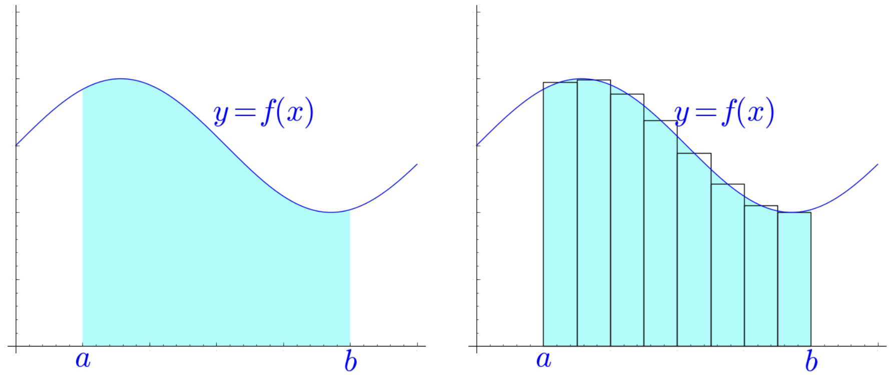
Math 232 Calculus 2 - Fall 2021

§6.2 - Area Between Curves

After completing this section, students should be able to

- Use an integral to compute the area between two curves.
- Decide if it is easier to integrate with respect to x or with respect to y when computing the area between two curves.
- Calculate the area between multiple curves by dividing it into several pieces.

Recall: to compute the area below a curve $y = f(x)$, between $x = a$ and $x = b$, we can divide up the region into rectangles.

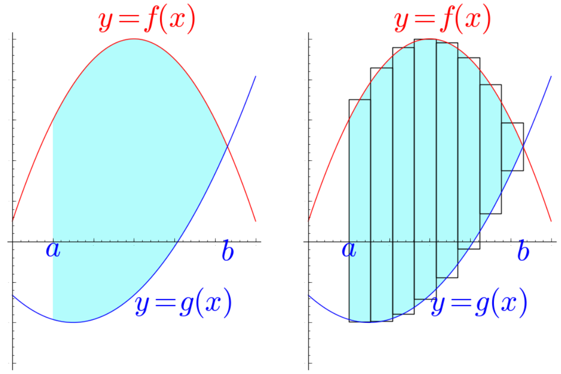


The area of one small rectangle is

The approximate area under the curve is

The exact area under the curve is

To compute the area between the curves $y = f(x)$ and $y = g(x)$, between $x = a$ and $x = b$, we can divide up the region into rectangles.



The area of one small rectangle is

    (f(x(subscript(i)^*))−(g(x(subscript(i)^*))Δx

The approximate area between the two curves is

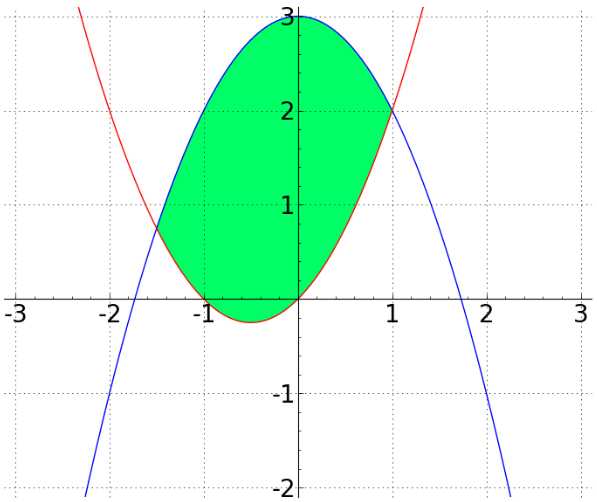
    lim as n −> infinity nΣi=1 f(x(subscript(i)^*))−(g(x(subscript(i)^*))Δx

The exact area between the two curves is

    ∫(b−a) f(x)−g(x)

This formula works as long as $f(x)$ ≥ $g(x)$.

