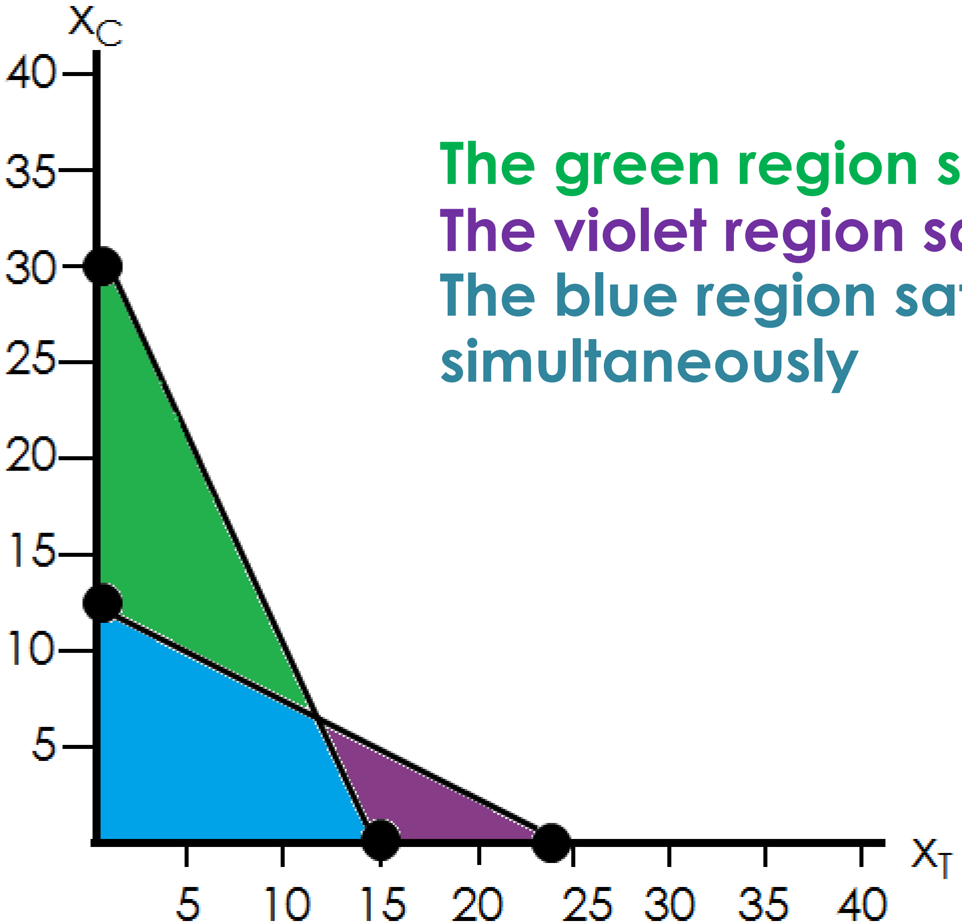
Graphical Interpretation of a Linear Program



The green region satisfies the first constraint
The violet region satisfies the second constraint
The blue region satisfies both constraints simultaneously

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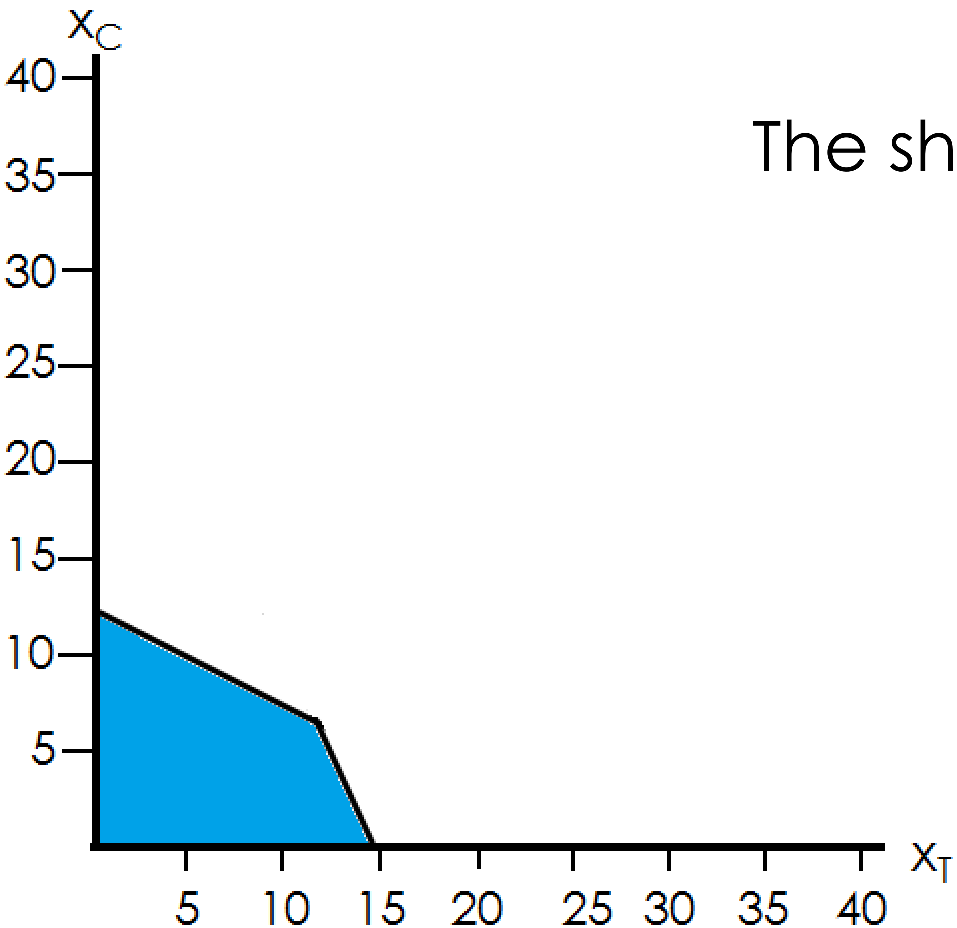
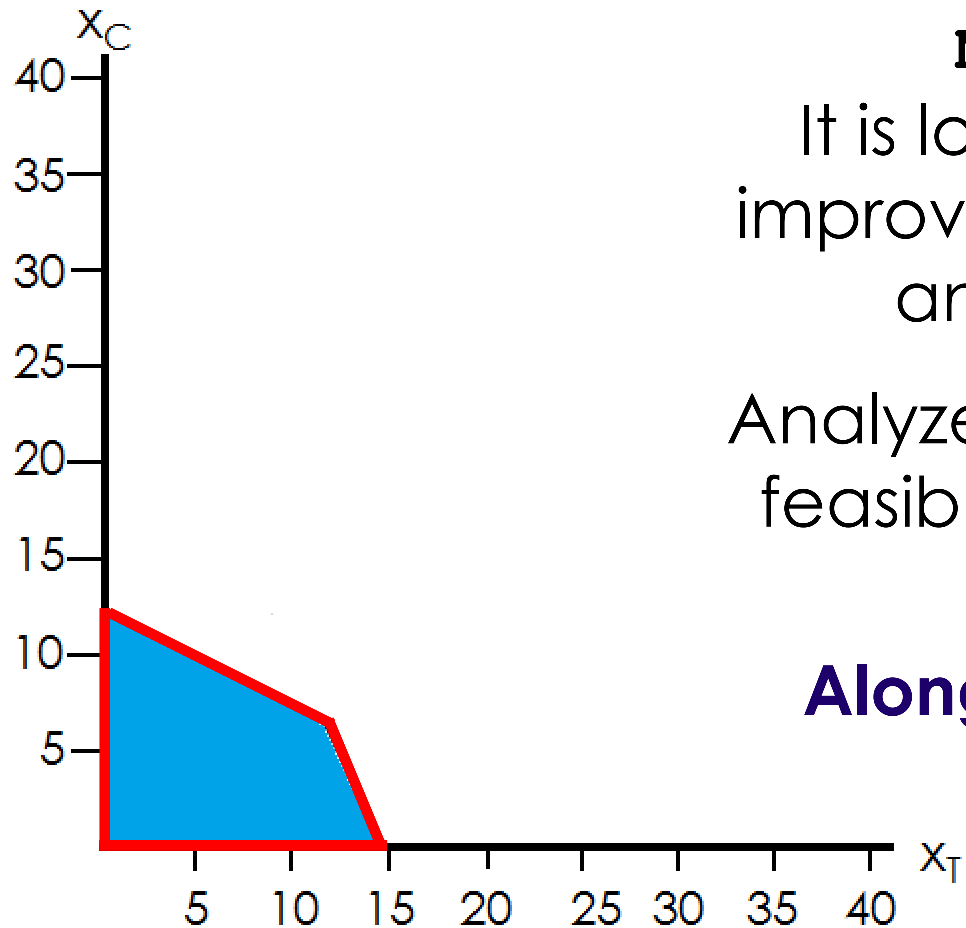


