## §10.1 - Sequences and Series Intro

After completing this section, students should be able to:

- Explain the difference between a sequence and a series.
- Use a recursive formula to write out the terms of a sequence.
- Use a closed form formula to write out the terms of a sequence.
- Translate a list of terms of a sequence into a recursive formula or a closed form formula.
- Explain what it means for a sequence to converge or diverge.
- Write out partial sums for a series.
- Explain what it means for a series to converge or diverge.
- Use numerical evidence to make a guess about whether a sequence converges.
- Use numerical evidence from partial sums to make a guess about whether a series converges.

**Definition.** A sequence is an ordered list of numbers.

**Example.** $3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, 9, \ldots$

A sequence is often denoted $\{a_1, a_2, a_3, \ldots\}$, or $\{a_n\}_{n=1}^{\infty}$, or $\{a_n\}$.

**Example.** For each sequence, write out the first three terms:

1. $\left\{\dfrac{3n+1}{(n+2)!}\right\}_{n=1}^{\infty}$

```
= {4/3!}, 7/4!, 10/5!
= 4/6, 7/24, 10/120
= 2/3, 7/24, 1/12
```

2. $\left\{(-1)^k \dfrac{k+3}{3^k}\right\}_{k=2}^{\infty}$

```
= {(−1)^2 5/3^2, −1^3 6/3^3, −1^4 7/3^4, 1^5 8/3^5
= 5/9, −6/27, 7/81, .........
```

**Definition.** Sometimes, a sequence is defined with a **recursive formula** (a formula that describes how to get the $n$th term from previous terms), such as

$$a_1 = 2, \quad a_n = 4 - \frac{1}{a_{n-1}}$$

**Example.** Write out the first three terms of this recursive sequence.

```
a1 = 2
a2 = 4−1/a1 = 4−1/2 = 7/2
a3 = 4−1/a2 =
```

**Note.** Sometimes it is possible to describe a sequence with either a recurvsive formula or a ”closed-form”, non-recursive formula.

**Example.** Write a formula for the general term $a_n$, starting with $n = 1$.

A. $\{7, 10, 13, 16, 19, \cdots\}$

repeated addition -> multiply -> slope
constant add -> y=mx+b

7= 3(1) + b
b=4

an = {3x+4} n=1

n starts at zero
y = mx+b
b=7

7=3(0)+b
an = 3n+7 n = 0

**Definition.** An **arithmetic sequence** is a sequence for which consecutive terms have the same common difference. repeated addition

If $a$ is the first term and $d$ is the common difference, then the arithmetic sequence has the form: repeated addition

an = a + dn

(starting with $n = 0$)

An arithmetic sequence can also be written:

an = a + d(n−1)

(with the index starting at $n = 1$.)

**Example.** For each sequence, write a formula for the general term $a_n$ (start with $n = 1$ or with $n = 0$).

B. $\{3, 0.3, 0.03, 0.003, 0.0003, \cdots\}$

```
repeated multiplication (repeated division) -> exponential y = ab^x
```

```
n = 0                                n=1
an = 3(.1)^n n=0                     an = 3(.1)^n-1 n=1

an = (15/2)(5/2)^n n=0               an = (15/2)(5/2)^n-1 n=1

an = -1^n * 3(2/3)^n n=0             an = -1^n-1 * 3(2/3)^n-1 n=1
```

C. $\left\{\frac{15}{2}, \frac{75}{4}, \frac{375}{8}, \frac{1875}{16}, \ldots\right\}$

D. $\{3, -2, \frac{4}{3}, -\frac{8}{9}, \ldots\}$

**Definition.** A **geometric sequence** is a sequence for which consecutive terms have the same common ratio.

If $a$ is the first term and $r$ is the common ratio, then a geometric sequence has the form:

```
a
```

(with the index starting at 0)

A geometric sequence can also be written:

(with the index starting at 1)

**Example.** For each sequence, write a formula for the general term $a_n$, starting with $n = 1$.

E. $\{-\frac{2}{9}, \frac{4}{16}, -\frac{8}{25}, \frac{16}{36}, \ldots\}$

**Review.** A sequence is ... list of ordered #’w

**Example.** Consider the sequence $\{3, 7, 11, 15, 19, \cdots\}$

(a) What are the next three terms in this sequence? 23, 27, 31

(b) What is a recursive formula for this sequence?

