§10.4 - The Divergence Test and the Integral Test

After completing this section, students should be able to:

- State the Divergence Test and use it to prove that a series diverges.
- Explain why the Divergence Test cannot by used to prove that a series converges.
- Determine whether it is appropriate to use the integral test.
- Use the integral test, when appropriate, to prove that a series converges.
- Use the p-test to prove that a series converges.
- Identify the Harmonic Series.
- Use an integral, when appropriate, to find a bound on the remainder of a series with positive terms after evaluating a partial sum, and to find bounds on the value of the sum based on partial sums and integrals.
- Use an integral, when appropriate, to determine how many terms are needed to approximate the sum of a series to within a specified level of accuracy.

Example. Does this series converge or or diverge?

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

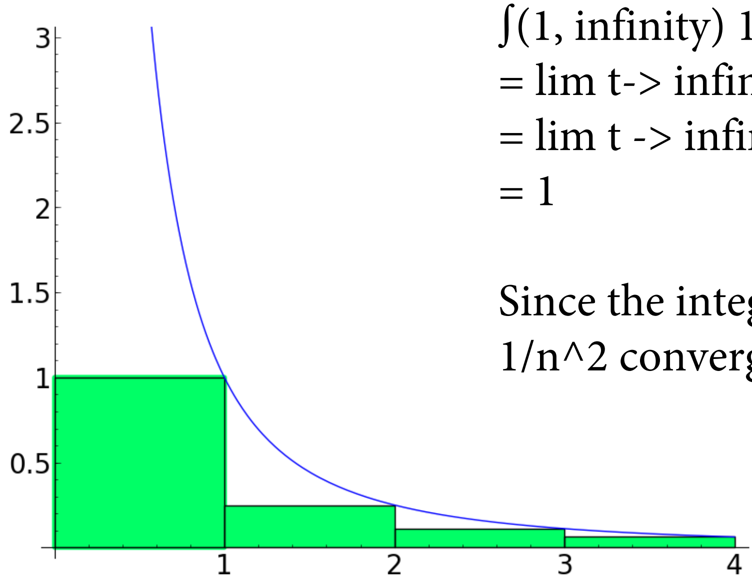
lin n-> infinity 1/n^2 -> 0

Technique
Telescoping
Limit laws
Partial sums

s1 = 1/1
s2 = 1/1 + 1/4 = 5/4
s3 = 1/1 + 1/4 + 1/9 = 49/9

converges

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is closely related to the improper integral $\int_1^{\infty} \frac{1}{x^2} dx$.



∫(1, infinity) 1/x^2 dx = lim t->infinity ∫(1,t) 1/x^2 dx
= lim t-> infinity -1/x [1, t]
= lim t -> infinity -1/t + 1/1
= 1

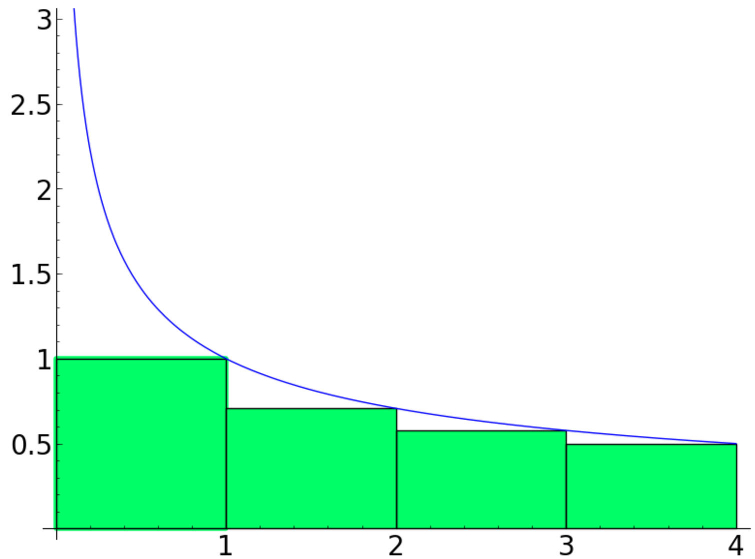
Since the integral of 1/x^2 converges then the series 1/n^2 converges