The shaded region defines our feasible solution



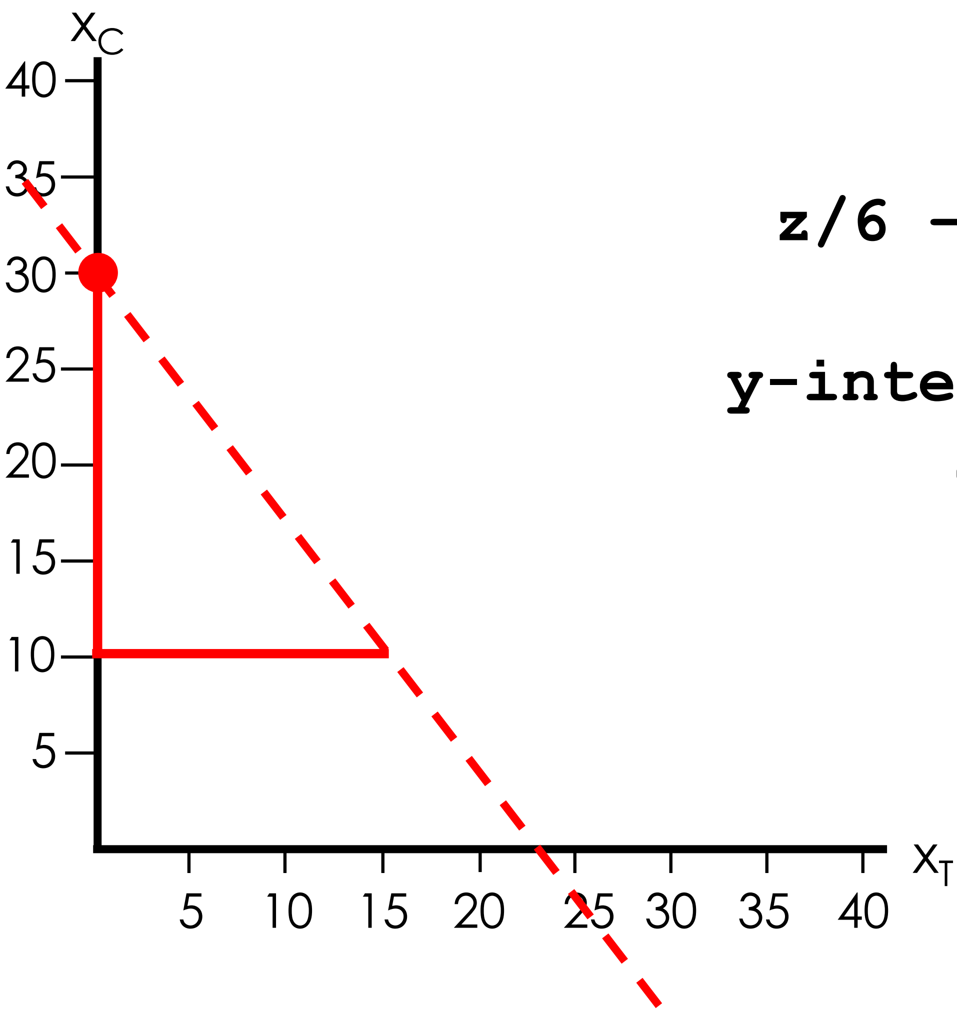
Given the objective function

$$\text{Max } z = 8x_T + 6x_C$$

It is logical to state that z will improve as x_T moves to the right and x_C moves upward

Analyze this: In which part of the feasible region will the optimal solution fall?

