S11.2 Properties of Power Series

After completing this section, students should be able to:

- Determine if an expression is a power series.
- Determine the center, radius, and interval of convergence of a power series.
- Create new power series out of old ones by multiplying by a power of x or composing with an expression like $3x^2$.
- Differentiate and integrate power series.

Informally, a power series is a series with a variable in it (often ”x”), that looks like a polynomial with infinitely many terms.

Example.

$$\sum_{n=0}^{\infty} \frac{(2n+1)x^n}{3^{n-1}} = 3 + 3x + \frac{5x^2}{3} + \frac{7x^3}{9} + \frac{9x^4}{27} + \frac{11x^5}{81} + \cdots$$

is a power series.

Example.

$$\sum_{n=0}^{\infty} \frac{(5^n)(x-6)^n}{n!} = 1 + 5(x-6) + \frac{5^2(x-6)^2}{2!} + \frac{5^3(x-6)^3}{3!} + \frac{5^4(x-6)^4}{4!} + \frac{5^5(x-6)^5}{5!} + \cdots$$

is a power series **centered at 6**.

Definition. A **power series centered at** a is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n =$$

where x is a variable, and the c_n's are constants called **coefficients**, and a is also a constant called **the center** .

Definition. A **power series centered at zero** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n =$$

Example. For what values of x does the power series $\sum_{n=0}^{\infty} n!\,(x-3)^n$ converge?

Ratio test ->   <1

lim n->infinity |an+1 / an| <1 converge
lim n->infinity |(n+1)! (x-3)^n+1 / n! (x-3)^n|
lim n->infinity |(x-3)(n+1)| =
lim n->infinity (n+1)|x-3| <1

converges only at x=3

Example. For what values of x does the power series $\sum_{n=0}^{\infty} \frac{(-2)^n(x+4)^n}{n!}$ converge?

lim n->infinity |(-2)^n+1 (x+4)^n+1 / (n+1)! * n! / (-2)^n (x+4)^n|
lim n->infinity |(-2) (x+4) / (n+1)|
lim n->infinity (2/n+1)|x+4| = 0
0<1

converges for all x's
Radius of convergence: R=infinity
Interval of convergence: (-infinity, infinity)

Example. For what values of x does the power series $\sum_{n=1}^{\infty} \frac{(-5x+2)^n}{n}$ converge?

center: (x-a) -< -5x+2 -> -5(x-2/5)
center: 2/5

lim n->infinity |(-5x+2)^n+1 / n+1 * n / (-5x+2)^n|
lim n->infinity |(-5x+2) n / n+1|
lim n->infinity n/n+1 |-5x+2|
= |-5x+2| < 1
-1 < -5x+2 < 1
-3 < -5x < -1
3/5 >= x > 1/5

Ratio test = 1 inconclusive, end points are the =1 so we have test

Test: 1/5
Σ(n=1, infinity) (-5(1/5)+2)^n / n
1^n / n
Diverges due to p-series

Test: 3/5
Σ(n=1, infinity) (-5(3/5)+2)^n / n
(-1)^n / n
By AST it converges at 3/5

Review. Which of the following are power series?

A. $\frac{1}{2} + \frac{(x+1)}{5} + \frac{(x+1)^2}{8} + \frac{(x+1)^3}{11} + \frac{(x+1)^4}{14} + \cdots$ Yes

B. $\frac{1}{x^2} + \frac{1}{x} + 1 + x + x^2 + x^3 + x^4 + \cdots$ No or yes, shift to make it work

C. $1 + 3 + 3^2 + 3^3 + 3^4 + \cdots$ No x's

D. None of these. No

Example. Find the *center* of any power series above.

Center for a: -1

Example. Find the center of the power series $\sum_{n=1}^{\infty} n^n(7+3x)^n$. For what values of x does it converge?

Hint: $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.

Center: 3x+7 -> 3(x+7/3)
Center: -7/3

Root test:
lim n->infinity n root|n^n(7+3x)^n|
lim n->infinity |n(7+3x)|
lim n->infinity n|7+3x| < 1. N is infinity

|7+3x| = 0
7+3x = 0
3x = -7
x=-7/3

converges only at -7/3

Example. Find the center of the power series $\sum_{n=0}^{\infty} \frac{(-5)^n(2x-3)^n}{\sqrt{3n+1}}$. For what values of x does it converge?

