GAMES 105
Fundamentals of Character Animation

Lecture 02:

Math Background

Libin Liu

School of Intelligence Science and Technology
Peking University

GAMES105 课程交流

VCL @ PKU

Outline

- Review of Linear Algebra
  - Vector and Matrix
  - Translation, Rotation, and Transformation

- Representations of 3D rotation
  - [回] Rotation matrices
  - [囘] Euler angles
  - [囬] Rotation vectors/Axis angles
  - [𡇈] Quaternions

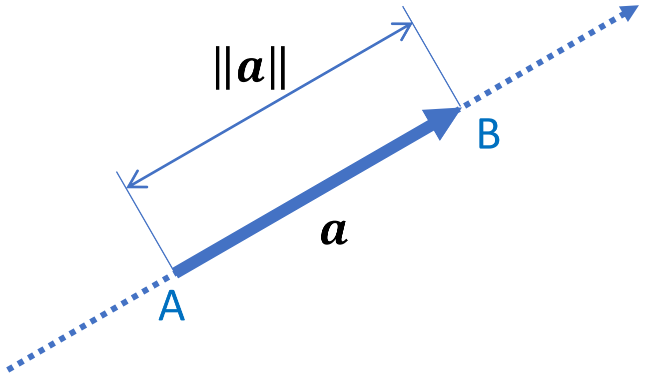
Review of Linear Algebra

Vectors and Matrices

* a few slides were modified from GAMES-101 and GAMES-103

Vector

- A quantity having both magnitude and direction



vector $\boldsymbol{a}$, written in **bold** letter

magnitude/length/norm: $\|\boldsymbol{a}\|$

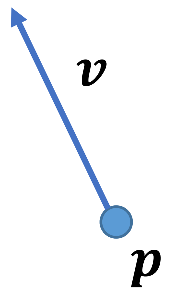
direction: $\dfrac{\boldsymbol{a}}{\|\boldsymbol{a}\|}$

$\|\boldsymbol{a}\| = 1$ → $\boldsymbol{a}$ is a unit vector

$\dfrac{\boldsymbol{a}}{\|\boldsymbol{a}\|}$ → normalize $\boldsymbol{a}$

Vector

- A quantity having both magnitude and direction
- Representing a location/velocity/abstract feature……

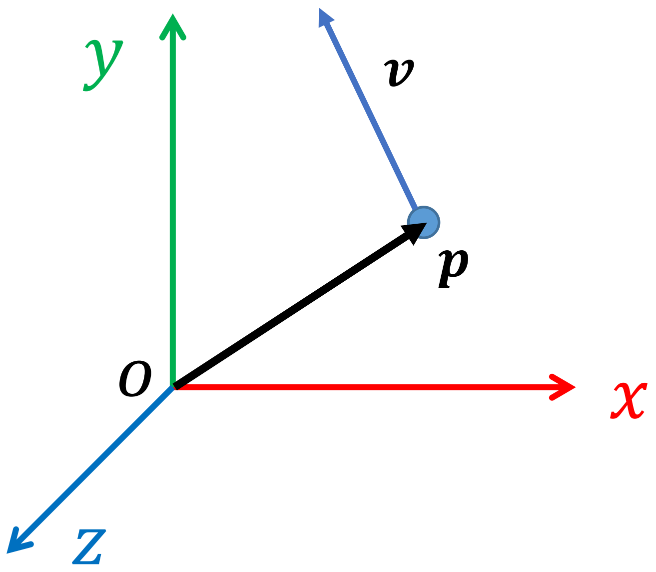


$\boldsymbol{p}$ position

$\boldsymbol{v}$ velocity

Vector

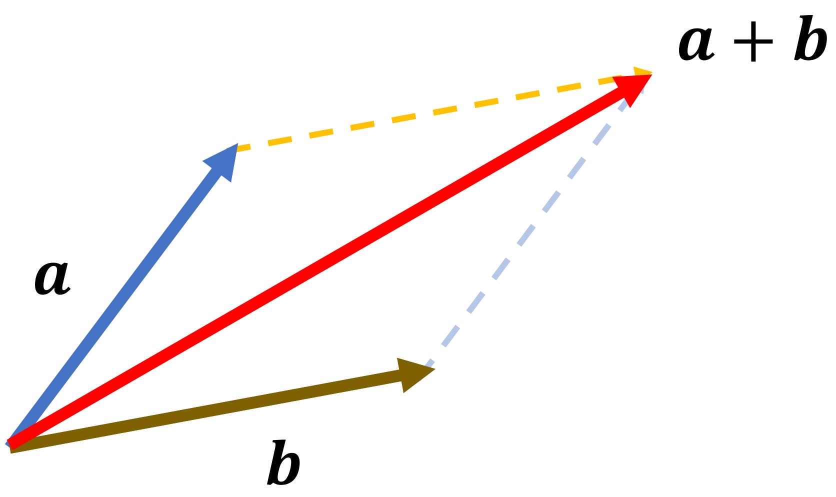
- A quantity having both magnitude and direction
- Representing a location/velocity/abstract feature……



$\boldsymbol{p}$ position

$\boldsymbol{v}$ velocity

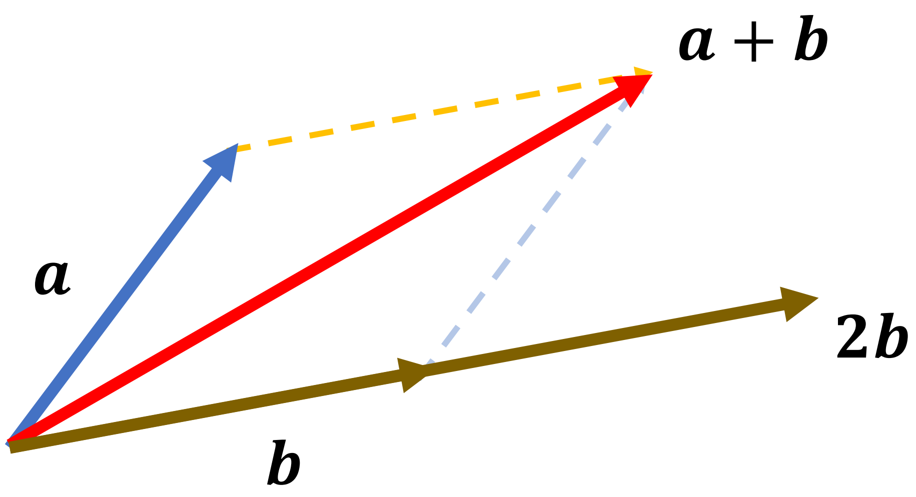
Vector Arithmetic



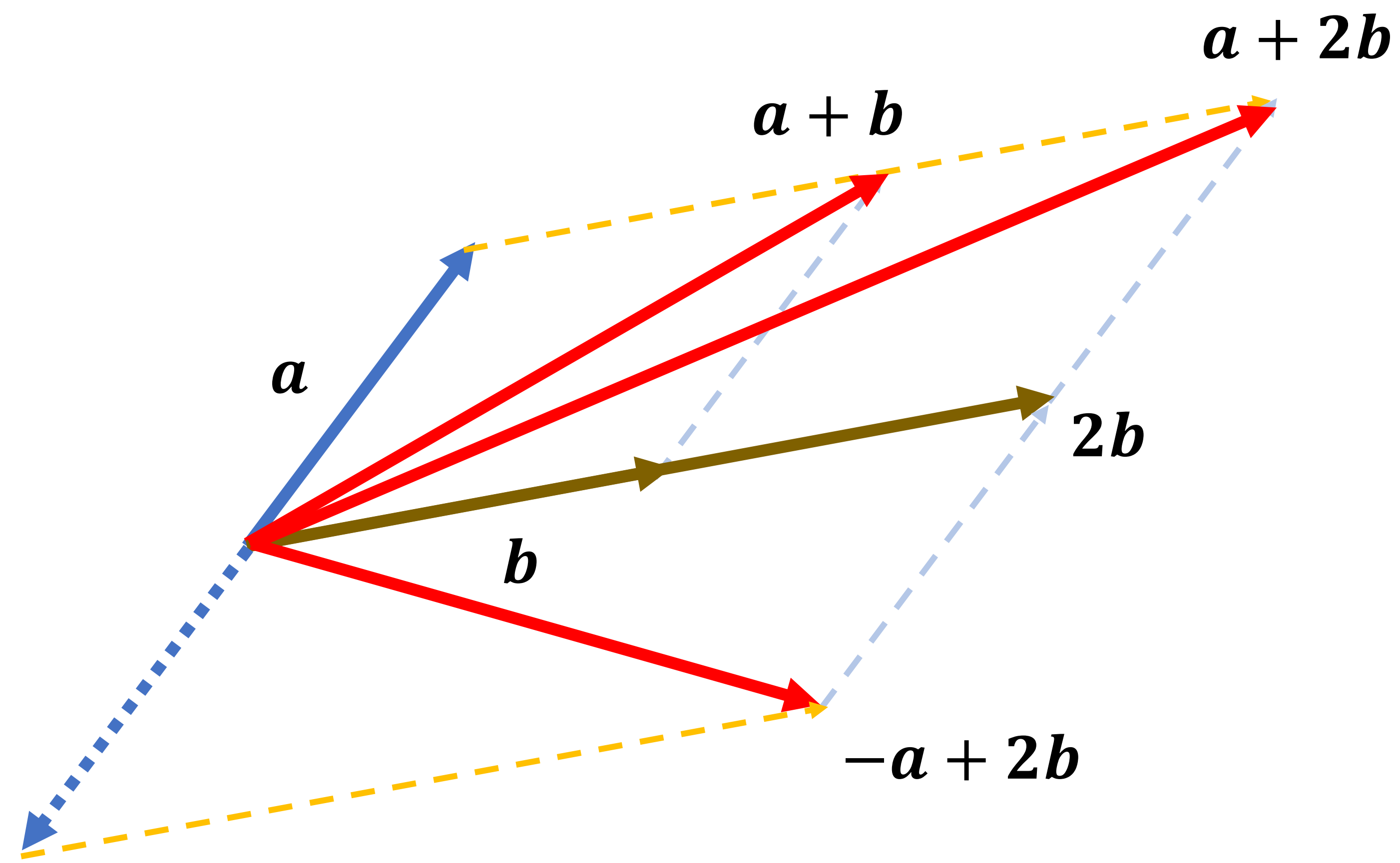
$$\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{b}+\boldsymbol{a}$$

*commutative

Vector Arithmetic

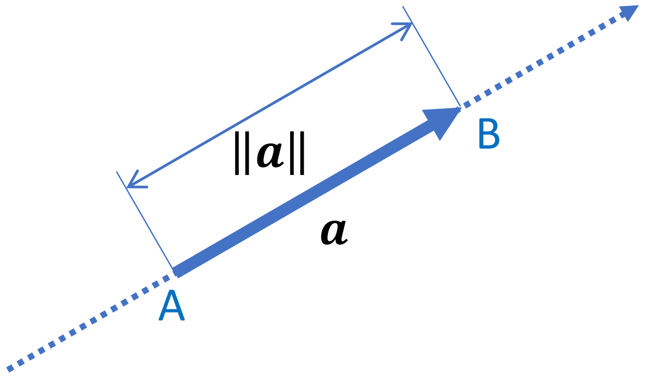


Vector Arithmetic



Vector Representation

- A vector can be represented as a [column] of numbers



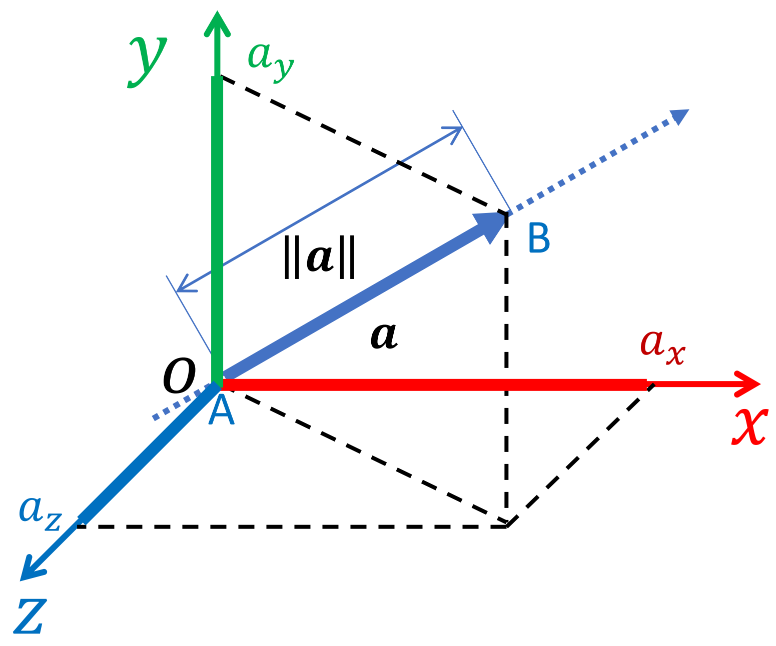
$$\boldsymbol{a} = (a_1, a_2, \dots, a_n)^T = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

magnitude/length/norm:

$$\|\boldsymbol{a}\|_2 = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

Vector Representation

- 3D vector in Cartesian coordinates



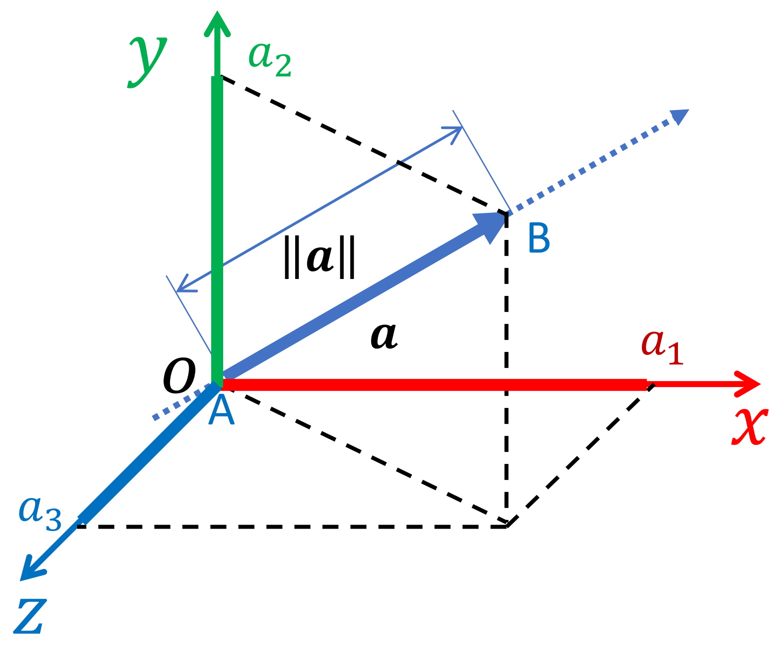
$$\boldsymbol{a} = (a_x, a_y, a_z)^T = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

magnitude/length/norm:

$$\|\boldsymbol{a}\|_2 = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

Vector Representation

- 3D vector in Cartesian coordinates



$$\boldsymbol{a} = (a_1, a_2, a_3)^T = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

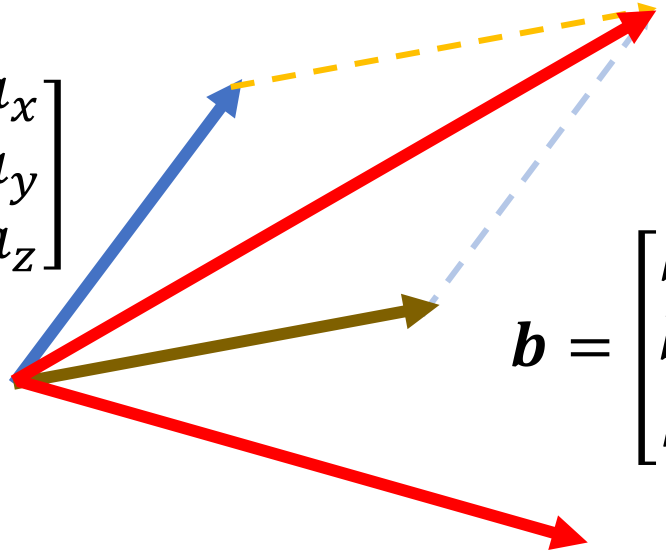
magnitude/length/norm:

$$\|\boldsymbol{a}\|_2 = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Vector Arithmetic

$$\boldsymbol{a} + \boldsymbol{b} = \begin{bmatrix} a_x + b_x \\ a_y + b_y \\ a_z + b_z \end{bmatrix}$$

$$\boldsymbol{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$



$$\boldsymbol{b} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

$$-\boldsymbol{a} + \boldsymbol{2b} = \begin{bmatrix} -a_x + 2b_x \\ -a_y + 2b_y \\ -a_z + 2b_z \end{bmatrix}$$

Dot Product

- Inner product/Scalar product

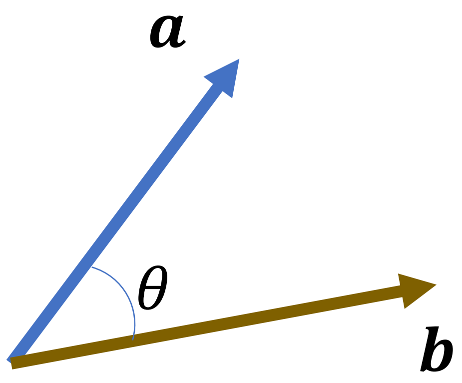
$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

- $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$
- $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$
- $\boldsymbol{a} \cdot \boldsymbol{a} = a_1 a_1 + a_2 a_2 + \cdots + a_n a_n = \|\boldsymbol{a}\|_2^2$

Geometric Meaning of Dot Product

- In Euclidean space,

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

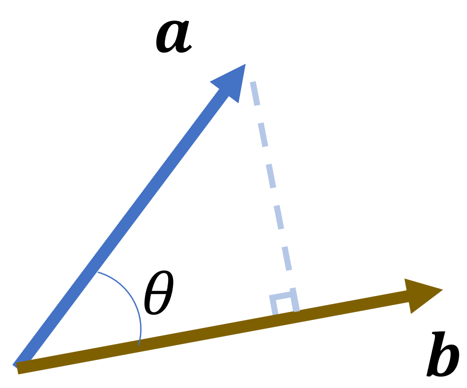


$$\boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\| \|\boldsymbol{b}\| \cos\theta$$

Geometric Meaning of Dot Product

- In Euclidean space,

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$



$$\boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\| \|\boldsymbol{b}\| \cos\theta$$

$$\theta = \arccos \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\|\boldsymbol{a}\| \|\boldsymbol{b}\|}$$

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{0}$$

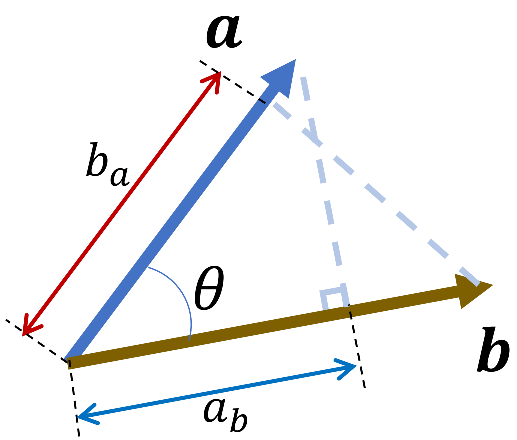
$$\Leftrightarrow \cos\theta = 0 \Leftrightarrow \boldsymbol{\theta} = \boldsymbol{90}^\circ$$

$$\Leftrightarrow \boldsymbol{a} \perp \boldsymbol{b}$$

Geometric Meaning of Dot Product

- In Euclidean space,

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$



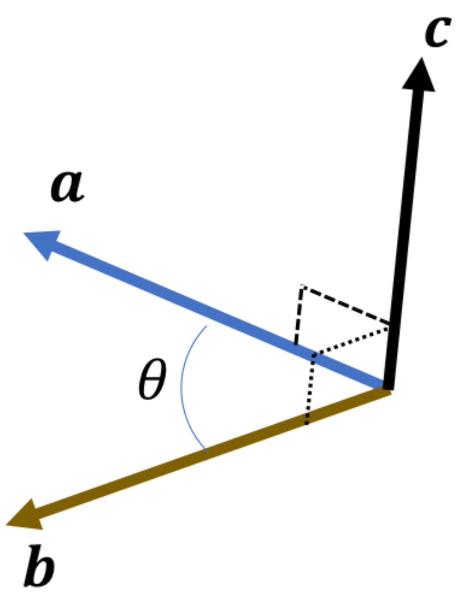
$$a_b = \|\boldsymbol{a}\| \cos\theta = \boldsymbol{a} \cdot \frac{\boldsymbol{b}}{\|\boldsymbol{b}\|}$$

$$b_a = \|\boldsymbol{b}\| \cos\theta = \boldsymbol{b} \cdot \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|}$$

$$\boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\| \|\boldsymbol{b}\| \cos\theta$$

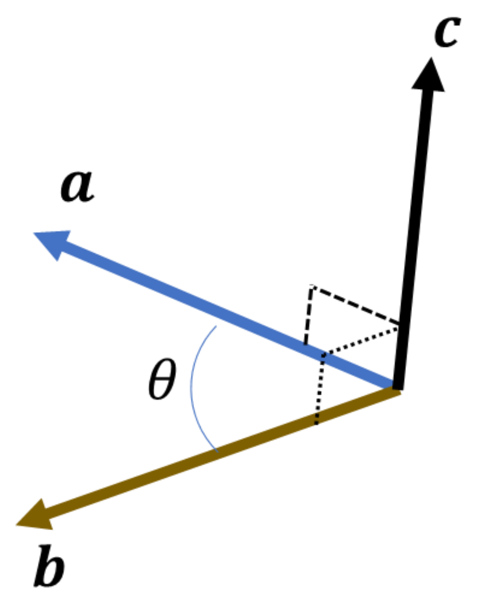
Cross Product of 3D Vectors

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix}$$



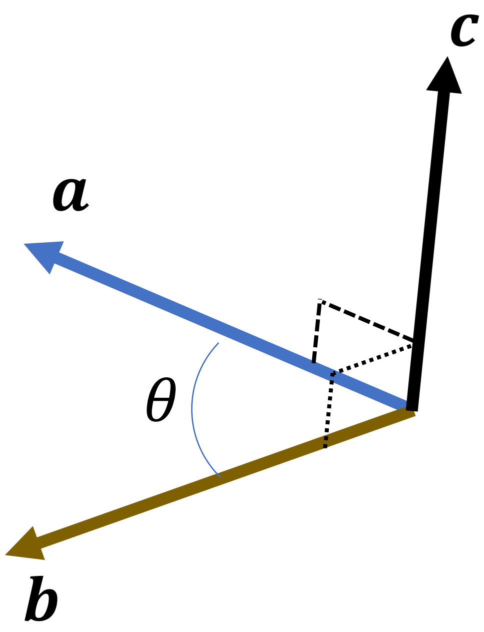
Cross Product of 3D Vectors

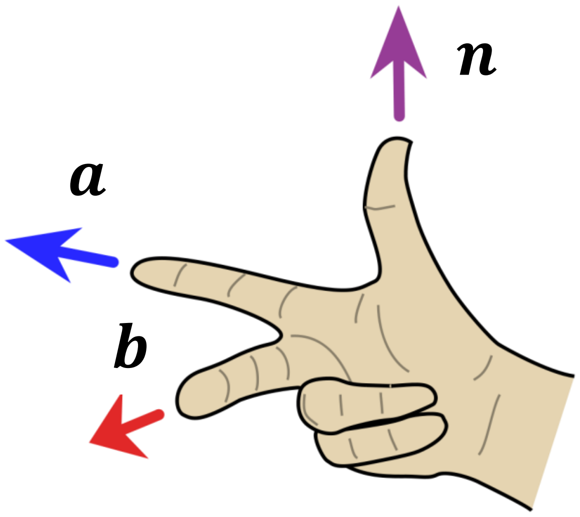
$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix} \quad \begin{array}{l} [x]: yz \\ [y]: zx \\ [z]: xy \end{array}$$



Cross Product of 3D Vectors

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix} \Rightarrow \boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = \|\boldsymbol{a}\|\|\boldsymbol{b}\| \sin(\theta)\, \boldsymbol{n}$$

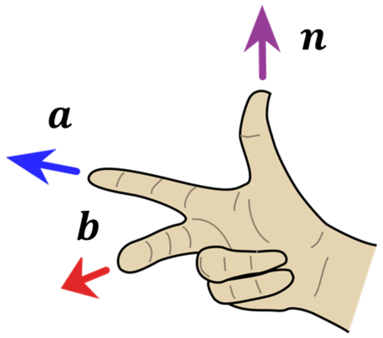




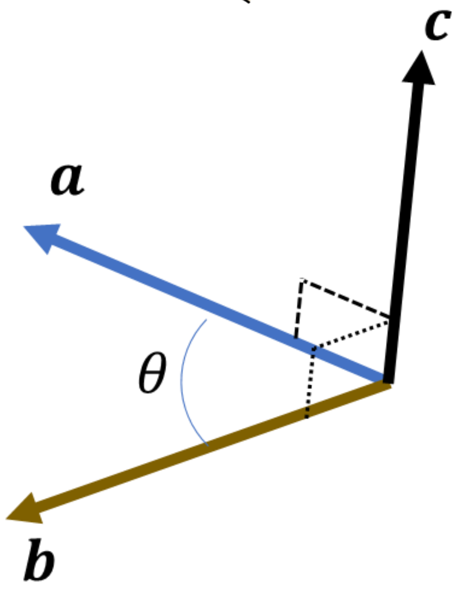
Cross Product of 3D Vectors

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix}$$

$[x]: yz$
$[y]: zx$
$[z]: xy$

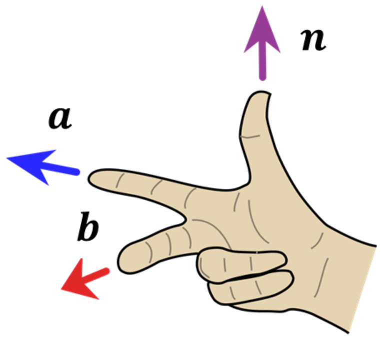


- $\boldsymbol{c} \cdot \boldsymbol{a} = \boldsymbol{c} \cdot \boldsymbol{b} = \boldsymbol{0}$
  - $\boldsymbol{c} \perp \boldsymbol{a},\ \boldsymbol{c} \perp \boldsymbol{b}$
- $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$
- $\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{d}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{d}$
- $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) \neq (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$



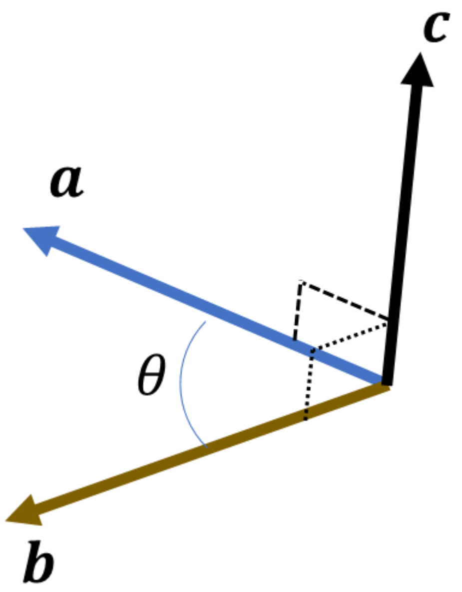
Cross Product of 3D Vectors

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix} \qquad \begin{matrix} [x]: yz \\ [y]: zx \\ [z]: xy \end{matrix}$$



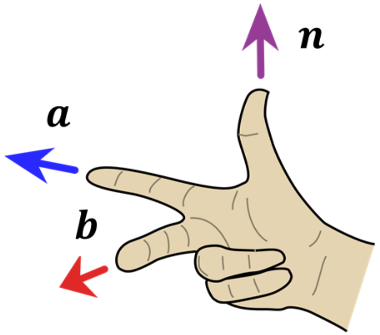
- Find a direction $\boldsymbol{n}$ perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$

$$\boldsymbol{n} = \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} \times \frac{\boldsymbol{b}}{\|\boldsymbol{b}\|}$$



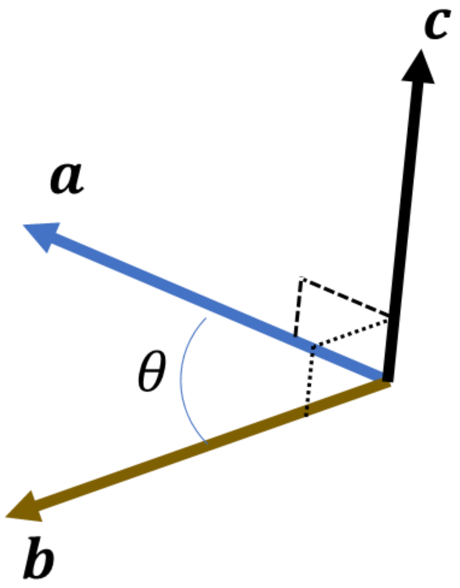
Cross Product of 3D Vectors

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix} \quad \begin{matrix} [x]: yz \\ [y]: zx \\ [z]: xy \end{matrix}$$



- Find a direction $\boldsymbol{n}$ perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$

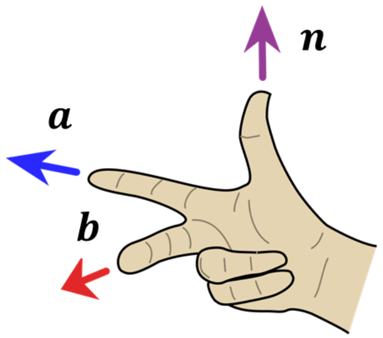
$$\boldsymbol{n} = \cancel{\frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} \times \frac{\boldsymbol{b}}{\|\boldsymbol{b}\|}}$$



Cross Product of 3D Vectors

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix}$$

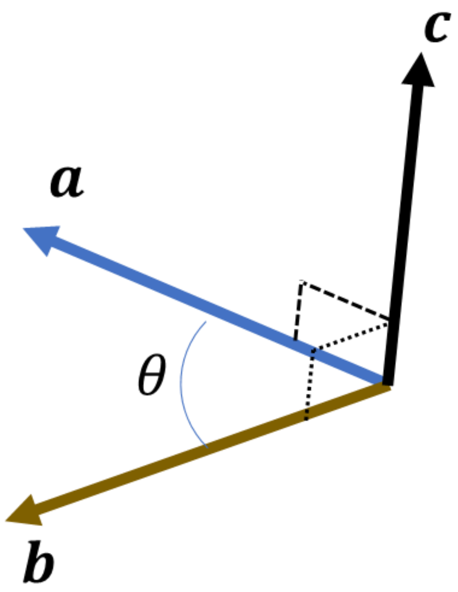
$[x]: yz$
$[y]: zx$
$[z]: xy$



- Find a direction $\boldsymbol{n}$ perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$

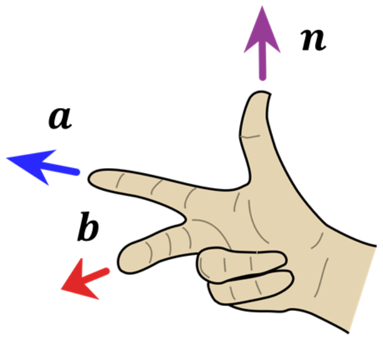
$$\boldsymbol{n} = \frac{\boldsymbol{a} \times \boldsymbol{b}}{\|\boldsymbol{a} \times \boldsymbol{b}\|}$$

$$\boxed{\begin{gathered} \boldsymbol{a} \neq 0, \boldsymbol{b} \neq 0 \\ \boldsymbol{a} \nparallel \boldsymbol{b} \end{gathered}}$$



Cross Product of 3D Vectors

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix} \quad \begin{matrix} [x]: yz \\ [y]: zx \\ [z]: xy \end{matrix}$$



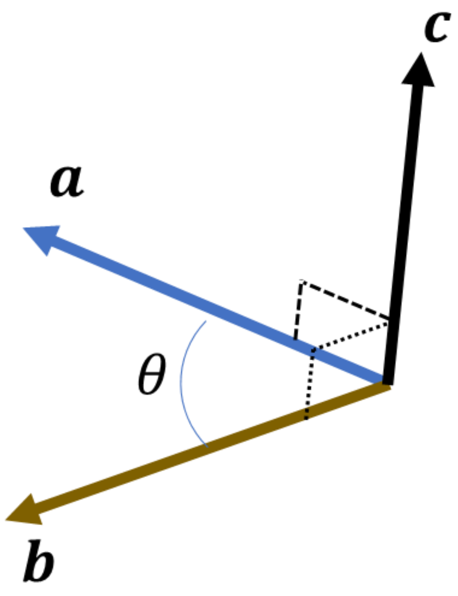
- Find a direction $\boldsymbol{n}$ perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$

$$\boldsymbol{n} = \frac{\boldsymbol{a} \times \boldsymbol{b}}{\|\boldsymbol{a} \times \boldsymbol{b}\|}$$

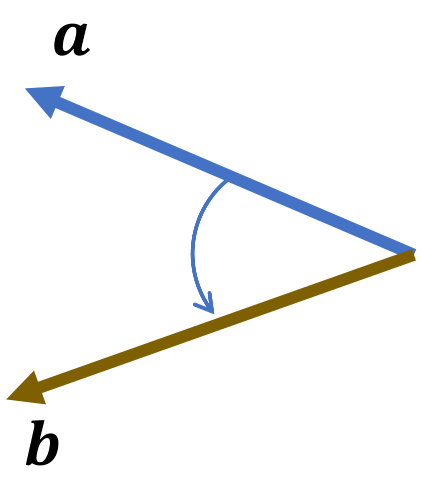
- Check if $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel

$$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{0} \ ?$$

$$\boxed{\boldsymbol{a} \neq \boldsymbol{0}, \boldsymbol{b} \neq \boldsymbol{0}}$$

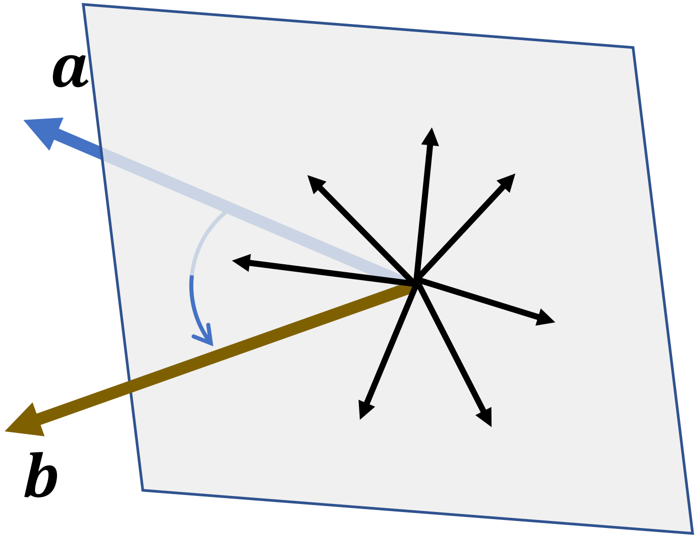


How to find the rotation between vectors?



