§8.9 -Improper Integrals

After completing this section, students should be able to:

- Determine if an integral is improper and explain why.
- Explain how to calculate an improper integral or determine that it diverges by taking a limit.
- Divide up an improper integral into several separate integrals in order to compute it, when it is improper in several ways.
- Calculate improper integrals or determine that they diverge.
- Choose appropriate functions to compare with integrands, when using the Comparison Theorem.
- Use the Comparison Theorem to determine if integrals converge or diverge without actually integrating.
- Give an example to show how failing to notice that an integral is improper and computing it as if it were proper can lead to nonsense.

Here are two examples of improper integrals:

$$\int_1^\infty \frac{1}{x^2}\,dx$$

and

$$\int_0^{\frac{\pi}{2}} \tan(x)\,dx$$

Question. What is so improper about them?

Infinity bound, bound at asymptote/asymptote in the middle of your bounds

Definition. An integral is called *improper* if either

(Type I)

or,

(Type II)

Discontinous integral

or both.

Type 1 Improper Integrals

To integrate over an infinite interval, we take the limit of the integrals over expanding finite intervals

Example. Find $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$

```
lim t-> infinity ∫(1, t) 1/x^2dx = -1/x (1, t)

= lim t-> infinity [-1/t - -1/1]
= lim t -> infinity [-1/t + 1]
= 0 + 1
= 1
```

Definition. The improper integral $\displaystyle\int_a^\infty f(x)\,dx$ is defined as ...

```
lim t -> infinity ∫(a, t) f(x)dx, where a < t
```

We say that $\displaystyle\int_a^\infty f(x)\,dx$ **converges** if ... `limit exists`

and **diverges** if ... `limit does not exist (+- infinity)`

Definition. Similarly, we define $\int_{-\infty}^{b} f(x)\,dx$ as ...

```
lim t -> -infinity ∫(-infinity, b) f(x)dx, where t < b
```

and say that $\int_{-\infty}^{b} f(x)\,dx$ **converges** if ... `limit exists`

and **diverges** ... `if limit does not exist (+- infinity)`

Example. Evaluate $\int_{-\infty}^{-1} \frac{1}{x}\,dx$ and determine if it converges or diverges.

```
lim t -> -infinity ∫(-infinity, -1) 1/x dx

= ln|x| [t, -1]
= ln(-1) - ln(t)
= lim t -> -infinity [-ln|t|]
= -infinity

diverges
```

Review. Which of the following are NOT *improper integrals*?

A. $\displaystyle\int_1^{\infty} e^{-x}\,dx$

B. $\displaystyle\int_0^{3} \frac{1}{x^2}\,dx$

C. $\displaystyle\int_{-5}^{5} \ln|x|\,dx$

D. $\displaystyle\int_{-\infty}^{0} \frac{4}{x+4}\,dx$

E. They are all improper integrals.

Example. Evaluate $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx$ and determine if it converges or diverges.

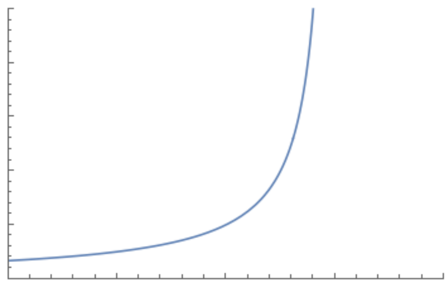
Diverges

Example. Find the area under the curve $y = e^{3x-2}$ to the left of $x = 2$.

```
∫(-infinity, 2) e^3x-2 dx

e^3x-2 / 3 [-infinity, 2]


= e^3(2)-2 / 3 - (e^3t-2 / 3)
= e^3(2)-2 / 3 - (e^-infinity / 3)
= e^3(2)-2 / 3 - (0)
= e^4 / 3

converges
```

Type 2 Improper integrals

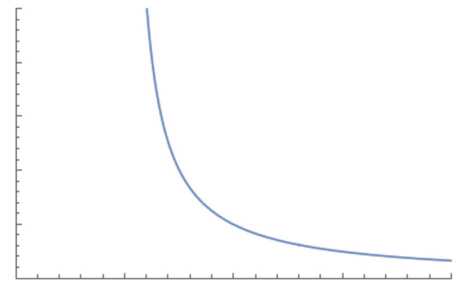
When the function we are integrating goes to infinity at one endpoint of an interval, we take a limit of integrals over expanding sub-intervals.