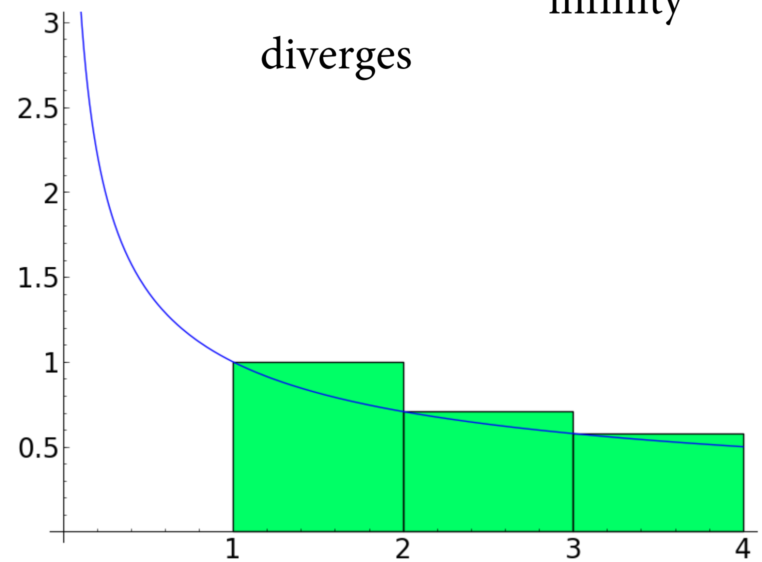
Example. Does this series converge or or diverge?

∫(1, infinity) 1/sqrt(x) dx
lim t -> infinity ∫(1, t) 1/sqrt(x) dx
lim t-> infinity 2x^1/2 [1, t]

