§11.1 - Approximating Series with Polynomials

Idea: Approximate a function with a polynomial.

Suppose we want to approximate a function $f(x)$ near $x = 0$. Assume that f's derivative, second derivative, third derivative, etc all exist at $x = 0$.

Warm up. Let $f(x)$ be a function whose derivatives all exist near $x = 0$. Suppose that $f(x)$ can be approximated by a degree 3 polynomial of the form

$$P_3(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$$

in such a way that the function and the polynomial have the same value at $x = 0$ and also have the same first through third derivatives at $x = 0$.

Write an expression for the polynomial coefficient c_3 in terms of $f^{(3)}(0)$.

f(x)=xc0+C1x+C2x^2+C3x^3
f'(x)=C1+2c2x+3C3x^2
f^2(x)=2C2+6C3x
f^3(x)=6C3

Warm up. Let $f(x)$ be a function whose derivatives all exist near $x = 5$. Suppose that $f(x)$ can be approximated by a degree 4 polynomial of the form

$$P_4(x) = c_0 + c_1(x-5) + c_2(x-5)^2 + c_3(x-5)^3 + c_4(x-5)^4$$

in such a way that the function and the polynomial have the same value at $x = 5$ and also have the same first through fourth derivatives at $x = 5$.

Suppose $f(5) = 1$, $f'(5) = 3$, $f''(5) = 7$, $f^{(3)}(5) = 13$, and $f^{(4)}(5) = -11$. What are the coefficients of the polynomial?

f(x) = C0 + C1 (x-5) + C2(x-5)^2 + C3(x-5)^3 + C4(x-5)^4 f(5) = C0 C0 = 1
f'(x) = C1 + 2C2(x-5) + 3C3(x-5)^2 + 4C4(x-5)^3 f'(5) = C1 C1 = 3
f^2(x) = 2C2 + 6C3(x-5) + 12C4(x-5)^2 f^2(5) = 2C2 C2 = 7/2
f^3(x) = 6C3 + 24C4(x-5) f^3(5) = 6C3 C3 = 13/6
f^4(x) = 24C4 f^4(5) = 24C4 C4 = -11/24

P4(x) = 1 + 3(x-5) + 7/2(x-5)^2 + 13/6(x-5)^3 - 11/24(x-5)^4

Note. For a function $f(x)$ whose derivatives all exist near a, suppose we have a degree n polynomial $P_n(x)$ such that $P_n(a) = f(a)$, $P_n'(a) = f'(a)$, $P_n''(a) = f''(a)$, $\cdots$ $P_n^{(n)}(a) = f^{(n)}(a)$.

If $P_n(x)$ is written in the form $c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots c_n(x-a)^n$, what are the coefficients $c_0, \cdots c_n$ in terms of f?

f^n(a)
f^n(a) =n!Cn

Cn = f^n(a) / n!

Definition. For the function $f(x)$ whose derivatives are all defined at $x = a$, the polynomial of the form

Pn(x) = C0 + C1(x-a) + C2(x-a)^2 + ... + Cn(x-a)^n
Pn(x) = f^0(a) /0! + f^1(a) /1! (x-a) + f^2(a) /2! (x-a)^2 + f^3(a) /3! (x-a)^3 + ... + f^n(a) /n! (x-a)^n

is called the n^{th} degree *Taylor polynomial* for f, centered at $x = a$.

In summation notation, the Taylor polynomial can be written as:

We use the conventions that:

- $f^{(0)}(a)$ means f(a)
- $0! =$ 1
- $(x-a)^0 =$ 1

C0 = f^0(a) / 0! (x-a)^0 = f(a) / 1 (1) = f(a)
C0 = f(a)

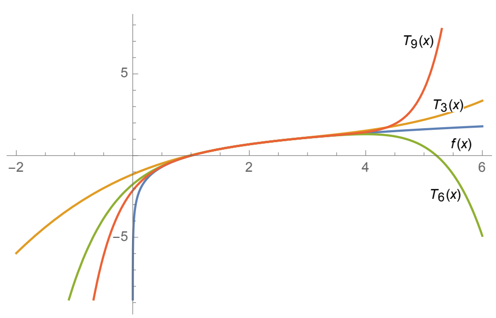
Example. For $f(x) = \ln(x)$,

(a) Find the 3rd degree Taylor polynomial centered at $a = 2$.

(b) Use it to approximate $\ln(2.1)$.

f(x) = ln(x)	f(2) = ln(2)	C0 = ln(2)
f'(x) = 1/x	f'(2) = 1/2	C1 = 1/2
f^2(x) = -1/x^2	f^2(2) = -1/4	C2 = -1/4/2! = -1/8
f^3(x) = 2/x^3	f^3(2) = 1/4	C2 = -1/4/3! = 1/24

f(x) = lnx ~~ ln2 + 1/2(x-2) - 1/8(x-2)^2 + 1/24(x-2)^3
ln(2.1) ~~ ln2 + 1/2(.1-2) - 1/8(.1-2)^2 + 1/24(.1-2)^3



Example. Find the 7th degree Taylor polynomials for $f(x) = \sin(x)$ and $g(x) = \cos(x)$, centered at $a = 0$.