Center: 3/2

lim n->infinity |(-5)^n+1 (2x-3)^n+1 / sqrt(3(n+1)+1) * sqrt(3n+1) / (-5)^n (2x-3)^n
lim n->infinity |-5 (2x-3) sqrt(3x+1/ sqrt(3n+4)|
lim n->infinity 5 sqrt(3x+1/ sqrt(3n+4) |2x-3|
= 5|2x-3| < 1
5|2x-3| < 1
|2x-3| < 1/5
-1/5 < 2x-3 < 1/5
14/5 < 2x < 16/5
7/5 < x < 8/5

Test x = 7/5
Plugging in and simplifying is 1/sqrt(3n+1)
Compare with 1/sqrt(n)
lim n->infinity sqrt(n) / sqrt(3n+1)
lim n->infinity sqrt(n/3n+1)
= sqrt(1/3) > 0
Diverges

Test x =8/5
Alt series
lim n->infinity > 0
Decreasing
Positive

Example. For what values of x does the power series $\sum_{n=0}^{\infty} \frac{x^{2n}}{(3n)!}$ converge?

Center: 0

lim n->infinity |x^2n+1 / (3(n+1))! * (3n)! / x^2n|
lim n->infinity |x^2n+2 * (3n)! / (3n+3)! x^2n|
lim n->infinity |x^2 / (3n+3)(3n+2)(3n+1)|
lim n-> infinity 1/(3n+3)(3n+2)(3n+1) |x^2|

Theorem. *For a given power series* $\sum_{n=0}^{\infty} c_n(x-a)^n$, *there are only three possibilities for convergence:*

(a) Series converges for all x
(b) Series converges only at the center
(c) There is a positive R such that the series converges if |x-a|<R *test the endpoints*

Definition. The **radius of convergence** is

(a) R = infinity
(b) R = 0
(c) R = 1/c

|x-a| < 1
C|x-a|<1
|x-a| < 1/c

Definition. The **interval of convergence** is the interval of all x-values for which the power series converges.

(a) (-infinity, infinity)
(b) {a}
(c) (a-R, a+R) check end points for [vs (

We can think of power series as functions.

Example. Consider $f(x) = \sum_{n=0}^{\infty} x^n =$ 1+x+x^2+x^3+....

(a) What is $f(\frac{1}{3})$?

f(1/3) = Σ(n=0, infinity) (1/3)^n = a/1-r = 1/1-1/3 = 1/2/3 = 3/2

(b) What is the domain of $f(x)$?

|r|<1 Geometric to converge
|x|<1
(-1,1)

(c) What is a *closed form* expression for $f(x)$?

f(x) = Σ(n=0, infinity) x^n
a/1-r
f(x) = 1/1-x

(d) What is the domain for the closed form expression?

1-x != 0 (-infinity, 1)U(1,infinity)
x!=1

We can think of the partial sums of $\sum_{n=0}^{\infty} x^n$ as a way to approximate the function $\frac{1}{1-x}$ with polynomials:

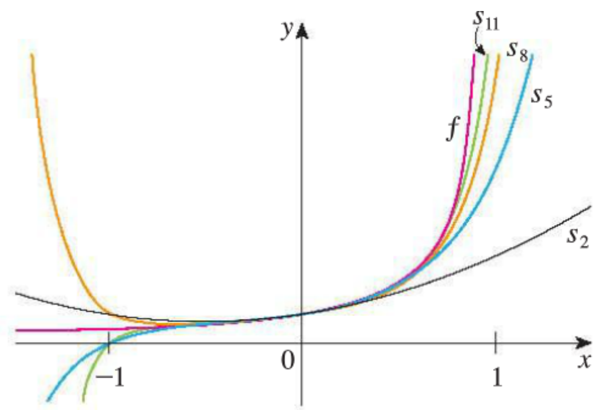
$s_0 =$ 1

$s_1 =$ 1+x

$s_2 =$ 1+x+x^2

$s_3 =$ 1+x+x^2+x^3

$s_n =$ 1+x+x^2+x^3+....+x^n



(-1,1) for geometric series
y=1/1-x

Example. Express $\dfrac{2}{x-3}$ as a power series and find the interval of convergence.