lim t-> infinity 2sqrt(t) - 2sqrt(1)
infinity
diverges

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{x}}$$



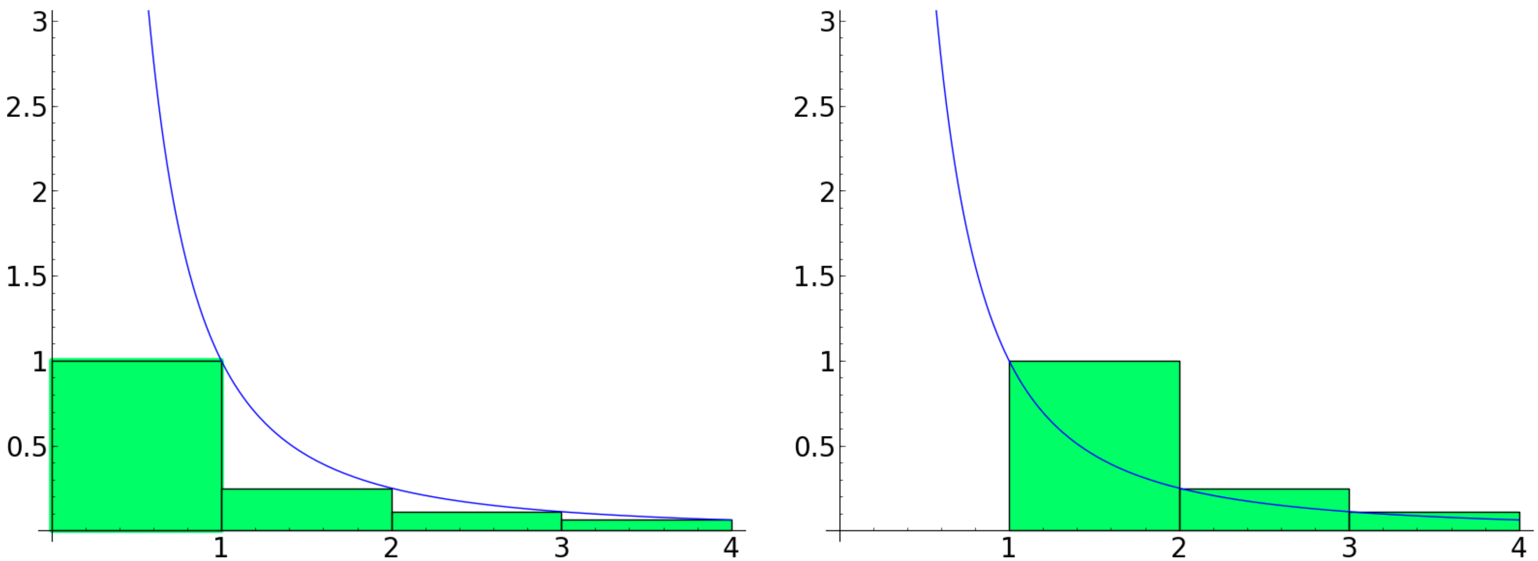


Since ∫(1, t) 1/sqrt(x) diverges the series 1sqrt(x) diverges

Theorem. *(The Integral Test) Suppose f is a* **continuous, positive, decreasing function** *on $[1,\infty)$ and $a_n = f(n)$. Then*

(a) If $\displaystyle\int_1^\infty f(x)\,dx$ converges, then $\displaystyle\sum_{n=1}^\infty a_n$ converges.

(b) If $\displaystyle\int_1^\infty f(x)\,dx$ diverges, then $\displaystyle\sum_{n=1}^\infty a_n$ diverges.



Example. Does $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converge or diverge?

lim n-> 1nn/n = lim n->infinity 1/n/1 = 0

Technique/Test
Geometric
Telescoping
Limit Laws
Partial Sums
Integral Test

Integral
Continuous [1,infinity)
Positive [1,infinity)
Decreasing [1, infinity)
f'(x) < 0
1/x * x - 1lnx / x^2 < 0
1 - lnx < 0
-lnx < -1
lnx > 1
x > e

Integral test
Σ(n=1, 2) lnn/n + Σ(n=3, infinity) lnn/n

Example. Does $\displaystyle\sum_{k=1}^{\infty} \frac{k}{k+1}$ converge or diverges?

lim k -> infinity k/k+1
lim k -> infinity 1/1
1

Diverges

Note. If the sequence of terms a_n do not converge to 0, then the series $\sum a_n$ ... Diverges

Theorem. *(The Divergence Test) If* lim n-> infinity != 0

then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ *diverges.*

Example. $\sum_{t=1}^{\infty} t\sin(1/t)$

-1 < sin1/t < 1
-t < tsin1/t < t
-infinity infinity

lim t-> infinity tsin1/t = lim t->infinity sin1/t / 1/t
= lim t -> infinity -1/t^2 cos1/t / -1/t^2

By divergence test, the series diverges

Example. $\sum_{t=1}^{\infty} (-1)^n$

lim t -> infinity (-1)^n
lim t -> infinity n(-1)^n-1

+- infinity
Diverges

Note. If the sequence of terms a_n **do** converge to 0, then the series $\sum a_n$ may or may not converge.

Review. We know that $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges to 1. Which of the following are true?

A. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

B. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = 1.$

C. Both of the above.

D. None of the above.

Example. Does $\sum_{n=1}^{\infty} \frac{n}{e^n}$ converge or diverge?

