§8.5 - Integrals of Rational Functions

After completing this section, students should be able to:

1. Recognize whether an integral is a good candidate for the method of partial fractions.
2. Rewrite a rational expression as a sum of appropriate partial fractions, performing long division first if the numerator has degree greater or equal to the denominator.
3. Compute an integral using the method of partial fractions.

Example. According to Wolfram Alpha,

$$\int \frac{3x+2}{x^2+2x-3}\,dx = \frac{5}{4}\ln|1-x| + \frac{7}{4}\ln|x+3| + C$$

Let's see where this answer came from. partial fractions

① Long ÷ → NO (numerator ≥ denominator degree wise)

② Create partial fractions Case 1: linear, distinct factors

$$\frac{3x+2}{x^2+2x-3} = \frac{3x+2}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3}$$

$$\frac{3x+2}{(x-1)(x+3)} = \frac{A}{x-1}\frac{(x+3)}{(x+3)} + \frac{B}{x+3}\frac{(x-1)}{(x-1)}$$

$$\frac{3x+2}{(x-1)(x+3)} = \frac{A(x+3)+B(x-1)}{(x-1)(x+3)}$$

⇒ common den ⇒ numerator are =

$$3x+2 = Ax+3A+Bx-B$$

$$3x = Ax+Bx \qquad 2 = 3A-B$$

$$3 = A+B$$

⇒ system of eq

$$3 = A + B$$
$$+ \quad 2 = 3A - B$$
$$5 = 4A$$
$$A = \frac{5}{4}$$

$$3 = A + B$$
$$3 = \frac{5}{4} + B$$
$$3 - \frac{5}{4} = B$$
$$\frac{7}{4} = B$$

③ Rewrite $\int$

$$\int \frac{3x+2}{x^2+2x-3}\,dx = \int \frac{\frac{5}{4}}{x-1} + \frac{\frac{7}{4}}{x+3}\,dx = \boxed{\frac{5}{4}\ln|x-1| + \frac{7}{4}\ln|x+3| + C}$$

$$\ln|1-x|$$
$$\ln|-1(x-1)|$$
$$\ln|x-1|$$

Review. True or $\boxed{\text{False:}}$ $\displaystyle\int \frac{1}{2x^2 - 7x - 4}\,dx = \ln|2x^2 - 7x - 4| + C$

↳ chain rule

↳ partial fractions

Example. Find $\displaystyle\int \frac{2x^2 + 7x + 19}{x^2 - 5x + 6}\,dx$

① long ÷ → yes

$$\begin{array}{r} 2 \\ x^2 - 5x + 6 \overline{\smash{)}\, 2x^2 + 7x + 19} \\ -\left(2x^2 - 10x + 12\right) \\ \hline 17x + 7 \end{array}$$

$$\frac{2x^2 + 7x + 19}{x^2 - 5x + 6} = 2 + \frac{17x + 7}{x^2 - 5x + 6}$$

② Rewrite Fraction w/partial fractions case 1

$$\frac{17x + 7}{x^2 - 5x + 6} = \frac{17x + 7}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$$

$$17x + 7 = A(x-3) + B(x-2)$$

$$17x + 7 = Ax - 3A + Bx - 2B$$

$17x = Ax + Bx$ $\qquad$ $7 = -3A - 2B$

$17 = A + B$

$2(17 = A + B)$
$7 = -3A - 2B$

$\rightarrow$

$$\begin{aligned} 34 &= 2A + 2B \\ +\ 7 &= -3A - 2B \\ \hline 41 &= -A \end{aligned}$$

$A = -41$

$17 = A + B$
$17 = -41 + B$
$+41 \quad +41$
$58 = B$

③ Rewrite integral

$$\int 2 + \frac{-41}{x-2} + \frac{58}{x-3}\, dx$$

$$\boxed{2x - 41 \ln|x-2| + 58 \ln|x-3| + C}$$

Example. How would you set up partial fractions to integrate this?

$$\int \frac{5x+7}{(x-2)(x+5)(x)}\,dx \qquad \text{case } 1$$

① long ÷ → NO

② partial fraction

$$\frac{5x+7}{(x-2)(x+5)(x)} = \frac{A}{x-2} + \frac{B}{x+5} + \frac{C}{x}$$

$$5x+7 = A(x+5)(x) + B(x-2)(x) + C(x-2)(x+5)$$

$$5x+7 = A(x^2+5x) + B(x^2-2x) + C(x^2+3x-10)$$

$$5x+7 = Ax^2 + 5Ax + Bx^2 - 2Bx + Cx^2 + 3Cx - 10C$$

$* \; 5x = 5Ax - 2Bx + 3Cx \qquad 7 = -10C \qquad 0 = A + B + C$

$5 = 5A - 2B + 3C$

Example. How would you set up partial fractions to integrate this? $\int \frac{4x^2+3x+7}{x^3-4x^2+4x}dx$

A. $\frac{4x^2+3x+7}{x(x-2)^2} = \frac{A}{x} + \frac{B}{x-2}$

B. $\frac{4x^2+3x+7}{x(x-2)^2} = \frac{A}{x} + \frac{B}{(x-2)^2}$

C. $\frac{4x^2+3x+7}{x(x-2)^2} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$

D. $\frac{4x^2+3x+7}{x(x-2)^2} = \frac{A}{x} + \frac{Bx+C}{(x-2)^2}$

① long ÷ → no

② partial fractions → case 2
linear, not all distinct factors

$$\frac{4x^2+3x+7}{x^3-4x^2+4x} = \frac{4x^2+3x+7}{x(x^2-4x+4)} =$$

$$\frac{4x^2+3x+7}{x(x-2)(x-2)} = \frac{4x^2+3x+7}{x(x-2)^2}$$

$$\frac{4x^2+3x+7}{x(x-2)^2} = \frac{A}{x} + \frac{B}{(x-2)^2} + \frac{C}{x-2}$$

$$\int \frac{x(5-3x)}{(3x-1)(x+7)^2}\,dx$$

① long $\div$ $\rightarrow$ no

② partial fractions $\rightarrow$ case 2

$$\frac{x(5-3x)}{(3x-1)(x+7)^2} = \frac{A}{3x-1} + \frac{B}{(x+7)^2} + \frac{C}{x+7}$$

Case 3: non linear, distinct factors

$$\int \frac{1}{x(x^2+2x+7)}\,dx = \int \frac{A}{x} + \frac{Bx+C}{x^2+2x+7}$$

case 4: non linear & not distinct

$$\int \frac{1}{x(x^2+2x+7)^2}\,dx = \int \frac{A}{x} + \frac{Bx+C}{(x^2+2x+7)^2} + \frac{Dx+E}{(x^2+2x+7)}$$

Test: Case 1 & 2!