Along the boundaries of the feasible region



Graphing: $z = 8x_T + 6x_C$

$$z = 8x_T + 6x_C$$

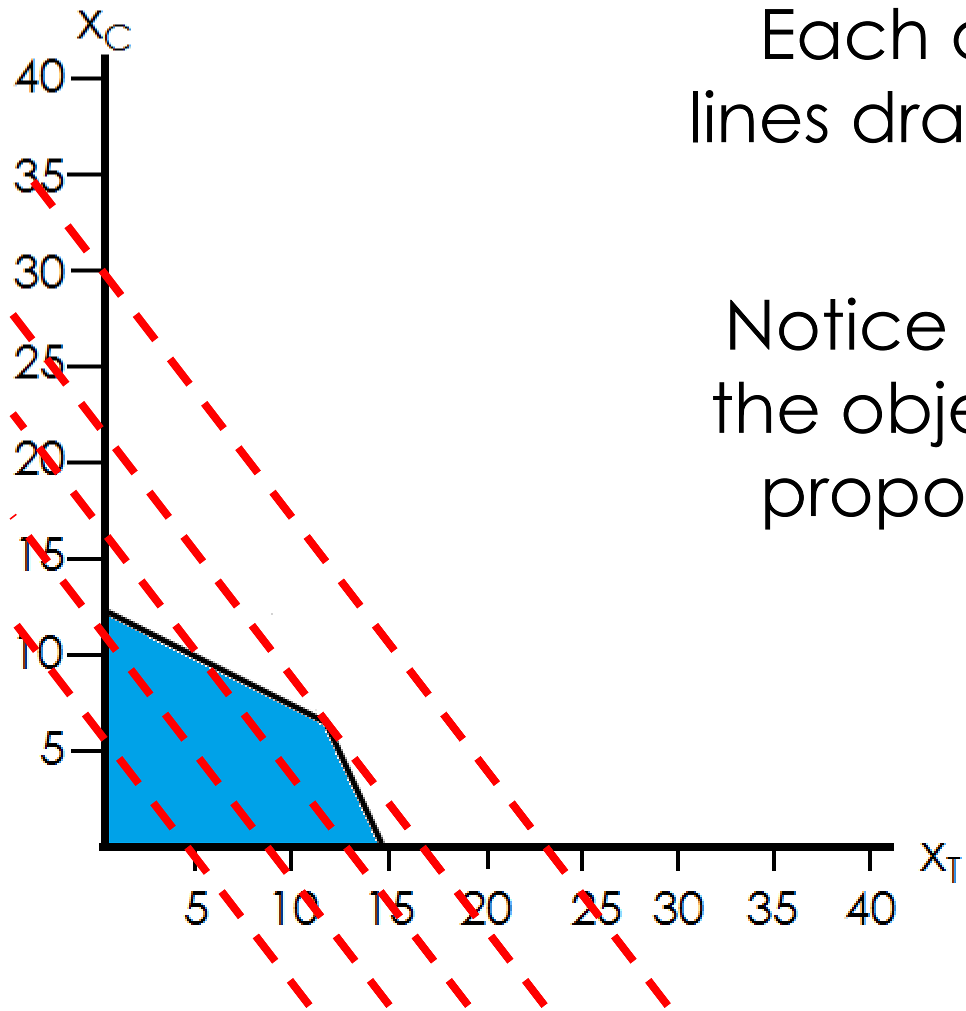
$$z - 8x_T = + 6x_C$$

$$z/6 - (8/6)x_T = x_C$$

$$b + mx = y$$

y-intercept, $b = z/6$

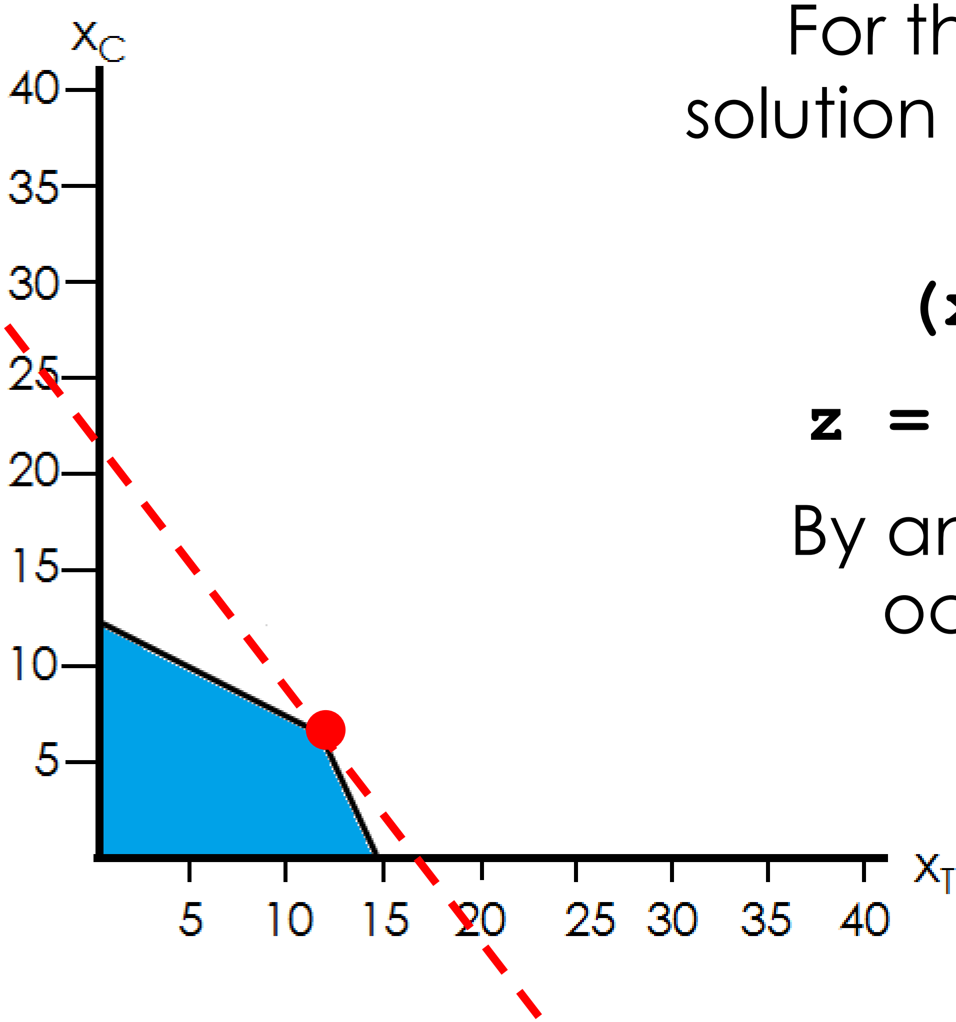
Slope, $m = -4/3$



Each of the objective function lines drawn represent an objective function value

Notice also that for this problem, the objective function z is directly proportional to the y-intercept