(c) What is a explicit (closed form) formula for this sequence?

n = 0 an = 3 + 4n
n=1 an = 3+4(n−1)

**Example.** Consider the sequence $\left\{-\frac{1}{2}, \frac{3}{10}, -\frac{9}{50}, \frac{27}{250}, \cdots\right\}$

(a) What are the next three two terms in this sequence?

```
-81/1250,
243/6250
```

(b) What is a recursive formula for this sequence?

```
an = (-3/5) an-1
```

(c) What is a explicit (closed form) formula for this sequence?

```
an = -1^n-1 (1/2)(3/5) n=0

an = -1^n (1/2)(3/5)^n-1 n=1
```

**Example.** Consider the sequence $\left\{(-1)^n \frac{2 \cdot 5}{n!}\right\}_{n=0}^{\infty}$

What are the first three terms in this sequence?

**Definition.** A sequence $\{a_n\}$ **converges** if: lim n->infinity an = L

Otherwise, the sequence diverges. In other words, a sequence diverges if:

lim n->infinity an = +- infinity or DNE

**Example.** Which of the following sequences converge?

A. $\{3, 7, 11, 15, 19, \cdots\}$ diverge lim n->infinity an = infinity

B. $\left\{-\frac{1}{2}, \frac{3}{10}, -\frac{9}{50}, \frac{27}{250}, \cdots\right\}$ converge lim n->infinity an = 0

C. $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots\right\}$ converge lim n->infinity an = 1

**Definition.** For any sequence $\{a_n\}_{n=1}^{\infty}$, the sum of its terms $a_1 + a_2 + a_3 + \cdots$ is a series. Often this series is written as

$$\sum_{n=1}^{\infty} a_n$$

**Example.** Consider the sequence $\left\{\frac{1}{2^n}\right\}_{n=1}^{\infty}$. If we add together all the terms, we get the series:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} =$$

What does it mean to add up infinitely many numbers?

sum up to a number. Therefore it converges

**Definition.** The **partial sums** of a series $\sum_{n=1}^{\infty} a_n$ are defined as the sequence $\{s_n\}_{n=1}^{\infty}$, where

$s_1 =$ a1

$s_2 =$ a1 + a2

$s_3 =$ a1 + a2 + a3

$s_n =$ a1 + a2 + a3 + a4 ...... + an

**Definition.** The series $\sum_{n=1}^{\infty} a_n$ is said to **converge** if :

sequence converge sequence(an)to zero

lim n-> sn = s

Otherwise, the series **diverges**.

**Note.** Associated with any series $\sum_{n=1}^{\infty} a_n$, there are actually two sequences of interest:

(a) {an}

(b) {sn}

**Example.** For the series $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$, write out the first 4 terms and the first 4 partial sums. Does the series appear to converge?

```
lim n-> infinity an = lim n-> infinity 1/n^2 + n = 0

s1 = 1/2
s2 = 1/2 + 1/6 = 2/3
s3 = 1/2 + 1/6 + 1/12 = 3/4
s4 = 1/2 + 1/6 + 1/12 + 1/20 = 4/5
sn = n/n+1

lim n->infinity n/n+1 = 1/1 = 1
lim n->infinity sn=1

Σ(n=1, infinity) 1/n^2+n converges
```

**Review.** What is the difference between the following two things?

- the sequence $\left\{\frac{1}{4^k}\right\}_{k=1}^{\infty}$ `List of terms`
- the series $\sum_{k=1}^{\infty} \frac{1}{4^k}$ `The summation of the list`

**Question.** What does it mean for the sequence $\left\{\frac{1}{4^k}\right\}_{k=1}^{\infty}$ to converge vs. diverge?

```
lim k->infinity an=L

lim k->infinity 1/4^k = 0 converges
```

**Question.** What does it mean for the series $\sum_{k=1}^{\infty} \frac{1}{4^k}$ to converge vs. diverge?

```
lim n->infinity an=0

lim n->infinity sn=s

converges
```

**Question.** Does the series $\sum_{k=1}^{\infty} \frac{1}{4^k}$ converge or diverge?

```
s1 = 1/4
s2 = 1/4 + 1/16 = 5/16
s3 = 1/4 + 1/16 + 1/64 = 21/64
s4 = 1/4 + 1/16 + 1/64 + 1/256 = 85/256

sn = 1/4^n

converges to 1/3
```

**Example.** Using your calculator, Excel, or any other methods, compute several partial sums for each of the following series and make conjectures about which series converge and which diverge.

A. $4 + 0.2 + 0.02 + 0.002 + \cdots$

4()

B. $\displaystyle\sum_{j=1}^{\infty} (-1)^j$

Diverges cuz the limit is DNE

C. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{k+1}$

1/2 + 2/3 + 3/4 + 4/5 + 5/6 + 6/7 + 7/8

According to l'hopital the limit is 1 and not 0
Diverges cuz it goes to infinity