§8.3 - Integrating Trig Functions

After completing this section, students should be able to:

- Compute integrals of powers of sine and cosine that include at least one odd power by converting sines to cosines or vice versa and using u-substitution.
- Compute integrals of even powers of sine and cosine using the trig identities $\cos^(\theta) = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$ and $\sin^(\theta) = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$
- Compute some powers of sec and tan by converting them to sine and cosine, or by applying u-substitution.
- Compute $\int \sec(x)\,dx$ and $\int \csc(x)\,dx$.
- Compute $\int \sec^2(x)\,dx$ and $\int \tan^2(x)\,dx$

Note. Here are some useful trig identities for the next few sections.

1. Pythagorean Identity: `sin^2(x) + cos^2(x) = 1`
2. Converted into tan and sec: `tan^2(x) + 1 = sec^2(x)`
3. Converted into cot and csc: `1 + cot^2(x) = csc^2(x)`
4. Even and Odd:
   ```
   even: cos(-theta) = cos(theta)
   odd:
   sin(-theta) = -sin(theta)
   tan(-theta) = -tan(theta)
   ```
5. Angle Sum Formula: $\sin(A+B) =$ `sinAcosB + cosAsinB`
6. Angle Sum Formula: $\cos(A+B) =$ `sinAcosB - cosAsinB`
7. Double Angle Formula: $\sin(2\theta) =$ `2sin(theta)cos(theta)`
8. Double Angle Formulas: $\cos(2\theta) =$
   ```
   cos^2(theta) - sin^2(theta)
   2cos^2(theta) - 1
   1 - 2sin^2(theta)
   ```
9. `cosAcosB = cos(A+B) + cos(A-B) / 2`
10. `sinAsinB = cos(A-B) - cos(A+B) / 2`
11. $\cos^2(\theta) =$ `1 + cos(2theta) / 2`
12. $\sin^2(\theta) =$ `1 - cos(2theta) / 2`

Example. Find $\displaystyle\int \sin^4(x)\cos(x)\,dx$

```
u = sinx
du = cosx dx

∫u^4 du
1/5(u^5)
1/5(sin^5(x)) + c
```

```
One of the powers is odd
```

Example. Find $\displaystyle\int \sin^4(x)\cos^3(x)\,dx$

```
∫sin^4(x)cos^2(x)cosx dx
∫sin^4(x)(1−sin^2(x))cosx dx

u = sinx
du = cosx dx

∫u^4(1−u^2)du
∫(u^4−u^6) du
1/5(u^5) − 1/7(u^7)
1/5(sin^5(x)−sin^7(x)) + c
```

Example. Find $\int \sin^5(x)\cos^2(x)\,dx$

```
∫sin^4(x)cos^2(x)sinx dx
∫(sin^2(x))^2 cos^2(x)sinx dx
∫(1−cos^2(x))^2 cos^2(x)sinx dx

u = cosx
du = −sinx dx
−du = sinx dx

−∫(1−u^2)^2 u^2 du
−∫(1−2u^2+u^4) u^2 du
−∫(u^2−2u^4+u^6) du
((−u^3)/3 + (2u^5)/5 − (u^7)/7)
(−cos^3(x)/3 + 2cos^5(x)/5 − (cos^7(x)/7)
```

Example. Consider

$$\int \cos^2(x)dx$$

.

According to a TI-89 calculator

$$\int \cos^2(x)\,dx = \frac{\sin(x)\cos(x)}{2} + \frac{x}{2}$$

.

According to the table in the back of the book,

$$\int \cos^2(x)\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x$$

.

Are these answers the same?

Compute $\int \cos^2(x)\, dx$ by hand. Hint: $\cos^2(x) = \dfrac{1+\cos(2x)}{2}$

∫cos^2(x)dx = ∫(1+cos(2x))/2 dx = 1/2∫1+cos2x dx
= 1/2([∫1dx + ∫cos2x dx]) = 1/2[x + sin2x/2]
= x/2 + sin2x/4 + c

Example. Compute $\int \sin^2(x)\, dx$ by hand.

