§10.5 - Comparison Tests for Series

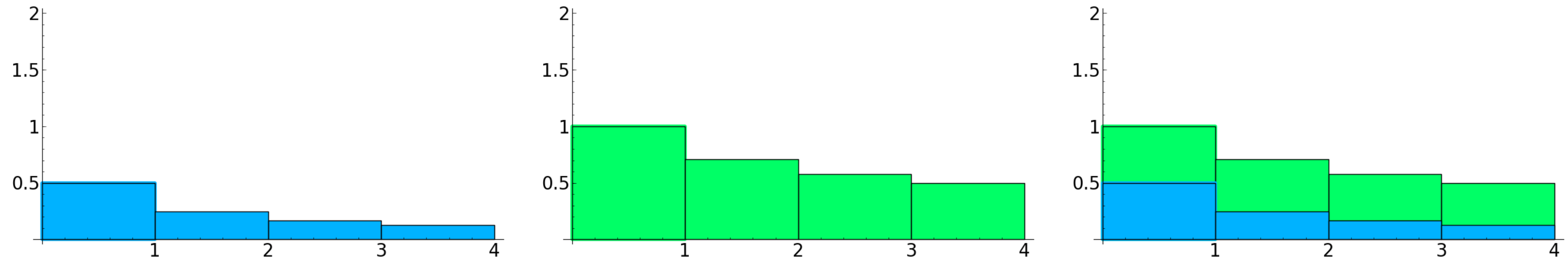
After completing this section, students should be able to:

- For the (ordinary) comparison test, give conditions that will guarantee convergence of a series and conditions that will guarantee divergence of a series, and justify why these conditions make sense.
- For the limit comparison test, state what values of the limit of the ratio of terms allows you to determine that a series converges or diverges, and what values are inconclusive.
- Determine what series to compare another series to, when using the comparison or limit comparison test.
- Identify situations that make it preferable to use the ordinary comparison test instead of the limit comparison test and vice versa.

Theorem. *(The Comparison Test for Series) Suppose that* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are series and*
$0 \leq a_n \leq b_n$ *for all* n*.*

(a) If ________ converges, then ________ converges.

(b) If ________ diverges, then ________ diverges.



Note. The following series are especially handy to compare to when using the comparison test.

(a) ______________________________ which converges when ________

(b) ______________________________ which converges when ________

Example. Does $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{5^n + n^2}$ converge or diverge?

Theorem. *(The Limit Comparison Test) Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

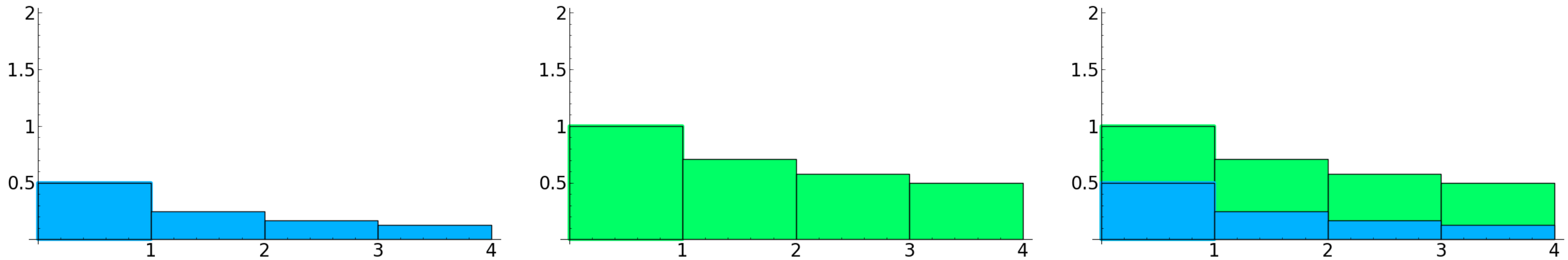
where L is a finite number and $L > 0$, then either both series converge or both diverge.

Example. Does $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{5^n - n^2}$ converge or diverge?

Review. The (Ordinary) Comparison Test for Series: Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms and $0 \leq a_n \leq b_n$ for all n.

(a) If ________ converges, then ________ converges.

(b) If ________ diverges, then ________ diverges.



Review. Suppose $\sum^{\infty} a_n$ and $\sum^{\infty} b_n$ are series with positive terms. Which of the following will allow us to conclude that $\sum^{\infty} b_n$ converges? (More than one answer may be correct.)

A. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n$ and $\displaystyle\sum^{\infty} a_n$ converges.

B. $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 0$ and $\sum^{\infty} a_n$ converges.

C. $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{1}{3}$ and $\sum^{\infty} a_n$ converges.

D. $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 5$ and $\sum^{\infty} a_n$ converges.

Review. The Limit Comparison Test: Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

where L ____,
then:

Advice on the Comparison Theorems:

Question. What series are especially handy to compare to when using the comparison test?

Question. How to decide whether to use the Ordinary Comparison Test or the Limit Comparison Test?

Example. Decide if the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{3n - 5}{\sqrt{n^3 + 2n}}$$

Example. Decide if $\displaystyle\sum_{n=3}^{\infty} \frac{n\sin^2(n)}{n^3+7n}$ converges or diverges.

Example. Decide if $\sum_{n=3}^{\infty} \frac{n \sin^2(n)}{n^3 - 7n}$ converges or diverges.

nsin^2(n) / n^3 - 7n Both are >= 1/n^2 converges
sin^2(n) / n^2-7

sin^2(n) / n^2-7
= lim n-> infinity (sin^2(n)/n^2-7) (n^2)
= lim n-> infinity n^2sin^2(n) / n^2-7

lim n->infinity sin^2(n) / n^2-7 / sin^2(n) / n^2+7
lim n-> infinity sin^2(n) / n^2-7 * n^2+7 / sin^2(n)
lim n-> infinity n^2+7 / n^2-7
lim n->infinity 2n/2n
= 1

Question. True or False: For $a_n, b_n > 0$, if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then the series $\sum a_n$ and $\sum b_n$ have the same convergence status.

Can anything be concluded if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$?

Question. Find the error: Consider the two series

$$\sum_{n=1}^{\infty} a_n = (-1) + (-2) + (-3) + (-4) + (-5) + (-6) \dots$$

and

$$\sum_{n=1}^{\infty} b_n = 2 + (-1) + (1/2) + (-1/4) + (1/8) + (-1/16) + \dots$$

Note that $\sum_{n=1}^{\infty} b_n$ is a geometric series with ratio $r = -1/2$.

Since $a_n \leq b_n$ for all n, and $\sum b_n$ converges, $\sum a_n$ also converges, by the Ordinary Comparison Test.

Error: Cannot use ordinary comparison test or limit comparison test, an, bn > 0

Note. Orders of magnitude:

Σan and Σbn

an, bn >= 0
Σan want to know
Σbn know

lim n->infinity an/bn = L

L>0, Σan + Σbn do the same thing (both converge or diverge)
L = 0 Σbn converges, Σan converges
L = infinity Σbn diverge, Σan diverge

Note. Review of the convergence tests for series so far:

(a)

(b)

(c)

(d)

(e)

(f)