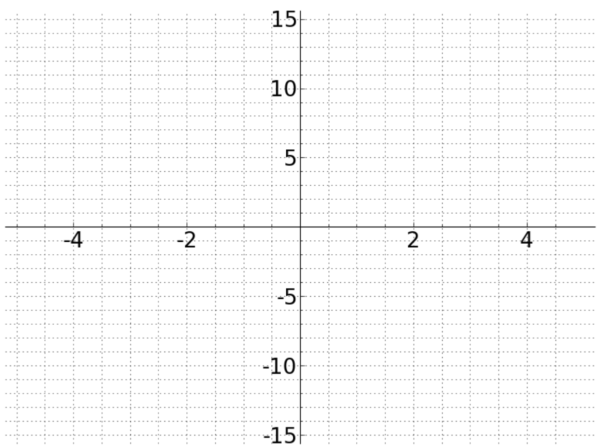
§12.1 - Parametric Equations

Definition. A **cartesian equation** for a curve is an equation in terms of x and y only.

Definition. **Parametric equations** for a curve give both x and y as functions of a third variable (usually t). The third variable is called the **parameter**.

Example. Graph $x = 1 - 2t$, $y = t^2 + 4$



t	x	y
-2	5	8
-1	3	5
0	1	4
1	-1	5
2	-3	8

Find a Cartesian equation for this curve.

x=1-2t
x-1=-2t
t=x-1/-2

y=t^2 + 4
y=(x-1/-2)^2 +4
y=(x-1)^2/4 + 4

Example. Plot each curve and find a Cartesian equation:

(a) $x = \cos(t)$, $y = \sin(t)$, for $0 \leq t \leq 2\pi$

(b) $x = \cos(-2t)$, $y = \sin(-2t)$, for $0 \leq t \leq 2\pi$

T	x=cos	y=sin
0	1	0
pi/2	0	1
pi	-1	0
3pi/2	0	-1
2pi	1	0

Unit Circle

Sin^2t + cos^2t = 1
y^2+x^2=1
x^2+y^2=1

T	x=cos(-2t)	y=sin(-2t)
0	1	0
pi/4	0	-1
pi/2	-1	0
pi	1	0
2pi	1	0

Unit Circle but twice as quick

sin^2(t)+cos^2(t)=1
sin^2(-2t)+cos^2(-2t)=1
y^2+x^2=1
x^2+y^2=1

Example. Write the following in parametric equations:

(a) $y = \sqrt{x^2 - x}$ for $x \leq 0$ and $x \geq 1$

Try simple sub: let t=x
polynomial/root

x=t
y=sqrt(t^2-t)

(b) $25x^2 + 36y^2 = 900$

Circle/ellipse -> set eq=1
25x^2 / 900 + 36y^2/900 = 1
x^2/36 + y^2/25 = 1
(x/6)^2 + (y/5)^2 = 1
x=6cost y=5sint

Example. Describe a circle with radius r and center (h, k):

a) with a Cartesian equation

b) with parametric equations

(x-h)^2 + (y-k)^2 = r^

(x-h)^2/r^2 + (y-k)^2/r^2 = 1
(x-h/r)^2 + (y-k/r)^2 = 1
cost sint

cost = x-h/r sint=y-k/r

x = rcost + h
y = rsint+k

Create a Cartesian and parametric with (-3,2) with a radius of 5

Cartesian eq
(x+3)^2 + (y-2)^2 = 25

Parametric
x = 5cost - 3
y = 5sint + 2

Example. Find a Cartesian equation for the curve.

Methods:
Poly like -> sub

(a) $x = 5\sqrt{t}, y = 3 + t^2$

x/5 = sqrt(t)
x^2/25 = t

y = 3+(x^2/25)^2
y=3+x^4/25^2

(b) $x = e^t, y = e^{-t}$

(c) $x = 5\cos(t) + 3, y = 2\sin(t) - 7$

sin/cos -> ellipse/circle Pythagorean theorem sin^2(x) + cos^2(x) = 1

x-3/5 = cost
y+7/2 = sint
(x-3/5)^2 + (y+7/2)^2 = 1
(x-3)^2/25 + (y+7)^2/4 = 1

Example. Find parametric equations for the curve.

(a) $x = -y^2 - 6y - 9$

let y = t

x = -t^2 -6t -9

Methods:

poly like -> direct sub

let t = x

(b) $4x^2 + 25y^2 = 100$

x^2/25 + y^2/4 = 1

(x/5)^2 + (y/2)^2 = 1

cost = x/5, sint = x/2

x = 5cost

y = 2sint

ellipse/circle -> sin/cos

sin^2(x)+cos^2(x) = 1

(c) $4(x-2)^2 + 25(y+1)^2 = 100$

ellipse/circle -> sin/cos

sin^2(x)+cos^2(x) = 1

(x-2)^2/25 + (y+1)^2/4 = 1

(x-2/5)^2 + (y+1/2)^2 = 1

cost = x-2/5, sint = y+1/2

x = 5cost +2

y = 2sint - 1

Example. What is the equation for a circle of radius 8 centered at the point (5, -2)

(a) in Cartesian coordinates ?

