10.2 Sequences

After completing this section, students should be able to:

- Define increasing, decreasing, non-decreasing, non-increasing, and monotonic.
- Define bounded.
- Use the first derivative to determine if sequences are increasing, decreasing and whether they are bounded.
- Determine if a sequence converges and find its limit by evaluating the limit of a function using Calculus 1 techniques.
- State the limit laws and use them to break apart limits and determine convergence.
- Recognize when limit laws don't apply due to component sequences diverging.
- Find the first term and common ratio of a geometric sequence and use the common ratio to determine if the sequence converges or diverges.
- State conditions involving boundedness and monotonic-ness that ensure that a sequence converges, and use this condition to prove that sequences converge.
- Use the squeeze theorem to prove that a sequence converges.
- Use the idea of the squeeze theorem to prove that a sequence diverges to ∞ or

$-\infty$

Definition. A sequence $\{a_n\}$ is **bounded above** if

A sequence $\{a_n\}$ is **bounded below** if:

Example. Which of these sequences are bounded?

A. $\{3, 0.3, 0.03, 0.003, 0.0003, \cdots\}$

B. $\{1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \cdots\}$

C. $\{3, -2, \frac{4}{3}, -\frac{8}{9}, \ldots\}$

Definition. A sequence $\{a_n\}$ is **increasing** if an+1> an

A sequence $\{a_n\}$ is **non-decreasing** if

A sequence $\{a_n\}$ is **decreasing** if

A sequence $\{a_n\}$ is **non-increasing** if

A sequence $\{a_n\}$ is **monotonic** if it is

Example. Which of these sequences are monotonic?

A. $\{3, 0.3, 0.03, 0.003, 0.0003, \cdots\}$

B. $\{1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \cdots\}$

C. $\{3, -2, \frac{4}{3}, -\frac{8}{9}, \ldots\}$

D. $\{-6, 5, -1, 4, 3, 7, 10, 17, \ldots\}$

strictly equal to

Question. What is the difference between increasing and non-decreasing? Decreasing and non-increasing? strict

equal to

Review. Give an example of a sequence that is

- monotonically increasing and bounded {2,5,8,11,....}

- monotonic non-increasing but not bounded

- not monotonic but bounded {3,2,9,4,27,8}

- not monotonic and not bounded {2,-3,4,-5,6,-7,8

Example. Is the sequence $\left\{\dfrac{n-5}{n^2}\right\}_{n=1}^{\infty}$ monotonic? Bounded?

-4,-3/4,-2/9,-1/16,0,1/36,2/49,3/64,......

bound below: an>-4
bound above: an<1

f'(x)
f(x) = x-5/x^2
f'(x)=1(x^2)-2x(x-5)/(x^2)^2
= x^2-2x^2+10x/x^4
=-x^2+10x/x^4

-x^2 + 10x = 0
-x(x-10)=0
x = 0,10

Review. Recall that a geometric sequence is a sequence that can be written in the form:

an = ar^n

n=0 start

Here, r represents ratio ________________ and a represents initial term ________________ .

What is an example of a geometric sequence?

an = {2*3^n}^infinity n=0

Example. Which of these are geometric sequences? Which of them converge?

- $\left\{\dfrac{(-1)^n 4^n}{5^{n+2}}\right\}_{n=0}^{\infty}$ Geometric {(-4)^n/5^n5^2 = {(-4/5)^n* 1/25} n=0 converge -> 0

- $\left\{\dfrac{5 \cdot 0.5^n}{3^{n-1}}\right\}_{2}^{\infty}$ Geometric {5*0*5^n / 3^n 3^-1} = {(5/3)^n *5(3)} = {(1/6)^n * 15} converge -> 0

- $\left\{4/3, 2, 3, \dfrac{9}{2}, \dfrac{27}{4} \ldots\right\}$ {(4/3)(3/2)^n} n=0 infinitely gets bigger, diverge to infinity

- $\{2, -4, 8, -16, 32, -64, \ldots\}$ {2(-2)^n} n=0 = diverge to +- infinity

Question. For which values of a and r does $\{a \cdot r^n\}_{n=0}^{\infty}$ converge?

lim n-> infinity ar^n = a lim n ->infinity r^n

|r| < 1 converge to zero
|r| > 1 diverge
r = 1 converges to a
r = -1 diverge to -a,+a

The following are some techniques for proving that a sequence converges:

Example. Does $\left\{ \frac{(-1)^t e^{t-1}}{3^{t+2}} \right\}_{t=0}^{\infty}$ converge or diverge?

{-1^t e^t e^-1 / 3^t 3^2} = {(-e/3)^t * 1/9e}
|r| < 1 -> converges

geometric sequences with |r|<1, so

Trick 1: Recognize geometric sequences

Example. Does $\left\{\dfrac{\ln(1+2e^n)}{n}\right\}_{n=1}^{\infty}$ converge or diverge?

lim n->infinity

Trick 2: Suppose $a_n = f(n)$ where $n = 1, 2, 3, \ldots$, for some function f defined on all positive real numbers. If $\lim_{x\to\infty} f(x) = L$ then ...

