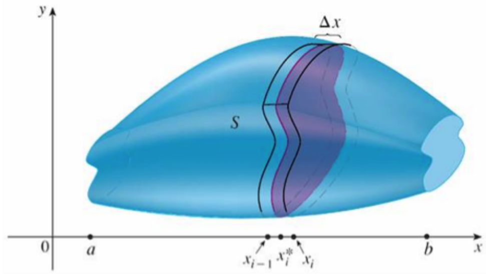
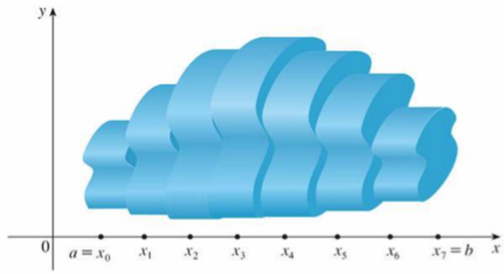
§6.3 - Volumes

After completing this section, students should be able to

- Calculate a volume by integrating the cross-sectional area.
- Calculate the volume of a solid of revolution using the disk / washer method.
- Identify the parts of the formula for the volume of a solid of revolution that correspond to cross-sectional area and thickness.
- Use calculus to derive fomulas for familar shapes such as pyramids and cones.

If you can break up a solid into n slabs, $S_1, S_2, \ldots S_n$, each with thickness Δx, then





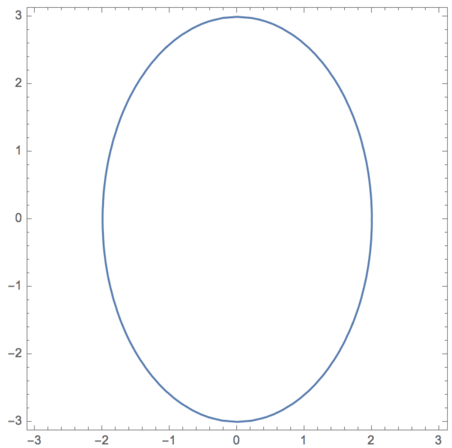
Volume of solid ≈

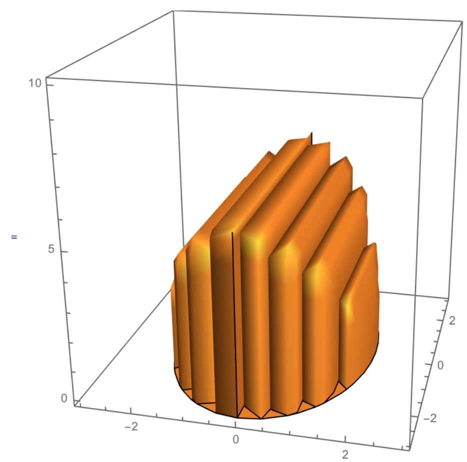
limit as n->infinity

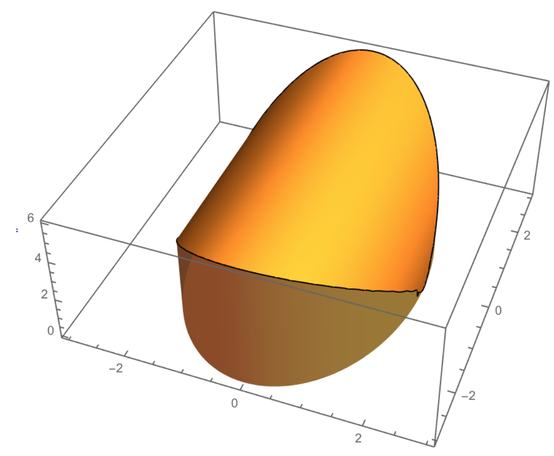
The thinner the slices, the better the approximation, so

Volume of solid =

Example. Find the volume of the solid whose base is the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ and whose cross sections perpendicular to the x-axis are squares.