b = z/6

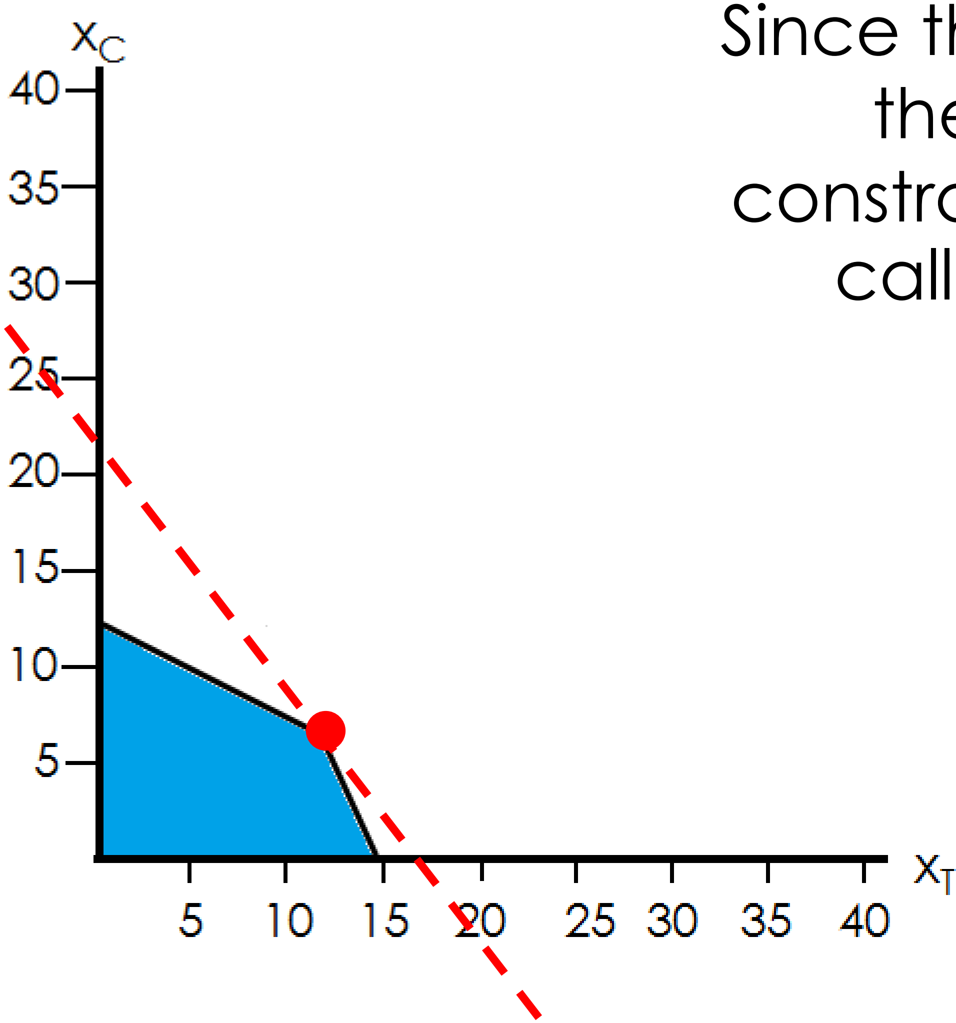


For this problem, the optimal solution is at the intersection of the two constraints

$$(x_T, x_C) = (12, 6)$$

$$z = 8(12) + 4(6) = 132$$

By analysis, optimal solutions occur on corner points



Since the optimal solution lies on the intersection of both constraints, both constraints are called binding constrains.

Binding Constraints

Constraints that define your optimal solution

Theorem

The feasible region of a linear program is convex

Convex Region

If any two points in a region is connected by a line segment, the entirety of the line segment is also within the region

