§10.8 - Strategy for Convergence Tests for Series

After completing this section, students should be able to:

- Identify appropriate tests to use to prove that a given series converges or diverges.
- Compare and contrast the conditions needed to apply particular convergence tests.

List as many convergence tests as you can. What conditions have to be satisfied?

(a) Divergence Test — lim $a_n \neq 0$, diverges

(b) Integral Test — pos, decreasing, cont.

(c) Geometric Series Test — $|r| < 1$ converges

(d) Ordinary Comparison Test — pos, larger converge, smaller diverge

(e) Limit Comparison Test — pos, $L > 0$ something

(f) Telescoping Test — partial sums

(g) p-series test — $p > 1$, converge

(h) alternating series test — b_n pos, decreasing, lim $= 0$

(i) Ratio Test — $L < 1$ converge

(j) Root Test — $L < 1$ converge

* (k) Absolute Convergence Test/Thm

* (l) Series laws — individual pieces have to converge

Question. The limit comparison test and the ratio test both involve ratios. How are they different?

Limit Comparison test:

① $\lim_{n\to\infty} \frac{a_n}{b_n} > 0$

do the same thing

② comparing 2 different series in the ratio

Ratio Test:

① $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

< 1 converges

> 1 diverges

$= 1$ inconclusive

② consecutive terms of the series in the ratio

Example. Which convergence test would you use for each of these examples? Carry out the convergence test if you have time.

D (a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^3}$

C (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n+3}$

D (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2+6n}}$

C (d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!} - \frac{1}{2^n}$

C (e) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{e^{n^2}}$

D (f) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n \ln n}$

(a) divergence test

$$\lim_{n\to\infty} \frac{2^n}{n^3} = \lim_{n\to\infty} \frac{2^n \ln 2}{3n^2} = \lim_{n\to\infty} \frac{2^n(\ln 2)^2}{6n}$$

$$= \lim_{n\to\infty} \frac{2^n(\ln 2)^3}{6} = \infty$$

By divergence test, $\sum \frac{2^n}{n^3}$ diverges

Comparison Test

$$\underbrace{\frac{1}{n^3}}_{\text{converges p-serie}} < \frac{2^n}{n^3}$$

$$\lim_{n\to\infty} \frac{\frac{2^n}{n^3}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{2^n}{n^3} \cdot n^3 = \lim_{n\to\infty} 2^n = \infty$$

inconclusive

$\frac{2^n}{n^3}$ Ratio: $\lim_{n\to\infty}\left|\frac{2^{n+1}}{(n+1)^3}\cdot\frac{n^3}{2^n}\right| = \lim_{n\to\infty}\left|\frac{2n^3}{(n+1)^3}\right|$

$= \lim_{n\to\infty}\frac{2n^3}{(n+1)^3} = \lim_{n\to\infty}\frac{6n^2}{3(n+1)^2} = \lim_{n\to\infty}\frac{12n}{6(n+1)} =$

$\lim_{n\to\infty}\frac{12}{6} = 2 > 1$

By Ratio Test, $\sum\frac{2^n}{n^3}$ diverges

(b) $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n+3}$

Alternating Series Test:

① $\frac{\ln n}{n+3}$ pos for $n \geq 1$

② $\lim_{n\to\infty} \frac{\ln n}{n+3} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$ ✓

③ decreasing : $\frac{\ln n \uparrow}{n+3 \uparrow}$ ← ↑ faster ⇒ smaller ⇒ decreasing ✓

$f'(x) < 0$

$\frac{\frac{1}{n}(n+3) - 1(\ln n)}{(n+3)^2} < 0$

$\frac{1}{n}(n+3) - \ln n < 0$

$1 + \frac{3}{n} - \ln n < 0$ how do we solve ???

just graphically confirm

By alternating series test, $\sum (-1)^n \frac{\ln n}{n+3}$ converges

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2+6n}}$

Divergence Test:

$\displaystyle\lim_{n\to\infty} \frac{1}{\sqrt[3]{n^2+6n}} = 0$ ..... has a chance