∫sin^2(x) = ∫((1−cos2x)/2) dx = 1/2∫1−cos2x dx
= 1/2([∫1dx − ∫cos2x dx]) = 1/2[x − sin2x/2]
= x/2 − sin2x/4 + c

Example. Compute $\int \sin^6(x)\, dx$ by hand.

Review. What tricks can be used to calculate $\int \cos^7(5x)\sin^4(5x)\,dx$?

```
theta = 5x
dtheta = 5dx
1/5 dtheta = dx

1/5∫cos^7(theta)sin^4(theta) dtheta
1/5∫cos^6(theta)sin^4(theta) cos(theta) dtheta
1/5∫(cos^2(theta))^3 sin^4(theta) cos(theta) dtheta
1/5∫(1−sin^2(theta)^3) sin^4(theta) cos(theta) dtheta
```

```
∫sin^2(x) cos^2(x) dx
∫sinx sinx cosx cosx dx
∫sinx cosx sinx cosx dx
∫(sinxcosx)^2 dx
∫(sin(2x)/2)^2 dx
∫1/4(sin^2 dx
```

Which of these integrals can be attacked in the same way, using the identity $\sin^2(x) + \cos^2(x) = 1$ and u-substitution?

A. $\displaystyle\int \sin^3(x)\cos^4(x)\,dx$

B. $\displaystyle\int \frac{\cos^3(\sqrt{x})}{\sqrt{x}}\,dx$

C. $\displaystyle\int \cos^2(x)\sin^4(x)\,dx$

D. $\displaystyle\int \sin^3(2x)\sqrt{\cos(2x)}\,dx$

E. $\displaystyle\int \tan^3(x)\,dx$

F. $\displaystyle\int \sin^2(x)\,dx$

Even powers of sine and cosine.

Review. What trig identities are most useful in evaluating $\int \cos^2(x)\sin^4(x)\,dx$?

```
1/2 angle identities
sin2x
sum angle formulas
```

Example. Compute $\int \cos^2(x)\sin^4(x)\,dx$ by hand.

```
∫cos^2xsin^4x dx
= ∫cos^2x sin^2x sin^2x dx
= ∫(sinxcosx)^2 sin^2x dx
= ∫(sin(2x)/2)^2 (1−cos(2x)/2) dx
= ∫sin^2(2x)/4 (1−cos(2x)/2) dx
= 1/8∫((1−cos4x)/2)(1−cos2x) dx
= 1/16∫(1−cos2x − cos4x) + (cosx)
= 1/16∫1−cos2x − cos4x + (cos6x + cos2x)/2 dx
= 1/16[x − sin(2x)/2 − sin(4x)/4 + sin(x)/12 + sin(2x)/4]
= x/16 − sin(2x)/32 − sin(4x)/64 + sin(x)/192 + sin(2x)/64] + c
```

Conclusions:

To find $\displaystyle\int \sin^m(x)\cos^n(x)\,dx$,

if m is odd and n is even:

```
sin is odd
let u = cosx
du = −sinx dx
sin^2(x) = 1−cos^2(x)
```

if n is odd and m is even:

```
cos is odd
let u = sinx
du = cosx dx
cos^2(x) = 1−sin^2(x)
```

if both m and n are odd:

```
Can do either 1 or 2
∫sin^3(x)cos^5(x) dx
∫sin^3(x)cos^4(x) cosx dx
∫sin^3(x)(1−sin^2(x))^2 cosx dx
```

if both m and n are even:

```
Create squares
Use half angle identities
sin(2x) identity
sum angle formulas
```

Note. Often the answers that you get when you integrate by hand do not look identical to the answers you will see if you use your calculator, Wolfram Alpha, or the integral table in the back of the book. Of course, the answers should be equivalent. Why do you think the answers look so different?

These integrals have their own special tricks.

Example. $\int \tan^2(x)\,dx$

Example. $\int \sec(x)\,dx$