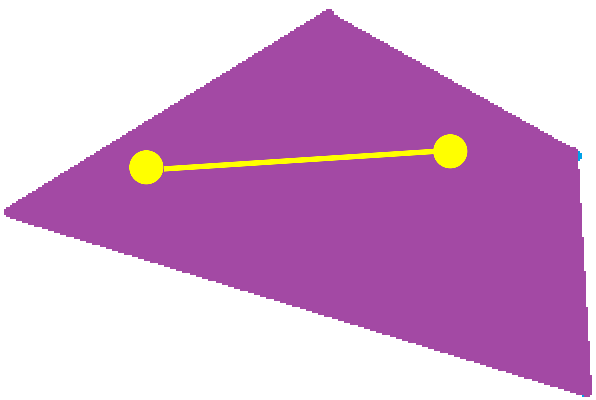
Given any $(x_a, x_b) \in R$

$$\lambda x_a + (1-\lambda)x_b \in R$$

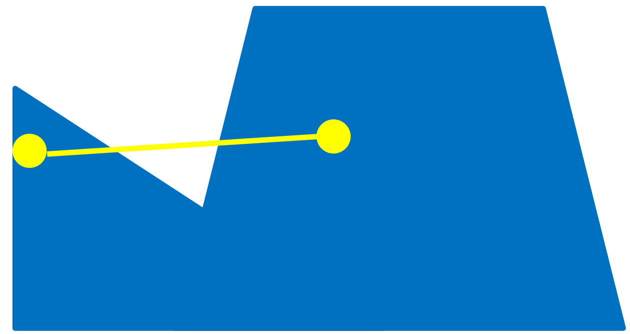
$$0 \leq \lambda \leq 1$$

Graphical Interpretation of a Linear Program

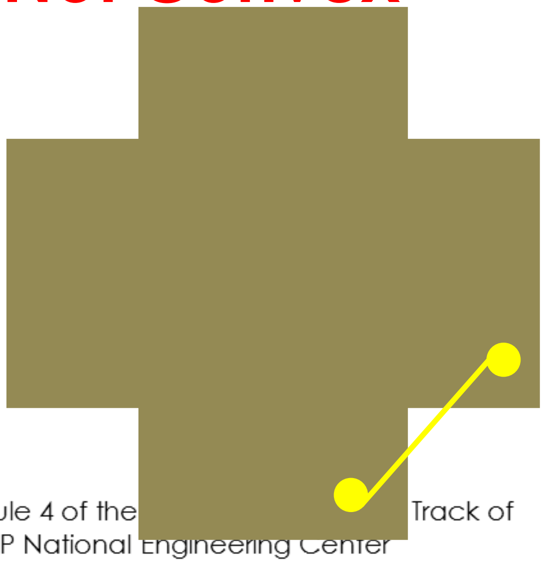
Convex



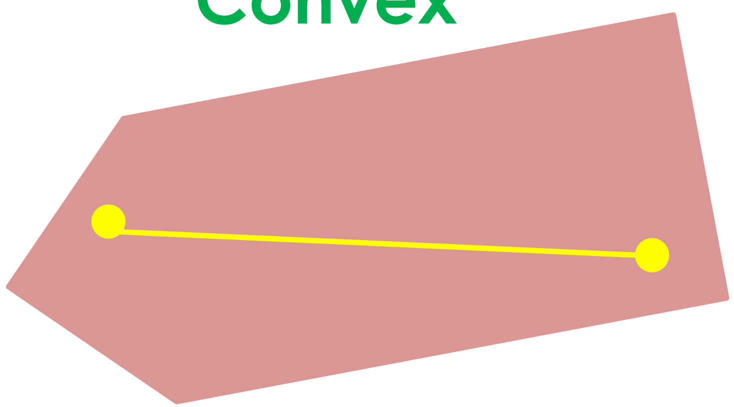
Not Convex

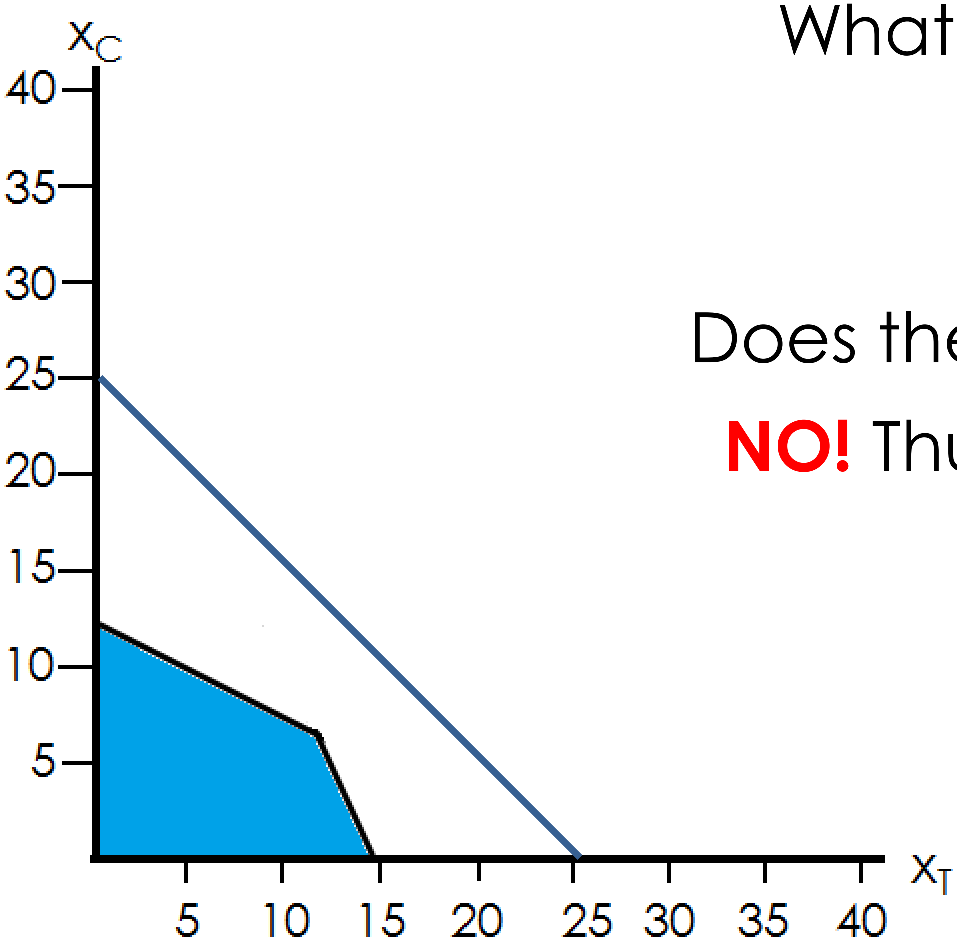


Not Convex



Convex





What happens if we add the constraint

$$x_T + x_C \leq 25$$

Does the feasible region change?

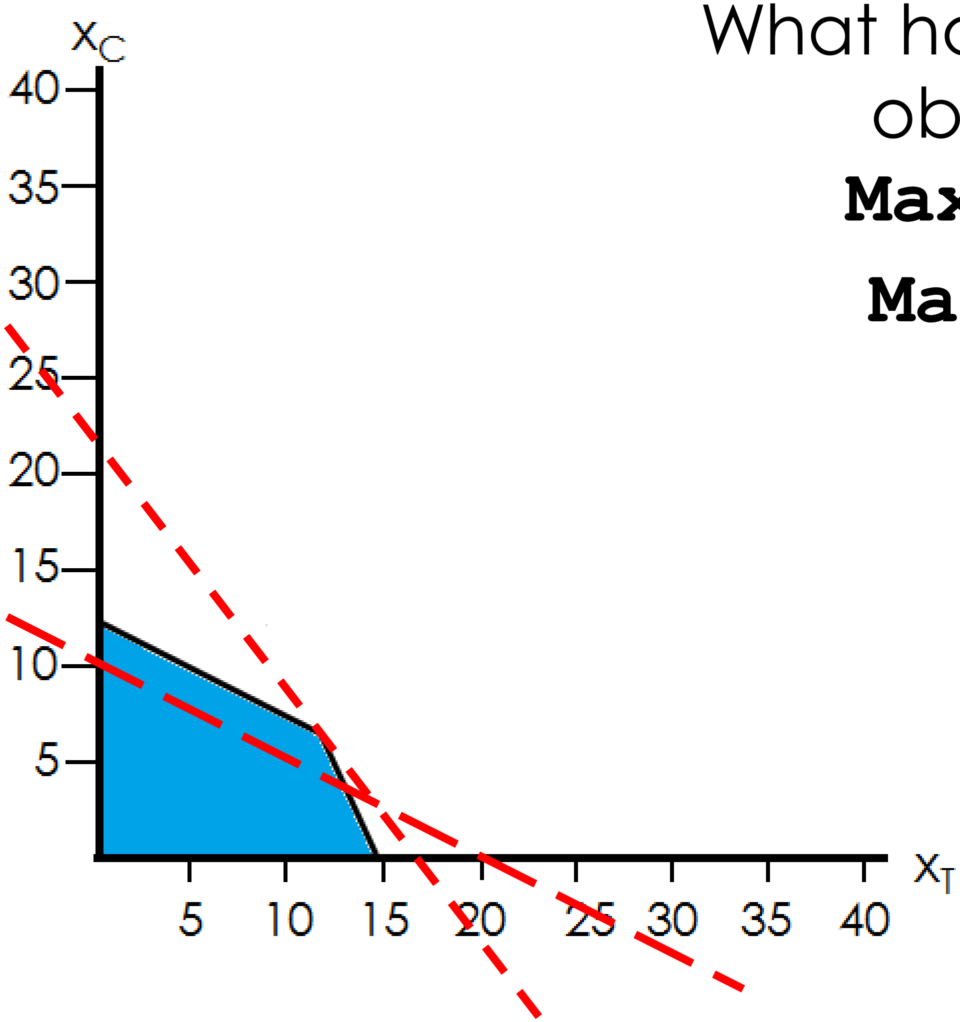
NO! Thus the constraint is said to be redundant