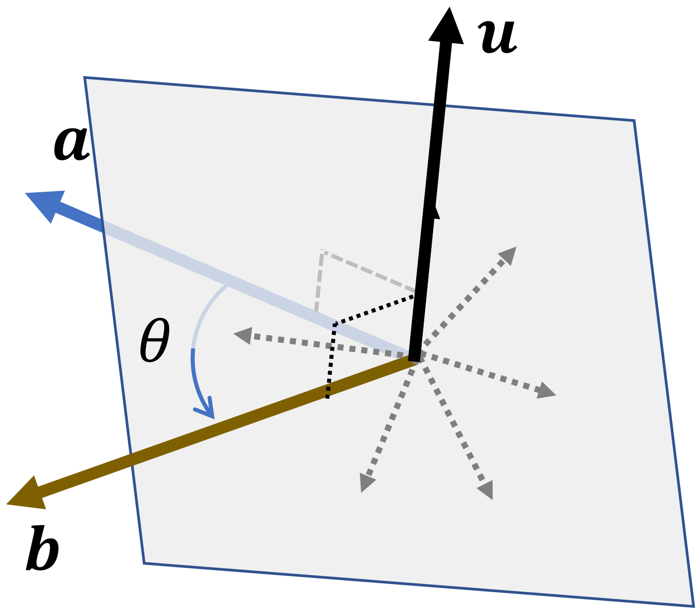
How to find the rotation between vectors?

Any vector in the bisecting plane can be the axis

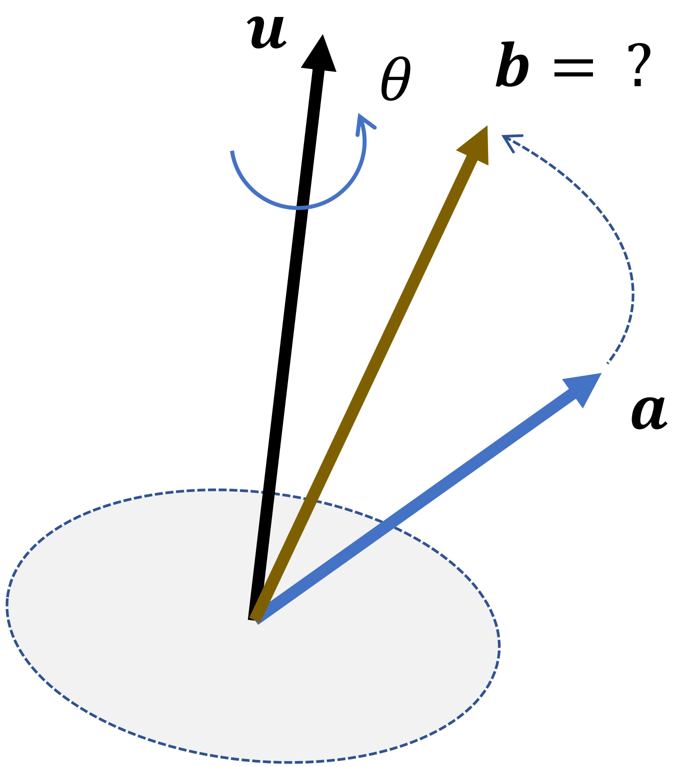


How to find the rotation between vectors?

The minimum rotation: $\boldsymbol{u} = \dfrac{\boldsymbol{a} \times \boldsymbol{b}}{\|\boldsymbol{a} \times \boldsymbol{b}\|}$ $\theta = \arg\cos \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\|\boldsymbol{a}\| \|\boldsymbol{b}\|}$

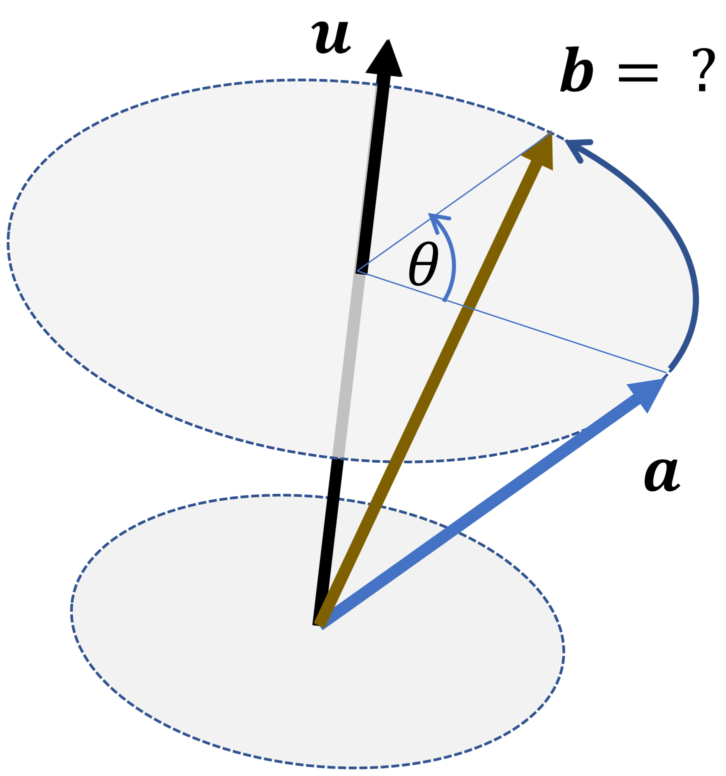


How to rotate a vectors?



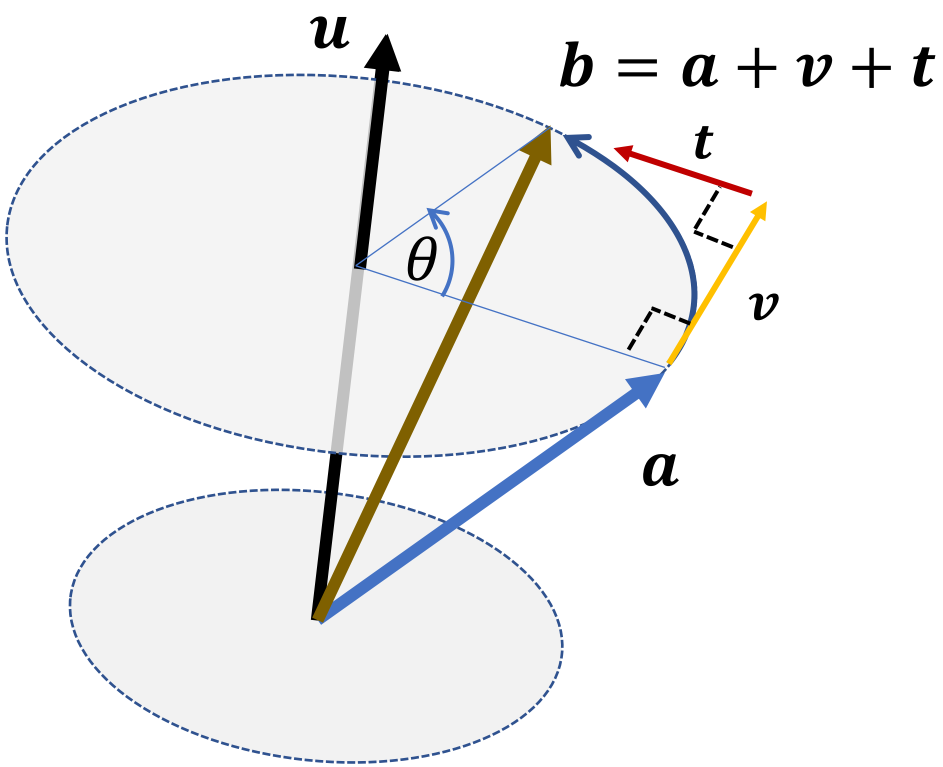
$$\|\boldsymbol{u}\| = \mathbf{1}$$

How to rotate a vectors?



$$\|\boldsymbol{u}\| = \mathbf{1}$$

How to rotate a vectors?

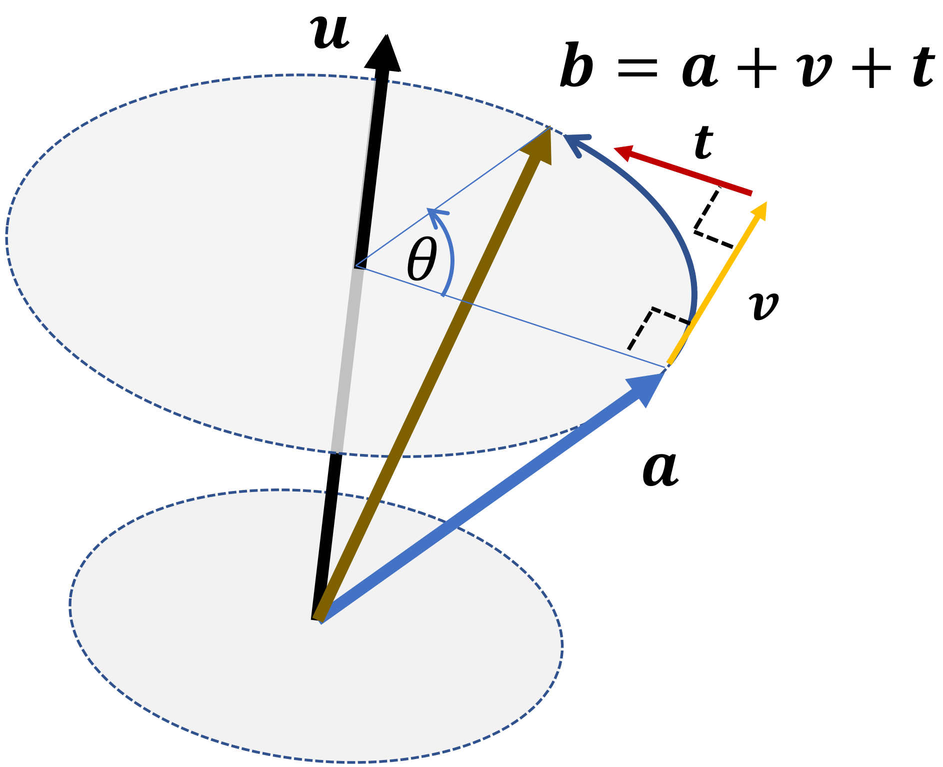


$\|\boldsymbol{u}\| = \mathbf{1}$

$$\boldsymbol{v} \leftarrow \boldsymbol{u} \times \boldsymbol{a}$$

$$\boldsymbol{t} \leftarrow \boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{u} \times (\boldsymbol{u} \times \boldsymbol{a})$$

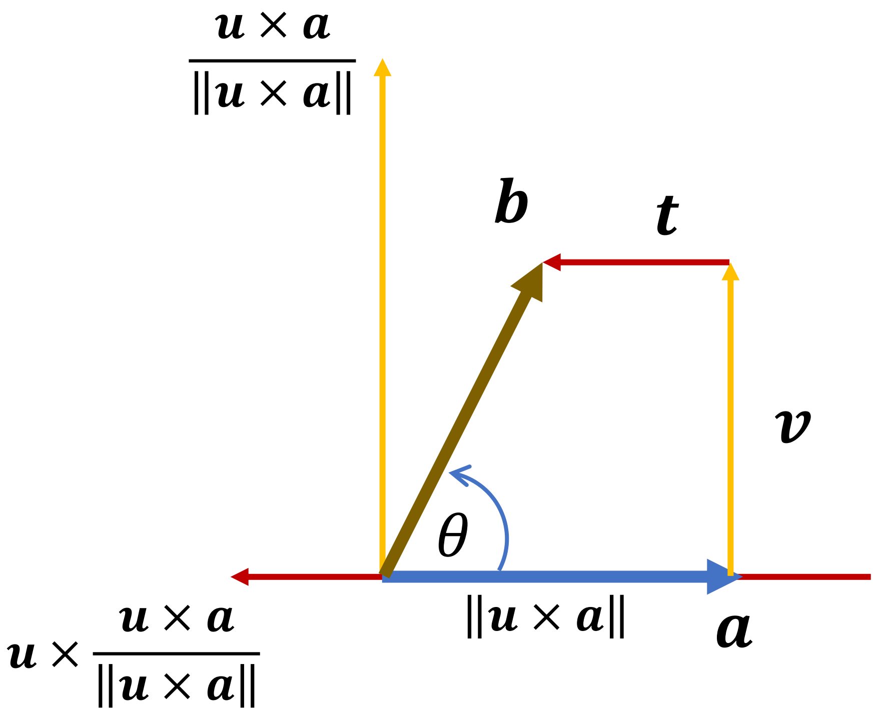
How to rotate a vectors?



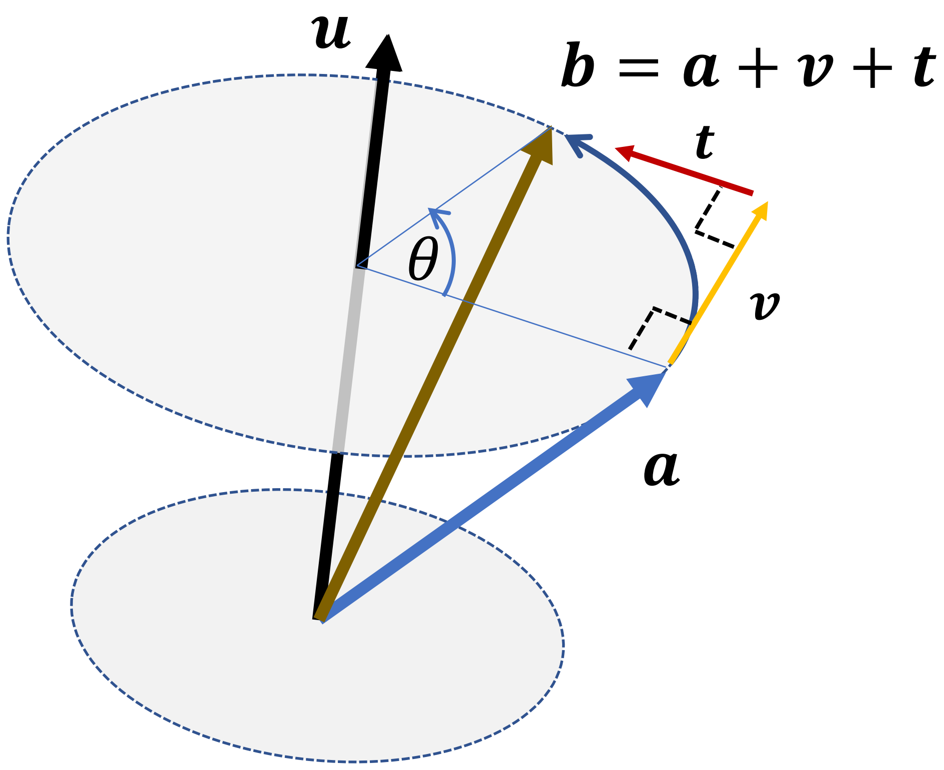
$$\|\boldsymbol{u}\| = \mathbf{1}$$

$$\boldsymbol{v} \leftarrow \boldsymbol{u} \times \boldsymbol{a}$$

$$\boldsymbol{t} \leftarrow \boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{u} \times (\boldsymbol{u} \times \boldsymbol{a})$$



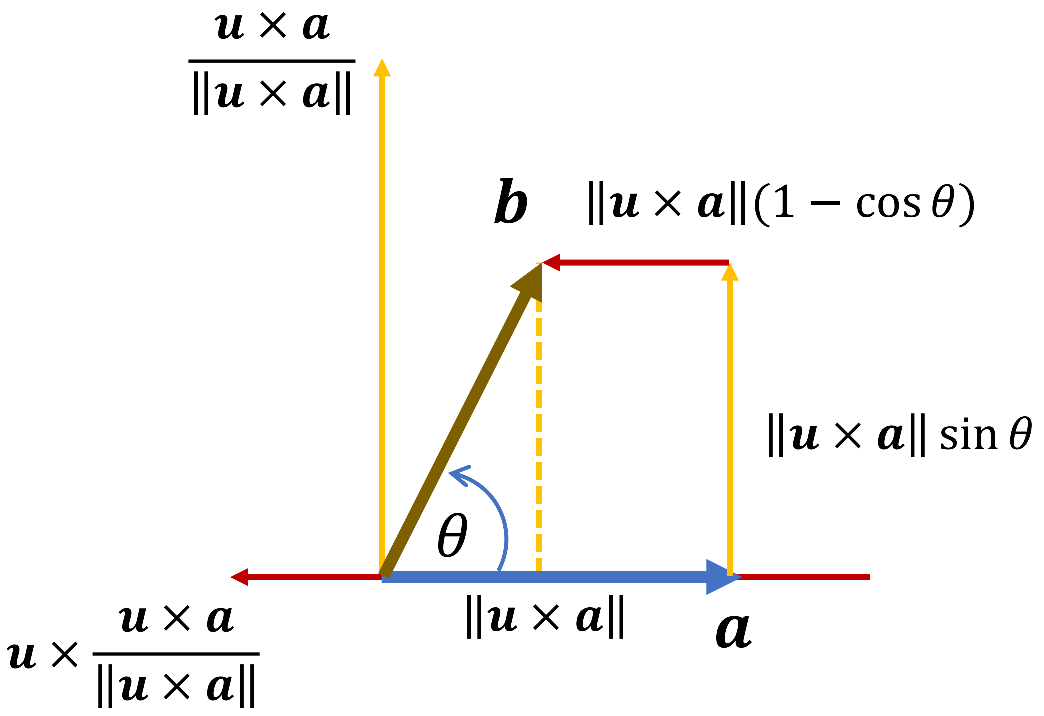
How to rotate a vectors?



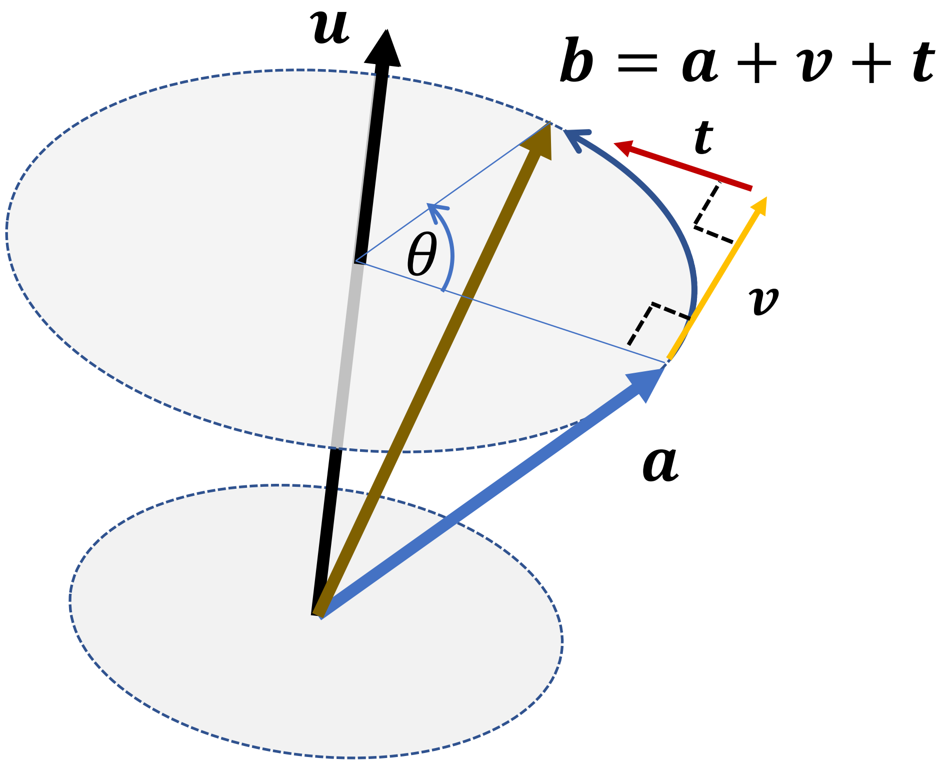
$\|\boldsymbol{u}\| = \boldsymbol{1}$

$$\boldsymbol{v} \leftarrow \boldsymbol{u} \times \boldsymbol{a}$$

$$\boldsymbol{t} \leftarrow \boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{u} \times (\boldsymbol{u} \times \boldsymbol{a})$$



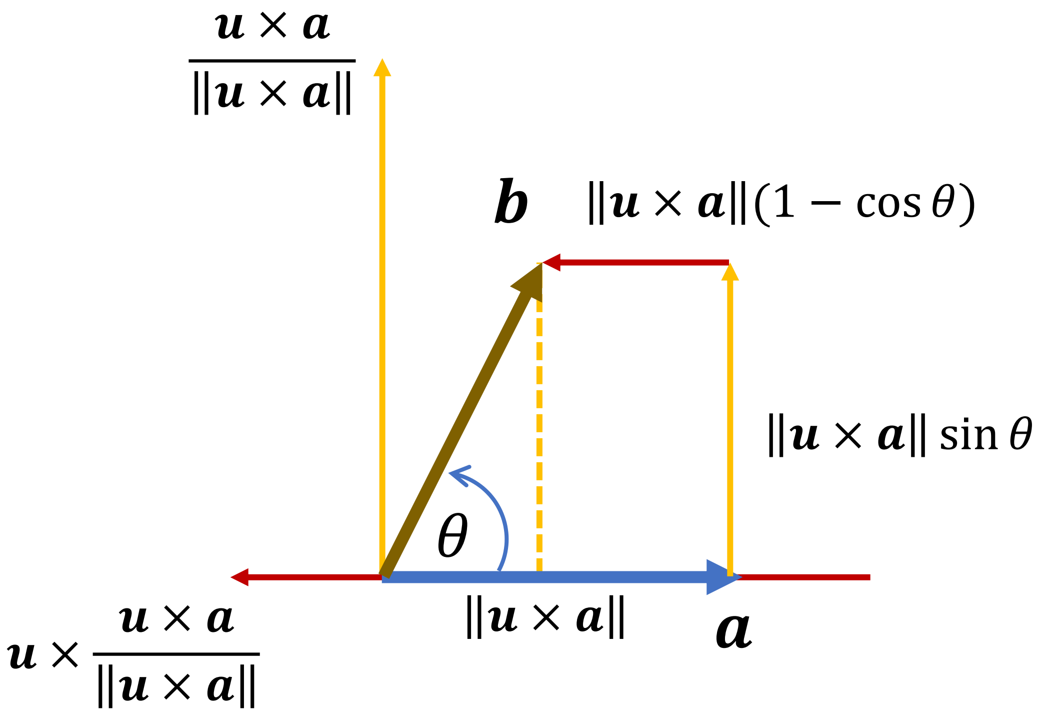
How to rotate a vectors?



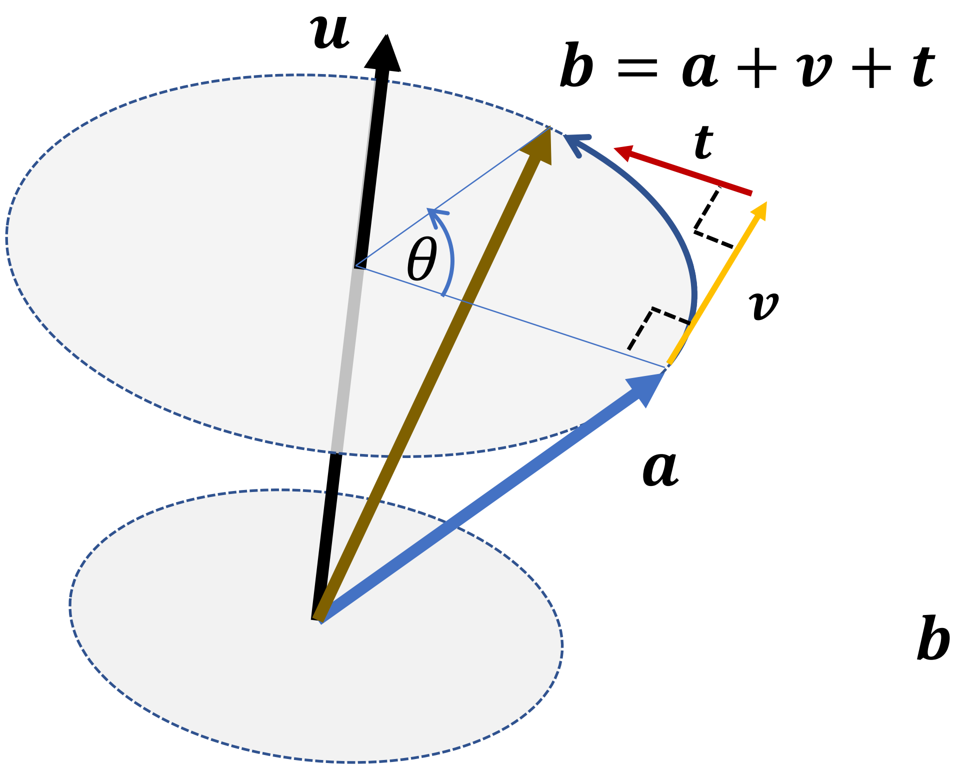
$\|\boldsymbol{u}\| = \mathbf{1}$

$$\boldsymbol{v} = (\sin\theta)\, \boldsymbol{u} \times \boldsymbol{a}$$

$$\boldsymbol{t} = (1 - \cos\theta)\, \boldsymbol{u} \times (\boldsymbol{u} \times \boldsymbol{a})$$



How to rotate a vectors?



$$\|\boldsymbol{u}\| = \mathbf{1}$$

$$\boldsymbol{v} = (\sin\theta)\,\boldsymbol{u} \times \boldsymbol{a}$$

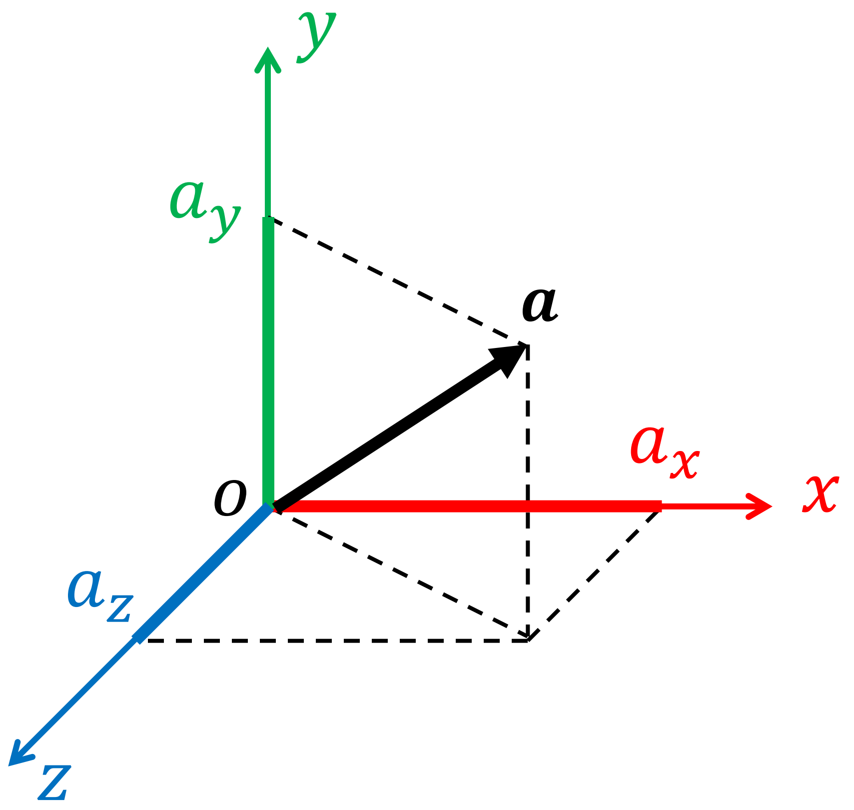
$$\boldsymbol{t} = (1 - \cos\theta)\,\boldsymbol{u} \times (\boldsymbol{u} \times \boldsymbol{a})$$

Rodrigues' rotation formula

$$\boldsymbol{b} = \boldsymbol{a} + (\sin\theta)\,\boldsymbol{u} \times \boldsymbol{a} + (1 - \cos\theta)\,\boldsymbol{u} \times (\boldsymbol{u} \times \boldsymbol{a})$$

Orthogonal Basis & Orthogonal Coordinates

$$\boldsymbol{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \in \mathbb{R}^3$$

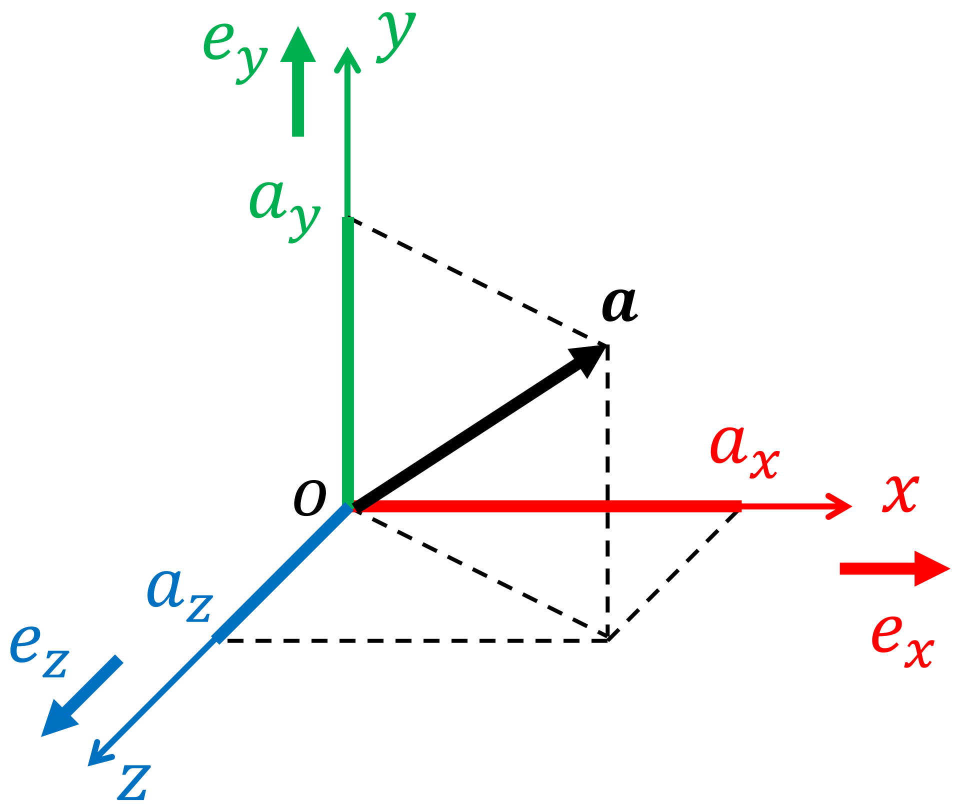


Orthogonal Basis & Orthogonal Coordinates

- $\|\boldsymbol{e_x}\| = \|\boldsymbol{e_y}\| = \|\boldsymbol{e_z}\| = \boldsymbol{1}$
- $\boldsymbol{e_x} \cdot \boldsymbol{e_y} = \boldsymbol{e_y} \cdot \boldsymbol{e_z} = \boldsymbol{e_z} \cdot \boldsymbol{e_x} = \boldsymbol{0}$
- $\boldsymbol{e_x} \times \boldsymbol{e_y} = \boldsymbol{e_z}$

  $\boldsymbol{e_y} \times \boldsymbol{e_z} = \boldsymbol{e_x}$

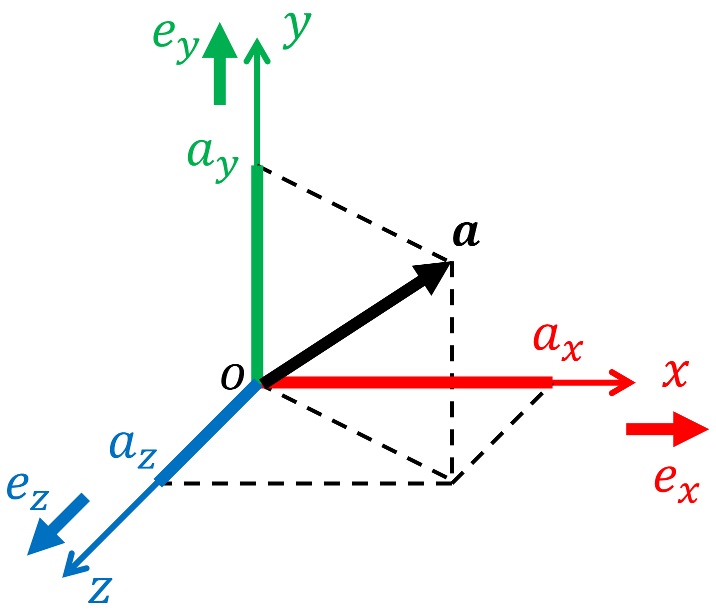
  $\boldsymbol{e_z} \times \boldsymbol{e_x} = \boldsymbol{e_y}$



Orthogonal Basis & Orthogonal Coordinates

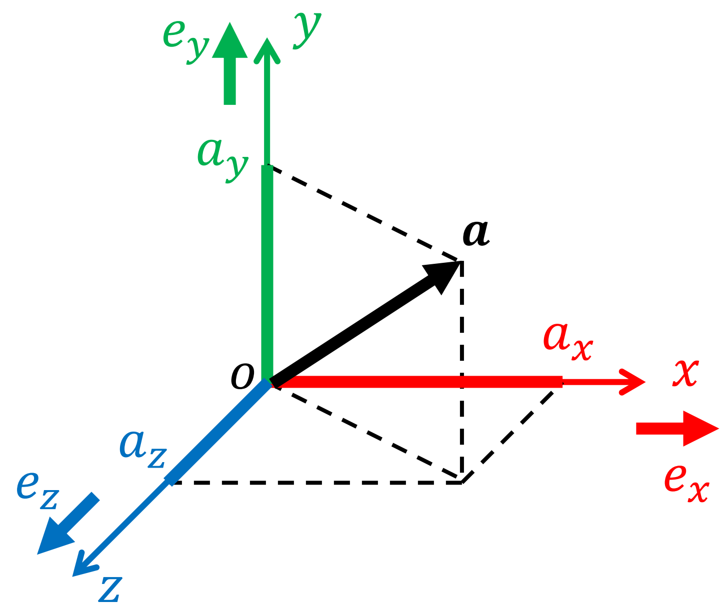
$$\boldsymbol{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \in \mathbb{R}^3$$

$$\boldsymbol{a} = a_x \boldsymbol{e_x} + a_y \boldsymbol{e_y} + a_z \boldsymbol{e_z}$$



Orthogonal Basis & Orthogonal Coordinates

$$\boldsymbol{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \in \mathbb{R}^3$$



$$\boldsymbol{a} = a_x \boldsymbol{e_x} + a_y \boldsymbol{e_y} + a_z \boldsymbol{e_z}$$

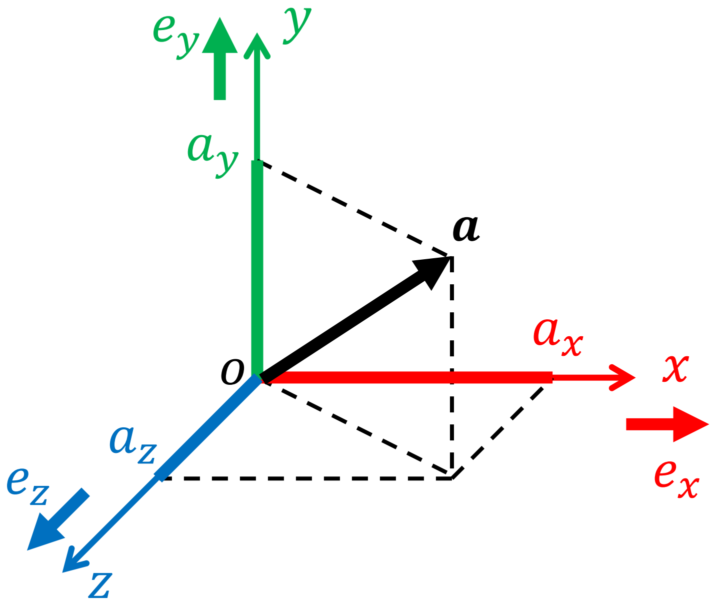
$$\boldsymbol{a} \cdot \boldsymbol{b} = (a_x \boldsymbol{e_x} + a_y \boldsymbol{e_y} + a_z \boldsymbol{e_z}) \cdot (b_x \boldsymbol{e_x} + b_y \boldsymbol{e_y} + b_z \boldsymbol{e_z})$$

$$= a_x b_x \boldsymbol{e_x} \cdot \boldsymbol{e_x} + a_y b_y \boldsymbol{e_y} \cdot \boldsymbol{e_y} + a_z b_z \boldsymbol{e_z} \cdot \boldsymbol{e_z}$$

$$+ \sum_{i \neq j} a_i b_j \boldsymbol{e_i} \cdot \boldsymbol{e_j}$$

Orthogonal Basis & Orthogonal Coordinates

$$\boldsymbol{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \in \mathbb{R}^3$$



$$\boldsymbol{a} = a_x \boldsymbol{e_x} + a_y \boldsymbol{e_y} + a_z \boldsymbol{e_z}$$

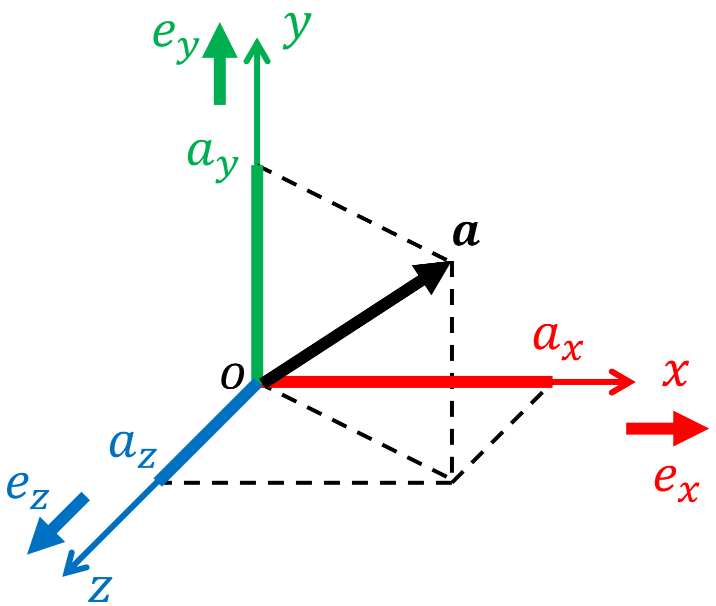
$$\boldsymbol{a} \cdot \boldsymbol{b} = (a_x \boldsymbol{e_x} + a_y \boldsymbol{e_y} + a_z \boldsymbol{e_z}) \cdot (b_x \boldsymbol{e_x} + b_y \boldsymbol{e_y} + b_z \boldsymbol{e_z})$$

$$= a_x b_x \boldsymbol{e_x} \cdot \boldsymbol{e_x} + a_y b_y \boldsymbol{e_y} \cdot \boldsymbol{e_y} + a_z b_z \boldsymbol{e_z} \cdot \boldsymbol{e_z}$$

$$\cancel{+ \sum_{i \neq j} a_i b_j \boldsymbol{e_i} \cdot \boldsymbol{e_j}}$$

Orthogonal Basis & Orthogonal Coordinates

$$\boldsymbol{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \in \mathbb{R}^3$$



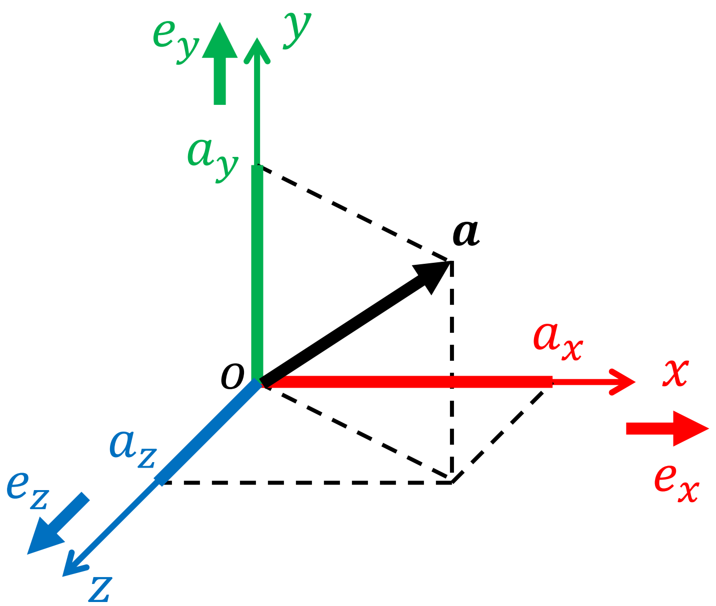
$$\boldsymbol{a} = a_x \boldsymbol{e_x} + a_y \boldsymbol{e_y} + a_z \boldsymbol{e_z}$$

$$\begin{aligned} \boldsymbol{a} \times \boldsymbol{b} &= \left(a_x \boldsymbol{e_x} + a_y \boldsymbol{e_y} + a_z \boldsymbol{e_z}\right) \times \left(b_x \boldsymbol{e_x} + b_y \boldsymbol{e_y} + b_z \boldsymbol{e_z}\right) \\ &= a_x b_x \boldsymbol{e_x} \times \boldsymbol{e_x} + a_x b_y \boldsymbol{e_x} \times \boldsymbol{e_y} + a_x b_z \boldsymbol{e_x} \times \boldsymbol{e_z} \\ &+ a_y b_x \boldsymbol{e_y} \times \boldsymbol{e_x} + a_y b_y \boldsymbol{e_y} \times \boldsymbol{e_y} + a_y b_z \boldsymbol{e_y} \times \boldsymbol{e_z} \\ &+ a_z b_x \boldsymbol{e_z} \times \boldsymbol{e_x} + a_z b_y \boldsymbol{e_z} \times \boldsymbol{e_y} + a_z b_z \boldsymbol{e_z} \times \boldsymbol{e_z} \end{aligned}$$

Orthogonal Basis & Orthogonal Coordinates

$$\boldsymbol{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \in \mathbb{R}^3$$

$$\boldsymbol{a} = a_x\boldsymbol{e_x} + a_y\boldsymbol{e_y} + a_z\boldsymbol{e_z}$$

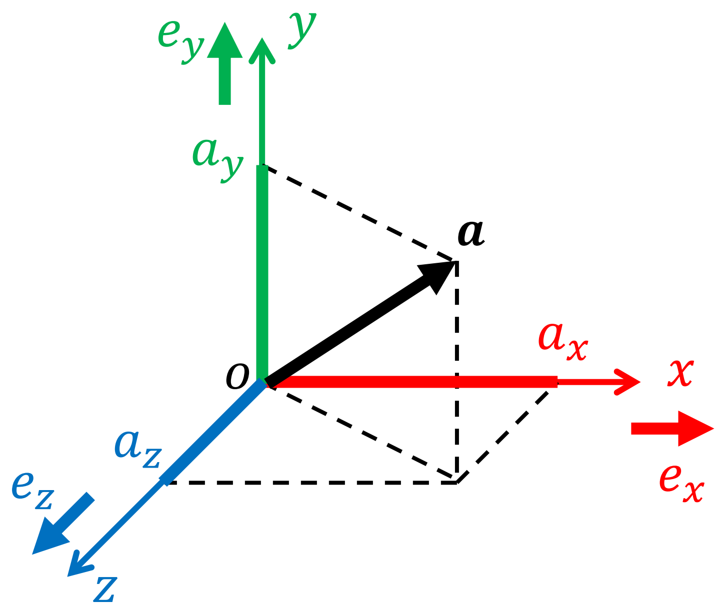


$$\begin{aligned}
\boldsymbol{a} \times \boldsymbol{b} = \cancel{a_x b_x \boldsymbol{e_x} \times \boldsymbol{e_x}} + a_x b_y \boldsymbol{e_x} \times \boldsymbol{e_y} + a_x b_z \boldsymbol{e_x} \times \boldsymbol{e_z} \\
+ a_y b_x \boldsymbol{e_y} \times \boldsymbol{e_x} + \cancel{a_y b_y \boldsymbol{e_y} \times \boldsymbol{e_y}} + a_y b_z \boldsymbol{e_y} \times \boldsymbol{e_z} \\
+ a_z b_x \boldsymbol{e_z} \times \boldsymbol{e_x} + a_z b_y \boldsymbol{e_z} \times \boldsymbol{e_y} + \cancel{a_z b_z \boldsymbol{e_z} \times \boldsymbol{e_z}}
\end{aligned}$$

Orthogonal Basis & Orthogonal Coordinates

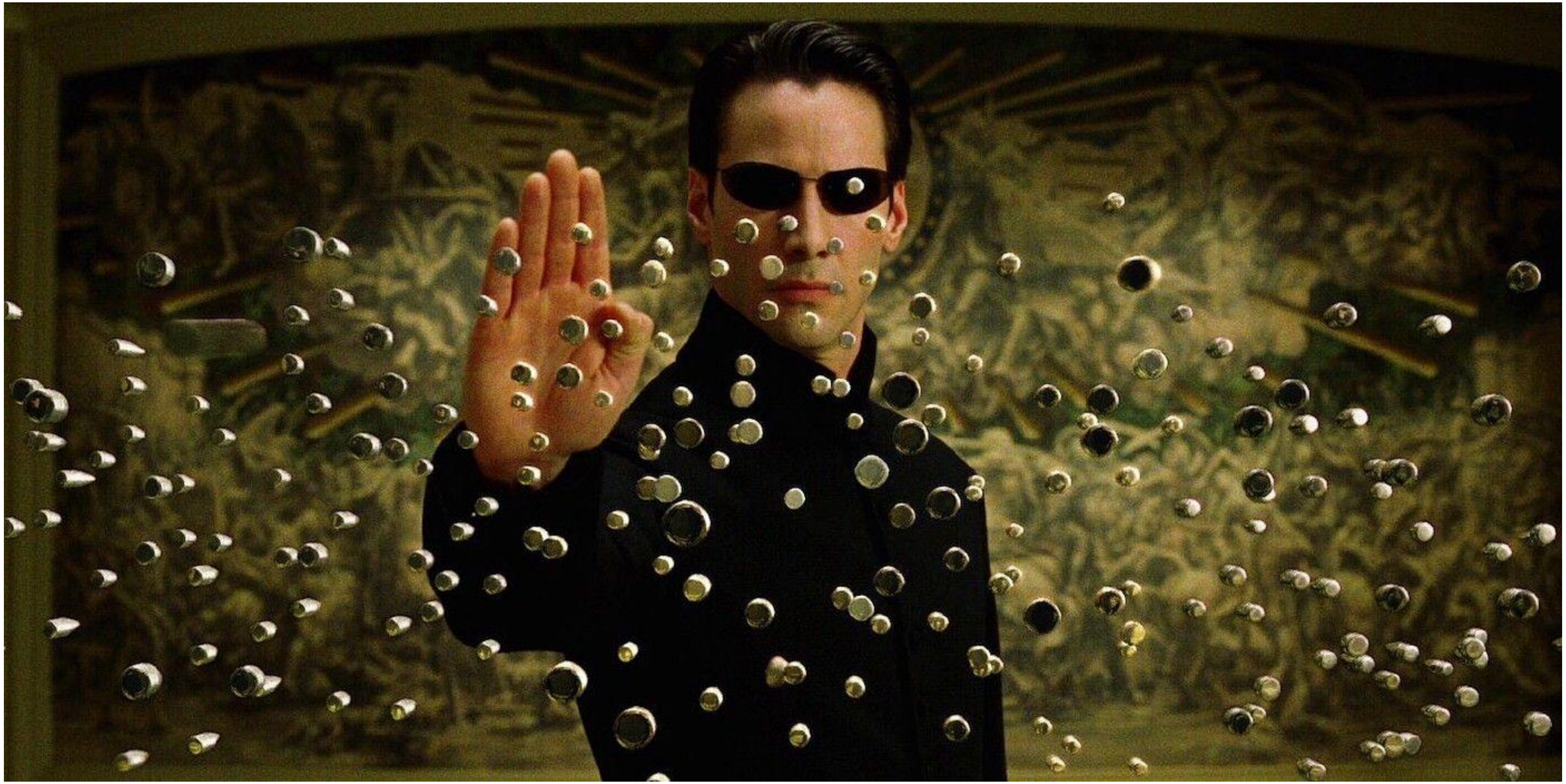
$$\boldsymbol{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \in \mathbb{R}^3$$

$$\boldsymbol{a} = a_x \boldsymbol{e_x} + a_y \boldsymbol{e_y} + a_z \boldsymbol{e_z}$$



$$\begin{aligned}\boldsymbol{a} \times \boldsymbol{b} = &\left(a_y b_z - a_z b_y\right)\boldsymbol{e_x} \\ &+ \left(a_z b_x - a_x b_z\right)\boldsymbol{e_y} \\ &+ \left(a_x b_y - a_y b_x\right)\boldsymbol{e_z}\end{aligned}$$

Matrix



The Matrix, 1999

Matrix

- A 2D array of numbers

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

$$= [\boldsymbol{a}_1 \quad \boldsymbol{a}_2 \quad \boldsymbol{a}_3] = \begin{bmatrix} \boldsymbol{a}_{1*} \\ \boldsymbol{a}_{2*} \\ \boldsymbol{a}_{3*} \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{x}_3 \end{bmatrix} \in \mathbb{R}^{3\times 1}$$

Matrix

- A 2D array of numbers

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

- Special matrices

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

identity

$$\begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$$

diagonal

$$\begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}$$

symmetric

$$\begin{bmatrix} 0 & -a & b \\ a & 0 & -c \\ -b & c & 0 \end{bmatrix}$$

skew-symmetric

Matrix Operation

- Transpose of a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

$$= [\boldsymbol{a}_1 \quad \boldsymbol{a}_2 \quad \boldsymbol{a}_3] = \begin{bmatrix} \boldsymbol{a}_{1*} \\ \boldsymbol{a}_{2*} \\ \boldsymbol{a}_{3*} \end{bmatrix}$$

Transpose

$$A^{\mathrm{T}} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_1^T \\ \boldsymbol{a}_2^T \\ \boldsymbol{a}_3^T \end{bmatrix} = [\boldsymbol{a}_{1*}^T \quad \boldsymbol{a}_{2*}^T \quad \boldsymbol{a}_{3*}^T]$$

Matrix Operation

- Transpose of a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

identity	diagonal	symmetric	skew-symmetric
$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$	$\begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}$	$\begin{bmatrix} 0 & -a & b \\ a & 0 & -c \\ -b & c & 0 \end{bmatrix}$
$A^T = A$	$A^T = A$	$A^T = A$	$A^T = -A$

Matrix Operation

- Scalar multiplication and matrix addition

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

$$sA = \begin{bmatrix} sa_{11} & sa_{12} & sa_{13} \\ sa_{21} & sa_{22} & sa_{23} \\ sa_{31} & sa_{32} & sa_{33} \end{bmatrix}$$

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \\ a_{31} + b_{31} & a_{32} + b_{32} & a_{33} + b_{33} \end{bmatrix}$$

Matrix Operation

- Matrix multiplication

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

$$C = AB = \begin{bmatrix} \boxed{a_{11} \quad a_{12} \quad a_{13}} \\ a_{21} \quad a_{22} \quad a_{23} \\ a_{31} \quad a_{32} \quad a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & \boxed{\begin{matrix} b_{12} \\ b_{22} \\ b_{32} \end{matrix}} & \begin{matrix} b_{13} \\ b_{23} \\ b_{33} \end{matrix} \end{bmatrix}$$

$$= \begin{bmatrix} * & \boxed{?} & * \\ * & * & * \\ * & * & * \end{bmatrix} = [c_{ij} = \boldsymbol{a}_{i*} \cdot \boldsymbol{b}_j]$$

Matrix Operation

- Matrix multiplication

$$\boxed{AB \neq BA}$$

$$ABC = (AB)C = A(BC)$$

$$A(B + C) = AB + AC$$

$$(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}} \qquad IA = A$$

- Inverse of a matrix

$$M = A^{-1} \Leftrightarrow AM = MA = I$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

Matrix Form of Dot Product

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_x b_x + a_y b_y + a_z b_z$$

$$= \boldsymbol{a}^T \boldsymbol{b} = \begin{bmatrix} a_x & a_y & a_z \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

$$= \boldsymbol{b}^T \boldsymbol{a}$$

Matrix Form of Cross Product

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = [\boldsymbol{a}]_\times \boldsymbol{b}$$

$$[\boldsymbol{a}]_\times + [\boldsymbol{a}]_\times^{\boldsymbol{T}} = \boldsymbol{0}$$ skew-symmetric

Matrix Form of Cross Product

$$\boldsymbol{a} \times \boldsymbol{b} = [\boldsymbol{a}]_{\times}\boldsymbol{b}$$

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = [\boldsymbol{a}]_{\times}([\boldsymbol{b}]_{\times}\boldsymbol{c})$$

$$= [\boldsymbol{a}]_{\times}[\boldsymbol{b}]_{\times}\boldsymbol{c}$$

$$\boldsymbol{a} \times (\boldsymbol{a} \times \boldsymbol{c}) = [\boldsymbol{a}]_{\times}^{2}\boldsymbol{b}$$

$$[\boldsymbol{a}]_{\times} + [\boldsymbol{a}]_{\times}^{\boldsymbol{T}} = \boldsymbol{0}$$ skew-symmetric

Matrix Form of Cross Product

$$\boldsymbol{a} \times \boldsymbol{b} = [\boldsymbol{a}]_\times \boldsymbol{b}$$

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = [\boldsymbol{a}]_\times ([\boldsymbol{b}]_\times \boldsymbol{c})$$

$$= [\boldsymbol{a}]_\times [\boldsymbol{b}]_\times \boldsymbol{c}$$

$$\boldsymbol{a} \times (\boldsymbol{a} \times \boldsymbol{c}) = [\boldsymbol{a}]_\times^2 \boldsymbol{b}$$

$$(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = [\boldsymbol{a}]_\times [\boldsymbol{b}]_\times \boldsymbol{c} \quad ???$$

$[\boldsymbol{a}]_\times + [\boldsymbol{a}]_\times^{\boldsymbol{T}} = \boldsymbol{0}$ skew-symmetric

Matrix Form of Cross Product

$$\boldsymbol{a} \times \boldsymbol{b} = [\boldsymbol{a}]_\times \boldsymbol{b}$$

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = [\boldsymbol{a}]_\times ([\boldsymbol{b}]_\times \boldsymbol{c})$$

$$= [\boldsymbol{a}]_\times [\boldsymbol{b}]_\times \boldsymbol{c}$$

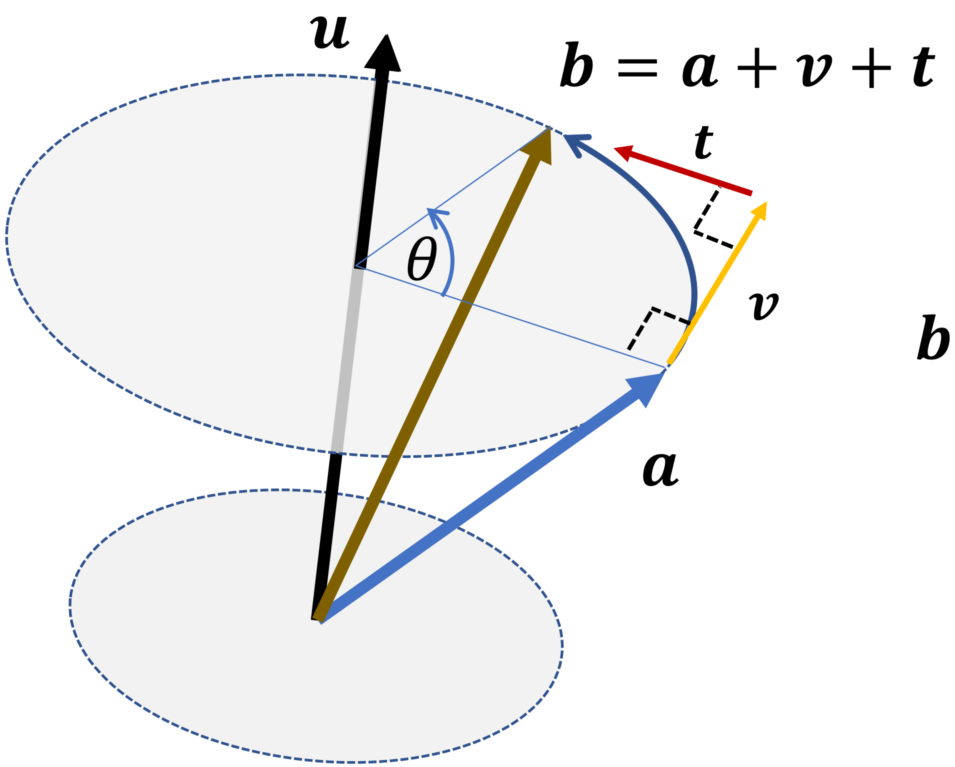
$$\boldsymbol{a} \times (\boldsymbol{a} \times \boldsymbol{c}) = [\boldsymbol{a}]_\times^2 \boldsymbol{b}$$

$(\boldsymbol{a} \times \boldsymbol{b}) \times \cancel{\boldsymbol{c} = [\boldsymbol{a}]_\times} [\boldsymbol{b}]_\times \boldsymbol{c}$???

$$(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = [\boldsymbol{a} \times \boldsymbol{b}]_\times \boldsymbol{c}$$

$[\boldsymbol{a}]_\times + [\boldsymbol{a}]_\times^{\boldsymbol{T}} = \boldsymbol{0}$ skew-symmetric

How to rotate a vectors?



$\|\boldsymbol{u}\| = \mathbf{1}$

$$\boldsymbol{v} = (\sin\theta)\, \boldsymbol{u} \times \boldsymbol{a}$$

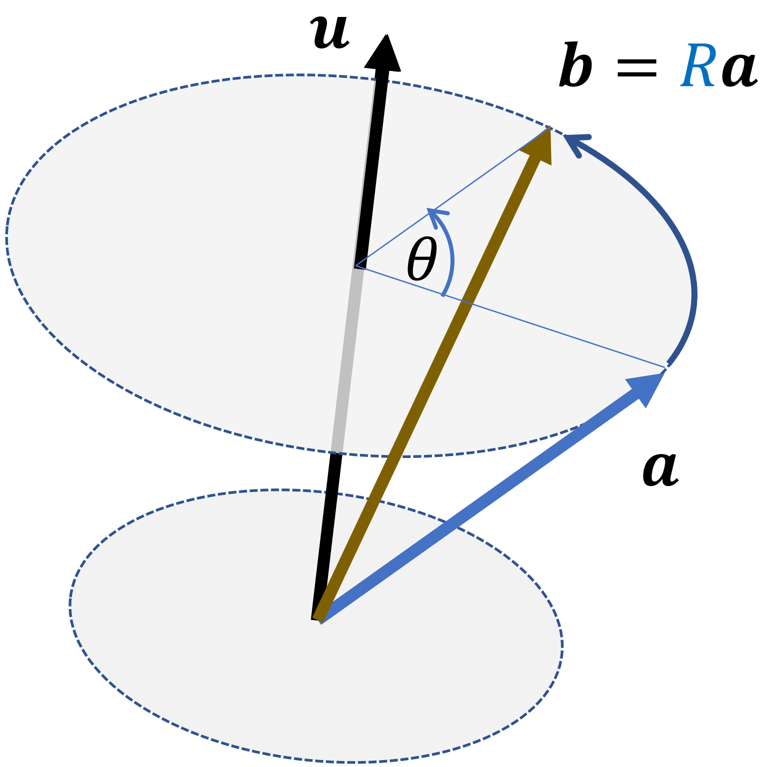
$$\boldsymbol{t} = (1 - \cos\theta)\, \boldsymbol{u} \times (\boldsymbol{u} \times \boldsymbol{a})$$

$$\boldsymbol{b} = \boldsymbol{a} + (\sin\theta)\, \boldsymbol{u} \times \boldsymbol{a} + (1 - \cos\theta)\, \boldsymbol{u} \times (\boldsymbol{u} \times \boldsymbol{a})$$

$$\boldsymbol{b} = \left(I + (\sin\theta)\ [\boldsymbol{u}]_\times + (1 - \cos\theta)\ [\boldsymbol{u}]_\times^{\mathbf{2}}\right)\boldsymbol{a}$$

$$= R\boldsymbol{a}$$

How to rotate a vectors?



$\|\boldsymbol{u}\| = \mathbf{1}$

Rodrigues' rotation formula

$$R = I + (\sin\theta)\ [\boldsymbol{u}]_\times + (1 - \cos\theta)\ [\boldsymbol{u}]_\times^{\mathbf{2}}$$

Orthogonal Matrix

- A matrix who columns (& rows) are orthogonal vectors

$$A = [\boldsymbol{a}_1 \quad \boldsymbol{a}_2 \quad \boldsymbol{a}_3] \qquad \boldsymbol{a}_i^T \boldsymbol{a}_j = \begin{cases} 1, & \text{if } i = j \\ 0. & \text{if } i \neq j \end{cases}$$

$$A^{\mathrm{T}} A = \begin{bmatrix} \boldsymbol{a}_1^T \\ \boldsymbol{a}_2^T \\ \boldsymbol{a}_3^T \end{bmatrix} [\boldsymbol{a}_1 \quad \boldsymbol{a}_2 \quad \boldsymbol{a}_3] = \begin{bmatrix} \boldsymbol{a}_1^T \boldsymbol{a}_1 & \boldsymbol{a}_1^T \boldsymbol{a}_2 & \boldsymbol{a}_1^T \boldsymbol{a}_3 \\ \boldsymbol{a}_2^T \boldsymbol{a}_1 & \boldsymbol{a}_2^T \boldsymbol{a}_2 & \boldsymbol{a}_2^T \boldsymbol{a}_3 \\ \boldsymbol{a}_3^T \boldsymbol{a}_1 & \boldsymbol{a}_3^T \boldsymbol{a}_2 & \boldsymbol{a}_3^T \boldsymbol{a}_3 \end{bmatrix} = \mathbf{I}$$

$$\boxed{A^{\mathrm{T}} = A^{-1}}$$

Determinant of a Matrix

$$A = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

$$\boxed{\det A = \begin{vmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{vmatrix}}$$

Determinant of a Matrix

$$A = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

$$\det A = \begin{vmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{vmatrix}$$
$$\begin{matrix} a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{matrix}$$

Determinant of a Matrix

$$A = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

$$\det A = \begin{vmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{vmatrix} \\ \begin{matrix} \quad a_{10} & a_{11} & a_{12} \\ \quad a_{20} & a_{21} & a_{22} \end{matrix}$$

Determinant of a Matrix

- $\det I = 1$
- $\det AB = \det \mathrm{A} * \det \mathrm{B}$
- $\det A^{\mathrm{T}} = \det A$
- If A is invertible, $\det A^{-1} = (\det A)^{-1}$
- If U is orthogonal, $\det U = \pm 1$

Cross Product as a Determinant

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix}$$

$$= \det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix}$$

Eigenvalues and Eigenvectors

For a matrix A, if a nonzero vector $\boldsymbol{x}$ satisfies

$$A\boldsymbol{x} = \lambda\boldsymbol{x}$$

Then:

λ: an eigenvalue of A

$\boldsymbol{x}$: an eigenvector of A

Eigenvalues and Eigenvectors

For a matrix A, if a nonzero vector $\boldsymbol{x}$ satisfies

$$A\boldsymbol{x} = \lambda\boldsymbol{x}$$

Then:

λ: an eigenvalue of A

$\boldsymbol{x}$: an eigenvector of A

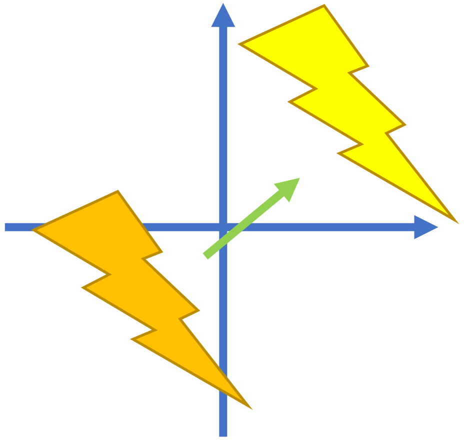
Especially, a 3×3 orthogonal matrix U

has at least one real eigenvalue: $\lambda = \det U = \pm 1$

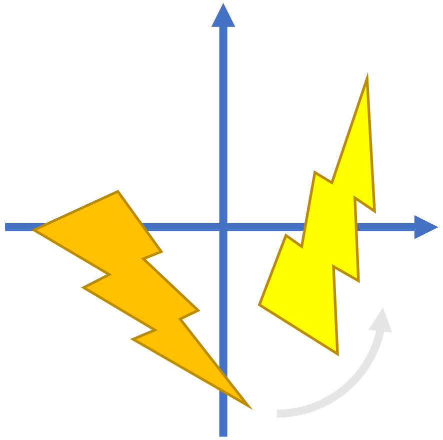
Rigid Transformation

Translation, rotation, and coordinate transformation

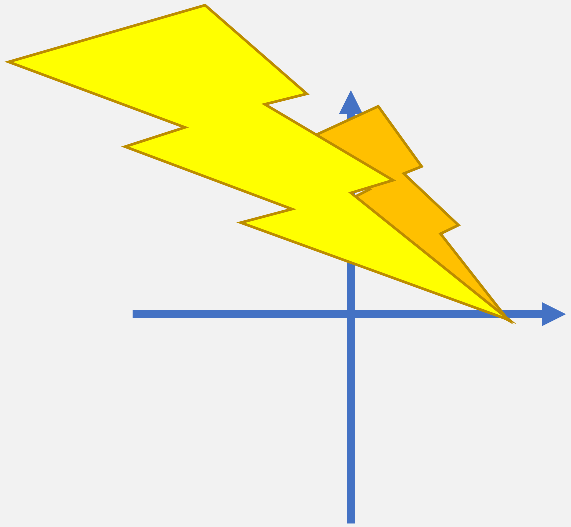
Rigid Transformation: Translation + Rotation



Translation

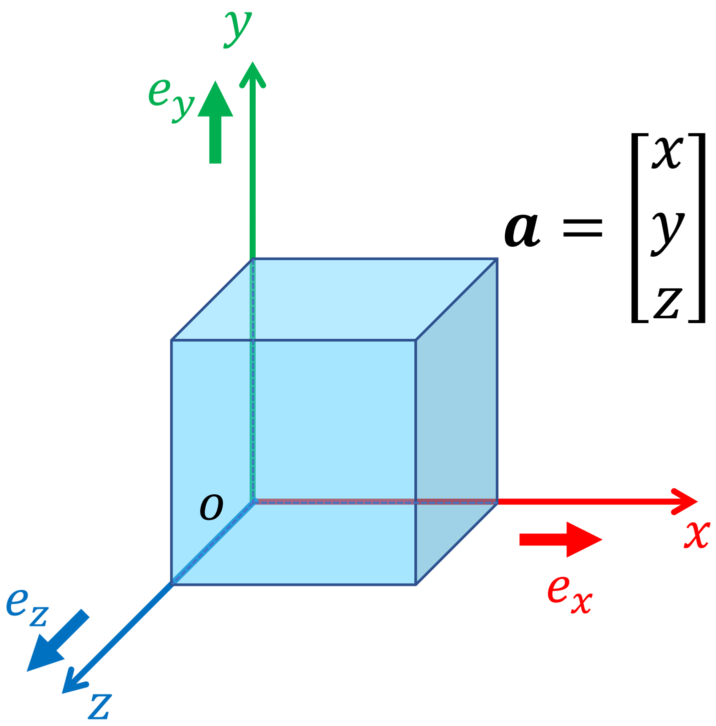


Rotation



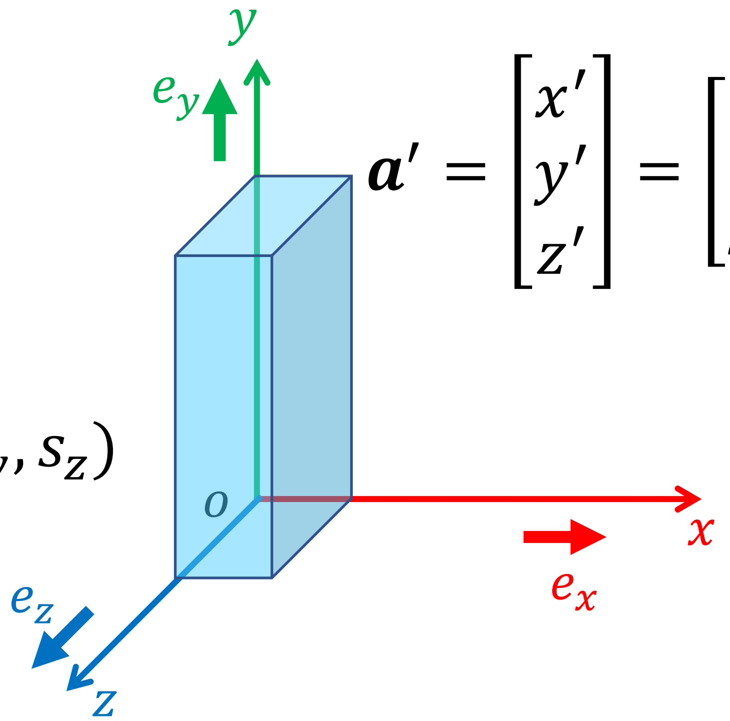
Scaling

Scaling



$$\boldsymbol{a} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

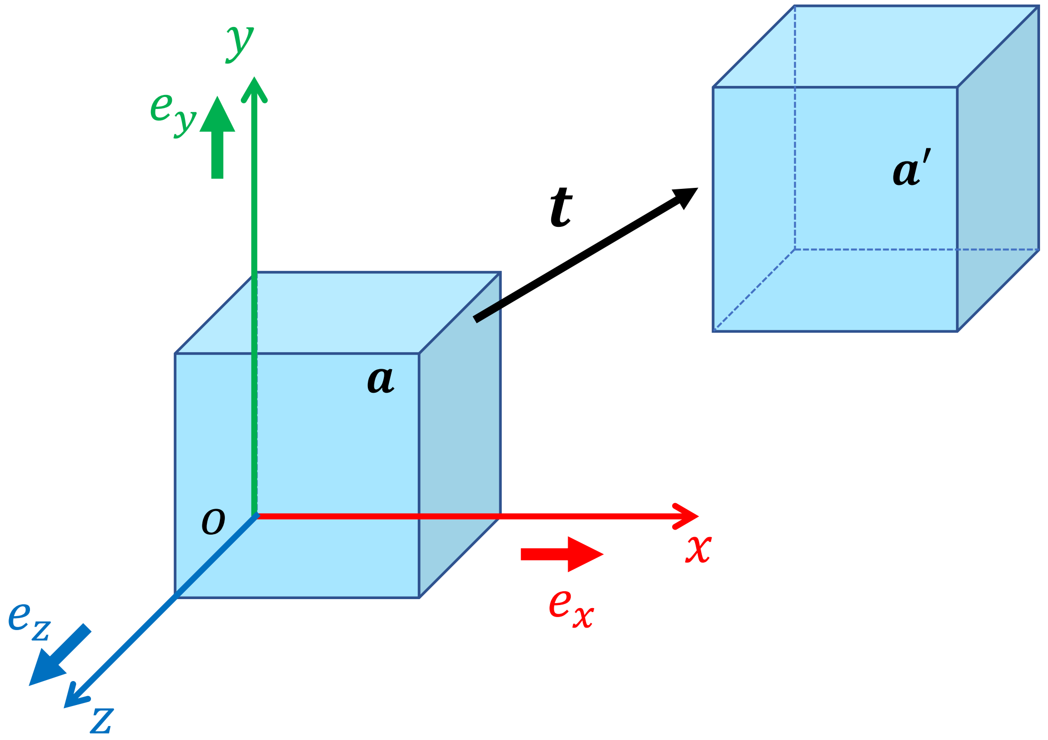
$$s = (s_x, s_y, s_z)$$



$$\boldsymbol{a}' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} s_x * x \\ s_y * y \\ s_z * z \end{bmatrix}$$

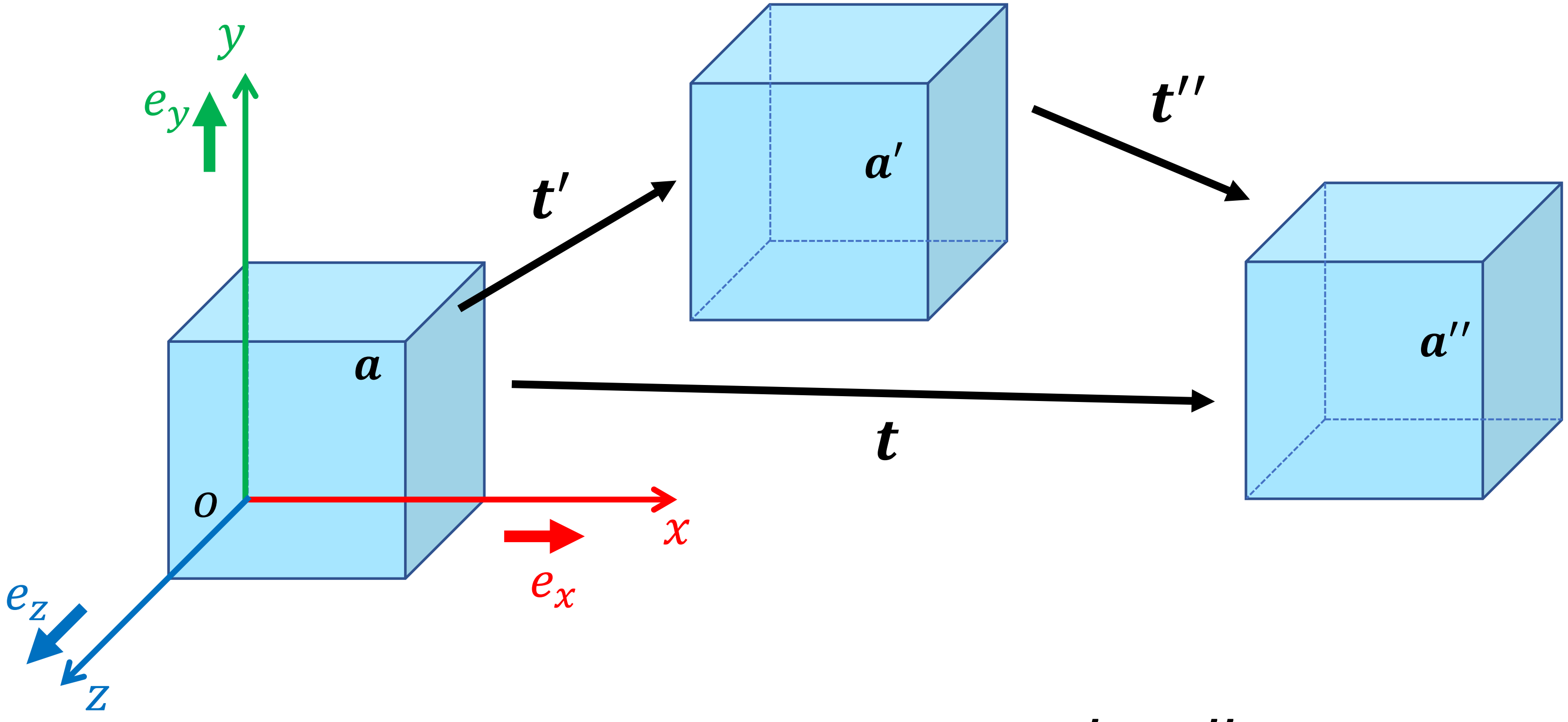
$$\boldsymbol{a}' = \begin{bmatrix} s_x & & \\ & s_y & \\ & & s_z \end{bmatrix} \boldsymbol{a}$$

Translation



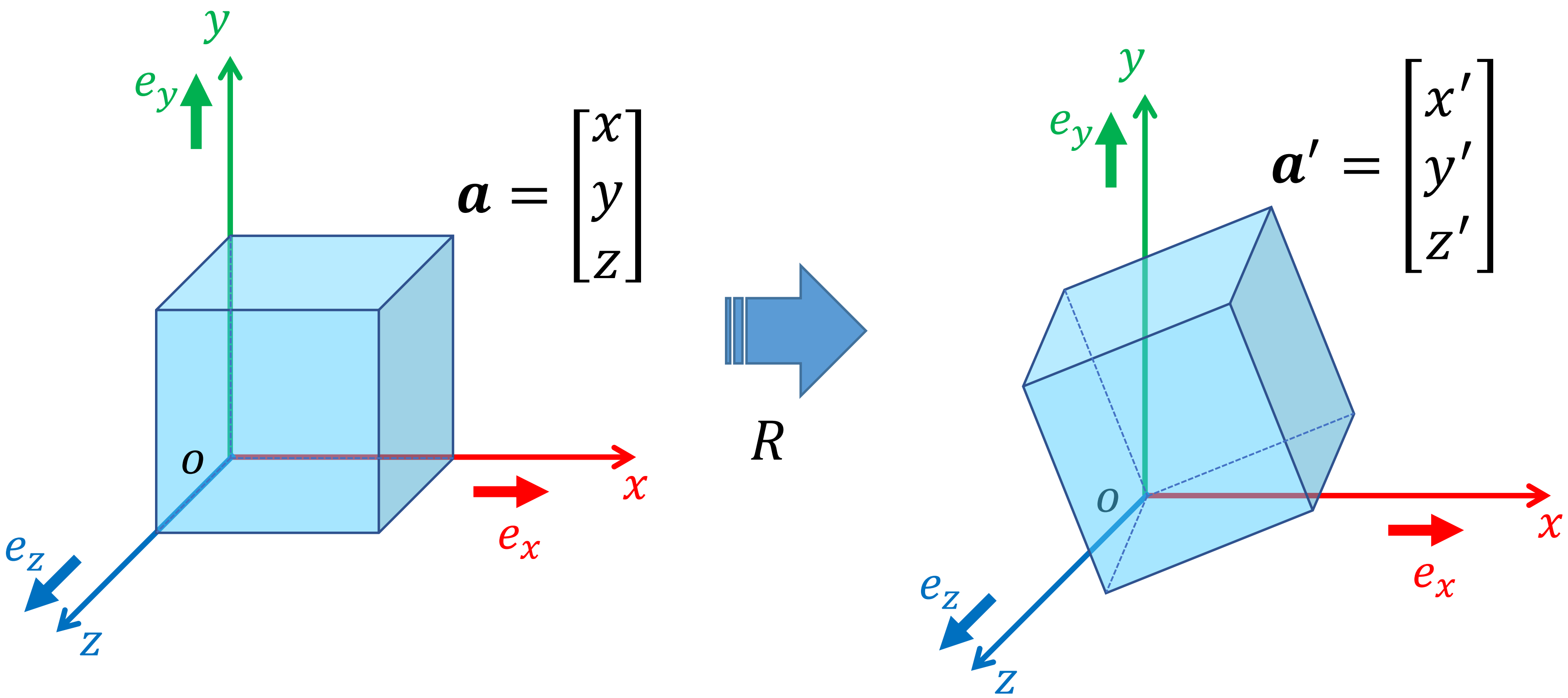
$$\boldsymbol{a}' = \boldsymbol{a} + \boldsymbol{t}$$

Combination of Translations



$$\boldsymbol{t} = \boldsymbol{t}' + \boldsymbol{t}''$$

Rotation



$$\boldsymbol{a}' = R\boldsymbol{a}$$

R: Rotation Matrix

Rotation Matrix

- Rotation matrix is orthogonal:

$$R^{-1} = R^{\mathrm{T}} \qquad R^{\mathrm{T}}R = RR^{\mathrm{T}} = I$$

- Determinant of R

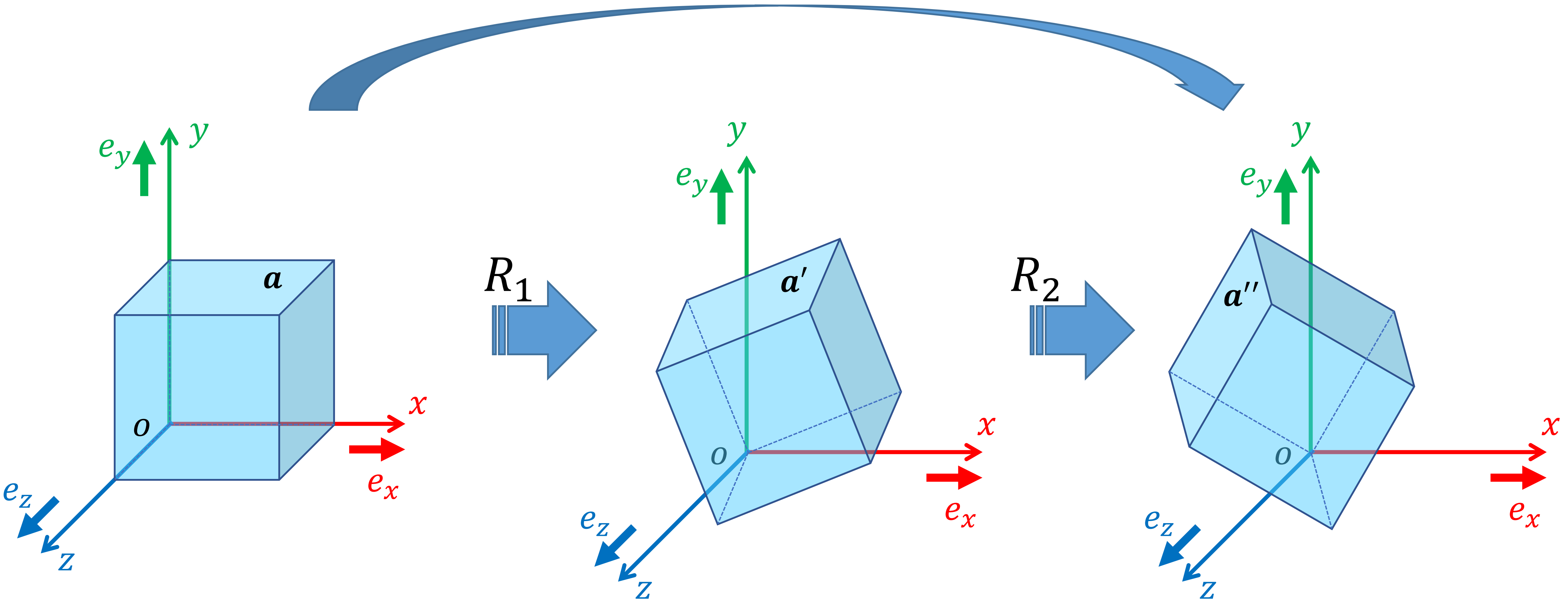
$$\det R = +1$$

- Rotation maintains length of vectors

$$||R\boldsymbol{x}|| = ||\boldsymbol{x}||$$

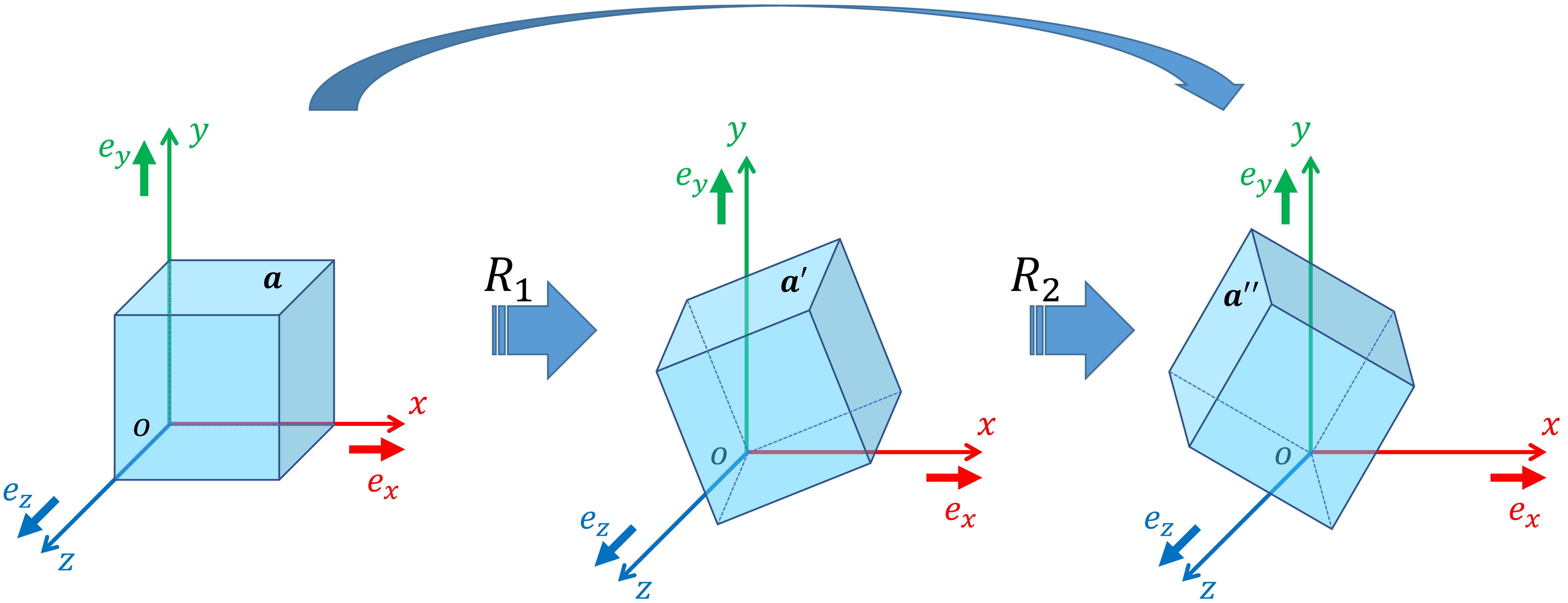
Combination of Rotations

$$R = R_1 R_2\ ???$$

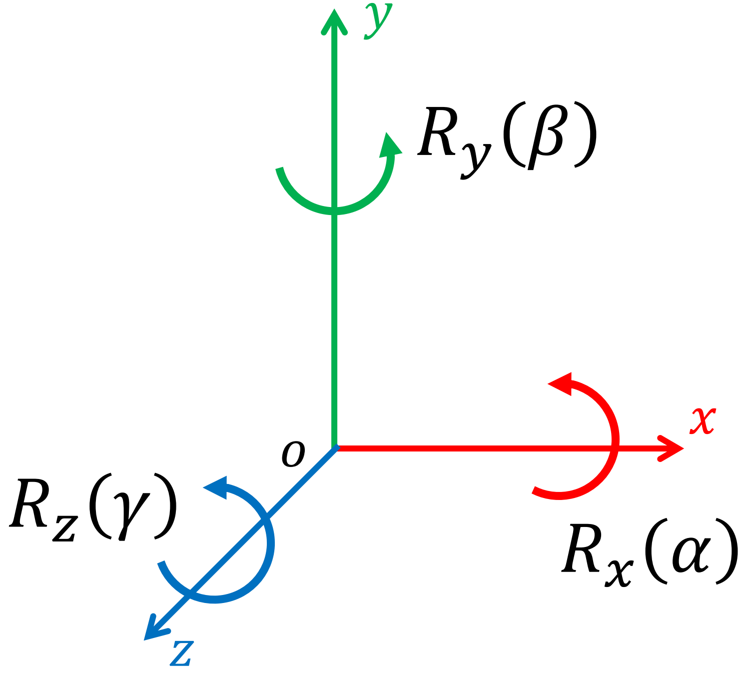


Combination of Rotations

$$R = R_2 R_1$$



Rotation around Coordinate Axes



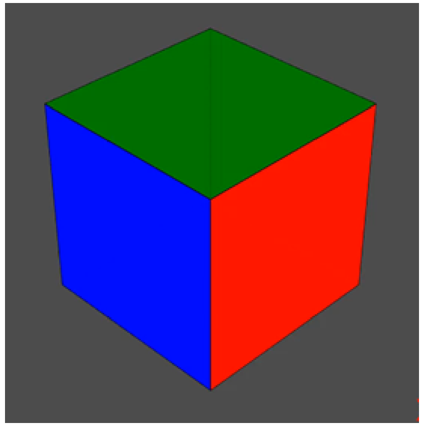
$$R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$$

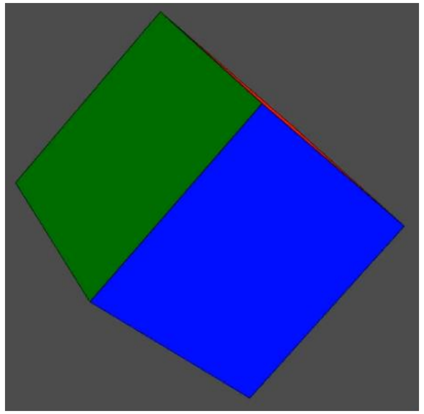
$$R_y(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}$$

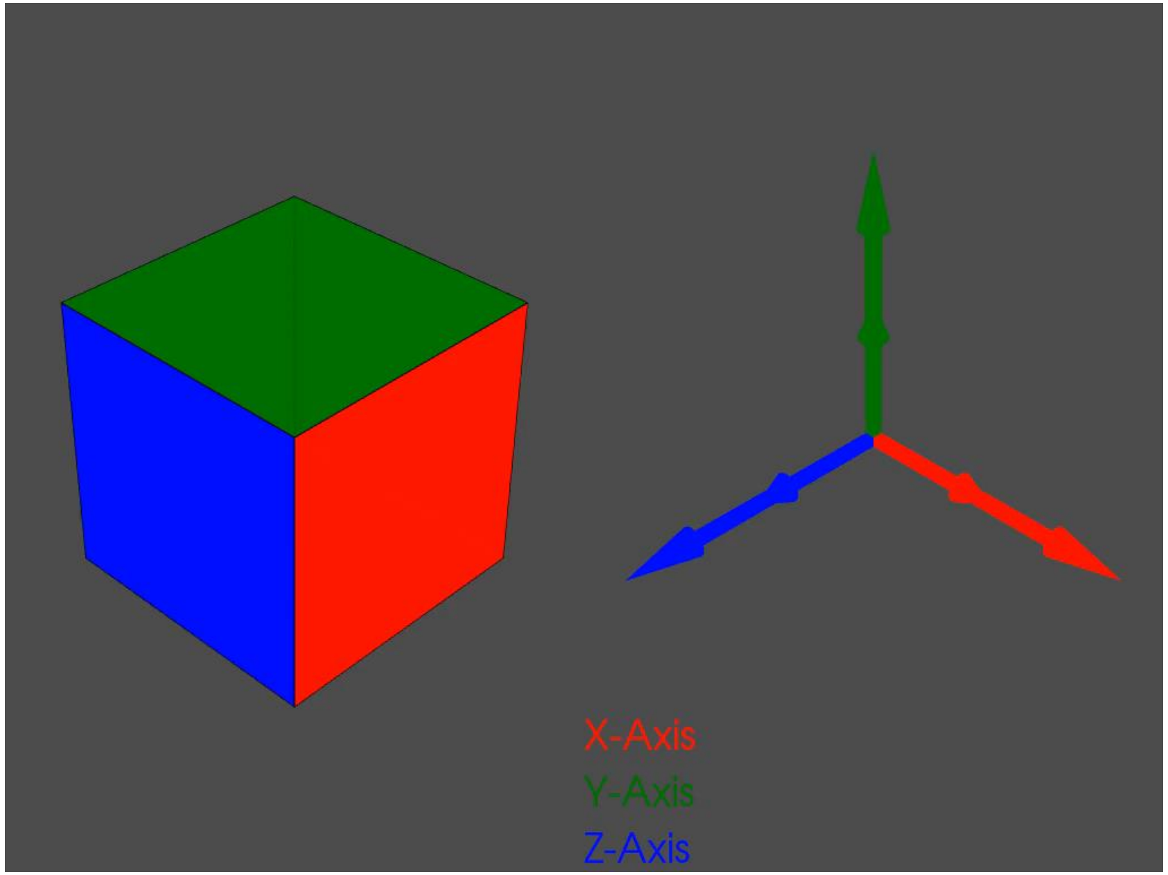
$$R_z(\gamma) = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Rotation around Coordinate Axes

$$R_z(72°)R_y(45°)R_x(60°)$$



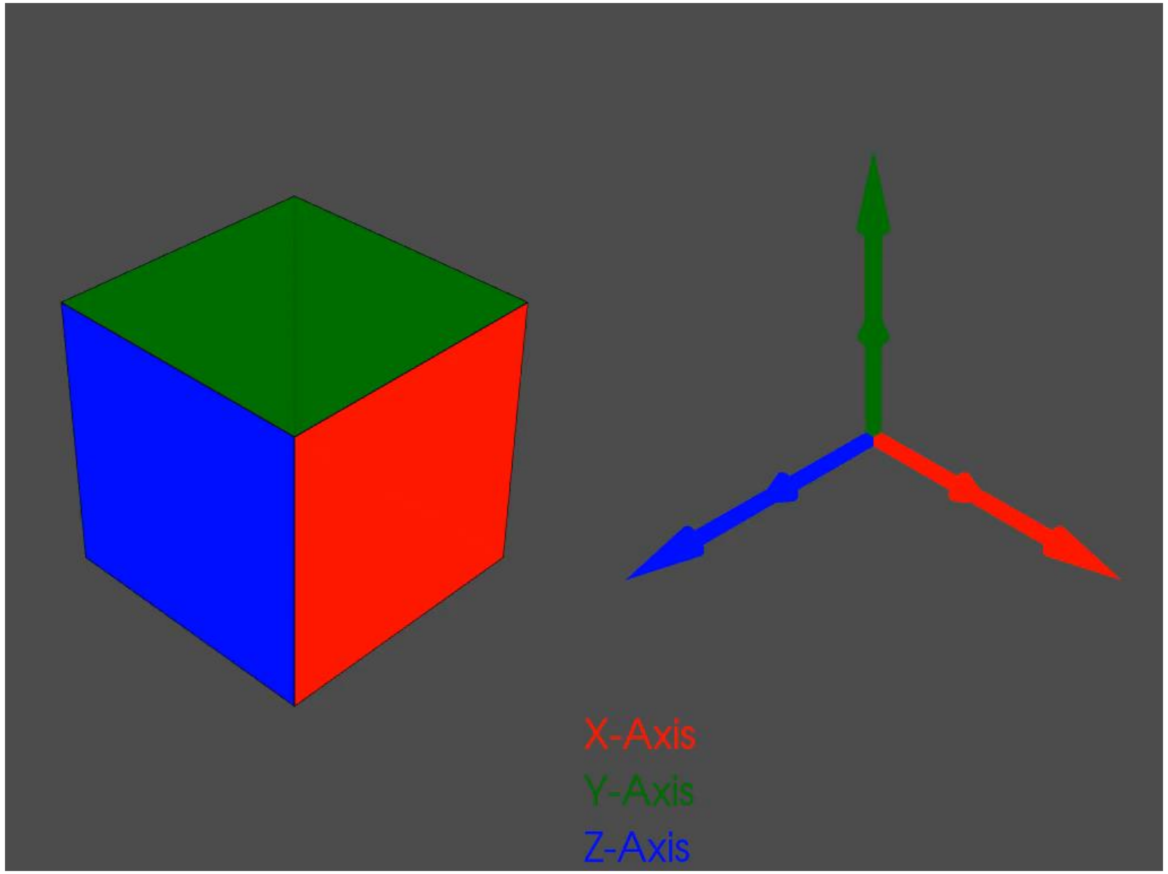


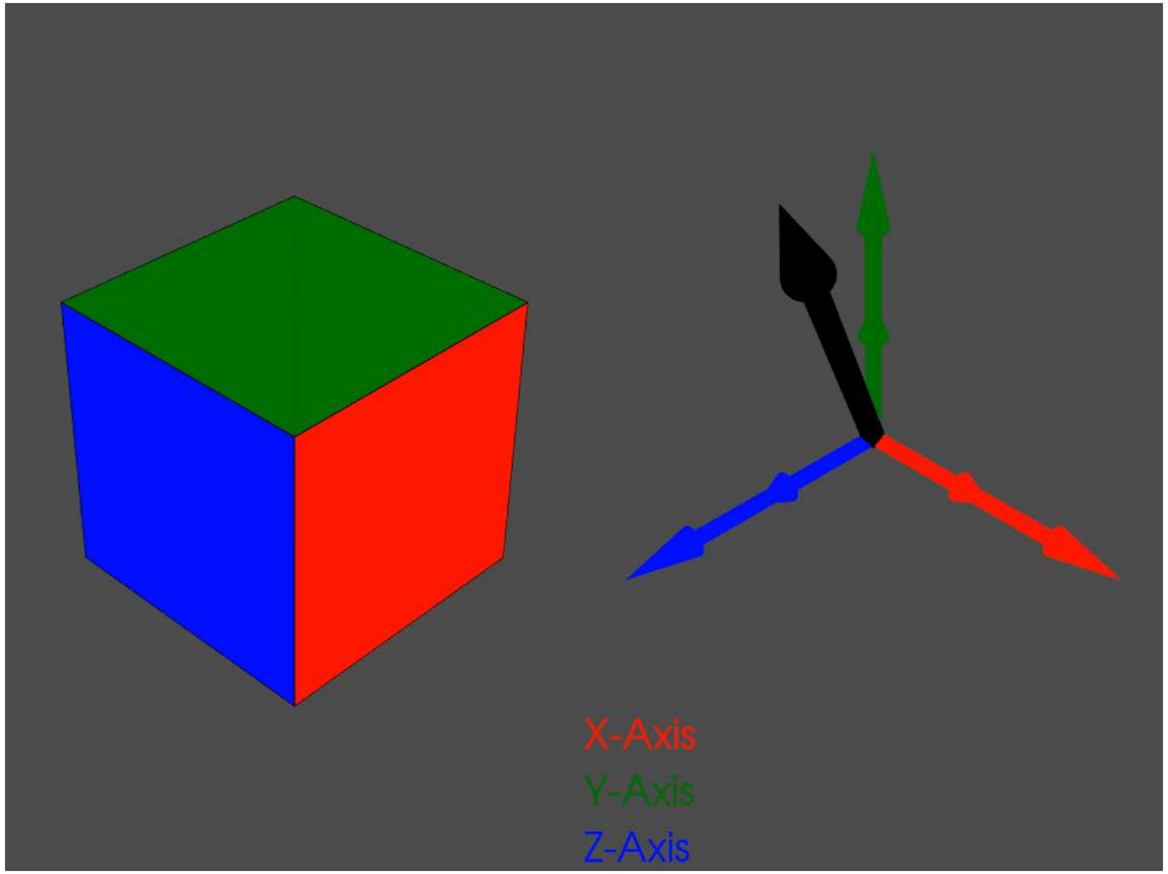


Rotation around Coordinate Axes

$R_z(72°)R_y(45°)R_x(60°)$

$u = (0.28, 0.83, 0.48) \quad \theta = 81.1°$





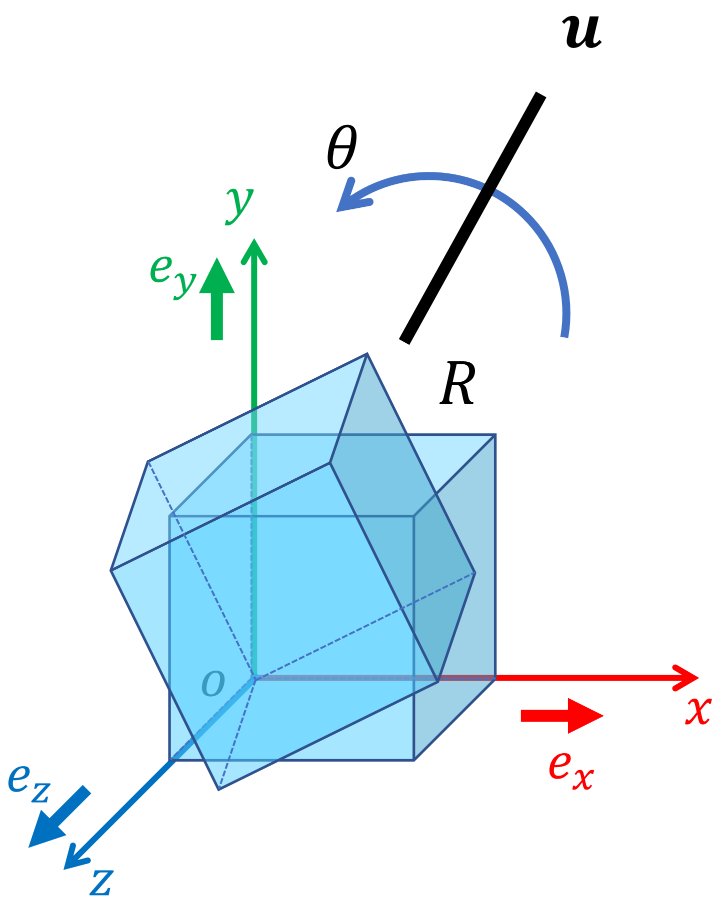
Rotation Axis and Angle

Rotation matrix R has a real eigenvalue: $+1$

$$R\boldsymbol{u} = \boldsymbol{u}$$

In other words, R can be considered as a rotation around axis $\boldsymbol{u}$ by some angle θ

How to find axis $\boldsymbol{u}$ and angle θ?



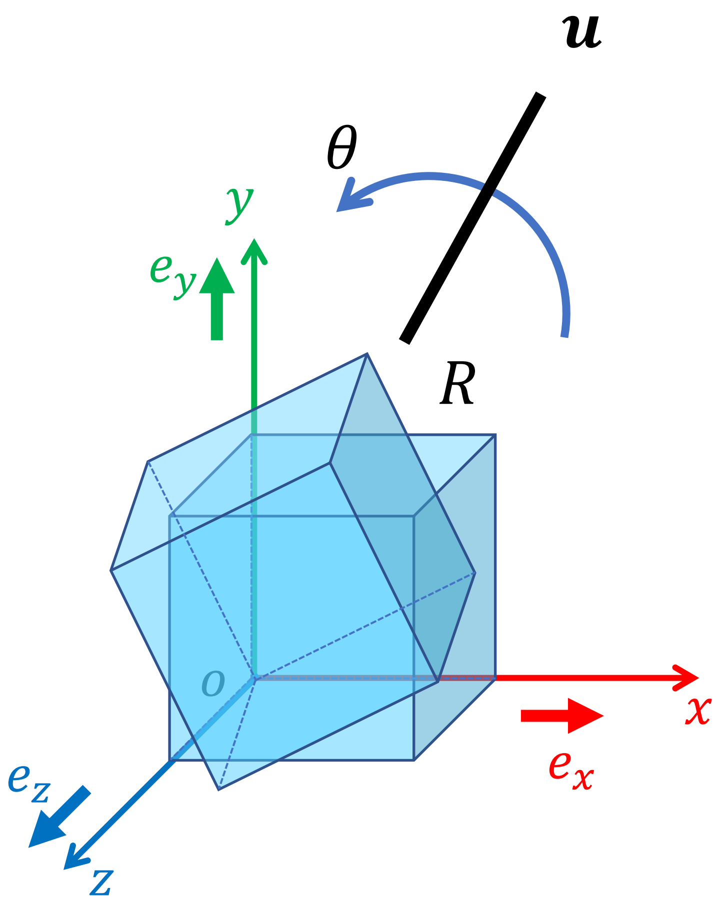
Rotation Axis and Angle

$$R\boldsymbol{u} = \boldsymbol{u} \quad \Rightarrow \quad \boldsymbol{u} = R^{\mathrm{T}}\boldsymbol{u}$$

$$\left(R - R^{\mathrm{T}}\right)\boldsymbol{u} = 0$$

$$\begin{bmatrix} 0 & -(r_{21} - r_{12}) & r_{13} - r_{31} \\ r_{21} - r_{12} & 0 & -(r_{32} - r_{23}) \\ -(r_{13} - r_{31}) & r_{32} - r_{23} & 0 \end{bmatrix} \boldsymbol{u} = 0$$

Skew-symmetric



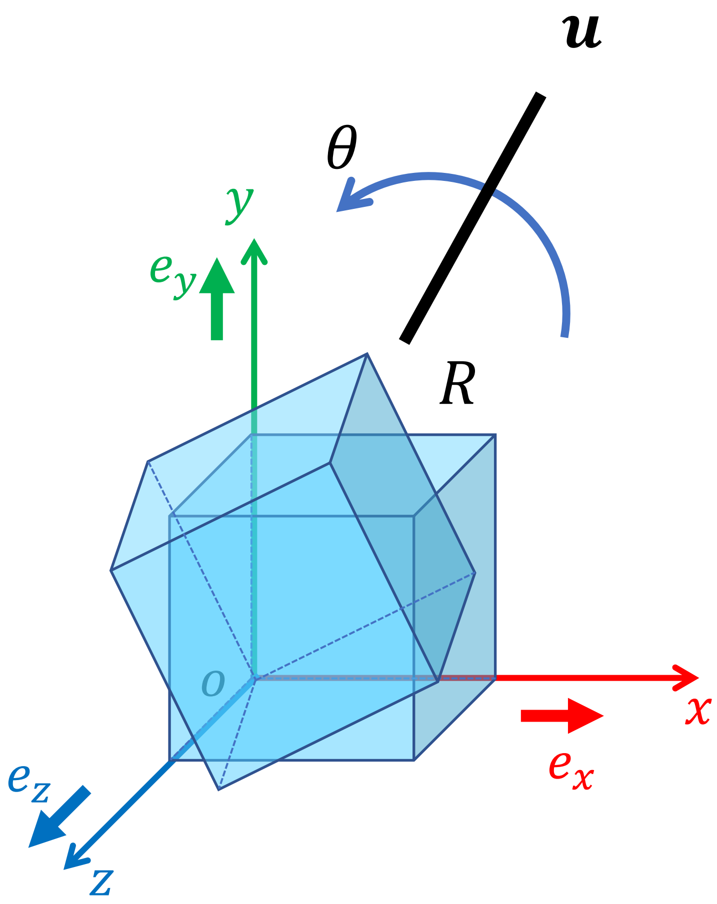
Rotation Axis and Angle

$$R\boldsymbol{u} = \boldsymbol{u} \quad \Rightarrow \quad \boldsymbol{u} = R^{\mathrm{T}}\boldsymbol{u}$$

$$(R - R^{\mathrm{T}})\boldsymbol{u} = 0$$

$$\begin{bmatrix} 0 & -(r_{21}-r_{12}) & r_{13}-r_{31} \\ r_{21}-r_{12} & & (r_{32}-r_{23}) \\ -(r_{13}-r_{31}) & r_{32}-r_{23} & 0 \end{bmatrix}\boldsymbol{u} = 0$$

$$\boldsymbol{u}' \times \boldsymbol{u} = 0$$



Skew-symmetric Matrix ⟺ Cross Product

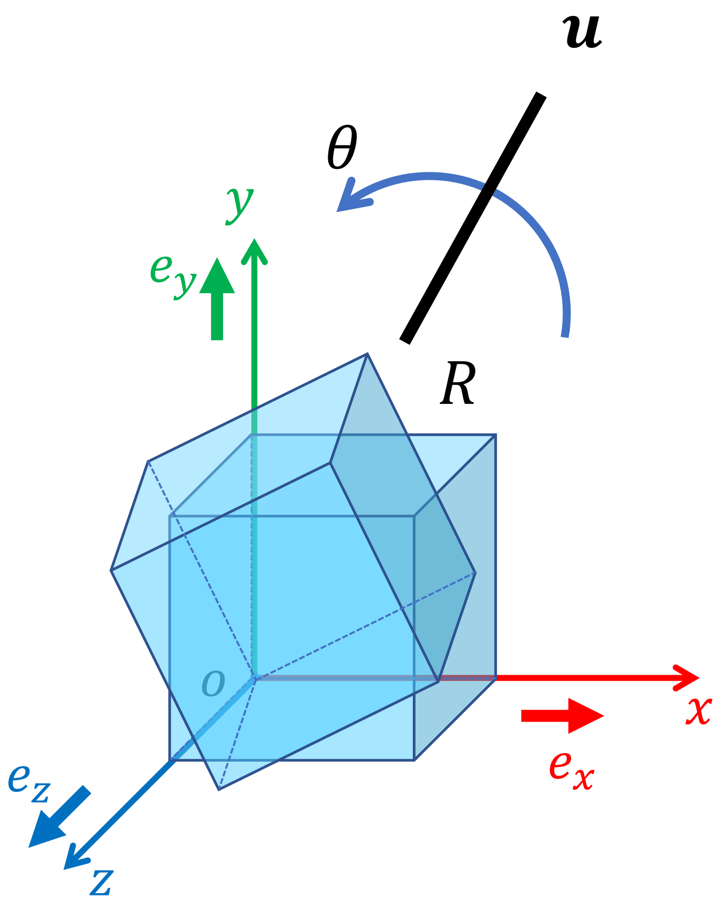
Rotation Axis and Angle

$$R\boldsymbol{u} = \boldsymbol{u} \quad \Rightarrow \quad \boldsymbol{u} = R^{\mathrm{T}}\boldsymbol{u}$$

$$(R - R^{\mathrm{T}})\boldsymbol{u} = 0$$

$$\boldsymbol{u} \leftarrow \boldsymbol{u}' = \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

When $R \neq R^T \Leftrightarrow \sin\theta \neq 0 \Leftrightarrow \theta \neq 0°$ or $180°$



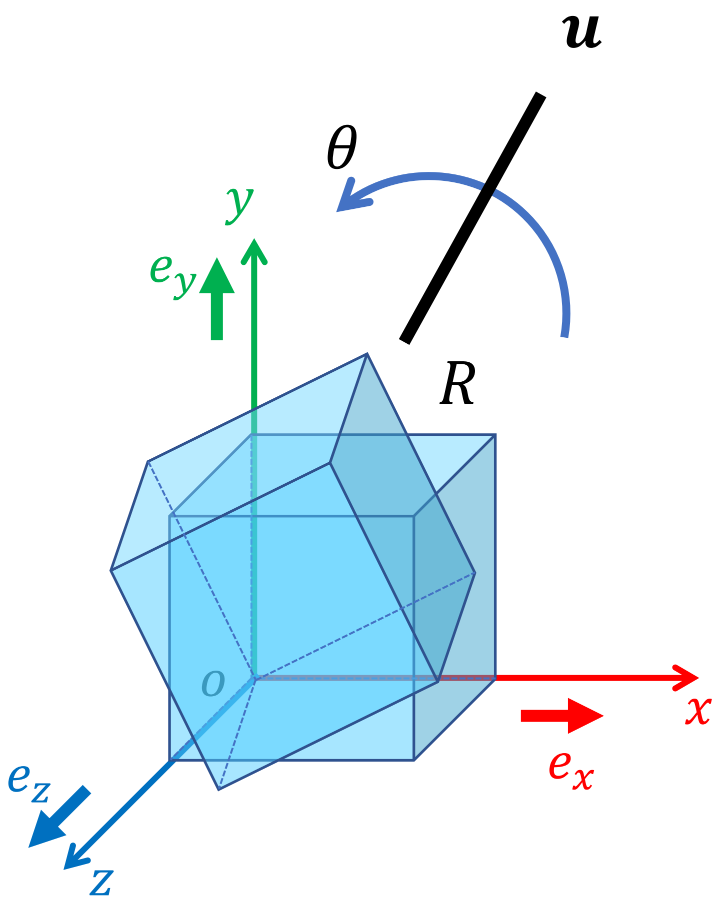
Rotation Axis and Angle

$$R = I + (\sin\theta)\ [\boldsymbol{u}]_\times + (1 - \cos\theta)\ [\boldsymbol{u}]_\times^2$$

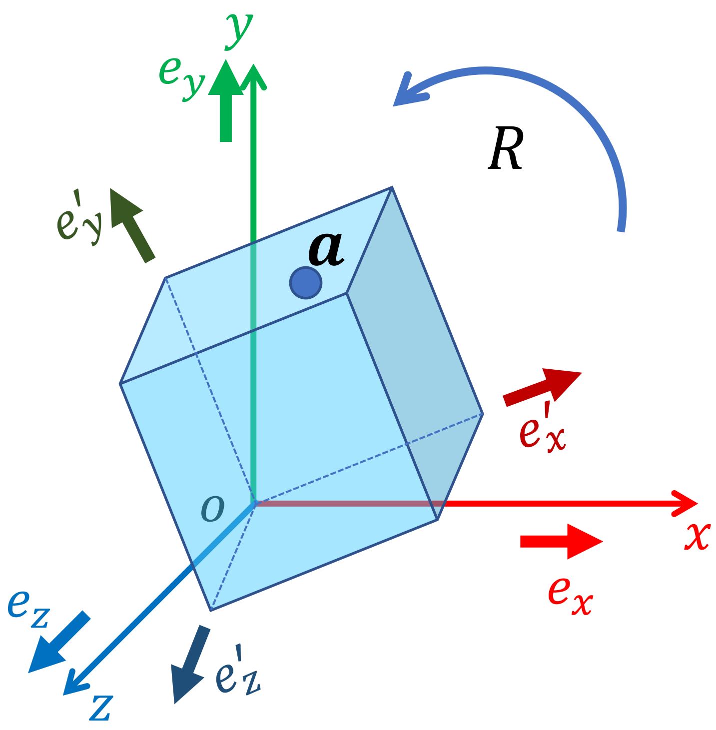
$$\boldsymbol{u} \leftarrow \boldsymbol{u}' = \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} \leftarrow R - R^{\mathrm{T}}$$

$$\|\boldsymbol{u}'\| = 2\sin\theta$$

When $R \neq R^T \Leftrightarrow \sin\theta \neq 0 \ \Leftrightarrow \theta \neq 0°$ or $180°$



Coordinate Transformation



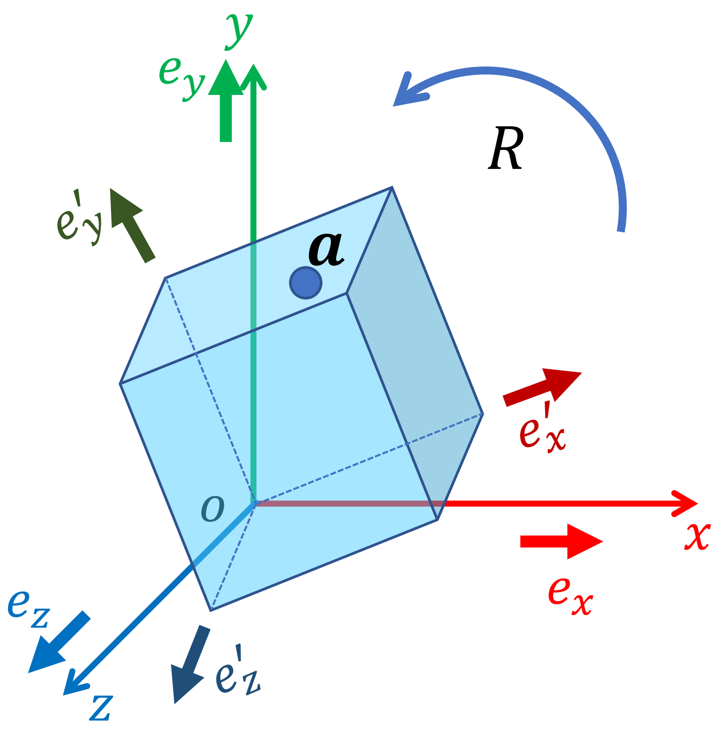
$(x', y', z')^T$: $\boldsymbol{a}$ in *object* system

$(x,\ y,\ z)^T$: $\boldsymbol{a}$ in *global* system

$$\boldsymbol{a} = \begin{bmatrix} | & | & | \\ \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ | & | & | \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} | & | & | \\ \boldsymbol{e}'_x & \boldsymbol{e}'_y & \boldsymbol{e}'_z \\ | & | & | \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

Coordinate Transformation



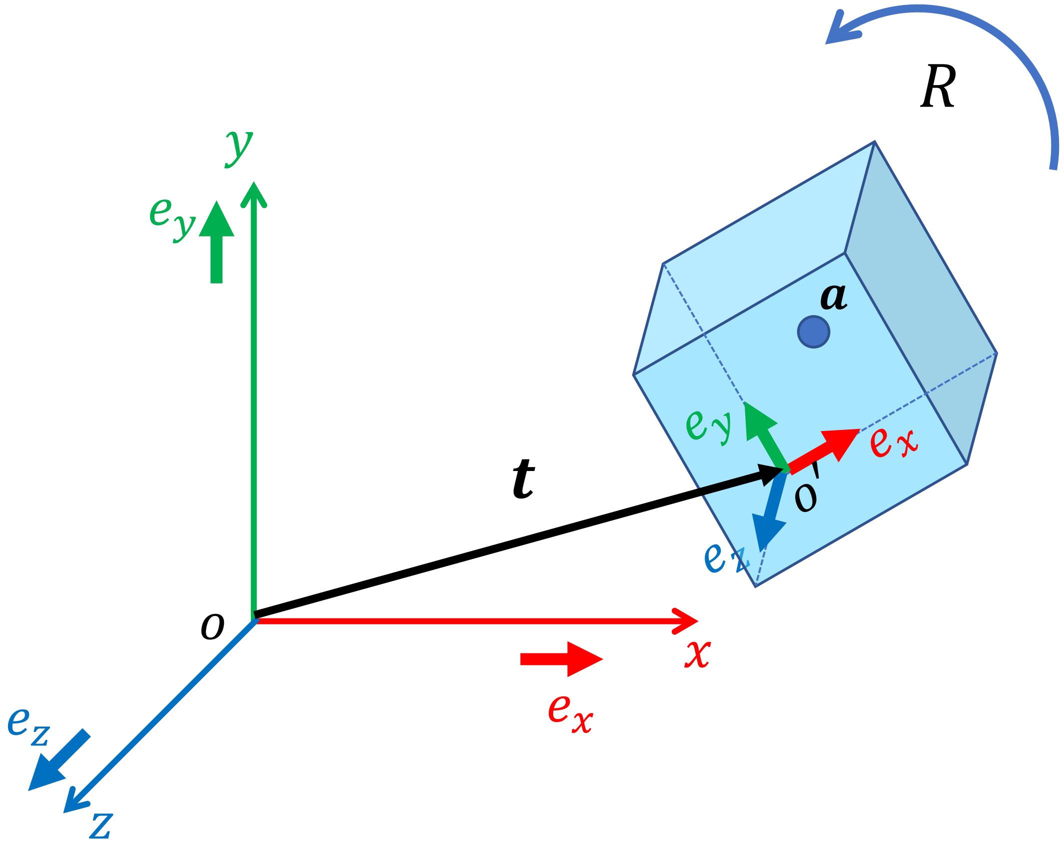
$(x', y', z')^{\boldsymbol{T}}$: $\boldsymbol{a}$ in *object* system

$(x,\ y,\ z)^{\boldsymbol{T}}$: $\boldsymbol{a}$ in *global* system

$$R = \begin{bmatrix} | & | & | \\ \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ | & | & | \end{bmatrix}^{-1} \begin{bmatrix} | & | & | \\ \boldsymbol{e}'_x & \boldsymbol{e}'_y & \boldsymbol{e}'_z \\ | & | & | \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = R \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

Coordinate Transformation



$(x', y', z')^{\boldsymbol{T}}$: $\boldsymbol{a}$ in *object* system

$(x,\ y,\ z)^{\boldsymbol{T}}$: $\boldsymbol{a}$ in *global* system

object → *global* $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = R \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} + \boldsymbol{t}$$

global → *object* $$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = R^{\mathrm{T}} \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \boldsymbol{t} \right)$$

Representations of 3D Rotation

回囘囬𡇈

Parameterization of Rotation

- A rotation matrix, 9 parameters: a_{ij}

$$R = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Parameterization of Rotation

- A rotation matrix, 9 parameters: a_{ij}

$$R = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$R^{\mathrm{T}} R = I$$

$$\begin{cases} a_{11}^2 + a_{21}^2 + a_{31}^2 = 1 \\ a_{12}^2 + a_{22}^2 + a_{32}^2 = 1 \\ a_{13}^2 + a_{23}^2 + a_{33}^2 = 1 \end{cases} \qquad \begin{cases} a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} = 0 \\ a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} = 0 \\ a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} = 0 \end{cases}$$

Parameterization of Rotation

- A rotation matrix, 9 parameters: a_{ij}

$$R = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$R^{\mathrm{T}}R = I$$

$$\begin{cases} a_{11}^2 + a_{21}^2 + a_{31}^2 = 1 \\ a_{12}^2 + a_{22}^2 + a_{32}^2 = 1 \\ a_{13}^2 + a_{23}^2 + a_{33}^2 = 1 \end{cases} \qquad \begin{cases} a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} = 0 \\ a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} = 0 \\ a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} = 0 \end{cases}$$

$$\boxed{\text{degrees of freedom (DoF)} = 3}$$

Parameterization of Rotation

- A rotation matrix, 9 parameters: a_{ij}

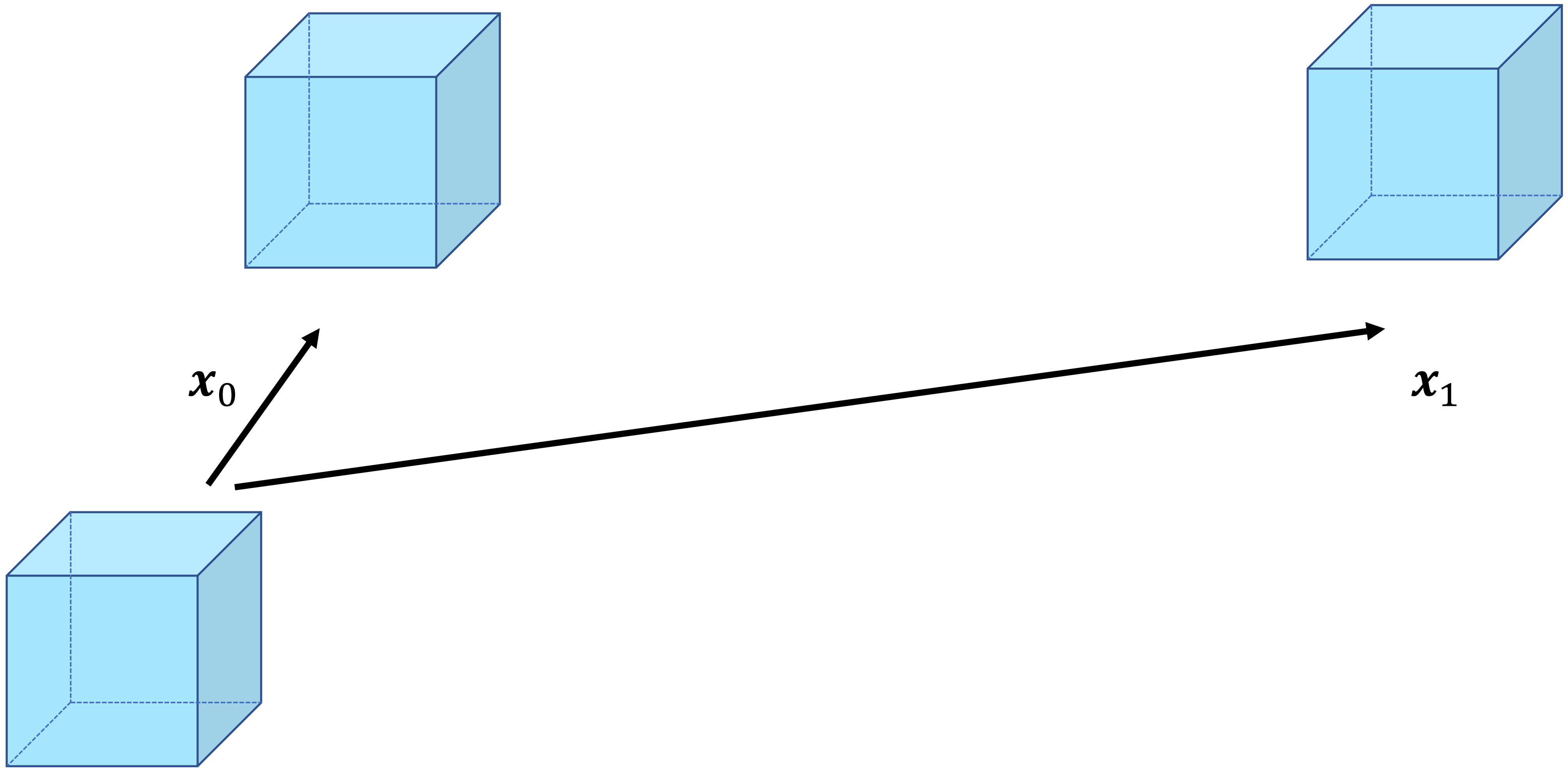
$$R = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$R^{\mathrm{T}}R = I \qquad \det R = 1$$

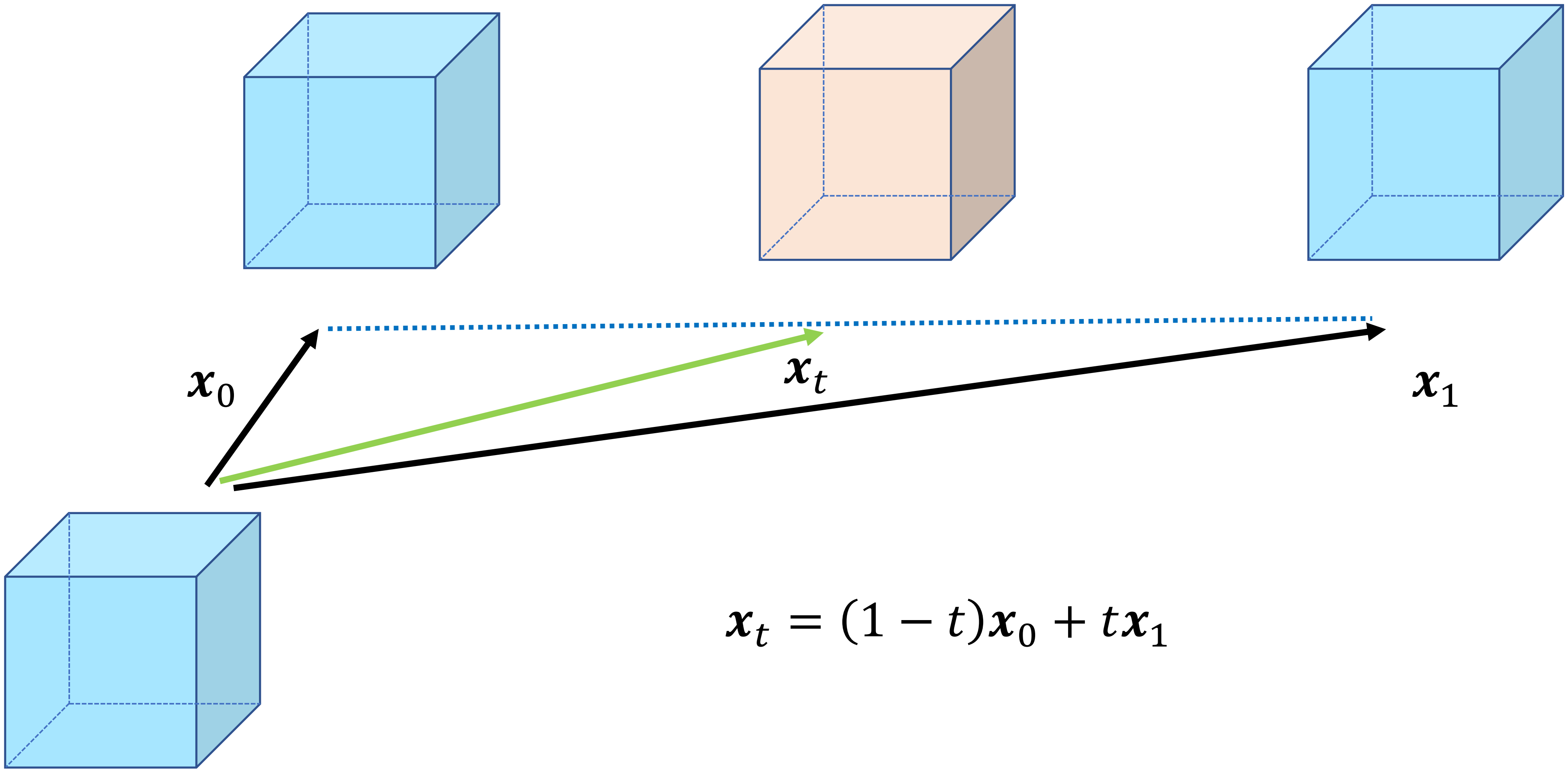
$$\begin{cases} a_{11}^2 + a_{21}^2 + a_{31}^2 = 1 \\ a_{12}^2 + a_{22}^2 + a_{32}^2 = 1 \\ a_{13}^2 + a_{23}^2 + a_{33}^2 = 1 \end{cases} \qquad \begin{cases} a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} = 0 \\ a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} = 0 \\ a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} = 0 \end{cases}$$

degrees of freedom (DoF) $= 3$

Interpolation of Translations

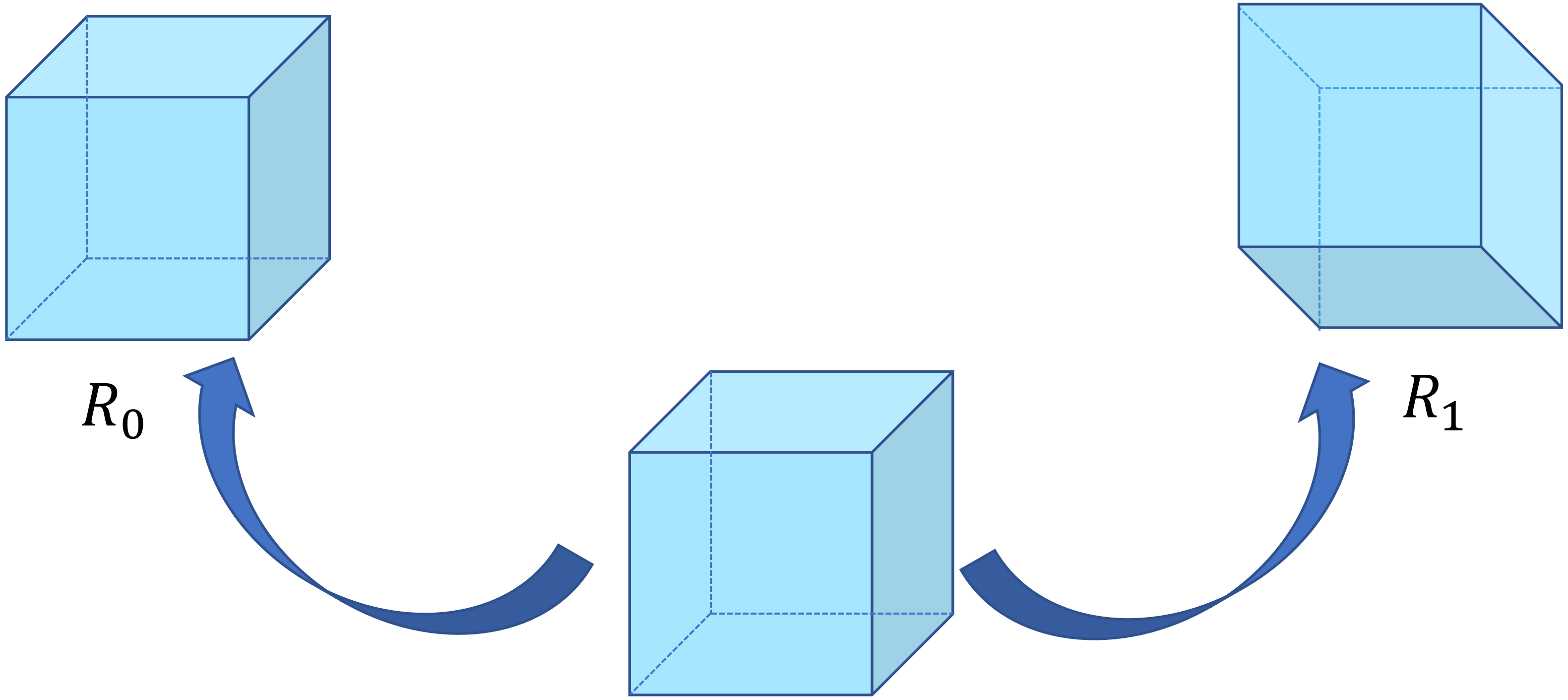


Interpolation of Translations

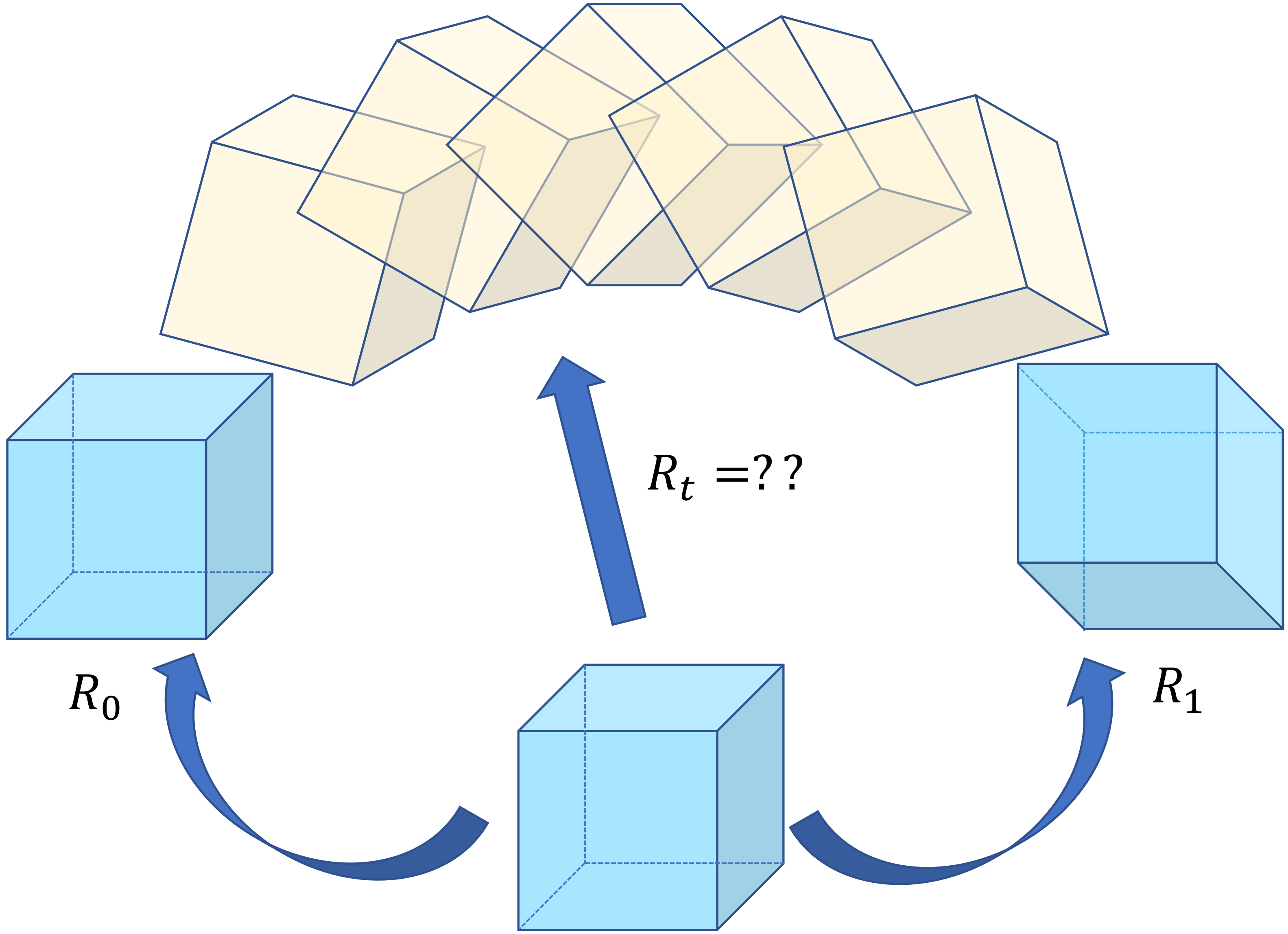


$$\boldsymbol{x}_t = (1-t)\boldsymbol{x}_0 + t\boldsymbol{x}_1$$

Interpolation of Rotations

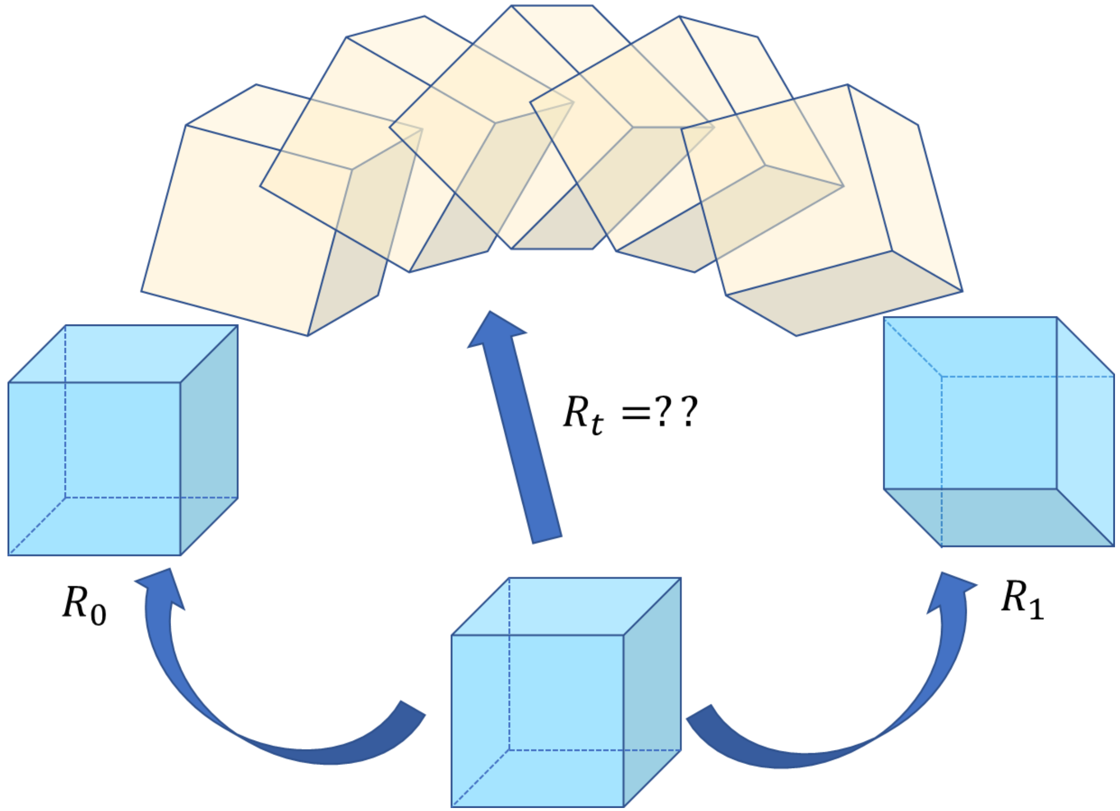


Interpolation of Rotations



Interpolation of Rotations

$$R_t = (1-t)R_0 + tR_1 \;\; ??$$



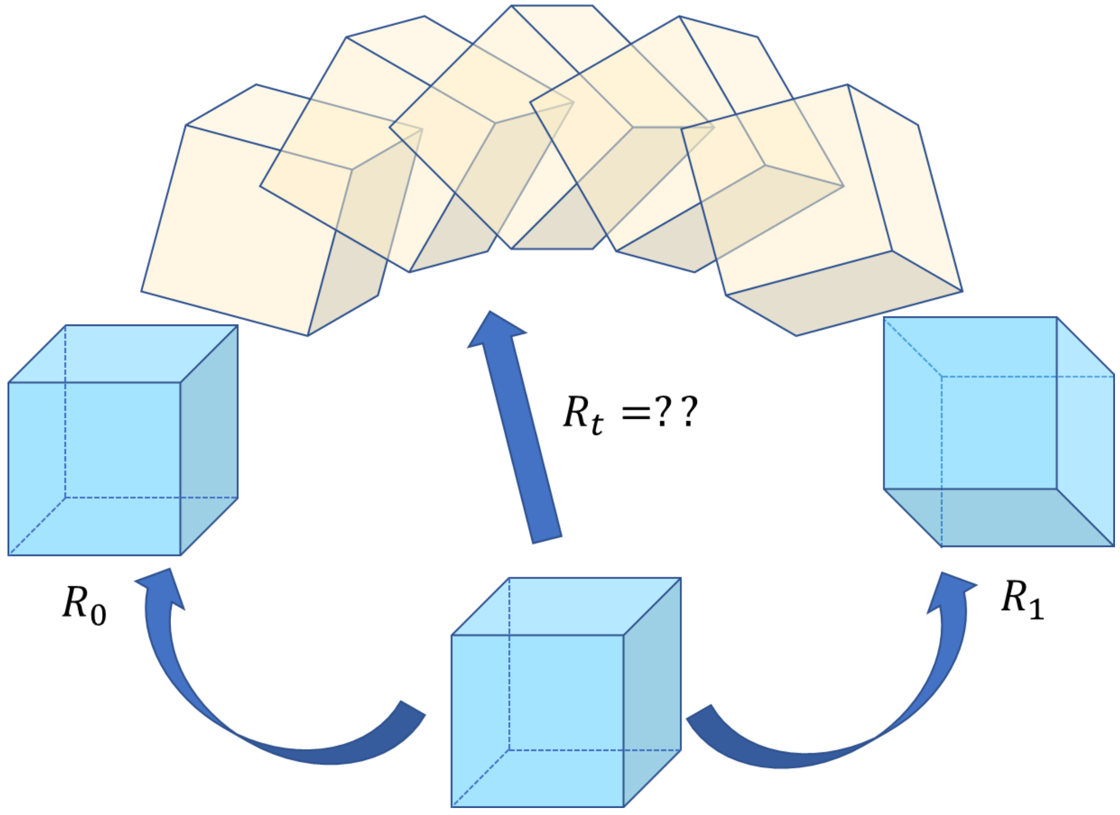
Interpolation of Rotations

$$R_t = (1-t)R_0 + tR_1 \;\; ??$$

$$R_0 = R_y(-90°) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

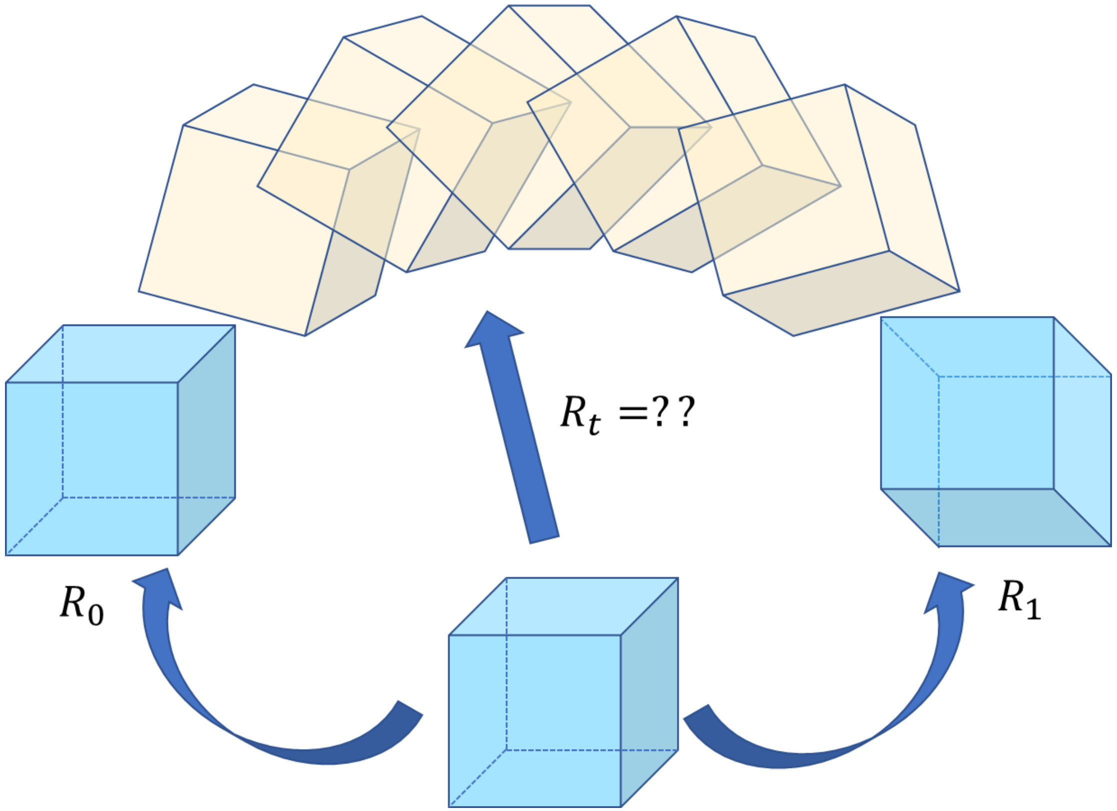
$$R_1 = R_y(+90°) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$R_{0.5} = 0.5(R_0 + R_1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Interpolation of Rotations

- What is good interpolation?
  - Rotation is valid at any time t
  - Constant rotational speed is preferred



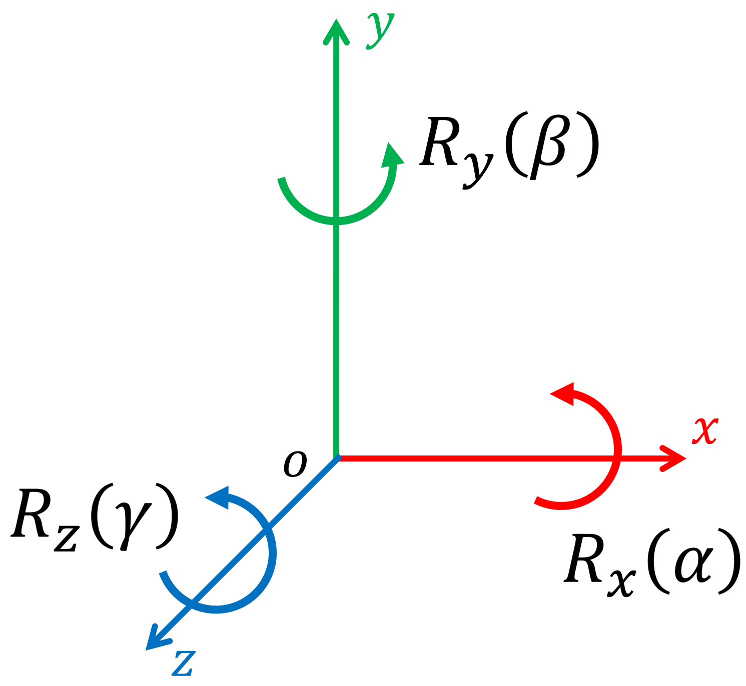
[回] Rotation Matrix

$$R = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad R^{\mathrm{T}} R = I$$

- Easy to compose? ✗
- Easy to apply? ✓
- Easy to interpolate? ✗

[回] Euler angles

- Basic rotations



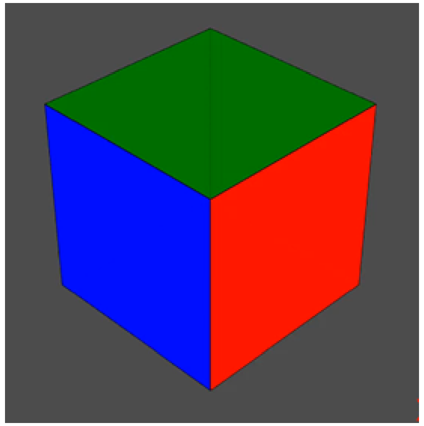
$$R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$$

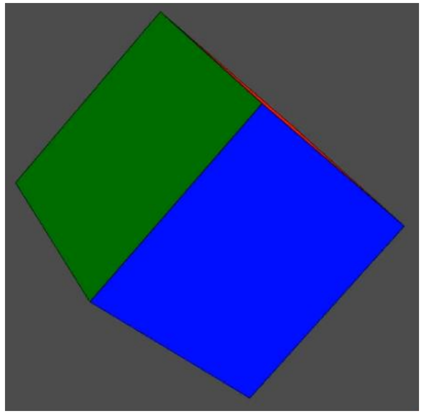
$$R_y(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}$$

$$R_z(\gamma) = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

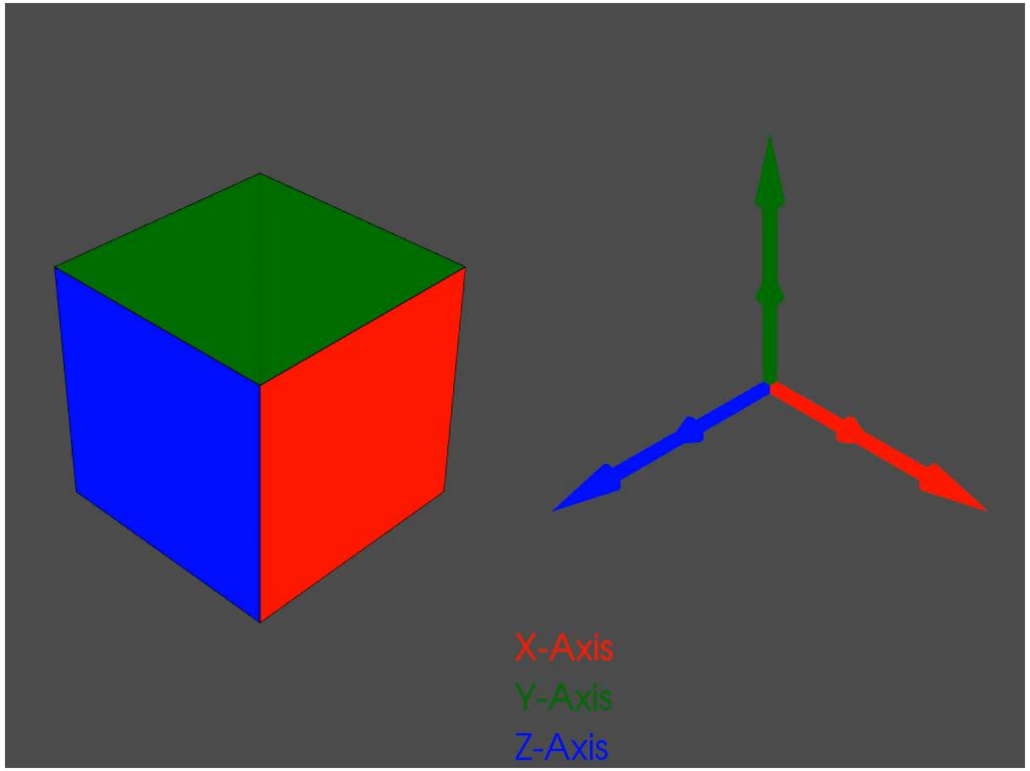
[回] Euler Angles

- Any rotation can be represented as a combination of three basic rotations



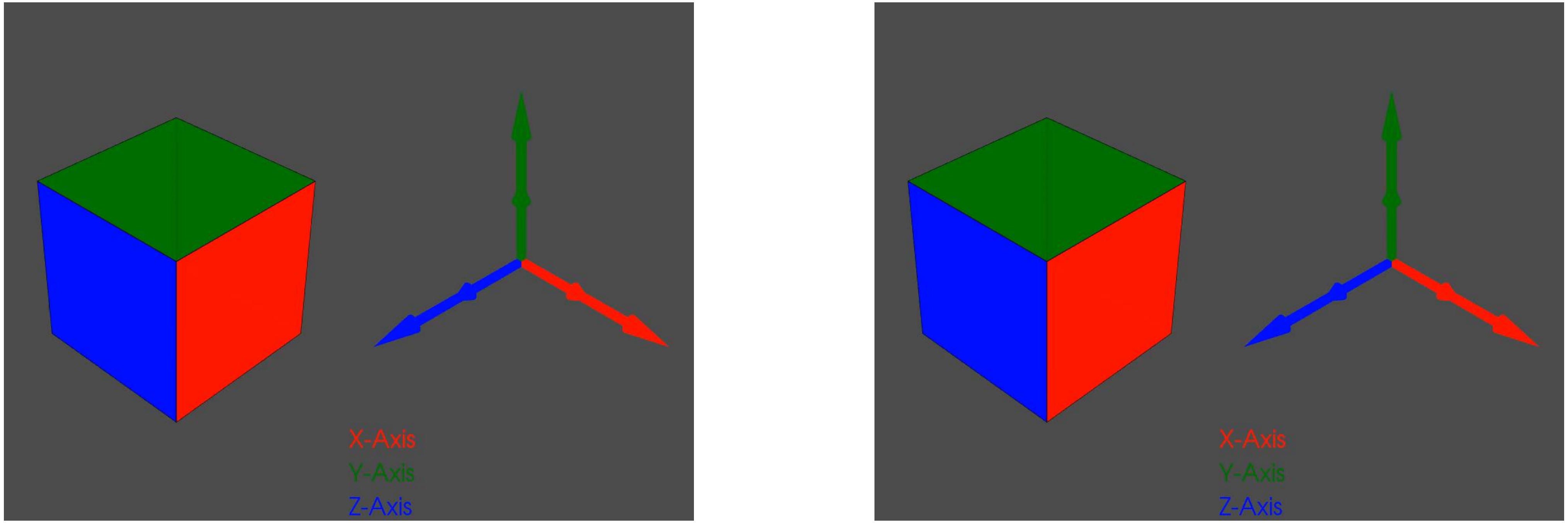


$$R_z(72^\circ)R_y(45^\circ)R_x(60^\circ)$$



[回] Euler Axes

- Any combination of three basic rotations are allowed
  - Excluding those rotate twice around the same axis
  - XYZ, XZY, YZX, YXZ, ZYX, ZXY, XYX, XZX, YXY, YZY, ZXZ, ZYZ

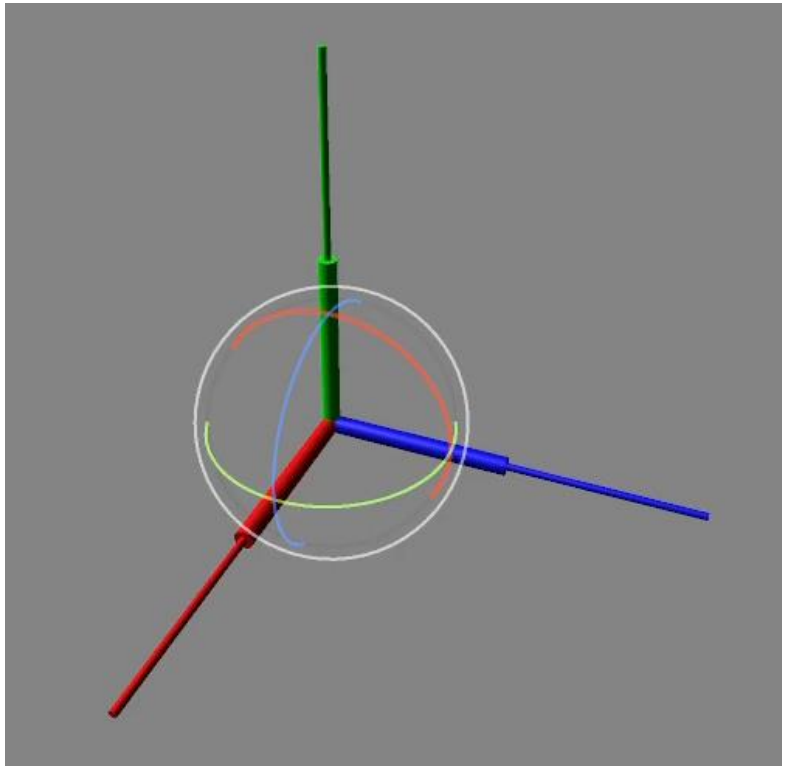


$R_z(72°)R_y(45°)R_x(60°)$ $R_x(69.2°)R_y(4.0°)R_z(42.4°)$

[回] Conventions of Euler Angles

intrinsic rotations:

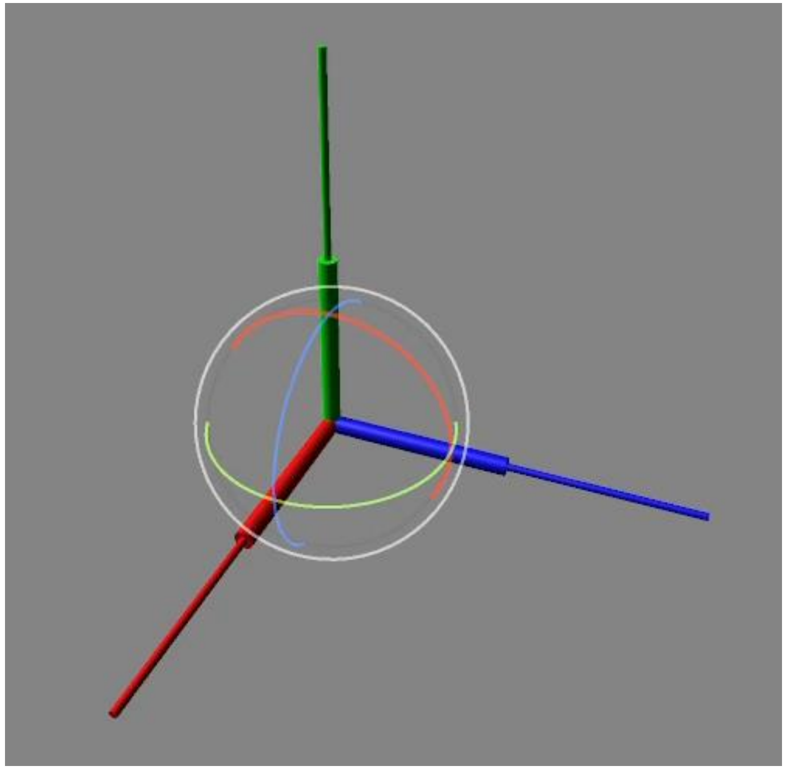
axes attached to the object



$R_x(\alpha)R_y(\beta)R_z(\gamma)$

extrinsic rotations:

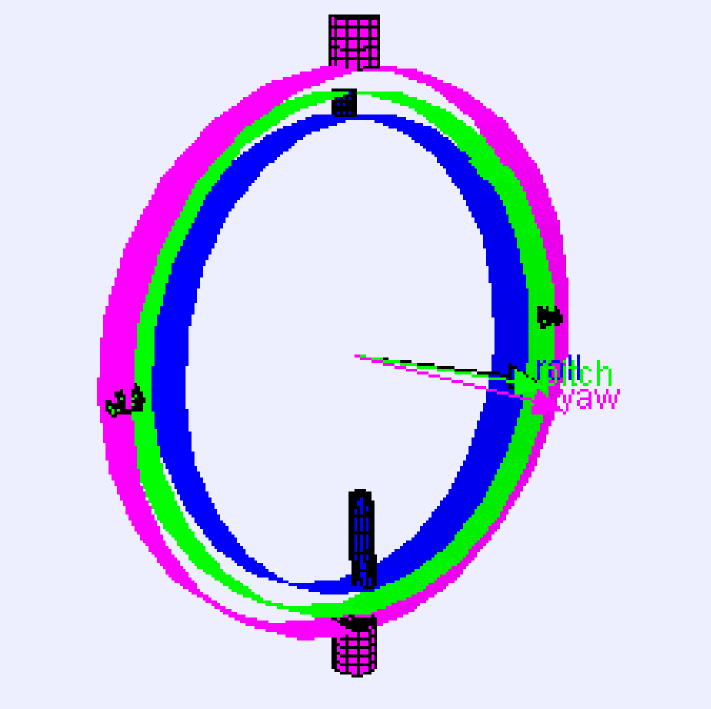
axes fixed to the world



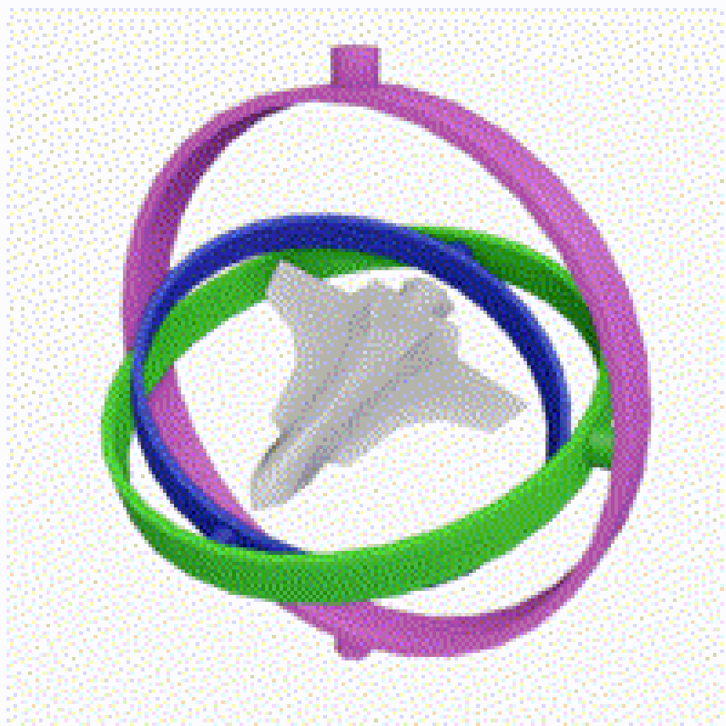
$R_z(\gamma)R_y(\beta)R_x(\alpha)$

[回] Gimbal Lock

- When two local axes are driven into a parallel configuration, one degree of freedom is “locked”



Normal Situation



Gimbal Lock

https://en.wikipedia.org/wiki/Gimbal_lock

[回] Euler Angles

$$R_x(\alpha)R_y(\beta)R_z(\gamma)$$

3 parameters: α, β, γ

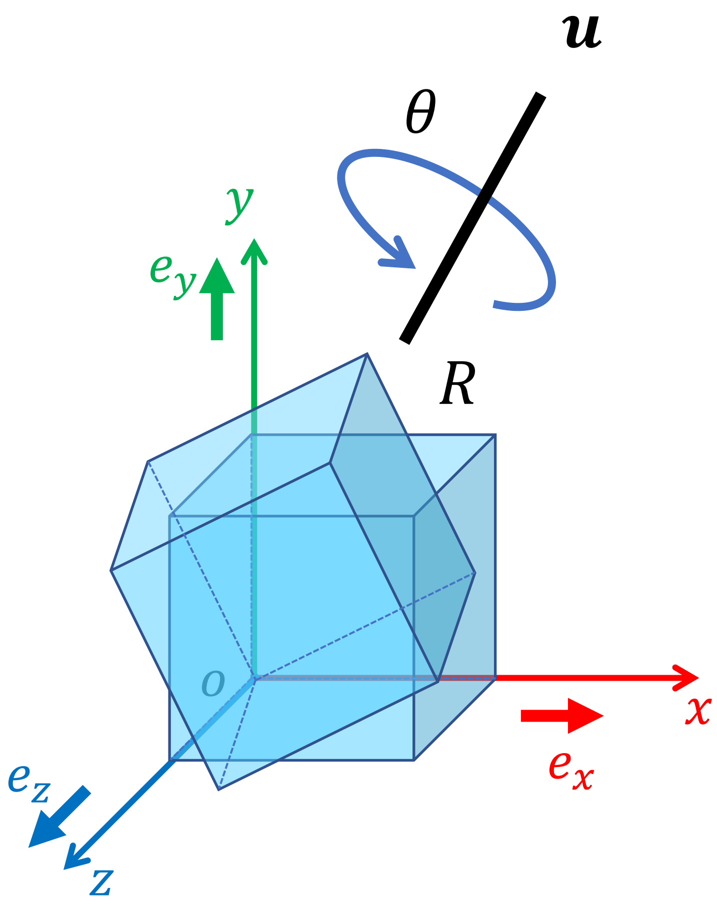
12 variations: XYZ, XZY, YZX, YXZ, ZYX, ZXY,
XYX, XZX, YXY, YZY, ZXZ, ZYZ

Intrinsic/Extrinsic rotations

- Easy to compose? ✓ But hard to create specific rotations
- Easy to apply? ✓ Need three matrix multiplications
- Easy to interpolate? ✓ Need to deal with singularities
- Gimbal lock ✗ rotational speed is not constant

[图] Rotation Vectors / Axis Angles

- Axis angle $(\boldsymbol{u}, \theta)$: represent a rotation using
  - A vector $\boldsymbol{u}$: rotation axis
  - A scalar θ: rotation angle

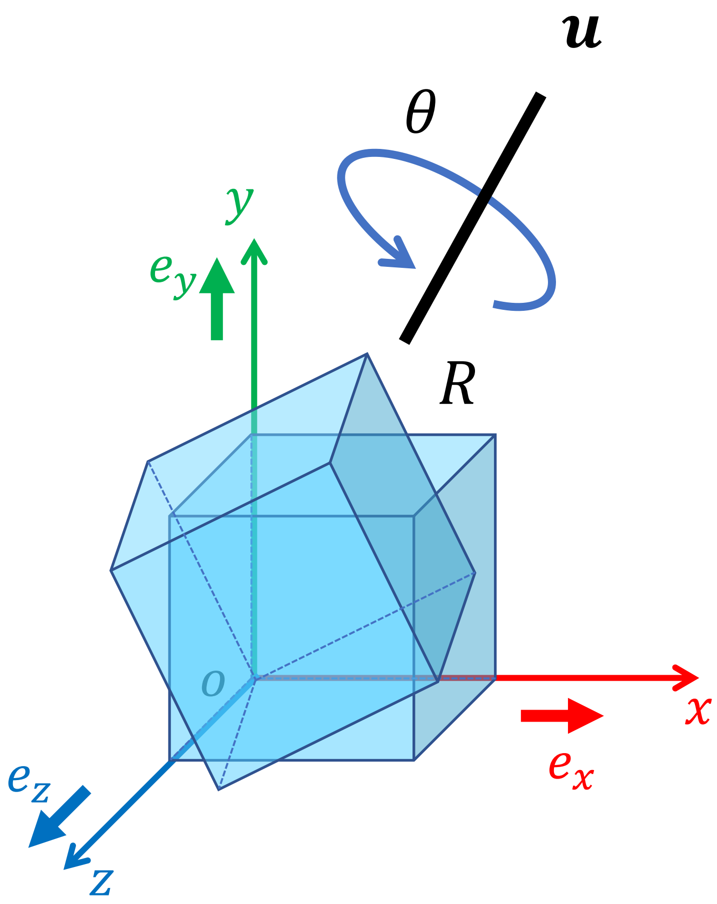


[附] Rotation Vectors / Axis Angles

- Axis angle $(\boldsymbol{u}, \theta)$: represent a rotation using
  - A vector $\boldsymbol{u}$: rotation axis
  - A scalar θ**:** rotation angle

- Rotation vector: represent a rotation as
  - $\boldsymbol{\theta} = \theta \boldsymbol{u}$
  - Obviously:

$$\theta = \|\boldsymbol{\theta}\| \qquad u = \frac{\boldsymbol{\theta}}{\|\boldsymbol{\theta}\|}$$



[田] Applying Rotation Vectors / Axis Angles

Rodrigues' rotation formula

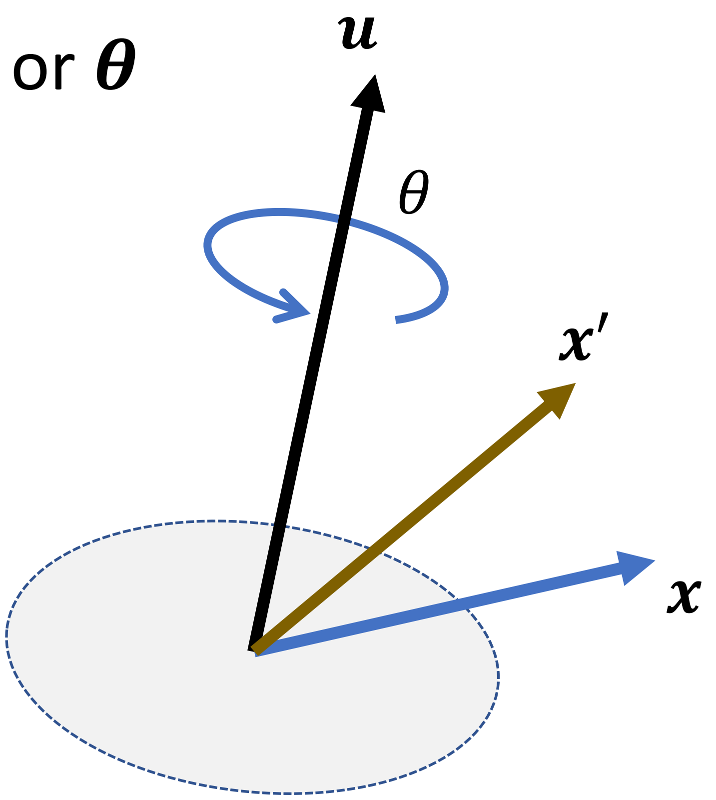
$$\boldsymbol{x}' = R\boldsymbol{x}$$

$$R = I + (\sin\theta)\ [\boldsymbol{u}]_\times + (1-\cos\theta)\ [\boldsymbol{u}]_\times^2$$

or

$$\boldsymbol{x}' = \boldsymbol{x} + (\sin\theta)\,\boldsymbol{u}\times\boldsymbol{x} + (1-\cos\theta)\,\boldsymbol{u}\times(\boldsymbol{u}\times\boldsymbol{x})$$

$(\boldsymbol{u}, \theta)$ or $\boldsymbol{\theta}$

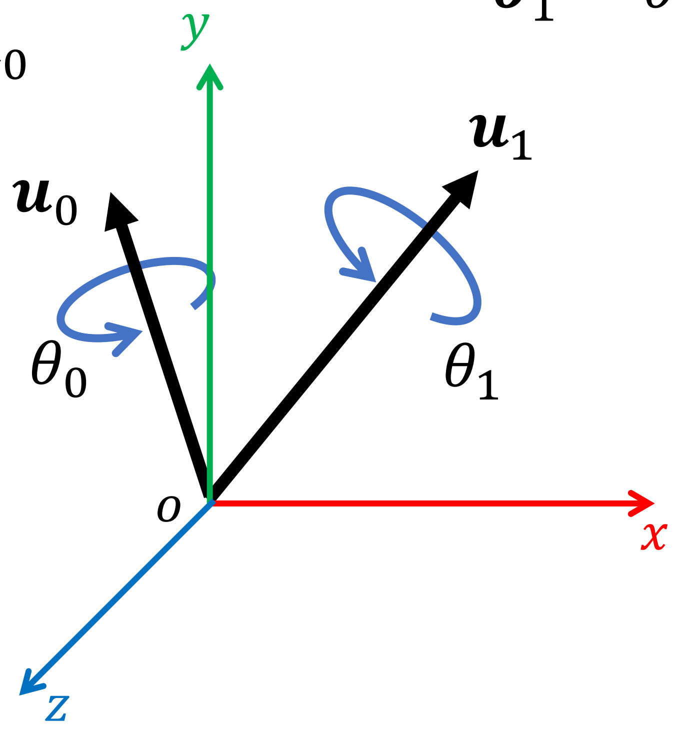


$\|\boldsymbol{u}\| = \mathbf{1}$

[附] Interpolating Rotation Vectors / Axis Angles

$\boldsymbol{\theta}_0 = \theta_0 \boldsymbol{u}_0$

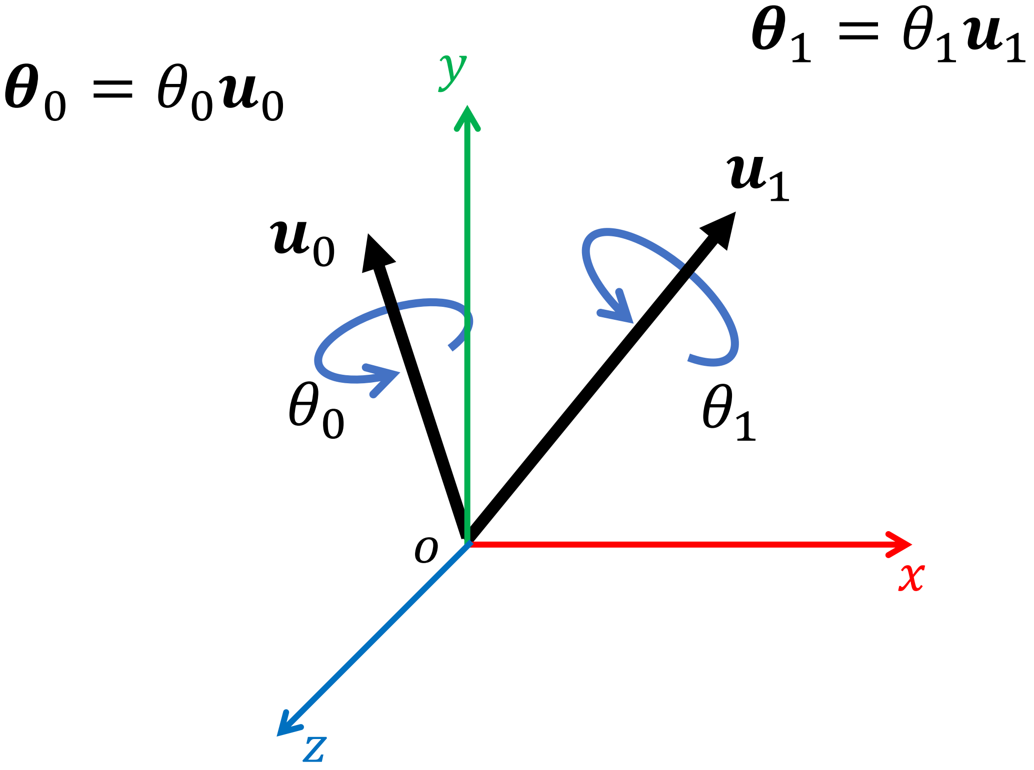
$\boldsymbol{\theta}_1 = \theta_1 \boldsymbol{u}_1$



Linear interpolation

$$\boldsymbol{\theta}_t = (1-t)\boldsymbol{\theta}_0 + t\boldsymbol{\theta}_1$$

[旧] Interpolating Rotation Vectors / Axis Angles

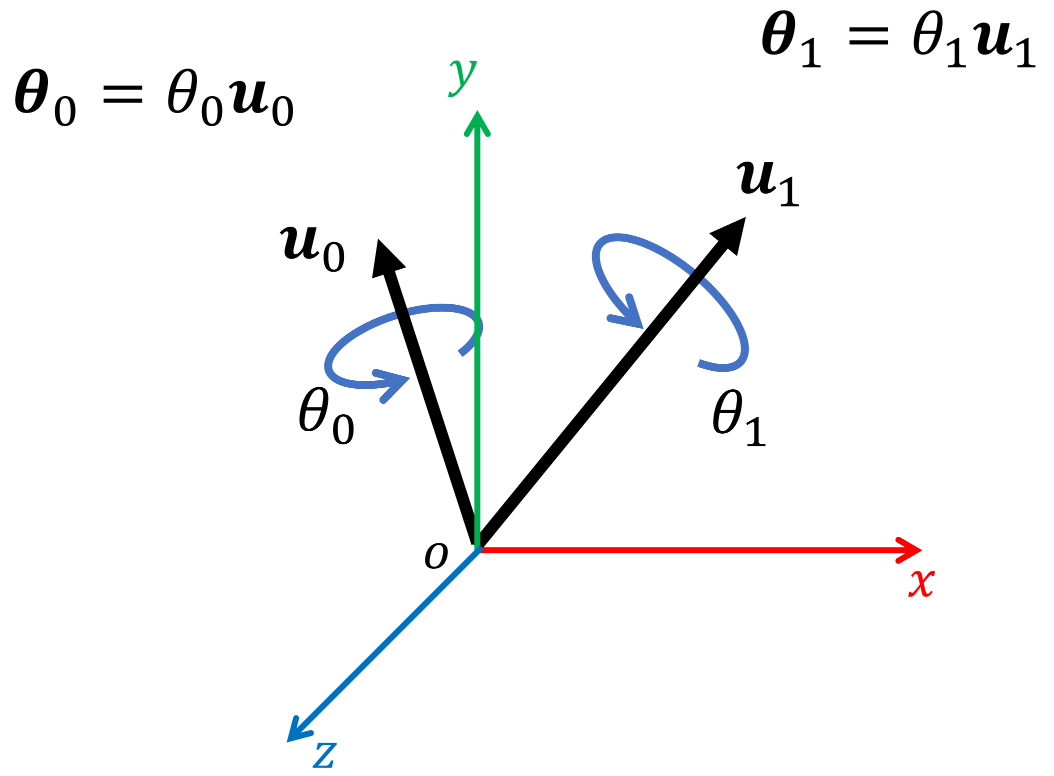


Linear interpolation

$$\boldsymbol{\theta}_t = (1-t)\boldsymbol{\theta}_0 + t\boldsymbol{\theta}_1$$

- $\boldsymbol{\theta}_t$ is valid ✓
- Constant speed? Not quite

[附] Interpolating Rotation Vectors / Axis Angles



Compute offset rotation

$$R(\delta\boldsymbol{\theta}) = R^T(\boldsymbol{\theta}_0)R(\boldsymbol{\theta}_1)$$

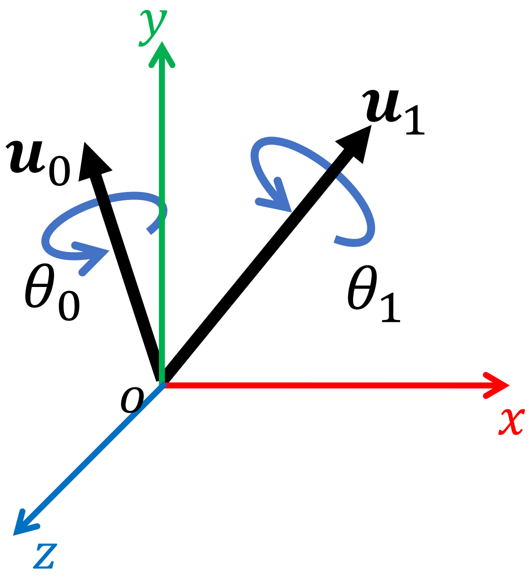
$$\delta\boldsymbol{\theta}_t = (1-t)\mathbf{0} + t\delta\boldsymbol{\theta}$$

$$R(\boldsymbol{\theta}_t) = R(\boldsymbol{\theta}_0)R(\delta\boldsymbol{\theta}_t)$$

- $\boldsymbol{\theta}_t$ is valid ✓
- Constant speed ✓

[田] Rotation Vectors / Axis Angles

$$(\boldsymbol{u}, \theta) \text{ or } \boldsymbol{\theta} = \theta \boldsymbol{u}$$



Representation is not unique

$$(\boldsymbol{u}, \theta), \quad (-\boldsymbol{u}, -\theta), \quad (\boldsymbol{u}, \theta + 2n\pi)$$

- Easy to compose? ✓ But hard to manipulate
- Easy to apply? ✗ Need to convert to matrix
- Easy to interpolate? ✓ Linear interpolation works, but not perfect need to deal with singularities
- No Gimbal lock ✓

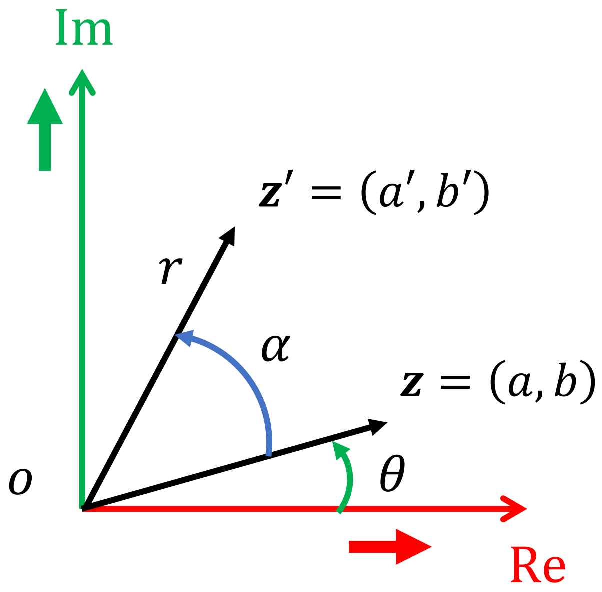
Quaternions

[圓] Quaternions

- Recall: a 2D rotation can be represented as a complex

$$\boldsymbol{z} = a + bi = re^{i\theta} \in \mathbb{C}, \qquad i^2 = -1$$

$$\begin{aligned} \boldsymbol{z}' &= re^{\boldsymbol{i}(\theta+\alpha)} \\ &= e^{\boldsymbol{i}\alpha} \times re^{\boldsymbol{i}\theta} \\ &= \boxed{e^{\boldsymbol{i}\alpha}}\,\boldsymbol{z} \end{aligned}$$



- How to deal with 3D rotation?