Goal: Rearrange 2/x-3 to 1/1-x = Σ(n=0,infinity) x^n

2/x-3 = 2(1/x-3)
2(1/-3+x)
2/3 (1/1-x/3)
-2/3 Σ(n=0, infinity) (x/3)^n
Σ(n=0, infinity) -2/3 * x^n / 3^n
Σ(n=0, infinity) -2x^n / 3^n+1

Recall |x| < 1 Interval of conv: (-3,3)
|x/3| < 1
|x| < 3
-3<x<3

Example. Find a power series representation of $\dfrac{x}{1+5x^2}$

Goal: Rearrange x/1+5x^2 to 1/1-x = Σ(n=0, infinity) x^n

x/1+5x^2 = x(1/1+5x^2) = x(1/1-(5x^2))
x / 1+5x^2 = x(1/1-(-5x^2))
x Σ(n=0, infinity) (-5x^2)^n
Σ(n=0, infinity) x(-5)^n x^2n
Σ(n=0, infinity) (-5)^n x^2n+1

|x|<1
|-5x^2|<1
5|x^2|

Review. $\frac{1}{1-x}$ can be represented by the power series:

Question. $\frac{1}{1-x}$ is equal to its power series:

A. when $x \neq 1$

B. when $x < 1$

C. when $-1 < x < 1$

D. for all real numbers

E. It is never exactly equal to its power series, only approximately equal.

Example. Express each of the following functions with a power series.

(a) $\dfrac{1}{1-x^4}$

1/1-x
Σ(n=0,infinity) x^n
Σ(n=0,infinity) (x^4)^n

(b) $\dfrac{1}{1+x^4}$

1/1-(-x^4)
Σ(n=0,infinity) (-x^4)^n
Σ(n=0,infinity) (-1)^n (x^4)^n

(c) $\dfrac{x^3}{1+x^4}$

x^3 (1/1+x^4)
x^3 (1/1-(-x^4))

Example. Find a power series representation of $f(x) = \frac{3}{2+5x}$. Find its radius of convergence.

Differentiation and Integration

Recall how to differentiate and integrate polynomials:

$$\frac{d}{dx}[5 + 3(x-2) + 4(x-2)^2 + 8(x-2)^3] =$$ 0+3+4*2(x-2)+8*3(x-2)^2

...

$$\int 5 + 3(x-2) + 4(x-2)^2 + 8(x-2)^3 \, dx =$$ 5(x-2)+3(x-2)^2/2 + 4(x-2)^3/3 + 8(x-2)^4/4 + c

Power series are also very easy to differentiate and integrate!

Theorem. *If the power series*

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

has a radius of convergence $R > 0$, then $f(x)$ is differentiable on the interval $(a-R, a+R)$ and

(i) $f'(x) =$ c1 + 2c2 (x-a) + 3c3 (x-a)^2 + ... = Σ(n=1,infinity) Cn n(x-a)^n-1

(ii) $\int f(x)\,dx =$ C0(x-a) + C1(x-a)^2 / 2 + C2(x-a)^3 / 3 + Cn n(x-a)^n+1 / n+1

The radius of convergence of the power series in (i) and (ii) are both R.

Question. Why does the summation sign for the derivative of a power series start at $n = 1$ instead of $n = 0$?

d/dx C0 = 0
start W/C1
n=1

Question. If a power series converges at the endpoints of its interval of convergence, do the derivative and integral power series also converge at the endpoints?

d/dx , S, original
Same radius of convergence and interval of convergence
endpoints we must check

Example. Find a power series representation for $\ln|x+2|$ and find its radius of convergence.

ln|x+2| = ∫(1/x+2) dx
∫1/2 * 1/1-(-x/2) dx

∫1/2 = Σ(n=0, infinity) (-x/2)^n dx
∫1/2 = Σ(n=0, infinity) (-1)^n x^n / 2^n dx
1/2 Σ(n=0, infinity) ((-1)^n * x^n+1 / 2^n*(n+1)) +c
Σ(n=0, infinity) ((-1)^n * x^n+1 / 2^n+1 * (n+1)) +c