Redundant Constraint

A constraint that does not contribute to the bounds of the feasible region

A constraint when omitted does not change the feasible region

Graphical Interpretation of a Linear Program



What happens when we alter the objective function from

Max z = $8x_T + 6x_C$ to

Max z = $3x_T + 6x_C$

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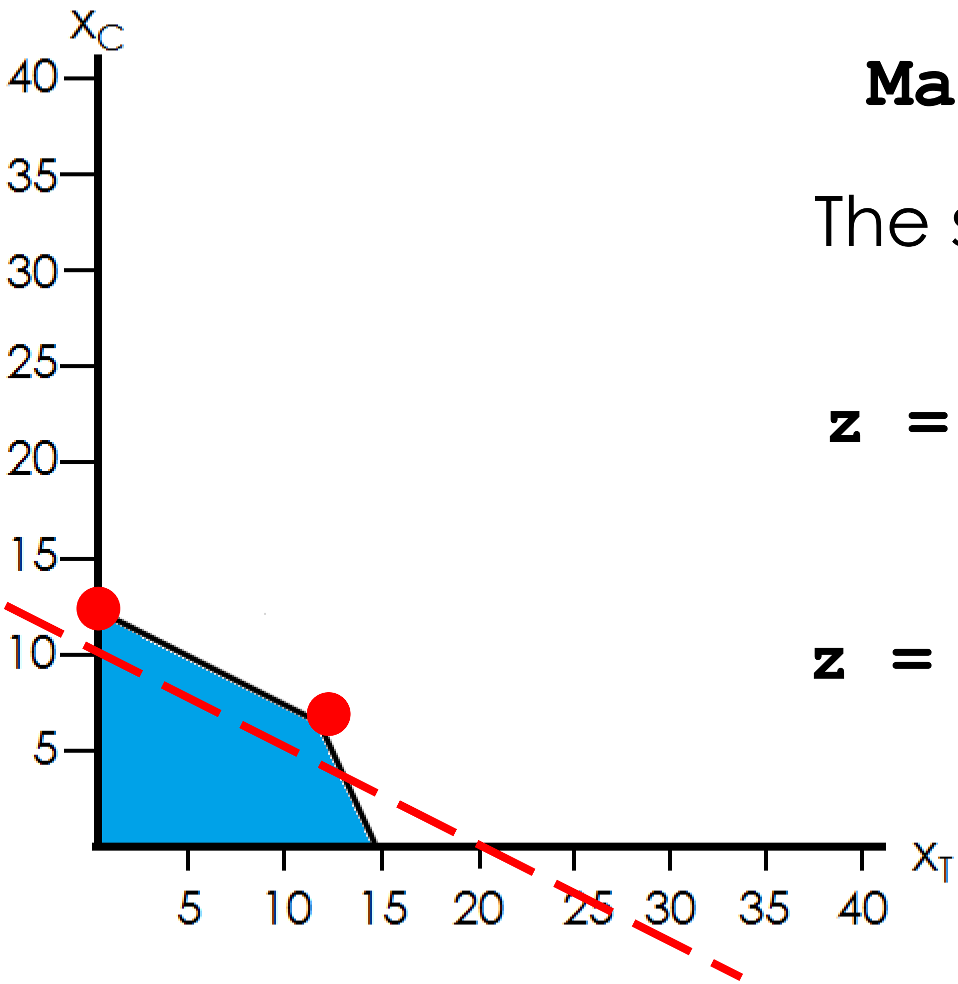


Max $z = 3x_T + 6x_C$

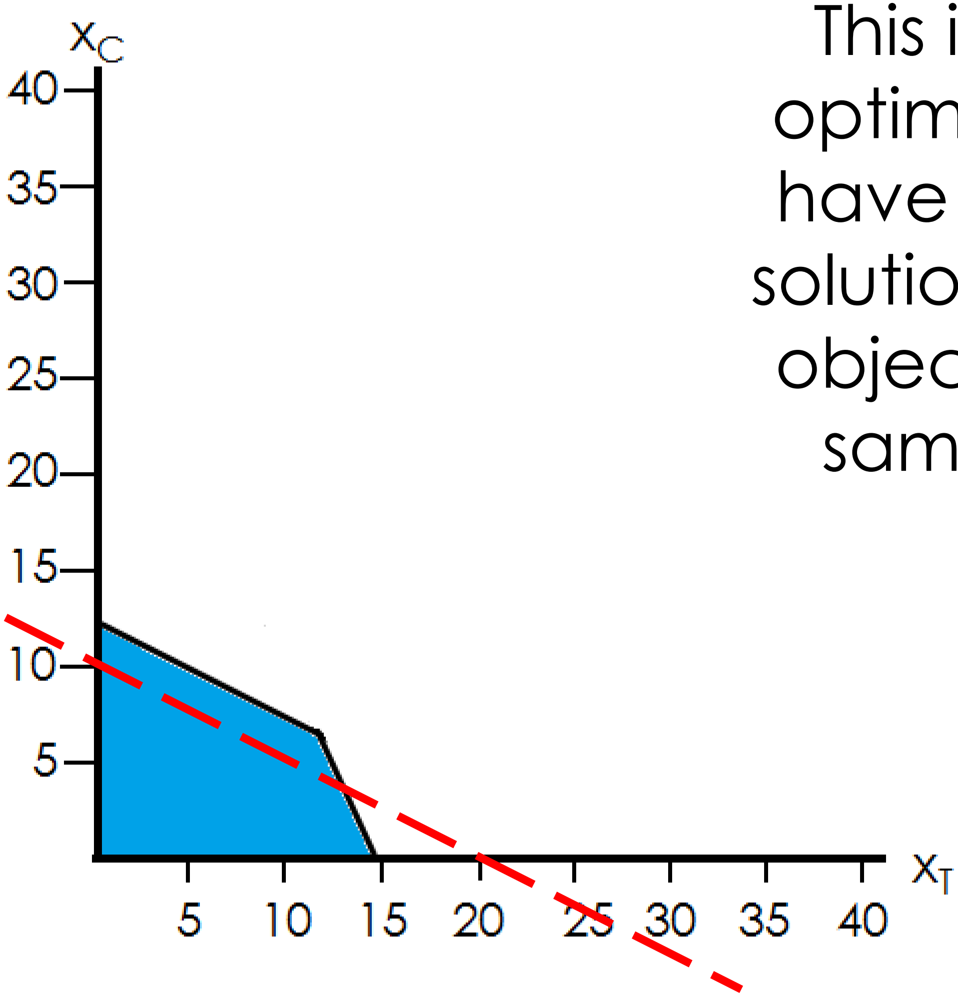
The solution **(12, 6)** is still optimal