(x-5)^2 + (y+2)^2 = 64

(b) in parametric equations?

x = 8cost + 5
y = 8sint - 2

Example. Find parametric equations for a line through the points $(2,5)$ and $(6,8)$.

(a) any way you want. Find equation, pt slope, y=mx+b, or y-y1=m(x-x1)

m = 8-5/6-2
=3/4
5 = 3/4(2)+b
5 = 3/2+b
b = 7/2
y = 3x/4 + 7/2

let x =t
y=3t/4 + 7/2

(b) so that the line is at $(2,5)$ when $t = 0$ and at $(6,8)$ when $t = 1$.

	x = x0 + at	y=y0 + bt
(2,5)	2=x0 + a(0)	5=y0+b(0)
t=0	2=x0	5=y0
	x=2+at	y=2+bt
(6,8)	6=2+a1	y=5+bt
t=1	4=a	b=3
	x=2+4t	y=5+3t

Example. A sailboat's position at time t is given by the equations $x = 3 - t$, $y = 2 - 4t$. A rowboat's position is give by the equations $x = 5 - 3t$, $y = -2 + t$.

(a) Do the boats collide? same place, same time

(b) Do the boats' paths cross? same place, but not the same time

Sail Boat
x = 3-t
t = -x+3
y = 2-4t
y = 2-4(-x+3)
y = 2+4x-12
y=4x-10

Row Boat
x=5-3t
x-5=-3t
t=-x/3 + 5/3
y = -2+t
y=-2+(-x/3+5/3)
y = -2 - x/3 + 5/3
y = -x/3 - 1/3

Cross Path?
4x-10 = -x/3 - 1/3
4x+1/3x = -1/3 + 10
13x/3 = 29/3
x=29/13

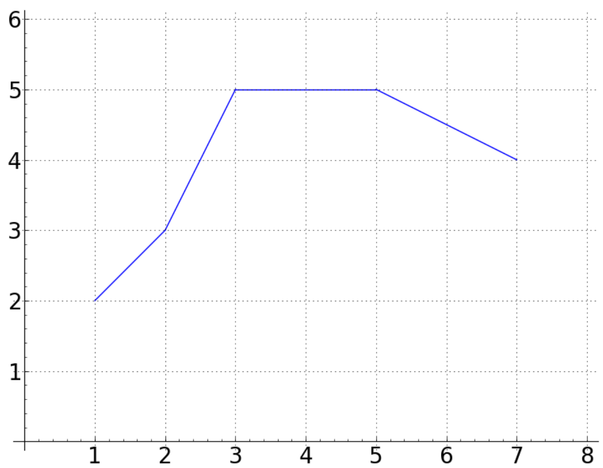
They do intersect

Collide?
t = -x+3
t = -29/13 + 3
t = 10/13

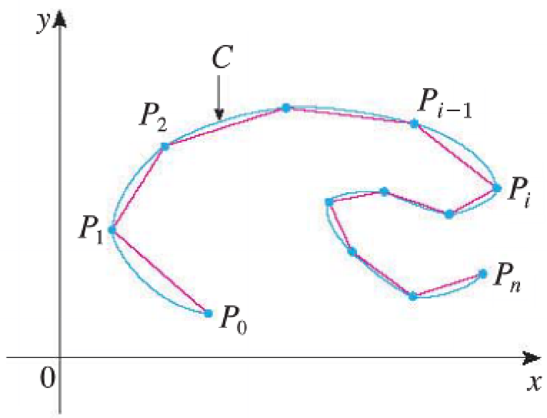
t = -x/3 + 5/3
t = -1/3(29/13) + 5/3
t = -29/39 + 5/3
= 36/39
= 12/13
No Collision

ARC LENGTH

Example. Find the length of this curve.



Note. In general, it is possible to approximate the length of a curve $x = f(t)$, $y = g(t)$ between $t = a$ and $t = b$ by dividing it up into n small pieces and approximating each curved piece with a line segment.



Arc length formula
L = ∫(b,a) sqrt(f'(x) - g'(x))

Arc length is given by the formula:

Set up an integral to express the arclength of the Lissajous figure

$$x = \cos(t), y = \sin(2t)$$

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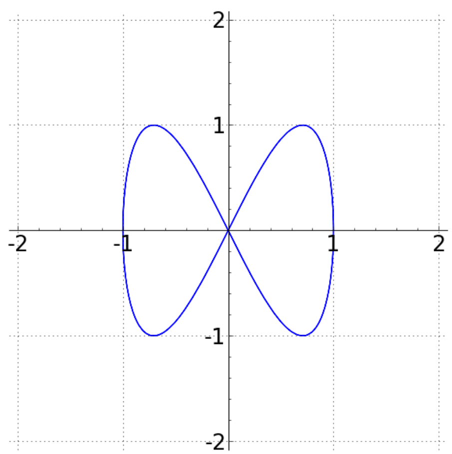
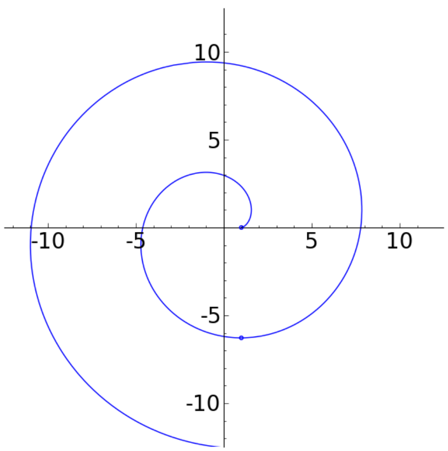


x1 = -sint
y1 = 2cos(2t)