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dx or dy

x-axis(vertical)

Area/Figure
For example for a square the area is A=S^2
S = 2y
A = S^2 -> (2y)^2 -> 4y^2


(x^2)/4 + (y^2)/9 = 1
(y^2)/9 = 1 - (x^2)/4
y^2 = 9 - (9x^2)/4
y = +- sqrt(9 - (9x^2)/4)

A = 4(sqrt(9 - (9x^2)/4))^2 -> 4(9 - (9x^2)/4) -> 36 - 9x^2

Volume = ∫(b, a) A(x)dx
V = ∫(2,-2) 36-9x^2 dx
  = 36x-3x^3 (2, -2)
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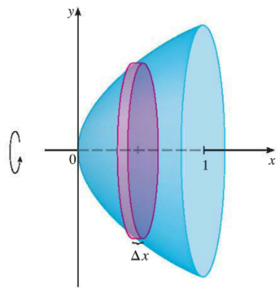
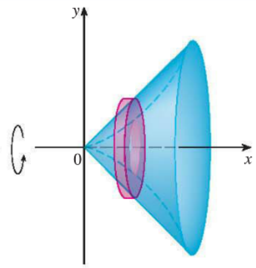
Volumes found by rotating a region around a line are called **solids of revolution**.





For solids of revolution, the cross sections have the shape of a disk______ or the shape of a washer______.

The area of the cross-sections can be described with the formulas

disk: A = pi(r^2)

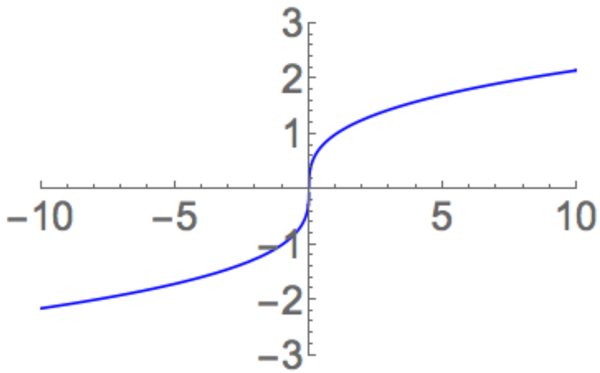
washer: A = pi(outer r^2) - pi(inner r^2) = pi(outer r^2 - inner r^2)

The volume of a solid of revolution can be described with the formulas:

disk: V = ∫(b, a) pi(r^2)dx
washer: V = ∫(b, a) pi(outer r^2 - inner r^2)dx

When the region is rotated around the x-axis, or any other horizontal line, then we integrate with respect to dx, vertical cut______.
When the region is rotated around the y-axis, or any other vertical line, then we integrate with respect to dy, horizontal cut______.

Example. Consider the region bounded by the curve $y = \sqrt[3]{x}$, the x-axis, and the line $x = 8$. What is the volume of the solid of revolution formed by rotating this region around the x-axis?



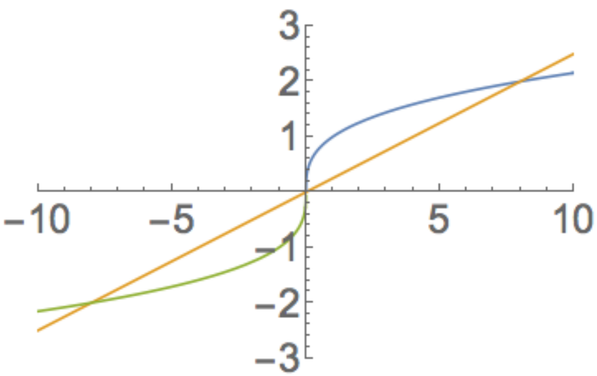

```
Rotation is a dx
x-axis -> vertical cut

Area/figure: Disk

A = pi(r^2)
A = pi(cube root(x))^2
A = pi(x^2/3)

Volume = ∫(b, a) A(x)dx
V = ∫(8, 0) pi(x^2/3)dx
  = (pi(x^5/3)) / 5/3
```

Example. Consider the region in the first quadrant bounded by the curves $y = \sqrt[3]{x}$ and $y = \frac{1}{4}x$. What is the volume of the solid of revolution formed by rotating this region around the x-axis? The y-axis?




```
Rotate dx
x-axis -> vertical cut
outer r = cube root(x)
inner r = 0.25x

Area/Figure: Washer
A = pi(outer r^2) - pi(inner r^2)
A = pi((cube root(x))^2) - pi((0.25x)^2)
A = pi(x^2/3) - pi((x^2)/16)

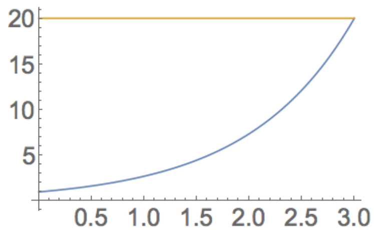
Volume = ∫(b, a) A(x)dx
V = ∫(8, 0) pi(x^2/3) - pi((x^2)/16)dx
V = pi((x^5/3)/5/3) - (pi(x^3)/48) (8,0)


Rotate dy
y-axis -> horizontal cut
outer r = 4y
inner r = y^3

Area/Figure: Washer
A = pi(outer r^2) - pi(inner r^2)
A = pi((4y)^2) - pi((y^3)^2)
A = pi(16y^2) - pi(y^6)

Volume ∫(b, a) A(x)dx
V = ∫(2, 0) pi(16^2) - pi(y^6)dx
V = pi((16y^3)/3)) - (pi(y^7)/7) (8,0)
```

