§8.4 - Trig Substitutions

After completing this section, students should be able to:

- Decide if an integral might be appropriate for computing using trig substitution.
- Determine what trig substitution should be used.
- Perform trig substitution to compute an integral, including converting back to original variables using a triangle and / or trig identities as needed.

The following three trig identities are useful for doing trig substitutions to solve some kinds of integrals with square roots in them.

$\sin^2(x) + \cos^2(x) = 1 \qquad \tan^2(x) + 1 = \sec^2(x) \qquad \cot^2(x) + 1 = \csc^2(x)$

new sub. rules

trig - right Δ trig

- trig inverse

Example. According to Wolfram Alpha,

$$\int \frac{x^2}{\sqrt{49-x^2}}\,dx = \frac{1}{2}\left(49\sin^{-1}\left(\frac{x}{7}\right) - x\sqrt{49-x^2}\right)$$

Let's see where that answer comes from using a trig substitution.

Goal: get rid of the sqrt $\to$ use trig subs

let $x = 7\sin\theta$ $\quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$dx = 7\cos\theta\,d\theta$

S.W. $\sqrt{49-x^2} = \sqrt{49-(7\sin\theta)^2} = \sqrt{49-49\sin^2\theta} = \sqrt{49(1-\sin^2\theta)} = \sqrt{49\cos^2\theta}$

$= 7\cos\theta$

Int: $\displaystyle\int \frac{(7\sin\theta)^2}{\cancel{7\cos\theta}}\,\cancel{7\cos\theta}\,d\theta = \int 49\sin^2\theta\,d\theta = 49\int \sin^2\theta\,d\theta =$

$\displaystyle 49\int \frac{1-\cos(2\theta)}{2}\,d\theta = \frac{49}{2}\int 1-\cos(2\theta)\,d\theta = \frac{49}{2}\left[\theta - \frac{1}{2}\sin(2\theta)\right] + C$

$\displaystyle = \frac{49}{2}\left[\theta - \frac{1}{2}\cdot 2\sin\theta\cos\theta\right] + C$

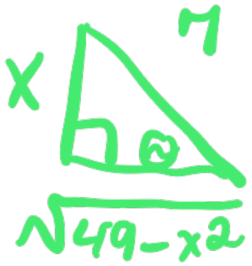
$\displaystyle = \frac{49}{2}\left[\sin^{-1}\left(\frac{x}{7}\right) - \left(\frac{x}{7}\right)\left(\frac{\sqrt{49-x^2}}{7}\right)\right] + C$

$\displaystyle = \frac{49}{2}\sin^{-1}\left(\frac{x}{7}\right) - \frac{1}{2}(x)\left(\sqrt{49-x^2}\right) + C$

Trig: $x = 7\sin\theta$

$\frac{x}{7} = \sin\theta$

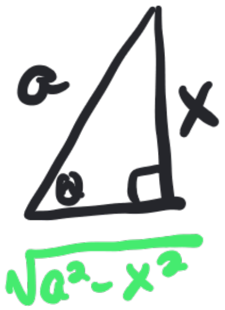
$\theta = \sin^{-1}\left(\frac{x}{7}\right)$



Which trig substitutions for which problems?

Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin\theta, \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec\theta, \quad 0 \leqslant \theta < \frac{\pi}{2}$ or $\pi \leqslant \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

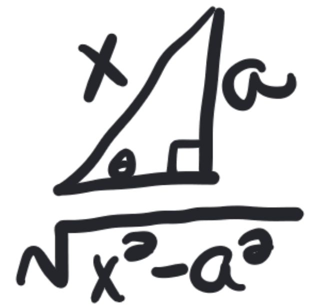


$\sin\theta = \frac{x}{a}$

$a\sin\theta = x$

$\tan\theta = \frac{x}{a}$

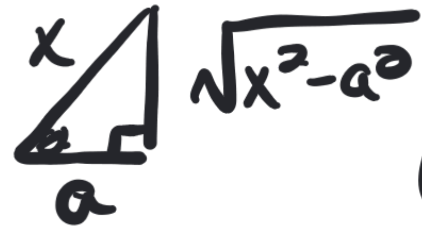
$a\tan\theta = x$



$\sin\theta = \frac{a}{x}$

$x\sin\theta = a$

$x = \frac{a}{\sin\theta} = a\csc\theta$



$\cos\theta = \frac{a}{x}$

$x = \frac{a}{\cos\theta} =$

$x = a\sec\theta$

Review. To compute $\int \frac{x^2}{\sqrt{49-x^2}}\,dx$, which substitution is most useful?

A. $u = 49 - x^2$

B. $x = \sin(\theta)$

C. $x = 7\sin(\theta)$

D. $x = \tan(\theta)$

E. $x = 49\tan(\theta)$

F. $x = 7\sec(\theta)$

Example. Find $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}}\,dx$. (Assume a is positive.)