L = ∫(b,a) sqrt(sin^2(t) + 4cos^2(2t))

Review. The length of a parametric curve $x = f(t)$, $y = g(t)$ from $t = a$ to $t = b$ is given by:

Example. Find the exact length of the curve $x = \cos(t) + t\sin(t)$, $y = \sin(t) - t\cos(t)$, from the point $(1, 0)$ to the point $(-1, \pi)$.



s.w
x1 = -sin(t) + sin(t) + cos(t) * t = tcos(t)
y1 = cos(t) -cos (t) + tsin(t) = tsin(t)

L = ∫(b,a) sqrt(t^2 cos^2(t) + t^2 sin^2(t)) dt
∫(b,a) t dt
t^2 / 2 [b,a]

x = cost + tsint
1 = cost + tsint
0 = sint - tcost

-1 = cost + tsint
pi = sint - tcost

a = 0
b = tbd

TANGENT LINES

The slope of the tangent line for a curve $y = p(x)$ (given in Cartesian coordinates) is:

If the curve is given by parametric equations $x = f(t)$, $y = g(t)$, then the slope of its tangent line is:

y' = p'(x)
dy/dx
dy/dt = dy/dx * dx/dt

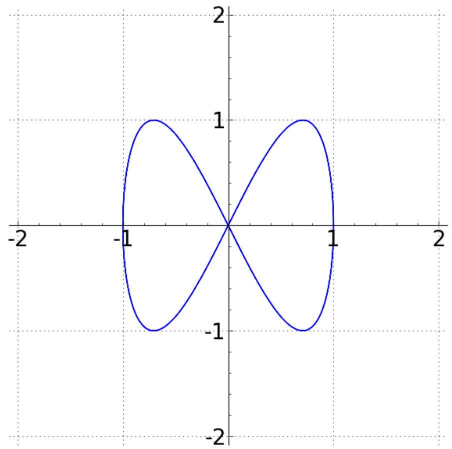
Example. For the Lissajous figure:

$$x = \cos(t), y = \sin(2t)$$

$$0 \leq t \leq 2\pi$$

(a) Find the slopes of the tangent lines at the center point $(0, 0)$.

(b) Find where the tangent line is horizontal.



dy/dx = dy/dt / dx/dt
= 2cos2t / -sint
formula for slope of tangent lines in terms of t

x=0
0=cost
t = pi/2, 3pi/2

y=0
0=sin2t
2t = 0,pi, 2pi, 3pi, 4pi, 5pi
t = 0, pi/2, pi, 3pi/2, 2pi

dy/dx, t= pi/2 = 2cost2pi/2 / -sin(pi/2) = -2/-1 = 1
dy/dx, t= 3pi/2 = 2cost6pi/2 / -sin(3pi/2) = -2/1 = -2

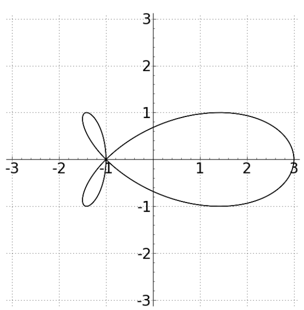
Review. The slope of the tangent line for a parametric curve $x = f(t)$, $y = g(t)$ is given by:

dy/dx = dy/dt / dx/dt

Example. The graph of the curve $x(t) = 2\cos(t) + \cos(2t)$, $y(t) = \sin(2t)$ for $0 \leq t \leq 2\pi$ is drawn below.

(a) Find the equations of the tangent lines at the point $(-1, 0)$ on the curve.

(b) Find the coordinates of all the points on the curve where the tangent line is vertical.



Only Part A

dy/dx = dy/dt / dx/dt = 2cost2t/-2sint - 2sin(2t)

point(-1,0)
x=-1
-1 = 2cost + cos(2t)
-1 = 2cost + 2cos^2t - 1
0 = cos^2t + 2cost
0 = 2cos(cost+1)

y=0
0=sin2t
2t = 0,pi,2pi,3pi,4pi
t = 0,pi/2,pi,3pi/2,2pi

0=2cost
t = pi/2,3pi/2
0 = cost + 1
-1 = cost
t = pi

t = pi/2 = 1
t = pi = DNE
t = 3pi/2 = -1

point (-1,0), m=1

y=mx+b
0=1(-1)+b
0=-1+b
b=1
y=x+1

point (-1,0), t=pi
slope DNE
x=-1

point(-1,0), t=3pi/2
y=-x+1