Section 10.6 - Alternating Series

After completing this section, students should be able to:

- Define an alternating series.
- Identify the conditions needed to guarantee that an alternating series converges.
- Bound the remainder when using a specified partial sum to approximate an alternating series.
- Determine how many terms are needed to approximate an alternating series within a specified level of accuracy.
- Explain the relationship between convergent, absolutely convergent, and conditionally convergent.
- Prove that a series $\sum_{n=1}^{\infty} a_n$ converges by showing that $\sum_{n=1}^{\infty} |a_n|$ converges and using the fact that absolutely convergent implies convergent.

Definition. An alternating series is a series whose terms are alternately positive and negative. It is often written as

$$\sum_{k=1}^{\infty}(-1)^{k-1}b_k$$

where the b_k are positive numbers.

Example. (The Alternating Harmonic Series)

Σ(n=1, infinity) (-1)^n+1

Does the Alternating Harmonic Series converge? Hint: look at "even" partial sums and "odd" partial sums separately.

Σ(n=1, infinity) (-1)^n+1
1/n = 1-1/2+1/3-1/4+1/5-1/6+....+-1/n+-

n odd: sn = 1 + 1/3 + 1/5 + 1/2n-1 ....infinity
n even: sn = -1/2 - 1/4 + 1/6 - 1/2n - ....-infinity
converge ln2

Note:
If you rearrange the terms, Σ will still converge but it will converge to a different sum

Theorem. *(Alternating Series Test) If the series*

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 \ldots$$

satisfies:

(a) bn > 0

(b) bn+1 <= bn non-increasing

(c) lim n->infinity bn=0

then the series is convergent.

Σ(n=1, infinity) (-1)^n+1 1/n

bn > 0 1/n > 0 for n>=1
bn+1 < bn 1/n+1 <= 1/n for n>=1
lim n->infinity bn=0 lim n->infinity 1/n=0

Converges by the AST

Example. Which of these series are guaranteed to converge by the Alternating Series Test?

A. $\frac{5}{\sqrt{2}} - \frac{5}{\sqrt{3}} + \frac{5}{\sqrt{4}} - \frac{5}{\sqrt{5}} + \frac{5}{\sqrt{6}} - \frac{5}{\sqrt{7}} + \cdots$

Σ(n=1, infinity) (-1)^n+1 5/sqrt(n+1)
bn > 0 5/sqrt(n+1) > 0 for n>=1
bn+1 <= bn 5/sqrt(n+1+1) <= 5/sqrt(n+1) for n>=1
lim n->infinity bn = 0 lim 5/sqrt(n+1) = 0. Converges by AST

B. $\frac{2}{2} - \frac{1}{2} + \frac{2}{3} - \frac{1}{3} + \frac{2}{4} - \frac{1}{4} + \frac{2}{5} - \frac{1}{5} + \cdots$

bn>0
bn+1 <= bn
2/4<= 1/3
1/2 <= 1/3. Failed
Cannot use AST

C. $\frac{1}{8} - \frac{1}{4} + \frac{1}{27} - \frac{1}{9} + \frac{1}{64} - \frac{1}{16} + \frac{1}{125} - \frac{1}{25} + \cdots$

bn > 0
bn+1 <= bn
1/9 <= 1/27. Failed
Cannot use AST

D. $2.1 - 2.01 + 2.001 - 2.0001 + 2.00001 \cdots$

bn > 0
lim n->infinity bn = 0. Failed because limit approaches 2

Question. Why is the condition $\lim_{n \to \infty} b_n = 0$ necessary?

Question. Why is the condition $b_{n+1} \leq b_n$ for all large n necessary?

Example. Does the series converge or diverge?

$$\sum_{n=1}^{\infty}(-1)^n \frac{n^2}{n^3-2}$$

bn > 0   n^2 / n^3 - 2 > 0

lim n->infinity bn=0
lim n-> infinity n^2/n^3-2
lim n-> infinity 2n/3n^2
lim n-> infinity 2/3n
0

bn+1 <= bn
f'(x) < 0
2n*(n^3-2) - n^2(3n^2)/(n^3-2)^2 < 0
2n^4 - 4n - 3n^3 / (n^3-2)^2 < 0
-n^4 - 4n < 0
-n^4 - 4n = 0
-n(n^3+4) = 0
n = 0, cube root(-4)
n>0 decreasing

by AST it converges

Example. Does the series converge or diverge?

$$\sum_{k=1}^{\infty}(-1)^k(1+k)^{1/k}$$

bn > 0 (1+k)^1/k > 0

lim n->infinity bn=0
lim k->infinity (1+k)^1/k
lnL = lim k->infinity ln((1+k)^1/k)
= lim k->infinity 1/k (ln)(1+k)
= lim k->infinity ln(1+k)/k
= lim k->infinity 1/1+k
lnL = 0/1

lnL = 0
L = e^0
L=1

Cannot use AST since limit doesn't equal 0

Bounding the Remainder

For the same type of series:

- series is alternating
- $\lim_{n\to\infty} b_n = 0$
- $b_{n+1} \le b_n$

We want to put a bound on the remainder. Call the sum of the infinite series s_∞ and the nth partial sum s_n.

(a) Write an equation for the nth remainder R_n.

Rn = S-Sn

(b) Find an upper bound on $|R_n|$:

$$|R_n| \le \underline{\text{ bn+1 }}$$

|Rn| = |S-Sn| <=

Example. Consider the series $-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\cdots$

If we add up the first 6 terms of this series, what is true about the remainder? (PollEv)

A. positive and < 0.01

B. positive and < 0.02

C. positive and < 0.05

D. negative with absolute value < 0.01

E. negative with absolute value < 0.02

F. negative with absolute value < 0.05

G. none of these.

|R6| <= bn = 1/64
|R6| <= 1/64 about 0.015625
|R6| <= 0.02

S6 = Σ(n=1, 6) = -1/4 + 1/9 - 1/6 + 1/25 - 1/36 + 1/49
S6 = -0.16876

Rn = S-Sn
R6 = S-S6
R6 = --0.16876
+0.16876
R6 = Pos

Example. How many terms of the series

$$-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\cdots$$

do we need to add up to approximate the limit to within 0.01?

$|R_n| \leq b_{n+1} < 0.01$

$|R_n| \leq b_{n+1} = 1/((n+1)+1)^2$

$= 1 / (n+2)^2 < 0.01$

$1/(n+2)^2 < 0.01$

$1 < 0.01(n+2)^2$

$100/(n+2)^2$

$100 < n^2+4n+4$

$0 < n^2+4n-96$

$(n-8)(n+12)$

$n=-12,8$

$n>8$

9 terms

Definition. A series $\sum a_n$ is called **absolutely convergent** if

the series of the absolute value Σ|an| is covergent

Example. Which of these series are convergent? Which are absolutely convergent ?

(a) $\sum_{m=0}^{\infty}(-0.8)^m$ convergent abs. convergent Σ|(-0.8)^m| = Σ(0.8^m)
Geom |r| < 1

(b) $\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}$ convergent abs. convergent P-series isn't > 1
Σ|1/sqrt(2)|
= Σ1/sqrt(k)

(c) $\sum_{j=5}^{\infty}(-1)^j\frac{1}{j}$ convergent abs. convergent Σ|(-1)^j 1/j| = Σ 1/j
p-series
Diverges

bn > 0
1/j > 0

lim n->infinity bn = 0
lim j->infinity 1/j = 0

decreasing

Question. Is it possible to have a series that is convergent but not absolutely convergent?

Yes, alternating harmonic series

Definition. A series $\sum a_n$ is called **conditionally convergent** if

it converges but not absolutely converges

Question. Is it possible to have a series that is absolutely convergent but not convergent?

No...

Theorm: If a series Σan is absolutely convergent, then the series is convergent

Review. Which of the following statements are true about a series $\sum^{\infty} a_n$?

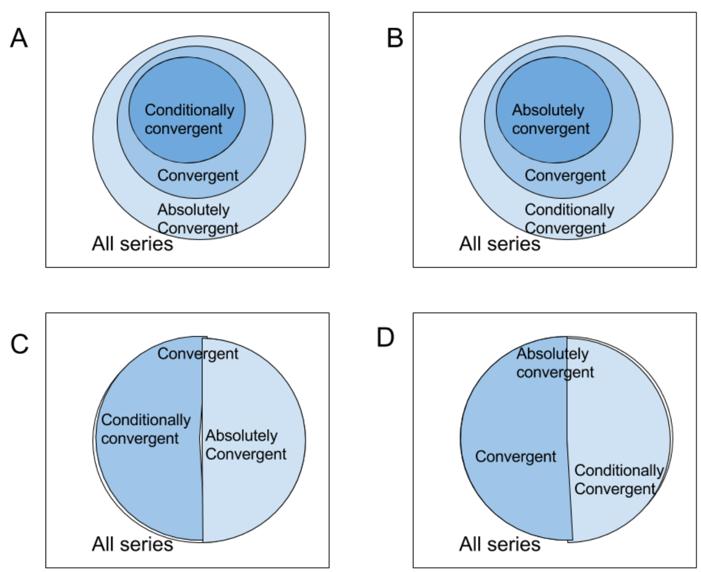
A. If the series is absolutely convergent, then it is convergent.

B. If the series is convergent, then it is absolutely convergent.

C. Both are true.

D. None of these statements are true.

Question. Which of the following Venn Diagrams represents the relationship between convergence, absolute convergence, and conditional convergence?



All series

Converge

Diverge

Conditionally

Absolutely

Example. Does this series converge or diverge? If it converges, does it converge absolutely or conditionally?

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n^2}$$

Σ|cos(npi/3) / n^2|
|cos(npi/3) / n^2| <= 1/n^2
P-series converges
By comparison test, Σ|cos(npi/3) / n^2| converges
Absolutely convergent
|cos(npi/3) / n^2| converges

Example. Does the series converge or diverge?

$$\sum_{n=2}^{\infty} \frac{\cos(n) + \sin(n)}{n^3}$$