§11.4 Working with Taylor Series

After completing this section, students should be able to

- Use Taylor series to find limits.
- Use Taylor series to compute approximate values of integrals.
- Use Taylor series to find the sum of a series.
- Use Taylor series to solve differential equations.
- List uses of Taylor series.

Question. What are Taylor series good for?

- Differetiate
- Integrate
- Represent Real #
- Represent functions as power series
- Evaulate limits
- Prove L'Hospital Rule

Find sum of infinite series (Newton)

Example. Use a Taylor series to evaluate $\lim_{x \to 0} \dfrac{e^{-x^2} - 1 + x^2}{x^4}$

e^x = Σ(n=0,infinity) x^n/n!

e^-x^2 = Σ(n=0,infinity) (-x^2)^n/n!

= Σ(n=0,infinity) (-1)^n (x^2n)/n!

= 1/1 - x^2/1! + x^4/2! - x^6/3! + ...

lim n->infinity e^-x^2 - 1+x^2 / x^4

lim n->infinity 1/1 - x^2/1! + x^4/2! - x^6/3! + ... + -1 + x^2/ x^4

lim n->infinity x^4/2! - x^6/3! + ... / x^4

lim n->infinity 1/2! - x^2/3! + ...   (-x^2/3! and onward goes to 0)

1/2

Example. Use Taylor series to prove L'Hospital's Rule.

L'Hospital: lim x->a f(x)/g(x) = f'(a)/g'(a)

g'(a) != 0
0/0 or infinity/infinity
f(a) = 0 + g(a) = 0

Proof: Suppose f(a)=g(a)=0 + g'(a) != 0
lim x->a f(x)/g(x)
lim x->a Σ(n=0,infinity) f^n(a)(x-a)^n/n! / Σ(n=0,infinity) g^n(a)(x-a)^n/n!
lim x->a Σ(n=0,infinity) f^n(a) / Σ(n=0,infinity) g^n(a)
Σ(n=0,infinity) f^n(a) / Σ(n=0,infinity) g^n(a)

First derivative -> taylor series n=1
lim x->a f(x)/g(x)
Σ(n=0,1) f^n(a) / Σ(n=0,1) g^n(a)
f(a) + f'(a) / g(a) + g'(a)
f'(a) / g'(a)

Example. (a) Find a power series representation for $e^{-\frac{x^2}{2}}$.

e^x = Σ(n=0, infinity) x^n/n!
e^(-x^2)/2 = Σ(n=0, infinity) ((-x^2)/2)^n/n!
Σ(n=0, infinity) (-1)^n (x^2n)/2^n n!

(b) Find a power series representation for $\int e^{-\frac{x^2}{2}}\,dx$.

∫e^(- x^2)/2 dx
∫Σ(n=0, infinity) (-1)^n (x^2n)/2^n n!
Σ(n=0, infinity) (-1)^n (x^2n+1)/2^n n! (2n+1)+ c

(c) Use the first three terms of your power series to estimate $\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-1}^{1} e^{-\frac{x^2}{2}}\,dx$.

1/sqrt(2pi)∫(-1,1) e^(- x^2)/2 dx
= 1/sqrt(2pi) Σ(n=0, 2) (-1)^n (x^2n+1)/2^n n! (2n+1) [-1,1]
= 1/sqrt(2pi) (x/1 - x^3/2*1*3 + x^5/2^2 * 2! *5) [-1,1]
= 1/sqrt(2pi) [(1-1/6+1/40) - (-1+1/6-1/40)]
= 1/sqrt(2pi) (2-2/6 + 2/40) ~~ 0.6849

(d) What does this number represent?

Area under the curve of e^(-x^2)/2 from -1 to 1

Example. Use the MacLaurin series for arctan(x) to show that

$$1 - 1/3 + 1/5 - 1/7 + \cdots = \frac{\pi}{4}$$

From Table: tan^-1(x) = Σ(n=0, infinity) (-1)^n (x^2n+1)/(2n+1)
= x-x^3/3+x^5/5-x^7/7

1-1/3+1/5-1/7 +........ = tan^-1(1) = pi/4

Example. Use a MacLaurin series from this table to find the sum of the Alternating Harmonic Series.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$

Note: AHS
Σ(n=0, infinity) (-1)^n+1/n
= 1-1/2+1/3-1/4

From table
ln(1+x) = (-1)^n-1 x^n/n
x=1

1-1/2+1/3-1/4+1/5
ln(2)

Example. Find a power series for the solution of the differential equation. Can you guess what function this power series represents?

$y'(t) = 6y + 9 \qquad y(0) = 2$

Center: a=0
Tay

Example. Find the Maclaurin series for $g(x) = e^{ix}$, where $i = \sqrt{-1}$.

Summary: What are Taylor Series good for?