Center: 0
find f(0) = ln|0+2| = ln(2)

Σ(n=0, infinity) ((-1)^n * x^n+1 / 2^n+1 * (n+1)) + ln(2)
|-x/2| < 1
1/2|x| < 1
-2<x<2

ROC: (2)
IOC: (-2,2]

Test: x=2
Σ(n=0, infinity) ((-1)^n * 2^n+1 / 2^n+1 * (n+1))
Σ(n=0, infinity) (-1)^n / (n+1)
Alt Series converges

Test: x=-2
Σ(n=0, infinity) ((-1)^n * -2^n+1 / 2^n+1 * (n+1))
Σ(n=0, infinity) (-1)^n * (-1)^n+1 * (2)^n+1 / 2^n+1 * (n+1)
Σ(n=0, infinity) (-1)^2n+1 / (n+1)
Σ(n=0, infinity) -1 / (n+1)
-Σ(n=0, infinity) 1 / (n+1) compare to 1/n
lim n->infinity 1/n+1/1/n = lim n-> infinity n/n+1 = 1>0
Diverges

Example. Find a power series representation for $\left(\frac{1}{4x-1}\right)^2$.

(1/4x-1)^2 = d/dx(1/4x-1) * -1/4
= -1/4 d/dx (1/-1+4x)
= -1/4 d/dx(1/-1 * 1/(1-4x))

SW (4x-1)^-1
d/dx (4x-1)^-1
-1*(4x-1)^-2 *4
-4(4x-1)^-2

-1/4 d/dx (-1 * Σ(n=0, infinity) (4x)^n)
= 1/4 d/dx (Σ(n=0, infinity) 4^n x^n)
= 1/4 Σ(n=1, infinity) 4^n* nx^n-1
= Σ(n=1, infinity) 1/4 * 4^n * nx^n-1
= Σ(n=1, infinity) 4^n-1 * nx^n-1
= Σ(n=1, infinity) n(4x)^n-1

Example. Find a power series representation for $\int \frac{x}{8+x^3}\,dx$ and use it to approximate $\int_0^1 \frac{x}{8+x^3}\,dx$, accurate to two decimal places.

1/1-x = Σ(n=0, infinity) x^n
∫ x/8+x^3 dx
= ∫x/8 (1/1+x^3/8) dx
= ∫x/8 (1/1-(-x^3/8)) dx

∫x/8 Σ(n=0, infinity) (-x^3/8)^n dx
∫x/8 Σ(n=0, infinity) ((-1)^n * x^3n / 8^n) dx
∫ Σ(n=0, infinity) ((-1)^n * x^3n+1 / 8^n+1) dx
Σ(n=0, infinity) ((-1)^n * x^3n+2 / 8^n+1 * (3n+2)) + c

Recall: |Rn| < bn+1 for all series
|Rn| < bn+1 < 0.01

1/8^n+1 * (3n+2) < 0.01
n=0, 1/8^0+1 * (3(0)+2) = 0.0625
n=1, 1/8^1+1 * (3(1)+2) = 0.003125, Valid

∫(0,1) x/8+x^3 dx
Σ(n=0, infinity) ((-1)^n * x^3n+2 / 8^n+1 * (3n+2)) [0,1]
Σ(n=0, infinity) ((-1)^n * 1^3n+2 / 8^n+1 * (3n+2)) - Σ(n=0, infinity) ((-1)^n * 0^3n+2 / 8^n+1 * (3n+2))
(-1)^0 (1)^2 / 8^1 (2) + (-1)^1 (1)^5 / 8^2 (5) = 0.0625 - 0.003125 =

Summary:

- We started by representing the function $\dfrac{1}{1-x}$ with the power series $\displaystyle\sum_{n=0}^{\infty} 1 + x + x^2 + x^3 + \cdots$
- We used the equation $\displaystyle\sum_{n=0}^{\infty} \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$ as a template to find the power series for many other related functions, by:
  - rearrange fractions
  - derivative
  - integral
- These same techniques can be used with other templates to build new power series out of old ones.