P7(x) = C0 + C1x + C2x^2 + ... + C7x^7
Cn = f^n(a) / n! = f^n(0) / n!

f(x) = sinx	f(0) = 0	C0 = 0
f'(x) = cosx	f'(0) = 1	C1 = 1/1! = 1
f^2(x) = -sinx	f^2(0) = 0	C2 = 0
f^3(x) = -cosx	f^3(0) = -1	C3 = -1/3! = -1/6

f^4(x) = 0
f^5(x) = 1/5! = 1/120
f^6(x) = 0
f^7(x) = -1/7! = -1/5040

sinx ~~ x - x^3/6 + x^5/120 - x^7/5040

f(x) = cosx	f(0) = 1	C0 = 1
f'(x) = -sinx	f'(0) = 0	C1 = 1/1! = 0
f^2(x) = -cosx	f^2(0) = -1	C2 = -1/2
f^3(x) = sinx	f^3(0) = 0	C3 = 0

f^4(x) = 1/4! = 1/24
f^5(x) = 0
f^6(x) = -1/6! = -1/720
f^7(x) = 0

cosx ~~ 1 - x^2/2 + x^4/24 - x^6/720

Example. Find the 4th Taylor polynomial for $f(x) = e^x$ centered at $a = 0$. What is the error when using it to approximate $e^{0.15}$?

P4(x) = C0 + C1x + C2x^2 + C3x^3 + C4x^4   cn = f^n(a) / n! = f^n(0) / n!
f(x) = e^x

e^x ~~ 1 + x + x^2/2 + x^3/6 + x^4/24
e^.15 ~~ 1 + .15 + 1/2(.15)^2 + 1/6(.15)^3 + 1/24(.15)^4

Example. Use polynomials of order 1, 2, and 3 to approximate $\sqrt{8}$.

f(x) = sqrt(x)
a = 9

f(x) = x^1/2
f'(x) = x^(-1/2)/2 f'(9) = 1/6 C1 = 1/6
f^2(x) = -x^(-3/2)/4 f^2(9) = -1/108 C2 = -1/216
f^3(x) = 3(x^(-5/2))/8 f^3(9) = 3/648 C2 = 1/3888

sqrt(x) ~~ 3 + 1/6(x-9) - 1/216 (x-9)^2 +) + 1/3888 (x-9)^3
sqrt(8) ~~ 3 + 1/6(-1) - 1/216 (-1)^2 +) + 1/3888 (-1)^3

Definition. For a function $f(x)$ and its Taylor polynomial $P_n(x)$, the **remainder** is written

$R_n(x) =$ f(x) - Pn(x)

Theorem. *(Taylor's Inequality) If $|f^{(n+1)}(c)| \leq M$ for all c betwen a and x inclusive, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality*

$|R_n(x)| \leq$ |f(x) - Pn(x)| <= M |x-a|^n+1 / (n+1)!

f^n(a) / n!   -> f^n+1 (a) / (n+1)! (x-a)^n+1

Example. Approximate $f(x) = \cos(x)$ by a Maclaurin polynomial of degree 4. Estimate the accuracy of the approximation when x is in the interval $[0, \pi/2]$.

P4(x) = 1- x^2 / 2 + x^4 / 24

Rn^x <= M |x-a|^n+1 / (n+1)!

R4^x <= M |x|^5 / 5! <= 1 (pi/2)^5 / 5!

Plug in n+a

|x| [o , pi/2]
|x| <= pi/2

M
|f^n+1 (c)| <= M
|f^5(c)| <= M
|-sinc| = |sinc| <= 1 for all C in [0, pi/2]

Example. Approximate $f(x) = \cos(x)$ by a Maclaurin polynomial of degree 4 (again). For what values of x is the approximation accurate to within 3 decimal places?

P4(x) = 1 - x^2 / 2 + x^4 / 24

|Rn(x)| <= 0.001
|Rn(x)| <= M |x-a|^n+1 / (n+1)!
|R4(x)| <= M|x|^5 / 5! <= 1 |x|^5 / 5! <= 0.001

|x|^5 / 5! <= 0.001
|x|^5 <= 0.12
- fifth root(0.12) <= x <= fifth root(0.12)

Check out the approximation graphically.

Example. How many terms of the Maclaurin series for e^x should be used to estimate $e^{0.5}$ to within 0.0001?

Pn(x) = 1 + x + x^2/2 + x^3/6 + x^4/24 + ... + x^n/n!

|Rn(x)| <= 0.0001
|Rn(x)| <= M |x-a|^n+1 / (n+1)! = M |x|^n+1 / (n+1)!

<= |0.5|^n+1 / (n+1)!

n = 4
|R4(x)| <= e^0.5 (0.5)^5 / 5! <= 0.000043

n=5 |R5(x)| <= e^0.5(0.5)^6 / 6! <= 0.000036

Five terms

Plug in n+a

|x|
|x| <= 0.5

M
|f^n+1(c)| = |e^c| <= e^.5 for all c in [-0.5, 0.5]

Extra Example. Approximate $f(x) = e^{x/3}$ by a Taylor polynomial of degree 2 at $a = 0$. Estimate the accuracy of the approximation when x is in the interval $[-0.5, 0.5]$.

f(x) = 1 + x/3 + x^2 / 18

|Rn(x)| <= M |x-a|^n+1 / (n+1)!
|R2(x)| <= M |x|^3 / 3! <= 1/27 e^1/6(0.5)^3 / 3!

<= 0.00091154

Plug in a+n

|x| [-0.5, 0.5]
|x| <= 0.5

M |f^n+1(c)| <= M
|f^3(c)|
= |1/27 e^c/3|
<= 1/27 e^.5/3
= 1/27 e^1/6 for c M[-0.5, 0.5]