B everything to the left of the limit

Definition. If $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to b^-$, then

$$\int_a^b f(x)\,dx =$$ lim t -> b- ∫(a, t) f(x)dx

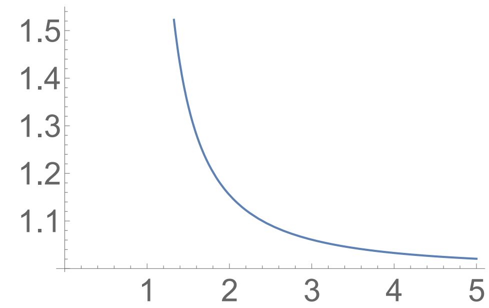


A everything to the right of the limit

Definition. If $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to a^+$, then

$$\int_a^b f(x)\,dx =$$ lim t -> A+ ∫(b, t) f(x)dx

Example. Find the area under the curve $y = \dfrac{x}{\sqrt{x^2 - 1}}$ between the lines $x = 1$ and $x = 2$.




```
A = ∫(1, 2) x/sqrt(x^2-1) dx
lim t -> 1+ ∫(1,2) x/sqrt(x^2-1) dx

u = x^2-1
du = 2xdx
1/2du = xdx

lim t->1+ 1/2∫(t,2) 1/sqrt(u) du
lim t->1+ 1/2 (u^1/2 / 1/2) [t,2]
lim t->1+ u^1/2 [t,2]
lim t->1+ sqrt(x^2-1)[t,2]
lim t->1+ sqrt(2^2 - 1)[t,2] - sqrt(t^2-1)
lim t->1+ sqrt(3) - sqrt(t^2-1)
= sqrt(3) - sqrt(0)
= sqrt(3)
```

Example. Find $\displaystyle\int_1^{10} \frac{4}{(x-3)^2}\,dx$.

```
∫(1, 10) 4/(x−3)^2 dx
= ∫(1, 3) 4/(x−3)^2 dx + ∫(3, 10) 4/(x−3)^2 dx
= lim t −> 3− ∫(1, A) 4/(x−3)^2 dx + lim A −> 3+ ∫(A, 10) 4/(x−3)^2

u = x−3
du = dx

= lim t −> 3− −4/x−3 [1, A] + lim A −> 3+ (−4/x−3) [A, 10]
= lim t −> 3− −4/A−3 + 4/−2 + lim A −> 3+ (−4/7) + 4/A−3
= lim t −> 3− infinity + 4/−2 + lim A −> 3+ (−4/7) + infinity
= infinity −> diverges
```

Note. Since $\dfrac{4}{(x-3)^2}$ blows up at $x = 3$, this integral must be computed as the sum of two indefinite integrals.

If you compute it without breaking it up YOU WILL GET THE WRONG ANSWER!

Question. For what values of $p > 0$ does $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converge?

```
lim t -> infinity ∫(1, t) 1/x^p dx

= lim t -> infinity [x^-p+1 / -p +1] (1, t)
= lim t -> infinity t^-p+1 / -p+1 - (1^-p+1 / -p+1)

p = neg?
lim t-> infinity t^-p+1 / -p+1
t^pos/pos -> infinity
diverge

0<p<1
lim t-> infinity t^-p+1 / -p+1
t^pos/pos -> infinity
diverge

p>1
lim t-> infinity t^-p+1 / -p+1
t^neg/neg -> 1/infinity -> 0
converge
```

Question. For what values of $p > 0$ does $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$ converge?

Question. For what values of $p > 0$ does $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converge?