$z = 3(12) + 6(6) = 72$

So is **(0, 12)**

$z = 3(0) + 6(12) = 72$

Graphical Interpretation of a Linear Program

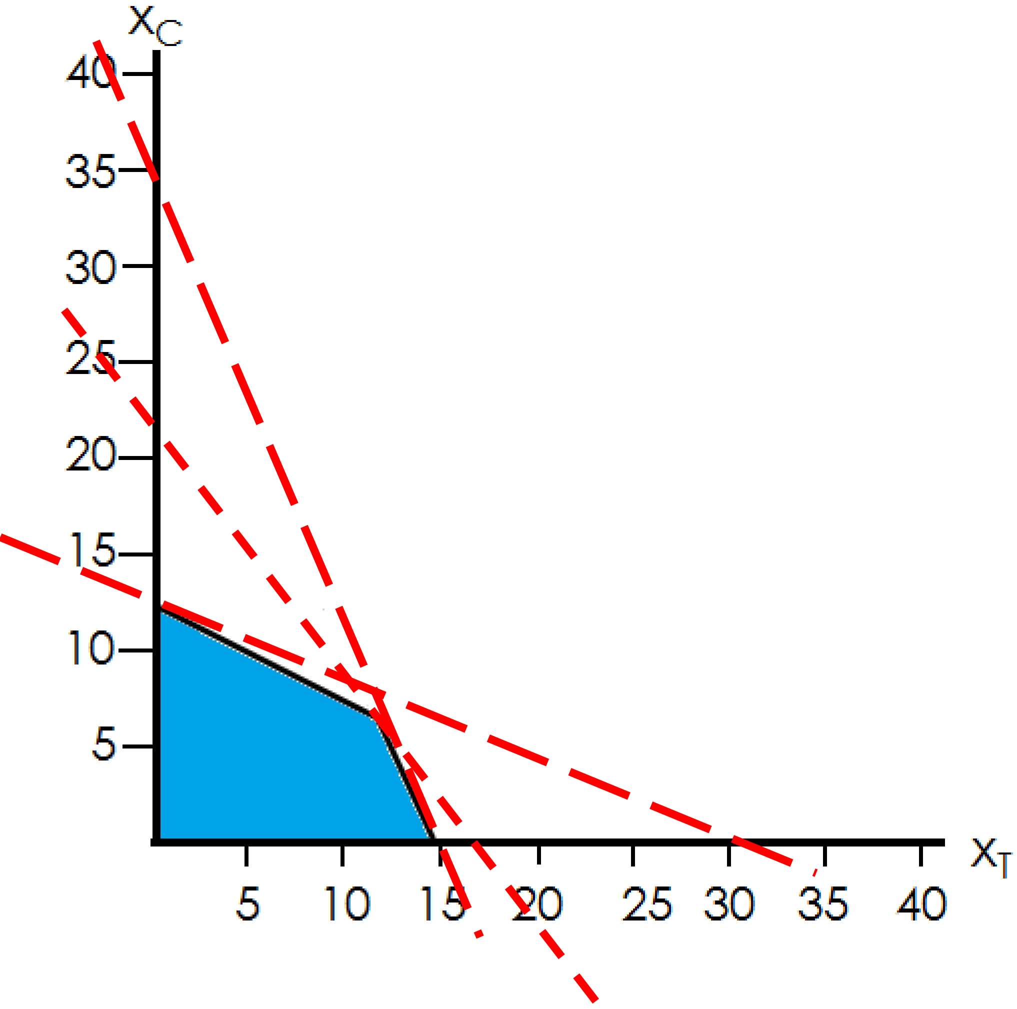


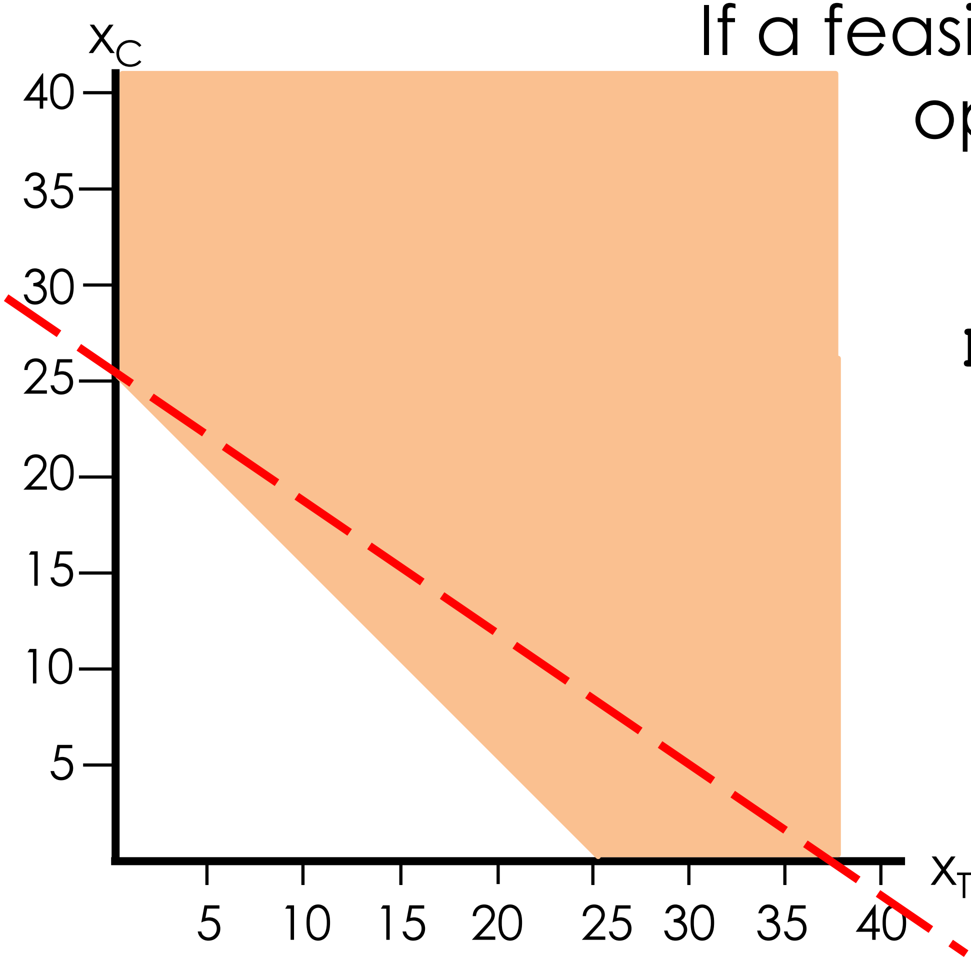
This is called an alternative optimal solution, wherein you have more than one optimal solution. This occurs when your objective function shares the same slope with a binding constraint