Example. The region between the curves $y = e^x$, $x = 0$, and $y = e^3$ is rotated around the x-axis, to make a solid of revolution. When computing the volume, what are the cross-sections and which variable do we integrate with respect to?



A. cross-sections are disks, integrate with respect to dx

B. cross-sections are disks, integrate with respect to dy

C. cross-sections are washers, integrate with respect to dx

D. cross-sections are washers, integrate with respect to dy

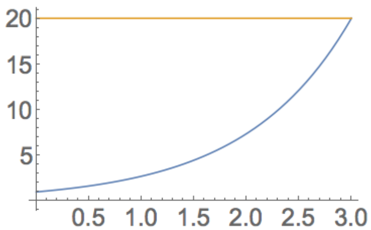
Set up an integral to calculate the volume.

```
Vertical cross section -> Dx

Area/Figure: Washer


Volume:
V = ∫(3, 0)pi(e^6)-pi(e^2x)dx
```

Example. The region between the curve $y = e^x$, $x = 0$, and $y = e^3$ is rotated around the y-axis, to make a solid of revolution. When computing the volume, what are the cross-sections and which variable do we integrate with respect to?



A. cross-sections are disks, integrate with respect to dx

B. cross-sections are disks, integrate with respect to dy

C. cross-sections are washers, integrate with respect to dx

D. cross-sections are washers, integrate with respect to dy

Set up an integral to calculate the volume.

```
Rotate -> dy
r = lny

Area/Figure: Disk
A = pi(r^2)
A = pi(lny)^2

Volume:
∫(e^3, 1) pi(lny)^2dy

Answer is D
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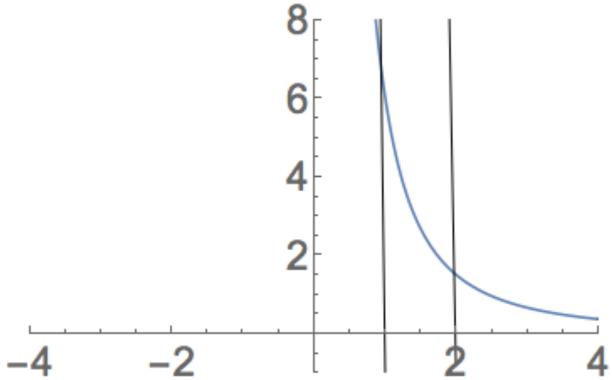
Extra Example. Consider the region bounded by $y = \frac{6}{x^2}$, $x = 1$, $x = 2$, and the x-axis.



Set up an integral to compute the volume of the solid obtained by rotating this region about the line $x = 3$.

```
Horizontal cross section -> dy
y = 6/(x^2)
x^2(y) = 6
x = sqrt(6/y)

Area/Figure: Washer
Bottom:
A = pi(outer r^2) - pi(inner r^2)
outer r = 3-1 = 2
inner r = 3-2 = 1
A = pi(2^2)-pi(1^2)
A = 3pi

Top:
A = pi(outer r^2) - pi(inner r^2)
outer r = 3-1 = 2
inner r = 3-sqrt(6/y)
A = pi(2^2)-pi(3-sqrt(6/y))^2
A = 4pi-pi(3-sqrt(6/y))^2

Volume
∫(1.5, 0)3pi dy + ∫(6, 1.5) 4pi-pi(3-sqrt(6/y))^2 dy
```

Example. Find the volume of the solid whose base is the region between $y = \sqrt{x}$, the x-axis, and the lines $x = 1$ and $x = 5$, and whose cross sections perpendicular to the x-axis are equilateral triangles.

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Extra Example. Find the volume of a pyramid with a square base of side length a and height h.

Extra Example. Find the volume of a cone with a circular base of radius a and height h.