So ... replace a_n with $f(x)$ and use l'Hospital's Rule or other tricks from Calculus 1 to show that $\lim_{x\to\infty} f(x)$ exists.

Example. Does $\left\{ \frac{\cos(n) + \sin(n)}{n^{2/3}} \right\}_{n=5}^{\infty}$ converge or diverge?

-sqrt(2) < cosn +sinn <sqrt(2)

-sqrt(2)/n^2/3 < (cosn +sinn)/n^2/3 <sqrt(2)/n^2/3

lim n-> -sqrt(2)/n^2/3 = 0 lim n->infinity sqrt(2)/n^2/3 = 0

By the squeeze theorem, converges to zero
{(cosn+sinn)/n^2/3}infinity n=5

Trick 3: Use the Squeeze Theorem: trap the sequence between two simpler sequences that converge to the same limit.

g(x) < f(x) < h(x)
lim x->infinity g(x)=L lim->infinity f(x) =L lim x->infinity h(x)=L

Example. $\{n + \sin(n)\}_{n=0}^{\infty}$

-1<sinn<1

-1+n<sinn +n<1+n

lim n->infinity -1+n = infinity   lim n->infinity 1+n = infinity

By squeeze theorem, {sinn+n}infinity n=0 diverges

Example. Does $0.1, 0.12, 0.123, 0.1234, \ldots, 0.12345678910, 0.1234567891011, 0.123456789101112,$ converge or diverge?

monotonic: increasing
lower bound: an >0.1
upper bound: an <0.2

The seq is bounded +monotonic, the sequence converges

bounded(on both ends) monotonic(always increasing or decreasing)

Trick 4: If $\{a_n\}$ is ______________ and ______________ , then it converges.

Trick 5: Use the Limit Laws

The usual limit laws about addition, subtractions, etc. hold for sequences as well as for functions. (See textbook.)

For example, if $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$, then

$\lim_{n\to\infty}(a_n + b_n) =$ lim n->infinity an + lim n->infinity bn

$\lim_{n\to\infty}(a_n b_n) =$ lim n->infinity an * lim n->infinity bn

$\lim_{n\to\infty}(c a_n) =$ c lim n->infinity an (c is a constant)

Example. Does $\left\{ \dfrac{k^2}{2k^2 - k} + \dfrac{4 \cdot \pi^k}{6^k} \right\}_{k=3}^{\infty}$ converge or diverge?

lim k->infinity k^2/2k^2-k
lim k->infinity 2k/4k-1
lim k->infinity 2/4=1/2

Question. Do the limit laws help establish the convergence of this sequence?

$$\left\{n + \frac{3-2n}{2}\right\}_{n=2}^{\infty}$$

lim n->infinity n doesn't work
lim n->infinity 3-2n/2

Algebra manipulation
n+3-2n/2
2n/2+3-2n/2
3/2

lim n->infinity 3/2 = 3/2

{n+3-2n/2}infinity n=2
coverges to 3/2

True or False:

(a) If $\{a_k\}$ converges, then so does $\{|a_k|\}$. True

(b) If $\{|a_k|\}$ converges, then so does $\{a_k\}$. False |(-1)^k| = 1^k -> 1
(-1)^k = 1, -1

(c) If $\{a_k\}$ converges to 0, then so does $\{|a_k|\}$. True

(d) If $\{|a_k|\}$ converges to 0, then so does $\{a_k\}$. True

Example. Does $\left\{\frac{(-1)^n}{n^2}\right\}$ converge or diverge?

lim n->infinity |(-1)^n/n^2| = lim n->infinity 1/n^2 = 0 |(-1)^n/n^2| converges to 0

since {|(-1)^n/n^2|} converge to 0, the {(-1)^n/n^2} converges to 0

True or False:

(a) Suppose $a_n = f(n)$ for some function f, where $n = 1, 2, 3, \ldots$. If $\lim_{x\to\infty} f(x) = L$ then $\lim_{n\to\infty} a_n = L$.

(b) Suppose $a_n = f(n)$ for some function f. If $\lim_{n\to\infty} a_n = L$, then $\lim_{x\to\infty} f(x) = L$.

Additional problems if additional time:

Do the following sequences converge or diverge? Justify your answer.

(a) $\left\{\dfrac{\cos(j)}{\ln(j+1)}\right\}_{j=1}^{\infty}$

-1<cosj<1

-1/ln(j+1) < cosj/ln(j+1) < 1/ln(j+1)

(b) $\left\{\dfrac{(-1)^t 4^{t-1}}{3^{2t}}\right\}_{t=3}^{\infty}$

(c) $\left\{\dfrac{\sqrt[3]{k}}{\ln(k)}\right\}_{k=2}^{\infty}$

(d) $\left\{\dfrac{3^n}{n!}\right\}_{n=1}^{\infty}$

(e) $\left\{\dfrac{n!}{3^n}\right\}_{n=1}^{\infty}$