Theorem

Each corner point is an optimal solution to a certain objective function

Graphical Interpretation of a Linear Program





If a feasible solution exists, does an optimal solution exist?

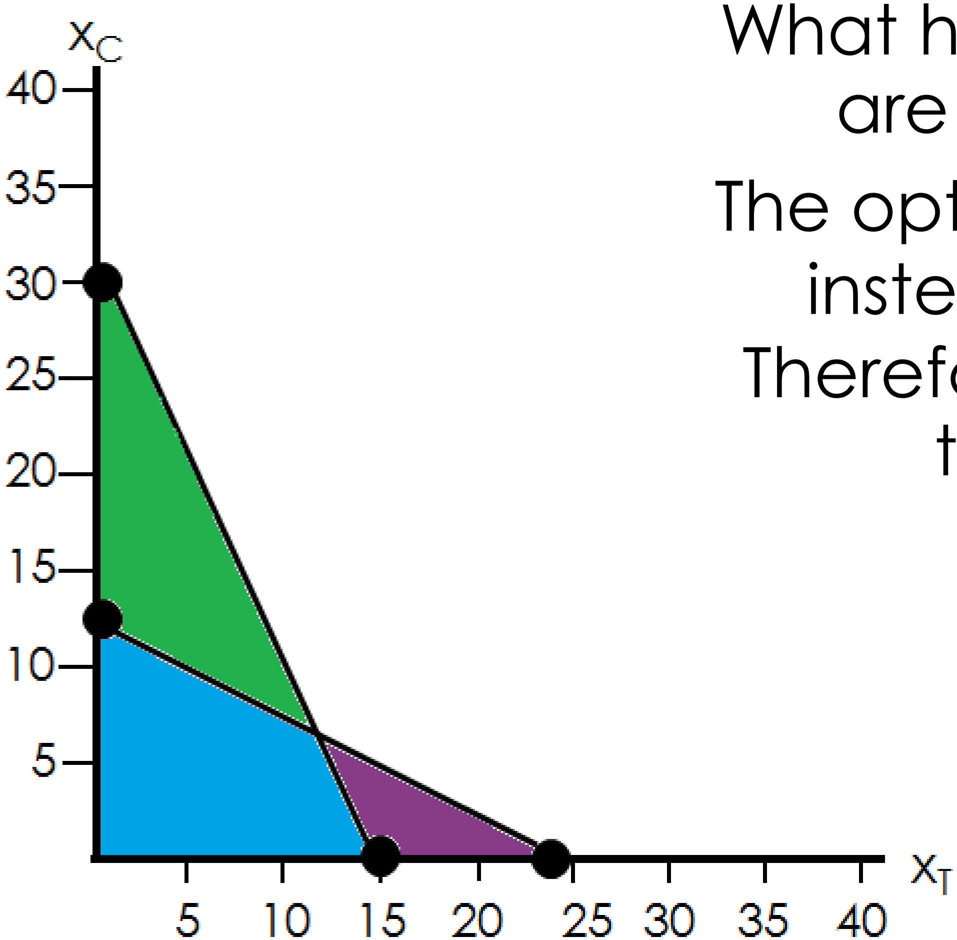
Not all the time

$$\text{Max } z = 5x_T + 8x_C$$
$$x_T + x_C \geq 25$$
$$x_T,\ x_C \geq 0$$

Theorem

A linear program that has a defined feasible solution has a defined optimal solution

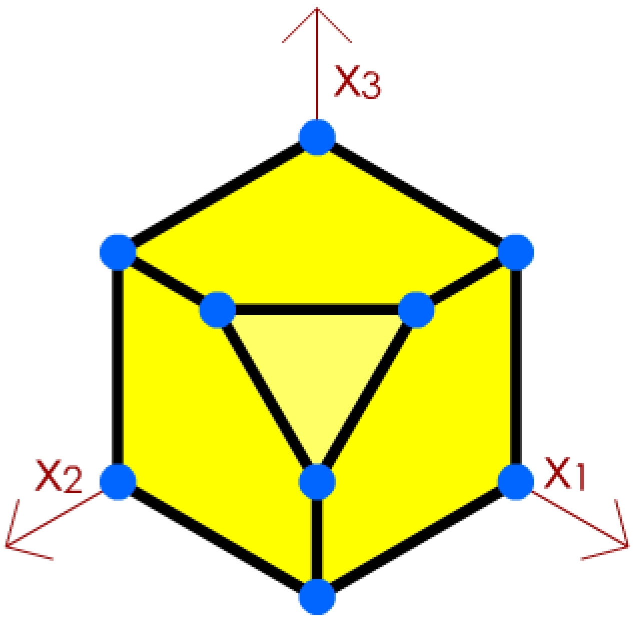
Graphical Interpretation of a Linear Program



What happens when constraints are of the ‘>’ or ‘<‘ type?

The optimum value will be a limit instead of a defined value.

Therefore LP’s **cannot** have this type of inequalities.



$$
\begin{array}{rcrcrcl}
x_1 & & & & & \le & 2 \\
 & & x_2 & & & \le & 2 \\
 & & & & x_3 & \le & 2 \\
x_1 & + & x_2 & + & x_3 & \le & 5 \\
x_1 & & & & & \ge & 0 \\
 & & x_2 & & & \ge & 0 \\
 & & & & x_3 & \ge & 0
\end{array}
$$

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Instructor, UP Diliman

Source: Sir Lu

Three types of Linear Problems

- One that has a defined optimal solution. This includes alternative optimal solutions
- One that has no feasible region
- One that has an objective function that can be improved indefinitely

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Instructor, UP Diliman

Next Session

Solutions Using a Software

Source: Taha

Fin.

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Instructor, UP Diliman