Example. Find the area between the curves $y = x^2 + x$ and $y = 3 - x^2$

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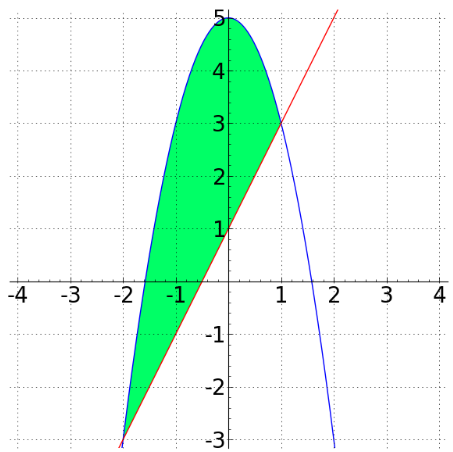
x^2 + x = 3 - x^2
2x^2 + x - 3 = 0
(2x+3)(x-1) = 0
x = 1, -1.5

∫(1,-1.5) (3-x^2)-(x^2+x)dx
∫(1,-1.5) 3-x^2-x^2-x dx
∫(1,-1.5) -2x^2-x+3 dx
(-2x^3)/3 - (x^2)/2 + 3x (1, -3/2)
```



Review. The area between the curves $y = 2x + 1$ and $y = 5 - 2x^2$ is given by:

A. $\displaystyle\int_{-2}^{1} 2x + 1 - 5 + 2x^2 \, dx$

B. $\displaystyle\int_{-2}^{1} 5 - 2x^2 - 2x + 1 \, dx$

C. $\displaystyle\int_{-2}^{1} 5 - 2x^2 - 2x - 1 \, dx$

D. $\displaystyle\int_{-3}^{5} 5 - 2x^2 + 2x + 1 \, dx$

E. None of these.

```
2x+1 = 5-2x^2
2x^2+2x-4 = 0
2(x^2+x-2) = 0
(x+2)(x-1) = 0
x = -2, 1

∫(1, -2) (5-2x^2) - (2x+1) dx
∫(1, -2) 5-2x^2-2x-1 dx

C is the answer
```

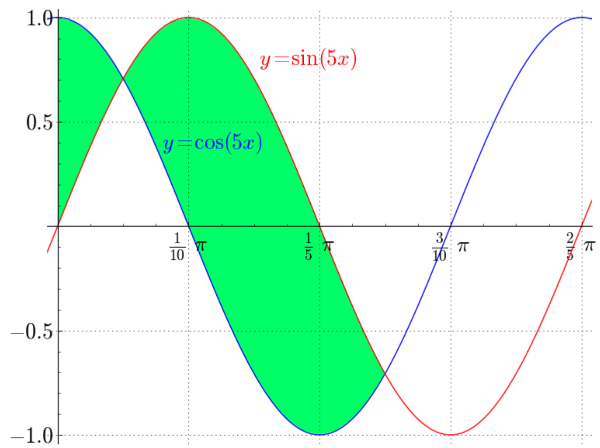
Example. The shaded area between the curves $y = \cos(5x)$, $y = \sin(5x)$, $x = 0$, and $x = \frac{\pi}{4}$ is given by:

A. $\displaystyle\int_0^{\pi/4} \sin(5x) - \cos(5x)\, dx$

B. $\displaystyle\int_0^{\pi/4} \cos(5x) - \sin(5x)\, dx$

C. Both of these answers are correct.

D. Neither of these answers are correct.



∫(pi/20, 0) cos(5x)−sin(5x) dx + ∫(pi/20, pi/4) sin(5x)−cos(5x) dx

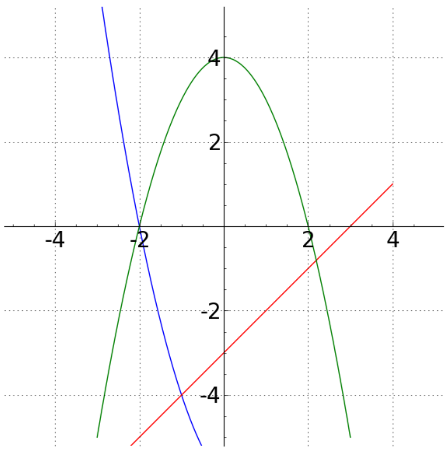
D is the answer

Extra Example. Set up the integral to find the shaded area bounded by the three curves in the figure shown.

- $f(x) = x^2 - x - 6$
- $g(x) = x - 3$
- $h(x) = -x^2 + 4$

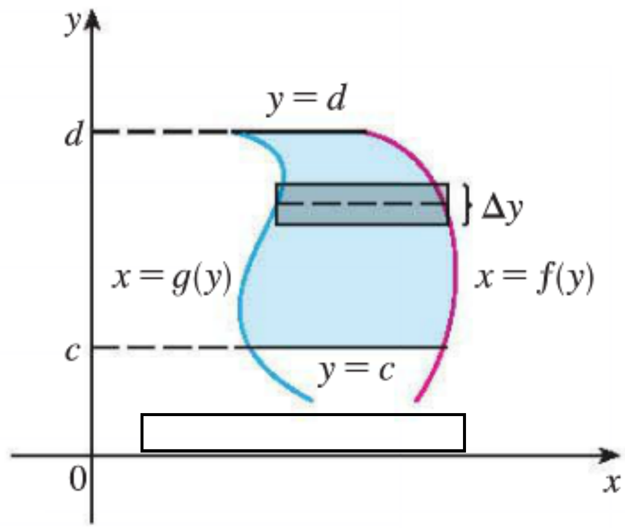
```
X-3, -x^2+4, x^2-x-6