Since ∫(2,infinity) x/e^x converges by the integral test the
Σ(n=1, infinity) n/e^n also converges

lim n -> infinity n/e^n
= lim n->infinity 1/e^n = 0

Technique/Test
Integral test

Positive, Continuous, Decreasing
1(e^x) - x(e^x) / e^x^2 < 0
e^x - e^x / e^x^2 < 0
e^x(1-x) / e^x^2 < 0
1-x/e^x < 0
1-x < 0
x>1

∫(1,infinity) x/e^x dx = ∫(1,2) x/e^x dx + ∫(2,infinity) x/e^x dx
lim t->infinity ∫(2,t) x/2^x dx = -xe^-x - ∫(2,t) -e^-x dx
u = x, du = dx, dv = e^-x, v = -e^-x
-xe^-x - e^-x [2,t] = -te^-t - e^-t - (-2e^-2 - e^-2)
-t/e^t - 1/e^t + 2/e^2 + 1/e^2
0 + 0 + 2/e^2 + 1/e^2

Converges

Example. Does the following series converge or diverge?

$$\frac{1}{5} + \frac{1}{8} + \frac{1}{11} + \frac{1}{14} + \cdots$$

Σ(n=1,infinity) 1/3n+2

Continuous, Decreasing, Decreasing
0(3n+2) - 1(3) / (3n+2)^2 < 0
n != -2/3
-3 < 0

∫(1, infinity) 1/3x+2 dx
u = 3x+2, du = 3, 1/3du = dx
∫(1, t) 1/u * 1/3 du
1/3 lnu[1,t] = 1/3 ln|3x+2| [1,t]
1/3ln(3t+2) - 1/3ln(five)
infininty - number

Diverges

Since diverges ∫(1, infinity) 1/3x+2 dx by integral test
Σ(n=1,infinity) 1/3n+2 diverges

Question. For what values of p does the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge?

lim n->infinity 1/n^p = 0

$p < 0$ not true
$p = 0$ not true
$p = 1$ possibly
$p > 1$ possibly
$0<p<1$ possibly

Test/Technique

∫(1, infinity) 1/x^p = ∫(1, t) 1/x^p = x^-p+1 / -p+1 [1,t]
t^-p+1 / -p+1 - 1/-p+1

$p = 1$ Diverge
$p > 1$ Converges, makes the power negative
$0<p<1$ Diverges, makes the power positive

P-series test
Σ(n=1, infinity) 1/n^p, then t converges when $p>1$, diverges otherwise

Definition. The Harmonic Series is the series:

Wave lengths of overtones of vibrating strings

Question. Does the Harmonic Series converge or diverge?

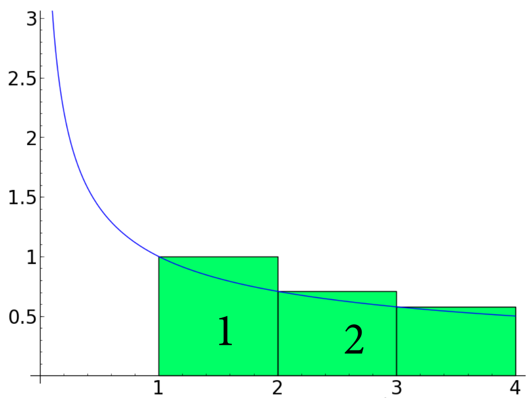
Diverges, p-series test, because $p>1$ converges, diverges otherwise

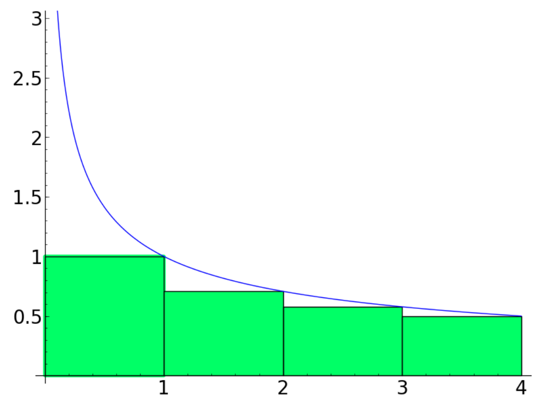
Bounding the Error

Definition. If $\sum_{n=1}^{\infty} a_n$ converges, and s_n is the nth partial sum, then for large enough n, s_n is a good approximation to the sum $s_\infty = \sum_{k=1}^{\infty} a_k$. Define R_n be the error, or remainder:

Remainder, actual, predicted

$R_n =$ S - Sn





R2 = a3 + an + ... >= ∫(3,infinity) f(x)dx

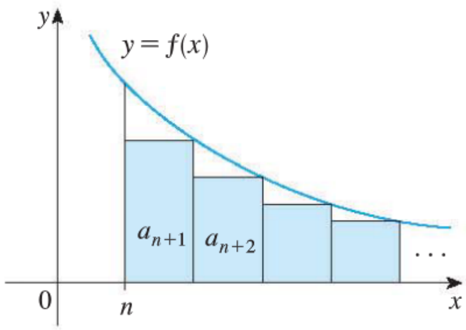
R2 = a3+an+...... =< ∫(2,infinity) f(x)dx

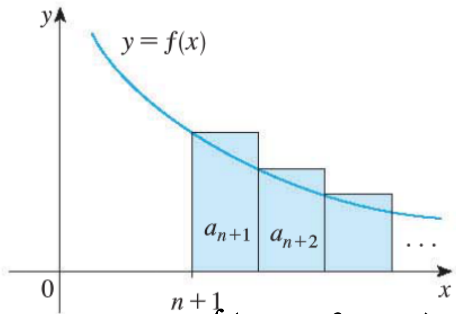
Use the pictures above to compare R_2 to $\int_2^\infty f(x)\,dx$ and $\int_2^\infty f(x)\,dx$ where $f(x)$ is the positive, decreasing function drawn with $a_n = f(n)$.

∫(3,a) f(x)dx =< R2 =< ∫(2,a) f(x)dx

Lower bound

Upper Bound





Rn >= ∫(n+1, infinity) f(x) dx

Rn >= ∫(n, infinity) f(x) dx

Use the pictures above to compare R_n to $\int_n^\infty f(x)\,dx$ and $\int_{n+1}^\infty f(x)\,dx$ where $f(x)$ is the positive, decreasing function drawn with $a_n = f(n)$.

Note. If $a_n = f(n)$ for a continuous, positive, decreasing function $f(x)$,

∫(n+1, infinity) f(x) dx $\leq R_n \leq$ ∫(n,infinity) f(x) dx

This is called the **Remainder Estimate for the Integral Test**

Example. (a) Put a bound on the remainder when you use the first three terms to approximate $\sum_{n=1}^{\infty} \frac{6}{n^2}$. Changed to 1/n^2

(b) Use the bound on the remainder to put bounds on the sum s_∞. Hint: $s_\infty = s_3 + R_3$.

(c) How many terms are needed to approximate the sum to within 10^{-3}? Note: by convention, this means $R_n < 0.0001$.

S3 = 1/1 + 1/4 + 1/9 = 49/36

R3 =< ∫(3, infinity) 1/x^2 dx
lim t -> infinity ∫(3, t) 1/x^2 dx
= lim t -> infinity -1/x [3,t]
= lim t -> infinity -1/t + 1/3 [3,t]
R3 =< 1/3
Size of the error using S3 = 49/36 is at most 1/3

Question. Which of the following are always true?

(a) Suppose f is a **continuous, positive, decreasing function** on $[1, \infty)$ and for $n \geq 1$, $a_n = f(n)$. Then $\sum_{n=1}^{\infty} a_n$ converges, if and only if $\int_1^{\infty} f(x)\,dx$ converges.

(b) Suppose f is a **continuous, positive, decreasing function** on $[5, \infty)$ and for $n \geq 5$, $a_n = f(n)$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_5^{\infty} f(x)\,dx$.

(c) Suppose f is a **continuous, positive function** on $[1, \infty)$ and for $n \geq 1$, $a_n = f(n)$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges.