§11.3 - Taylor Series

After completing this section, students should be able to:

- Use the definition of Taylor series to find a Taylor series for a function and write it in summation notation.
- Determine the interval of convergence for a Taylor series.
- Build new Taylor series out of old by substituting a power of x, or multiplying by a power of x, differentiating, or integrating.
- Use the binomial series to approximate square roots and other roots.
- Prove that the MacLaurin series for e^x actually converges to e^x, and likewise for the Maclaurin series for $\sin(x)$ and $\cos(x)$ and closely related series like $\sin(2x)$.

Definition. Suppose a function $f(x)$ has derivatives $f^{(k)}(a)$ of all orders at the point a. The power series

Σ(n=0, infinity) (f^n(a)/n!) (x-a)^n

is called the Taylor series of $f(x)$ centered at a.

We use the conventions that:

- $f^{(0)}(a)$ means f(a)
- $0! =$ 1
- $(x-a)^0 =$ 1

Definition. The power series centered at a=0

Σ(n=0, infinity) (f^n(a)/n!) (x)^n

is called the Maclaurin series for $f(x)$.

Question. What is the difference between a Taylor series and a Maclaurin series?

center is a

centered at a=0

Question. What is the difference between a Taylor series and a Taylor polynomial?

Taylor polynomial

specifc n(7th degree polynomial)

Σ(i=0, infinity) (f^i(a)/i!) (x-a)^i = Pn(x)

Note: Taylor polynomials are partial sums of the Taylor series

Example. Find the Taylor series for $f(x) = \dfrac{1}{x}$ centered at $a = 5$.

f(x) 1/x   f(5) = 1/5   C0 = 1/5
f'(x) = -1/x^2   f'(5) = -1/5^2   C1 = -1/25
f''(x) = 2/x^3   f''(5) = 2/5^3   C2 = 2/125/2!
f^3(x) = -6/x^4   f^3(5) = -6/5^4   C3 = -6/5^4/3!
f^4(x) = 24/x^5   f^4(5) = 24/5^5   C3 = 24/5^5/4!

n=0
initial thoughts
Cn = (-1)^n * n!/5^n+1/n!

f(x) = 1/x ~~ Σ(n=0, infinity) (-1)^n

Example. Find the Maclaurin series for $f(x) = \sin(x)$ and $g(x) = \cos(x)$. Find the radius of convergence.

f(x) = sinx f(0) = 0 C0 = 0
f'(x) = cos(x) f'(0) = 1 C1=1/1!
f^2(x) = -sin(x) f"(0) = 0 C2 = 0/2!
f^3(x) = -cosx f^3(0) = -1 C3=-1/3!

sin(x) = Σ(n=0,infinity) (-1)^n x^2n+1 / (2n+1)n!
cos(x) = Σ(n=0,infinity) (-1)^n x^2n / (2n)n!

lim n->infinity |(-1)^n+1 x^2(n+1)+1 / (2(n+1)+1! * (2n+1)!/(-1)^n x^2n+1|
lim n->infinity |(-1) x^2n+3 (2n+1)! / (2n+3)! x^2n+1 |
lim n->infinity |x^2 / (2n+3)(2n+2)|
lim n->infinity 1/(2n+3)(2n+2) |x^2|

R=infinity
ROC = (-infinity, infinity)

Question. If a function has derivatives of all orders at $x = a$, then it is possible to write down the Taylor series for f centered at a. But how do we know that it actually converges to f?

Note. The Taylor series for f centered at a converges to f on an interval I if and only if ...

Goal: show lim n->infinity Rn=0
Recall: |Rn(x) <= |f^n+1(c) (x-a)^n+1 /(n+1)!| <= 1|x|^n+1 / (n+1)!
= |x|^n+1 / (n+1)!
= |x|^n |x|^1 / (n+1) n!

Limit: lim n->infinity |Rn(x)| <= lim n->infinity |x|^n |x| / n! (n+1) = 0
lim n->infinity = 0

Question. Does the power series of $\sin(x)$ actually converge to $\sin(x)$ on its radius of convergence?

Example. Find the Maclaurin series for $f(x) = e^x$. What is the radius of convergence?

f(x) = e^x   f(0) = e^0 = 1   C0 = 1
f'(x) = e^x   f'(0) = 1   C1 = 1/1!
f"(x) = e^x   f"(0) = 1   C2 = 1/2!

f(x) = e^x = Σ(n=0, infinity) 1/n! x^n
Σ(n=0, infinity) x^n/n!

lim n->infinity |x^n+1 / (n+1)! * n!/x^n|
lim n->infinity |x/n+1|
lim n->infinity 1/n+1 |x|
0 |x| = 0
0 < 1

R = infinity

Example. Use the Maclaurin series for $f(x) = e^x$ to find the Maclaurin series for $g(x) = x^3 e^{-x^2}$.

e^x = Σ(n=0, infinity) x^n/n!
e^-x^2 = Σ(n=0, infinity) (-x^2)^n / n!
Σ(n=0, infinity) (-1)^n x^2n / n!
g(x) = x^3 e^-x^2 = x^3 Σ(n=0, infinity) (-1)^n x^n / n! = Σ(

Example. Find the Taylor series for $f(x) = (1 + x)^{\pi}$ centered at $x = 0$.

f(x) = (1+x)^pi   f(0) = 1^pi = 1   C0 = 1
f'(x) = pi(1+x)^pi-1   f'(0) = pi(1)^pi-1 = pi   C1 = pi/1!
f''(x) = pi(pi-1)(+x)^pi-2   f''(0) = pi(pi-1)(1)^pi-2 = pi(pi-1)   C2 = pi(pi-1)/2!
f^3(x) = pi(pi-1)(pi-2)^pi-3   f^3(0) = pi(pi-1)(pi-2)^pi-3 = pi(pi-1)(pi-2)   C3=pi(pi-1)(pi-2)/3!

Definition. The expression $\dfrac{p(p-1)(p-2)\ldots(p-n+1)}{n!}$ is written as ___(PN)___, pronounced ___(P choose N)___, and is also called a ___combination___ .

Note. $\binom{p}{0}$

Example. Write the Taylor series for $f(x) = (1+x)^{\pi}$ using choose notation.

Σ(n=0, infinity) (piN)x^n

Definition. The **binomial series** is the Maclaurin series for $(1+x)^p$, where k is any real number. That is, the binomial series is the series:

$(1{+}x)^p =$ Σ(n=0, infinity) ()

This series converges when $|x|<1$.

but p affects the end points
P >= 0, [-1,1] both endpoints
-1<p<0 (-1,1] only pos 1
p <= 1 (-1,1) neither endpoints

Example. Find the Maclaurin series for $\frac{1}{\sqrt{1+2x^3}}$. (1+2x^3)^-1/2
~~ (1+x)^p

(1+x)^p = Σ(n=0, infinity) (n choose p) x^n
Σ(n=0, infinity) (n choose -1/2) (2x^3)^n
Σ(n=0, infinity) (n choose -1/2) (2)^n (x^3n)

Recall |x|<1, p=-1/2
(-1,1]

|2x^3|<1, p=-1/2
include to Right endpoint
|x^3|<1/2
|x|^3<1/2
|x|< cube root(1/2)
(-cube root(1/2), cube root(1/2)]

Example. Find a Maclaurin series for $f(x) = \dfrac{1}{1-x}$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^k + \cdots = \sum_{k=0}^{\infty} x^k, \quad \text{for } |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-1)^k x^k + \cdots = \sum_{k=0}^{\infty} (-1)^k x^k, \quad \text{for } |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \text{for } |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \quad \text{for } |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \quad \text{for } |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{k+1} x^k}{k} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}, \quad \text{for } -1 < x \le 1$$

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^k}{k} + \cdots = \sum_{k=1}^{\infty} \frac{x^k}{k}, \quad \text{for } -1 \le x < 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^k x^{2k+1}}{2k+1} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}, \quad \text{for } |x| \le 1$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}, \quad \text{for } |x| < \infty$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}, \quad \text{for } |x| < \infty$$

$$(1+x)^p = \sum_{k=0}^{\infty} \binom{p}{k} x^k, \text{ for } |x| < 1 \text{ and } \binom{p}{k} = \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!}, \binom{p}{0} = 1$$

Question. Is it possible for a function to be represented by two different power series with the same center? That is, if $f(x) = \sum_{n=1}^{\infty} c_n(x-a)^n = \sum_{n=1}^{\infty} d_n(x-a)^n$, does it necessarily follow that $c_n = d_n$ for all n?

There is only 1 power series representation for a given function about a given center

May be several ways to find power series, but they simplify to the same power series in the end

Extra Example. If $P(x) = \sum_{n=0}^{\infty} \frac{5}{n!}(x-2)^n = 5 + \frac{5}{1!}(x-2) + \frac{5}{2!}(x-2)^2 + \cdots$, find $P'''(2)$.

A. 5

B. $\frac{5}{2!}$

C. $\frac{5}{3!}$

D. $\frac{5 \cdot 2^3}{3!}$

E. None of these.

P^1(x) = Σ(n=1,infinity) 5/n! * n (x-2)^n-1
P^2(x) = Σ(n=2,infinity) 5/n! * n *(n-1) * (x-2)^n-2
P^3(x) = Σ(n=3,infinity) 5/n! * n *(n-1) * (n-2) * (x-2)^n-3

P^3(2) = Σ(n=3,infinity) 5/n! * n *(n-1) * (2-2)^n-3
= 5/n! * 3 * 2 * 0^2-2
= 5 * 0^0 0^0 simplifies to 1
=5

Extra Example. Find a power series $P(x)$ such that $P^{(n)}(5) = n$ for all $n \geq 0$.

A. $\sum_{n=1}^{\infty} n(x-5)^n$

B. $\sum_{n=1}^{\infty} \frac{(x-5)^n}{(n-1)!}$

C. $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n!}$

D. None of these