∫(-1, -2) (-x^2+4)-(x^2-x-6)dx + ∫(2.19, -1) (-x^2+4)-(x-3)dx
```



Note ∫dy, as we go bottom to top, R->L

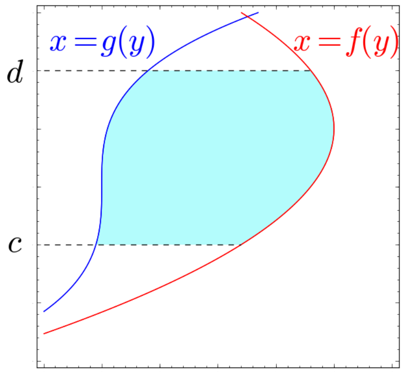
Note. The area between two curves $x = f(y)$ and $x = g(y)$ between $y = c$ and $y = d$ is given by:

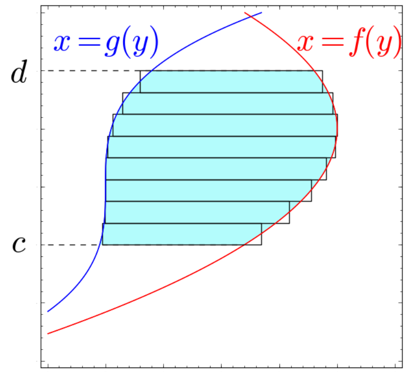


Area = ∫(d, c) f(y) - g(y) dy
R-L

This formula works as long as $f(y)$ ___≥___ $g(y)$.

To compute the area between the curves $x = f(y)$ and $x = g(y)$, between $y = c$ and $y = d$, we can again divide up the region into rectangles.





The area of one small rectangle is

```
(f(y(subscript(i)^*))−(g(y(subscript(i)^*))Δy
```

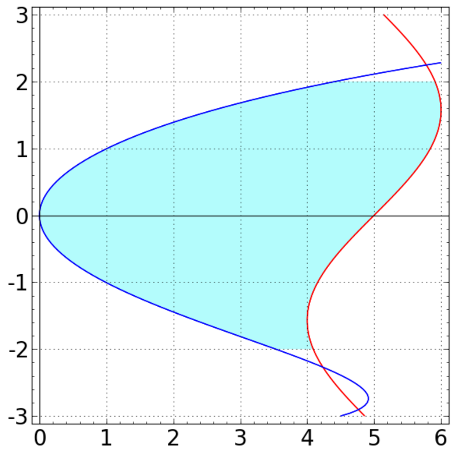
The approximate area between the two curves is