Ordinary Comparison Test

$$\frac{1}{\sqrt[3]{n^2}} = \underbrace{\frac{1}{n^{\frac{2}{3}}}}_{\text{p-series diverge}} > \frac{1}{\sqrt[3]{n^2+6n}}$$

both are positive $n \geq 1$

OCT does not work

Limit Comparison Test

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt[3]{n^2+6n}}}{\frac{1}{\sqrt[3]{n^2}}} = \lim_{n\to\infty} \frac{\sqrt[3]{n^2}}{\sqrt[3]{n^2+6n}} = \lim_{n\to\infty} \sqrt[3]{\frac{n^2}{n^2+6n}} = \lim_{n\to\infty} \sqrt[3]{\frac{\cancel{n^2}(1)}{\cancel{n^2}(1+\frac{6}{n})}}$$

$$= \lim_{n\to\infty} \sqrt[3]{\frac{1}{1+\underbrace{\frac{6}{n}}_{0}}} = \sqrt[3]{1} = 1 > 0$$ same thing

By Limit Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2+6n}}$ diverges

② $\sum_{n=1}^{\infty} \frac{1}{n!} - \frac{1}{2^n}$

$\sum_{n=1}^{\infty} \frac{1}{n!} - \sum \frac{1}{2^n}$ *only can do if both parts converge*

$\sum \frac{1}{2^n} = \sum \left(\frac{1}{2}\right)^n$ Geometric Series

$|r| < 1$ converges

$\sum \frac{1}{n!} \rightarrow$ ratio test $\lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} \right| = \lim_{n\to\infty} \left| \frac{1}{(n+1)!} \cdot \frac{n!}{1} \right|$

$= \lim_{n\to\infty} \left| \frac{1}{n+1} \right| = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$

converges

By series/limit laws, since both pieces converge, so does $\sum \frac{1}{n!} - \frac{1}{2^n}$ converges

(e) $\sum_{n=1}^{\infty} \frac{n^2}{e^{n^2}}$

Ratio Test

$$\lim_{n\to\infty}\left|\frac{(n+1)^2}{e^{(n+1)^2}}\cdot\frac{e^{n^2}}{n^2}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2}{e^{n^2+2n+1}}\cdot\frac{e^{n^2}}{n^2}\right| =$$

$$\lim_{n\to\infty}\left|\frac{(n+1)^2}{n^2}\cdot\frac{\cancel{e^{n^2}}}{\cancel{e^{n^2}}e^{2n}e^{1}}\right| = \lim_{n\to\infty}\frac{(n+1)^2}{n^2e^{2n+1}} = \lim_{n\to\infty}\frac{(n+1)^2}{n^2}\cdot\underbrace{\lim_{n\to\infty}\frac{1}{e^{2n+1}}}_{0}$$

$$= 0 < 1$$

By Ratio Test, $\sum \frac{n^2}{e^{n^2}}$ converges

(f) $\displaystyle\sum_{n=2}^{\infty} \frac{3}{n\ln n}$

Integral Test:

① $\frac{3}{n\ln n}$ pos for $n \geq 2$ ✓

② continuous for $n \geq 2$ ✓

③ decreasing for $n \geq 2$ ✓

$f'(x) < 0$

$$\frac{0 \cdot n\ln n - 3(\ln n + \frac{1}{n} \cdot n)}{(n\ln n)^2} < 0$$

$$-3\ln n - 3 < 0$$

$$-3\ln n < 3$$

$$\ln n > -1$$

$e^\wedge \qquad e^\wedge$

$$n > \frac{1}{e}$$

$$\int_2^\infty \frac{3}{x\ln x}\,dx$$

$$= \lim_{t\to\infty} \int_2^t \frac{3}{x\ln x}\,dx$$

let $u = \ln x$

$du = \frac{1}{x}\,dx$

$$= \lim_{t\to\infty} \int_2^t \frac{3}{u}\,du$$

$$= \lim_{t\to\infty} 3\ln|u| \Big]_2^t$$

$$= \lim_{t\to\infty} 3\ln|\ln x| \Big]_2^t$$

$$= \lim_{t\to\infty} \underbrace{3\ln|\ln t|}_{\infty} - \underbrace{3\ln|\ln 2|}_{\#}$$

$\boxed{\int_2^\infty \frac{3}{x\ln x}\,dx \text{ diverges, so by integral test } \sum \frac{3}{n\ln n} \text{ diverges}}$