Goal: Trig sub $\to$ get rid of $\sqrt{\ }$

$x = a\tan\theta \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$dx = a\sec^2\theta\,d\theta$

S.W. $\sqrt{x^2+a^2} = \sqrt{(a\tan\theta)^2 + a^2} = \sqrt{a^2\tan^2\theta + a^2} = \sqrt{a^2(1+\tan^2\theta)} = \sqrt{a^2\sec^2\theta}$

$= a\sec\theta$

int: $\displaystyle\int \frac{1}{\cancel{a}\cancel{\sec\theta}}\,\cancel{a}\sec^{\cancel{2}}\theta\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$

$= \ln\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right| + C$

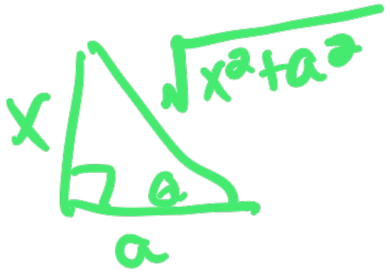
$= \ln\left|\frac{\sqrt{x^2+a^2}+x}{a}\right| + C$

$= \ln\left|\sqrt{x^2+a^2}+x\right| - \ln|a| + C$

$= \ln\left|\sqrt{x^2+a^2}+x\right| + K$ #

trig $\quad x = a\tan\theta$

$\frac{x}{a} = \tan\theta$



Example. Compute the integral $\displaystyle\int_{1/3}^{2/3} \frac{\sqrt{9x^2-1}}{x}\,dx$

$$\sqrt{9(x^2-\tfrac{1}{9})} = 3\sqrt{x^2-\tfrac{1}{9}}$$
$$3\sqrt{x^2-(\tfrac{1}{3})^2}$$

Goal: $x=\frac{1}{3}\sec\theta$ $\qquad 0\le\theta<\frac{\pi}{2}$

$dx=\frac{1}{3}\sec\theta\tan\theta\,d\theta$

S.W: $3\sqrt{x^2-\frac{1}{9}} = 3\sqrt{(\frac{1}{3}\sec\theta)^2-\frac{1}{9}} = 3\sqrt{\frac{1}{9}\sec^2\theta-\frac{1}{9}} = 3\sqrt{\frac{1}{9}(\sec^2\theta-1)} =$

$= 3\sqrt{\frac{1}{9}\tan^2\theta} = 3\cdot\frac{1}{3}\tan\theta = \tan\theta$

int: $\displaystyle\int_{\frac{1}{3}}^{\frac{2}{3}} \frac{\tan\theta}{\cancel{\frac{1}{3}}\cancel{\sec\theta}}\cdot\cancel{\tfrac{1}{3}}\cancel{\sec\theta}\tan\theta\,d\theta = \int_{\frac{1}{3}}^{\frac{2}{3}}\tan^2\theta\,d\theta = \int_{\frac{1}{3}}^{\frac{2}{3}}\sec^2\theta-1\,d\theta$

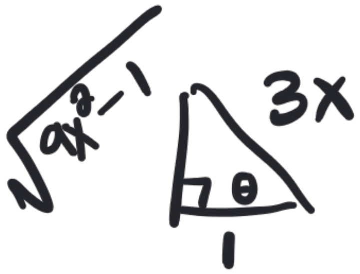
$\displaystyle = \tan^2\theta-\theta\Big]_{\frac{1}{3}}^{\frac{2}{3}} = \left(\frac{\sqrt{9x^2-1}}{1}\right)^2-\sec^{-1}(3x)\Big]_{\frac{1}{3}}^{\frac{2}{3}}$

$$= \boxed{9x^2-1 \;-\sec^{-1}(3x)\Big]_{\frac{1}{3}}^{\frac{2}{3}}}$$

trig: $x=\frac{1}{3}\sec\theta$

$3x=\sec\theta$

$\theta=\sec^{-1}(3x)$



What trig substitutions would be most useful for these integrals?

1. $\displaystyle\int \frac{2}{\sqrt{4+x^2}}\,dx$ $\quad x = 2\tan\theta$

2. $\displaystyle\int (100x^2-1)^{3/2}\,dx$ $\quad x = \frac{1}{10}\sec\theta$

3. $\displaystyle\int x\sqrt{4-\frac{x^2}{9}}\,dx$ $\quad$ u-sub

4. $\displaystyle\int (25-x^2)^2\,dx$ $\quad$ – polynomial

5. $\displaystyle\int \sqrt{-x^2-6x+7}\,dx$ $\quad \rightarrow$ complete sq

$\sqrt{7-6x-x^2}$

$\sqrt{7-(x^2+6x)}$

$\sqrt{7-(x^2+6x+9)+9}$

$\sqrt{16-(x+3)^2}$

let $x+3 = 4\sin\theta$

$x = 4\sin\theta - 3$

$dx = 4\cos\theta\, d\theta$