```
lim as n −> infinity nΣi=1 f(y(subscript(i)^*))−(g(y(subscript(i)^*))Δy
```

The exact area between the two curves

```
∫(b, a) f(y)−g(y)
```

Example. Find the area between the curves $f(y) = \sin(y) + 5$, $g(y) = \dfrac{y^2\sqrt{36 + y^3}}{6}$, $y = -2$, and $y = 2$.



Example. The area between the curves $y = x^2$ and $y = 3x^2$, and $y = 4$ is given by:

A. $\displaystyle\int_0^2 x^2 - 3x^2\,dx$

B. $\displaystyle\int_0^2 3x^2 - x^2\,dx$

C. $\displaystyle\int_0^2 \sqrt{y} - \sqrt{\frac{y}{3}}\,dy$

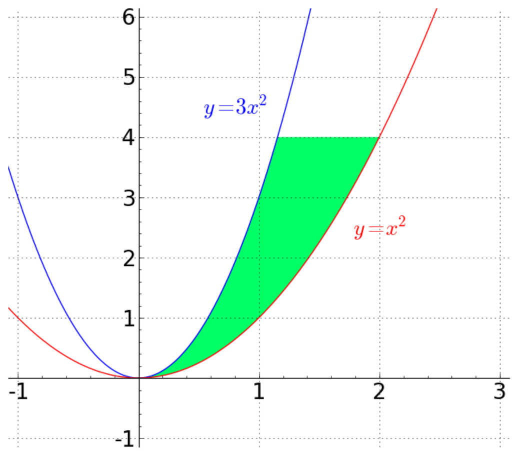
D. $\displaystyle\int_0^4 \sqrt{y} - \sqrt{\frac{y}{3}}\,dy$

E. $\displaystyle\int_0^4 \sqrt{\frac{y}{3}} - \sqrt{y}\,dy$

dy, R->L

∫(4, 0) sqrt(y)-sqrt(y/3) dy

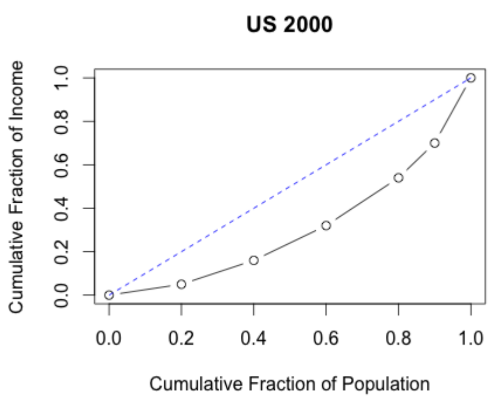
D is the answer



Extra Example. In the year 2000, the US income distribution was: (data from World Bank, see `http://wdi.worldbank.org/table/2.9`)

Income Category	Fraction of Population	Fraction of Total Income	Cumulative Fraction of Population	Cumulative Fraction of Income
Bottom 20%	0.20	0.05	0.20	0.05
2nd 20%	0.20	0.11	0.40	0.16
3th 20%	0.20	0.16	0.60	0.32
4th 20%	0.20	0.22	0.80	0.54
Next 10%	0.10	0.16	0.90	0.70
Highest 10%	10	0.30	1.00	1.00

The Lorenz curve plots the cumulative fraction of population on the x-axis and the cumulative fraction of income received on the y-axis.



The Gini index is the area between the Lorenz curve and the line $y = x$, multiplied by 2.

Estimate the Gini index for the US in the year 2000 using the midpoint rule.

Extra Example. Find the area between the curves $2x = y^2 - 4$ and $y = -3x + 2$ that lies above the line $y = -1$