## §10.7 - Ratio and Root Tests

After completing this section, students should be able to:

- Use the ratio test to determine if a series converges or diverges.
- Use the root test to determine if a series converges or diverges.
- Give an example of a series for which the ratio test and the root test are both inconclusive.

Recall: for a geometric series $\sum ar^n$

$|r| < 1$ -> converges

**Theorem.** *(The Ratio Test) For a series $\sum a_n$ :*

*(a) If* $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$*, then* $\sum_{n=1}^{\infty} a_n$ *is* absolutely convergent -> convergent.

*(b) If* $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ *or* $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$*, then* $\sum_{n=1}^{\infty} a_n$ *is* divergent.

*(c) If* $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$*, then* $\sum_{n=1}^{\infty} a_n$ inconclusive, need a different test.

**Example.** Apply the ratio test to

$$\sum_{n=1}^{\infty} \frac{n^2(-10)^n}{n!}$$

lim n->infinity |an+1 / an|
lim n->infinity |(n+1)^2 (-10)^n+1 / (n+1)!     * n!/ n^2(-10)^n|
clean up/cancel
    Canceling powers
    Expanding factorials

lim n->infinity |(n+1)^2 (-10)^n+1 / (n+1)(n!)        * n!/n^2(-10)^n |
lim n->infinity |(n+1)(-10)/n^2| = lim n->infinity 10n + 10 / n^2
0

absolutely convergent
convergent

**Review.** In which of these situations can we conclude that the series $\sum^{\infty} a_n$ converges?

A. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$ <1 absolutely convergent, converges

B. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0.3$ <1 absolutely convergent, converges

C. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$ inconclusive

D. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 17$ >1 diverges

E. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$ >1 diverges

**Review.** (The Ratio Test) For a series $\sum a_n$ :

(a) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum_{n=1}^{\infty} a_n$ is ________________________ .

(b) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then $\sum_{n=1}^{\infty} a_n$ is ________________________.

(c) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$ or DNE , then ________________________ .

**Example.** Apply the ratio test to

$$\sum_{n=1}^{\infty} \frac{(1.1)^n}{(2n)!}$$

lim n->infinity (1.1)^n+1 / (2n+2)!  * (2n!) / (1.1)^n
lim n->infinity 1.1^n+1 / (2n+2)(2n+1)(2n)!    * (2n!) / 1.1^n
lim n-> infinity 1.1 / (2n+2)(2n+1)
0

Converges

**Example.** Apply the ratio test to the series

$$\sum_{n=2}^{\infty} \frac{3}{n^2 - n}$$

lim n-> infinity 3/(n+1)^2 - (n+1)    *  n^2-n/3
lim n-> infinity 1/ n^2 + 2n +1 -n -1         n^2-n
lim n-> infinity n^2-n/n^2 + n
= 1
inconclusive

limit comparison
1/n^2
converges with p-series since p>1

**Extra Example.** Apply the ratio test to the series

$$a_1 = 1, a_n = \frac{\sin n}{n} a_{n-1}$$

Σsin n/n     an-1

lim n->infinity |sin(n+1)/n+1   *an+1-1 / 1     * n / sinn * an-1|
lim n->infinity |sin(n+1)/n+1   *an   * n / sinn * an-1 or 1/an|
lim n->infinity |sin(n+1)/n+1   *an   * 1/an|
lim n->infinity |sin(n+1)/n+1|
= 0

absolutely convergent
convergent

**Theorem.** *(The Root Test)*

*(a) If* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ *or* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$*, then* $\sum_{n=1}^{\infty} a_n$ diverges.

*(b) If* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$*, then* $\sum_{n=1}^{\infty} a_n$ absolutely convergent -> convergent.

*(c) If* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$*, then* $\sum_{n=1}^{\infty} a_n$ inconclusive.

**Example.** Determine the convergence of

$$\sum_{n=1}^{\infty} \frac{5^n}{n^n}$$

Σ(5/n)^n
lim n->infinity n root(|an|)
lim n->infinity n root(|(5/n)^n|)
lim n->infinity 5/n
0/1
0

Converges

## Rearrangements

**Definition.** A **rearrangement** of a series $\sum a_n$ is a series obtained by rearranging its terms.

**Fact.** If $\sum a_n$ is absolutely convergent with sum $s$, then any rearrangement of $\sum a_n$ also has sum $s$.

But if $\sum a_n$ is any conditionally convergent series, then it can be rearranged to give a different sum.

**Example.** Find a way to rearrange the Alternating Harmonic Series so that the rearrangement diverges.

Σ(-1)^n / n
not absolutely convergent -> abs value: Σ1/n
AHS can be rearranged so it diverges