Theorem. *Comparison Theorem for Integrals: Suppose* $0 \leq g(x) \leq f(x)$ *on* (a, b) *(where a or b could be* $-\infty$ *or* ∞*).*

(a) If $\displaystyle\int_a^b f(x)\,dx$ $\underline{\texttt{converges}}$*, then* $\displaystyle\int_a^b g(x)\,dx$ $\underline{\texttt{converges}}$ *also.*

(b) If $\displaystyle\int_a^b g(x)\,dx$ $\underline{\texttt{diverges}}$*, then* $\displaystyle\int_a^b f(x)\,dx$ $\underline{\texttt{diverges}}$ *also.*

Example. Does $\displaystyle\int_2^\infty \frac{2+\sin(x)}{\sqrt{x}}\,dx$ converge or diverge?

```
−1 < sinx < 1
1/sqrt(x) < 2+sinx < 3
1/sqrt(x) < 2+sinx / sqrt(x) < 3/ sqrt(x) = 3 ∗ 1/sqrtx
∫(2, infinity) 1/sqrt(x) diverges 3∫(2, infinity) 1/sqrt(x) diverges

diverges
```

Review. If $0 \leq f(x) \leq g(x)$ on the interval $[a, \infty)$, then which of the following are true?

A. If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ converges.

B. If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ diverges.

C. If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ converges.

D. If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges.

E. None of these are true.

Example. Does $\displaystyle\int_1^\infty \frac{\cos(x)+7}{4x^3+5x-2}\,dx$ converge or diverge?

```
-1 < cosx < 1
6 < cosx + 7 < 8

6/(4x^3+5x-2) < cosx + 7 /(4x^3+5x-2) < 8/(4x^3+5x-2)
8/4x^3
2/x^3

2∫1/x^3 dx converges
2/x^3 is larger than cosx + 7/(4x^3+5x-2)
∫(1,infinity) 2/x^3 converges. By comparison theorm ∫(1,infinity) cosx + 7/(4x^3+5x-2) converges
```

Example. Does $\int_7^\infty \frac{3x^2 + 2x}{\sqrt{x^6 - 1}}\, dx$ converge or diverge?

sqrt(x^6−1) < sqrt(x^6) = x^6/2 = x^3

1/sqrt(x^6−1) < 1/x^3
converges but cmp thm doesn't work

∫(7, infinity) 3x^2+2x/sqrt(x^6−1)
∫(7, infinity) 3x^2/sqrt(x^6−1) + ∫(7, infinity) 2x/sqrt(x^6−1)
1/sqrt(x^6−1) > 3x^2/x^3 = 3 ∗ 1/x
diverge

3/x < 3x^2/sqrt(x^6−1) + 3∫(7y, infinity) 1/x diverges, by comparison thm, ∫(7, infinity) 3x^2+2x/sqrt(x^6−1) diverges

Example. Does $\int_0^\infty e^{-x^2}\,dx$ converge or diverge?

e^-x^2 < e^-x
1/e^x^2 < 1/e^x for x>1

∫(0, 1) e^-x^2dx + ∫(1, infinity) e^-x^2 dx
∫(1, infinity) e^-xdx = lim t->infinity ∫(1, t) e^-xdx = lim t-> infinity -e^-x[1,t]
= lim t -> infinity -1/e^t + 1/e = 0 + 1/e
e^-x > e^-x^2 for x>1 + ∫(1, infinity) -e^-xdx converges, by the comparison thm ∫(0, infinty) e^-x^2dx converges

Question. What are some useful functions to compare to when using the comparison test?

```
∫(1, infinity) 1/x^p
converges p>1
diverges p<1

-1<trig<1
```

Question. True or False: Since $-\frac{1}{x} \leq \frac{1}{x^2}$ for $1 < x < \infty$, and $\int_1^\infty \frac{1}{x^2}\,dx$ converges, the Comparison Theorem guarantees that $\int_1^\infty -\frac{1}{x}\,dx$ also converges.

```
False, has to be greater than 0

-1/x x>1 -> not positive
```