[回] Quaternions

- Extending complex numbers

$$z = a + bi + \boldsymbol{cj} + \boldsymbol{dk} + ????$$

$$i^2 = -1$$

$$j^2 = -1, j \neq i$$

$$k^2 = -1, k \neq i, j$$

[目] Quaternions

- Extending complex numbers

$$\boldsymbol{q} = a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k} \in \mathbb{H}, a, b, c, d \in \mathbb{R}$$

- $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$
- $\boldsymbol{ij} = \boldsymbol{k}, \boldsymbol{ji} = -\boldsymbol{k}$ (*cross product)
- $\boldsymbol{jk} = \boldsymbol{i}, \boldsymbol{kj} = -\boldsymbol{i}$
- $\boldsymbol{ki} = \boldsymbol{j}, \boldsymbol{ik} = -\boldsymbol{j}$

William Rowan Hamilton

[圄] Quaternion Arithmetic

$$\boldsymbol{q} = a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k} \in \mathbb{H}, a, b, c, d \in \mathbb{R}$$

Conjugation: $\boldsymbol{q}^* = a - b\boldsymbol{i} - c\boldsymbol{j} - d\boldsymbol{k}$

Scalar product: $t\boldsymbol{q} = ta + tb\boldsymbol{i} + tc\boldsymbol{j} + td\boldsymbol{k}$

Addition: $\boldsymbol{q_1} + \boldsymbol{q_2} = (a_1 + a_2) + (b_1 + b_2)\boldsymbol{i} + (c_1 + c_2)\boldsymbol{j} + (d_1 + d_2)\boldsymbol{k}$

Dot product: $\boldsymbol{q_1} \cdot \boldsymbol{q_2} = a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2$

Norm: $\|\boldsymbol{q}\| = \sqrt{a^2 + b^2 + c^2 + d^2} = \sqrt{\boldsymbol{q} \cdot \boldsymbol{q}}$

[回] Quaternion Multiplication

$$\boldsymbol{q_1}\ \boldsymbol{q_2} = (a_1 + b_1\boldsymbol{i} + c_1\boldsymbol{j} + d_1\boldsymbol{k}) * (a_2 + b_2\boldsymbol{i} + c_2\boldsymbol{j} + d_2\boldsymbol{k})$$

$$\begin{aligned}\boldsymbol{q_1}\boldsymbol{q_2} = & \ a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2 \\ & +(b_1a_2 + a_1b_2 - d_1c_2 + c_1d_2)\boldsymbol{i} \\ & +(c_1a_2 + d_1b_2 + a_1c_2 - b_1d_2)\boldsymbol{j} \\ & +(d_1a_2 - c_1b_2 + b_1c_2 + a_1d_2)\boldsymbol{k}\end{aligned}$$

note:

- $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$
- $\boldsymbol{ij} = \boldsymbol{k}, \boldsymbol{ji} = -\boldsymbol{k}$ (*cross product)
- $\boldsymbol{jk} = \boldsymbol{i}, \boldsymbol{kj} = -\boldsymbol{i}$
- $\boldsymbol{ki} = \boldsymbol{j}, \boldsymbol{ik} = -\boldsymbol{j}$

[回] Quaternions

$$\boldsymbol{q} = w + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \quad \Rightarrow \quad \boldsymbol{q} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} w \\ \boldsymbol{v} \end{bmatrix}$$

$$q = [w, \boldsymbol{v}]^{\mathrm{T}} \in \mathbb{H},\ w \in \mathbb{R}, \boldsymbol{v} \in \mathbb{R}^3$$

$w = [w, \mathbf{0}]^{\mathrm{T}}$: scalar quaternion

$\boldsymbol{v} = [0, \boldsymbol{v}]^{\mathrm{T}}$: pure quaternion

[囙] Quaternion Arithmetic

$$\boldsymbol{q} = w + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \quad \Rightarrow \quad \boldsymbol{q} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} w \\ \boldsymbol{v} \end{bmatrix}$$

Conjugation: $\boldsymbol{q}^* = [w, -\boldsymbol{v}]^{\mathrm{T}}$

Scalar product: $t\boldsymbol{q} = [tw, t\boldsymbol{v}]^{\mathrm{T}}$

Addition: $\boldsymbol{q_1} + \boldsymbol{q_2} = [w_1 + w_2, \boldsymbol{v_1} + \boldsymbol{v_2}]^{\mathrm{T}}$

Dot product: $\boldsymbol{q_1} \cdot \boldsymbol{q_2} = w_1 w_2 + \boldsymbol{v_1} \cdot \boldsymbol{v_2}$

Norm: $\|\boldsymbol{q}\| = \sqrt{w_1 w_2 + \boldsymbol{v_1} \cdot \boldsymbol{v_2}} = \sqrt{\boldsymbol{q} \cdot \boldsymbol{q}}$

[圓] Quaternion Multiplication

$$\boldsymbol{q_1}\boldsymbol{q_2} = \begin{bmatrix} w_1 \\ \boldsymbol{v_1} \end{bmatrix} \begin{bmatrix} w_2 \\ \boldsymbol{v_2} \end{bmatrix} = \begin{bmatrix} w_1 w_2 - \boldsymbol{v}_1 \cdot \boldsymbol{v}_2 \\ w_1 \boldsymbol{v}_2 + w_2 \boldsymbol{v}_1 + \boldsymbol{v}_1 \times \boldsymbol{v}_2 \end{bmatrix}$$

[圓] Quaternion Multiplication

$$\boldsymbol{q}_1\boldsymbol{q}_2 = \begin{bmatrix} w_1 \\ \boldsymbol{v}_1 \end{bmatrix} \begin{bmatrix} w_2 \\ \boldsymbol{v}_2 \end{bmatrix} = \begin{bmatrix} w_1 w_2 - \boldsymbol{v}_1 \cdot \boldsymbol{v}_2 \\ w_1 \boldsymbol{v}_2 + w_2 \boldsymbol{v}_1 + \boldsymbol{v}_1 \times \boldsymbol{v}_2 \end{bmatrix}$$

Non-Commutativity:

$$\boldsymbol{q}_1\boldsymbol{q}_2 \neq \boldsymbol{q}_2\boldsymbol{q}_1$$

Associativity:

$$\boldsymbol{q}_1\boldsymbol{q}_2\boldsymbol{q}_3 = (\boldsymbol{q}_1\boldsymbol{q}_2)\boldsymbol{q}_3 = \boldsymbol{q}_1(\boldsymbol{q}_2\boldsymbol{q}_3)$$

[圓] Quaternion Multiplication

$$\boldsymbol{q_1 q_2} = \begin{bmatrix} w_1 \\ \boldsymbol{v}_1 \end{bmatrix} \begin{bmatrix} w_2 \\ \boldsymbol{v}_2 \end{bmatrix} = \begin{bmatrix} w_1 w_2 - \boldsymbol{v}_1 \cdot \boldsymbol{v}_2 \\ w_1 \boldsymbol{v}_2 + w_2 \boldsymbol{v}_1 + \boldsymbol{v}_1 \times \boldsymbol{v}_2 \end{bmatrix}$$

Conjugation:

$$(\boldsymbol{q_1 q_2})^* = \boldsymbol{q_2^* q_1^*}$$

Norm:

$$\|\boldsymbol{q}\|^2 = \boldsymbol{q}^* \boldsymbol{q} = \boldsymbol{q} \boldsymbol{q}^*$$

Reciprocal:

$$\boldsymbol{q q^{-1} = 1}$$

$$\boldsymbol{q^{-1} q = 1}$$

$$\Rightarrow \quad \boldsymbol{q^{-1}} = \frac{\boldsymbol{q}^*}{\|\boldsymbol{q}\|^2}$$

[囸] Unit Quaternions

$$\boldsymbol{q} = \begin{bmatrix} w \\ \boldsymbol{v} \end{bmatrix} \qquad \|\boldsymbol{q}\| = 1$$

For any non-zero quaternion $\tilde{\boldsymbol{q}}$:

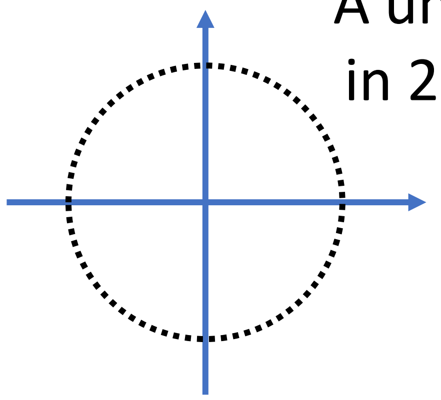
$$\boldsymbol{q} = \frac{\tilde{\boldsymbol{q}}}{\|\tilde{\boldsymbol{q}}\|}$$

Reciprocal:

$$\boldsymbol{q}^{-\mathbf{1}} = \boldsymbol{q}^{*} = \begin{bmatrix} w \\ -\boldsymbol{v} \end{bmatrix} \quad \Leftrightarrow \quad R^{-1} = R^{\mathrm{T}}$$

[圓] Unit Quaternions

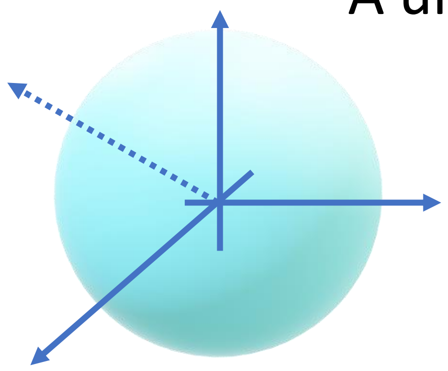
$$\boldsymbol{q} = \begin{bmatrix} w \\ \boldsymbol{v} \end{bmatrix} \qquad \|\boldsymbol{q}\| = 1$$



A unit circle in 2D space

unit complex number

$$z = \cos\theta + i\sin\theta$$



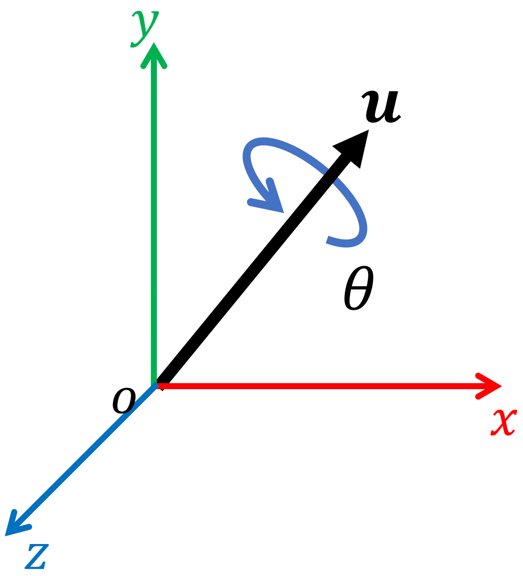
A unit hypersphere in 4D space

unit quaternion

$$\boldsymbol{q} = \left[\cos\frac{\theta}{2}, \boldsymbol{u}\sin\frac{\theta}{2}\right] \quad \|\boldsymbol{u}\| = 1$$

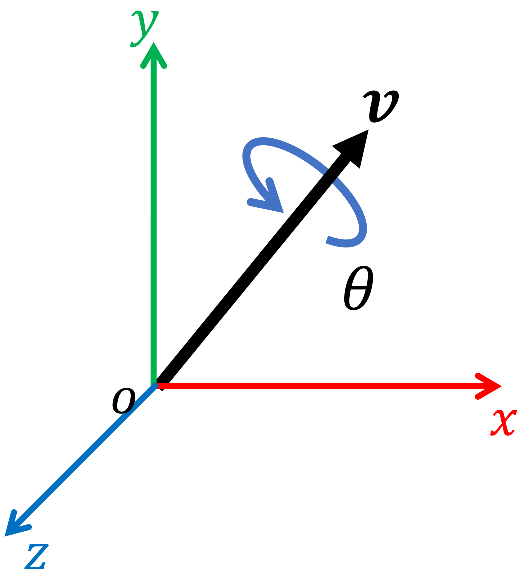
[回] Unit Quaternions

$$\boldsymbol{q} = \begin{bmatrix} w \\ \boldsymbol{v} \end{bmatrix} = \left[\cos\frac{\theta}{2}, \boldsymbol{u}\sin\frac{\theta}{2}\right] \qquad \|\boldsymbol{u}\| = 1$$



same information as axis angles $(\boldsymbol{u}, \theta)$
But in a different form

[圓] Unit Quaternions as 3D Rotations



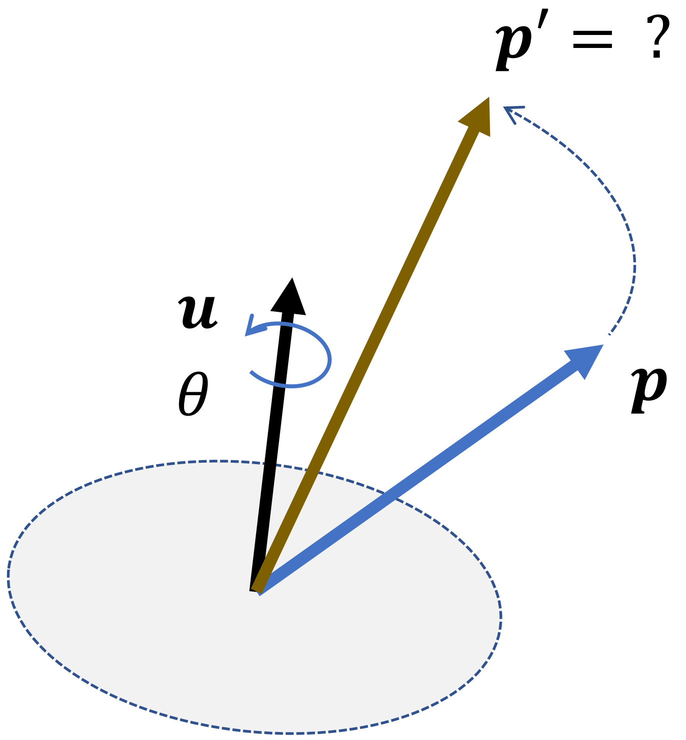
Any 3D rotation $(\boldsymbol{v}, \theta)$ can be represented as a unit quaternion

$$\boldsymbol{q} = \begin{bmatrix} w \\ \boldsymbol{v} \end{bmatrix} = \left[\cos\frac{\theta}{2}, \boldsymbol{u}\sin\frac{\theta}{2}\right]$$

Angle: $\theta = 2\arg\cos w$

Axis: $\boldsymbol{u} = \dfrac{\boldsymbol{v}}{\|\boldsymbol{v}\|}$

[回] Rotation a Vector Using Unit Quaternions



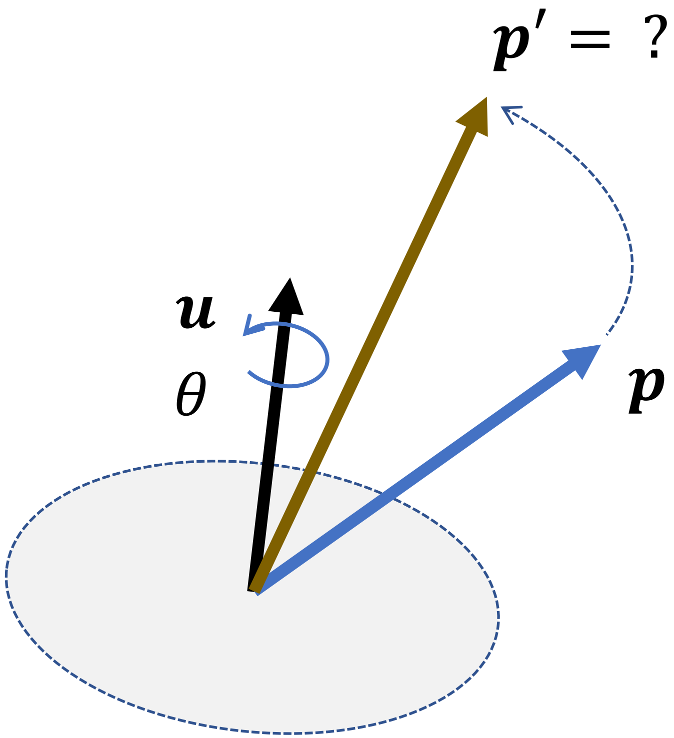
Unit quaternion: $\boldsymbol{q} = \begin{bmatrix} w \\ \boldsymbol{v} \end{bmatrix} = \left[\cos\frac{\theta}{2}, \boldsymbol{u}\sin\frac{\theta}{2}\right]$

3D vector: $\boldsymbol{p}$ Rotation result: $\boldsymbol{p}'$

Then the rotation can be applied by quaternion multiplication:

$$\begin{bmatrix} 0 \\ \boldsymbol{p}' \end{bmatrix} = \boldsymbol{q} \begin{bmatrix} 0 \\ \boldsymbol{p} \end{bmatrix} \boldsymbol{q}^*$$

[回] Rotation a Vector Using Unit Quaternions



Unit quaternion: $\boldsymbol{q} = \begin{bmatrix} w \\ \boldsymbol{v} \end{bmatrix} = \left[\cos\frac{\theta}{2}, \boldsymbol{u}\sin\frac{\theta}{2}\right]$

3D vector: $\boldsymbol{p}$ Rotation result: $\boldsymbol{p}'$

Then the rotation can be applied by quaternion multiplication:

$$\begin{bmatrix} 0 \\ \boldsymbol{p}' \end{bmatrix} = \boldsymbol{q} \begin{bmatrix} 0 \\ \boldsymbol{p} \end{bmatrix} \boldsymbol{q}^* = (-\boldsymbol{q}) \begin{bmatrix} 0 \\ \boldsymbol{p} \end{bmatrix} (-\boldsymbol{q})^*$$

$\boldsymbol{q}$ and $-\boldsymbol{q}$ represent the same rotation

[圓] Combination of Rotations

Unit quaternion: $\boldsymbol{q_1}, \boldsymbol{q_2}$

3D vector: $\boldsymbol{p}$

$$\begin{bmatrix} 0 \\ \boldsymbol{p}' \end{bmatrix} = \boldsymbol{q}_1 \begin{bmatrix} 0 \\ \boldsymbol{p} \end{bmatrix} \boldsymbol{q}_1^*$$

$$\begin{bmatrix} 0 \\ \boldsymbol{p}'' \end{bmatrix} = \boldsymbol{q}_2 \begin{bmatrix} 0 \\ \boldsymbol{p}' \end{bmatrix} \boldsymbol{q}_2^* = \boldsymbol{q}_2 \left(\boldsymbol{q}_1 \begin{bmatrix} 0 \\ \boldsymbol{p} \end{bmatrix} \boldsymbol{q}_1^* \right) \boldsymbol{q}_2^* = (\boldsymbol{q}_2 \boldsymbol{q}_1) \begin{bmatrix} 0 \\ \boldsymbol{p} \end{bmatrix} (\boldsymbol{q}_2 \boldsymbol{q}_1)^*$$

$$= \boldsymbol{q} \begin{bmatrix} 0 \\ \boldsymbol{p} \end{bmatrix} \boldsymbol{q}^*$$

[圓] Combination of Rotations

Unit quaternion: $\boldsymbol{q_1}, \boldsymbol{q_2}$ ⇨ Combined rotation: $\boldsymbol{q} = \boldsymbol{q_2 q_1}$

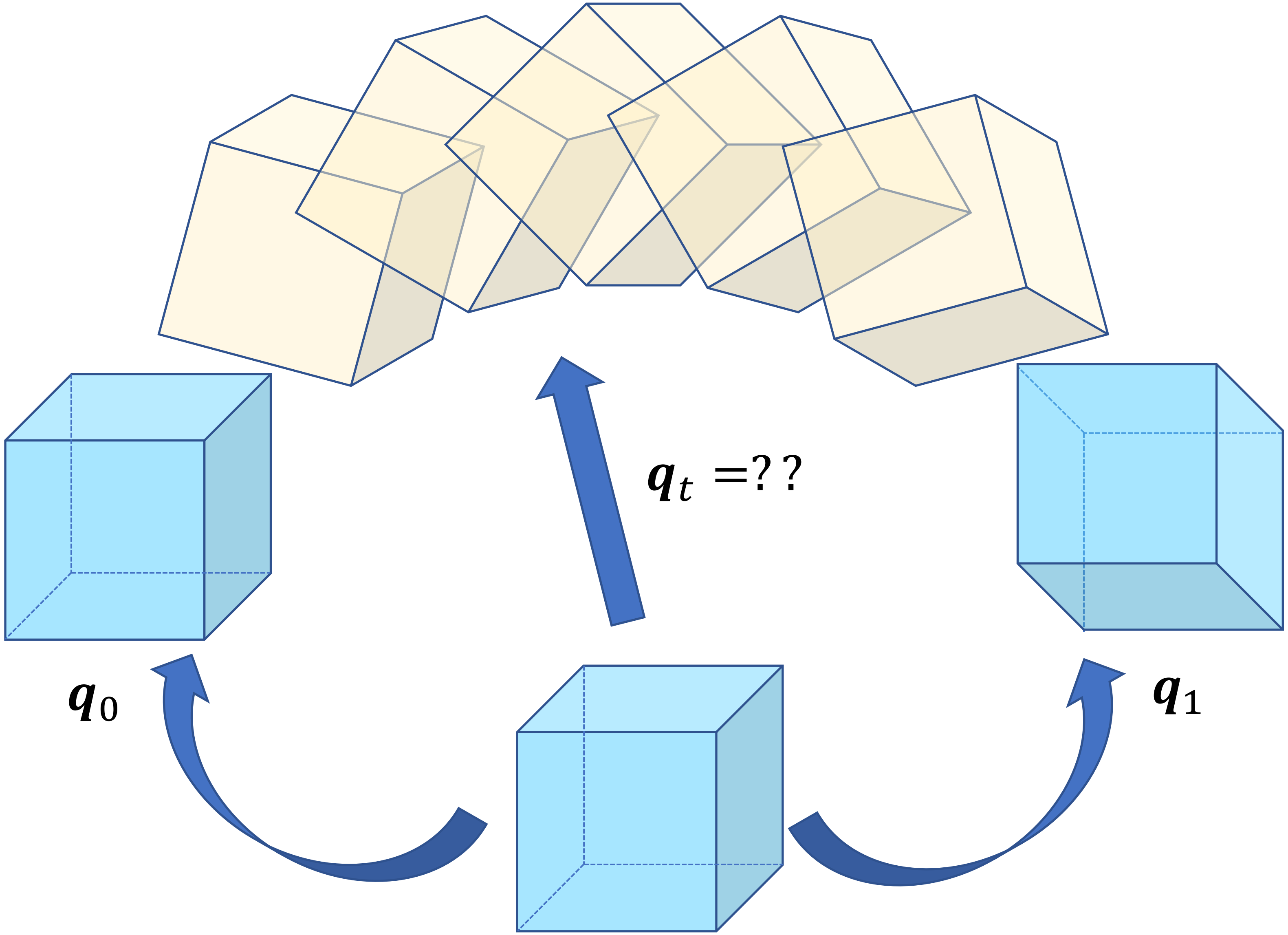
3D vector: $\boldsymbol{p}$

$$\begin{bmatrix} 0 \\ \boldsymbol{p}' \end{bmatrix} = \boldsymbol{q}_1 \begin{bmatrix} 0 \\ \boldsymbol{p} \end{bmatrix} \boldsymbol{q}_1^*$$

$$\begin{bmatrix} 0 \\ \boldsymbol{p}'' \end{bmatrix} = \boldsymbol{q}_2 \begin{bmatrix} 0 \\ \boldsymbol{p}' \end{bmatrix} \boldsymbol{q}_2^* = \boldsymbol{q}_2 \left(\boldsymbol{q}_1 \begin{bmatrix} 0 \\ \boldsymbol{p} \end{bmatrix} \boldsymbol{q}_1^* \right) \boldsymbol{q}_2^* = (\boldsymbol{q}_2 \boldsymbol{q}_1) \begin{bmatrix} 0 \\ \boldsymbol{p} \end{bmatrix} (\boldsymbol{q}_2 \boldsymbol{q}_1)^*$$

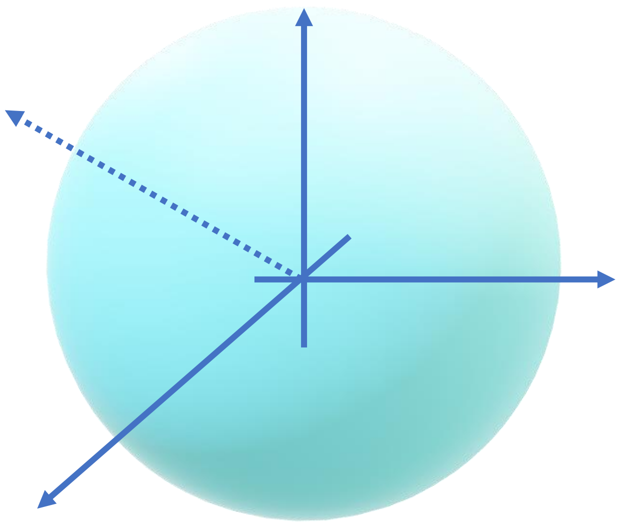
$$= \boldsymbol{q} \begin{bmatrix} 0 \\ \boldsymbol{p} \end{bmatrix} \boldsymbol{q}^*$$

[圓] Quaternion Interpolation



[回] Quaternion Interpolation

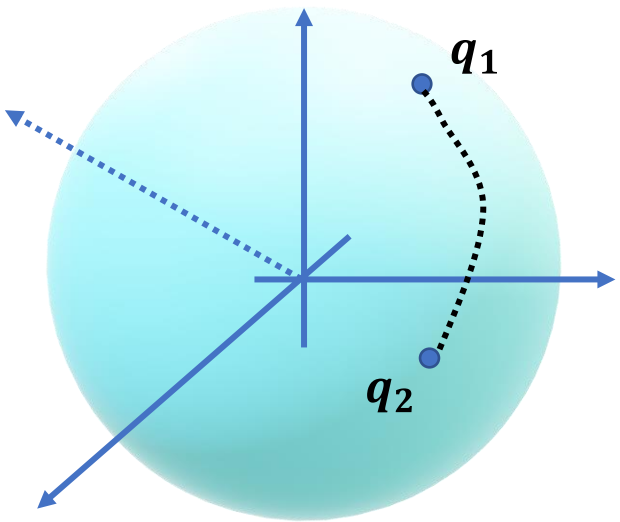
$$\boldsymbol{q} = \begin{bmatrix} w \\ \boldsymbol{v} \end{bmatrix} \qquad \|\boldsymbol{q}\| = 1$$



A unit hypersphere
in 4D space

[回] Quaternion Interpolation

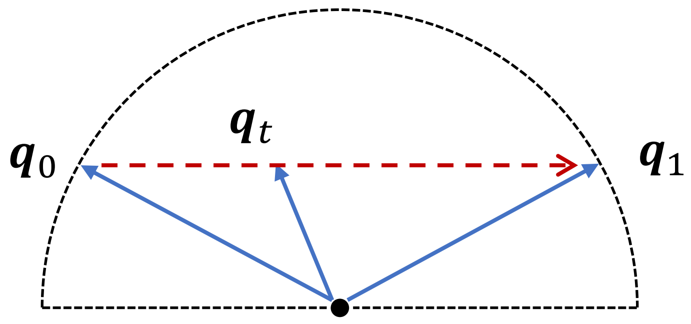
$$\boldsymbol{q} = \begin{bmatrix} w \\ \boldsymbol{v} \end{bmatrix} \qquad \|\boldsymbol{q}\| = 1$$



A unit hypersphere
in 4D space

[圓] Linear Interpolation

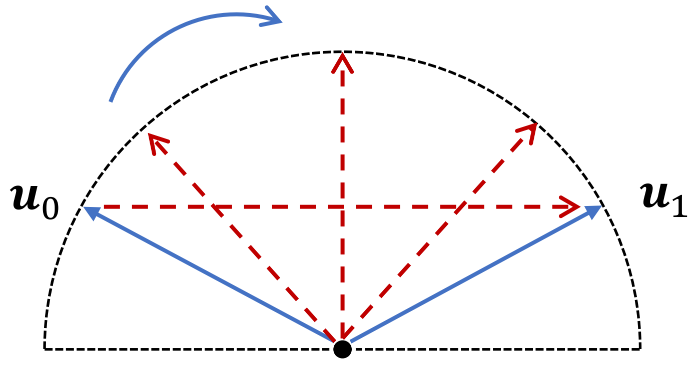
$$\boldsymbol{q_t} = (1-t)\boldsymbol{q}_0 + t\boldsymbol{q}_1$$



$\boldsymbol{q_t}$ is not a unit quaternion

[圓] Linear Interpolation + Projection

$$\tilde{\boldsymbol{q}}_t = (1-t)\boldsymbol{q}_0 + t\boldsymbol{q}_1 \qquad \boldsymbol{q}_t = \frac{\tilde{\boldsymbol{q}}_t}{\|\tilde{\boldsymbol{q}}_t\|}$$

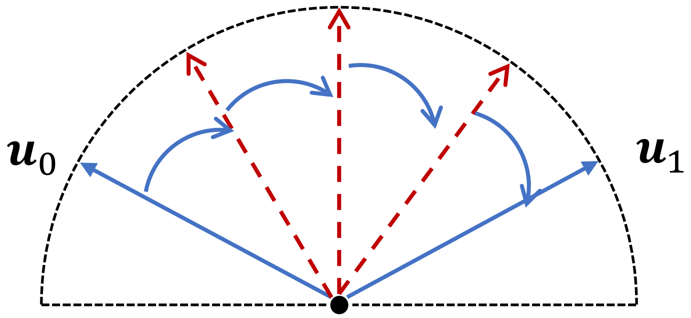


$\boldsymbol{q}_t$ is a unit quaternion

Rotational speed is not constant

[圓] SLERP: Spherical Linear Interpolation

$$\boldsymbol{q}_t = a(t)\boldsymbol{q}_0 + b(t)\boldsymbol{q}_1$$



[回] SLERP: Spherical Linear Interpolation

$$r = a(t)p + b(t)q$$

Consider the angle θ between p, q: $\cos\theta = p \cdot q$

We have:

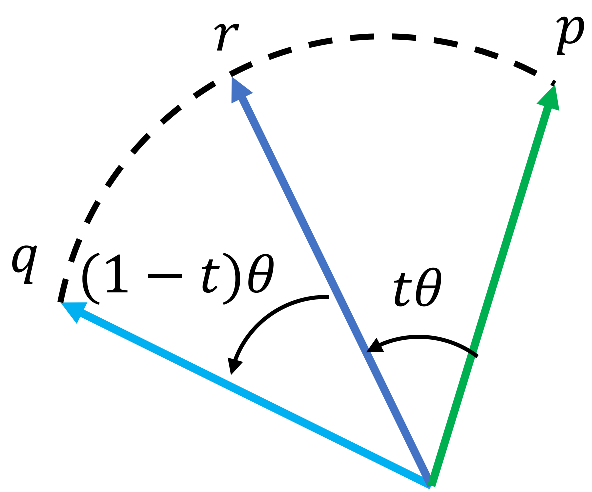
$$p \cdot r = a(t)p \cdot p + b(t)q \cdot p$$
$$\Rightarrow \quad \cos t\theta = a(t) + b(t)\cos\theta$$

similarly

$$q \cdot r = a(t)q \cdot p + b(t)$$
$$\Rightarrow \quad \cos(1-t)\theta = a(t)\cos\theta + b(t)$$

then we have：

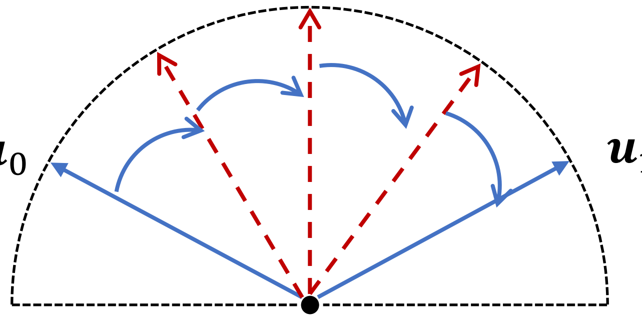
$$a(t) = \frac{\sin[(1-t)\theta]}{\sin\theta}, \quad b(t) = \frac{\sin t\theta}{\sin\theta}$$



[圓] SLERP: Spherical Linear Interpolation

$$\boldsymbol{q}_t = \frac{\sin[(1-t)\theta]}{\sin\theta}\boldsymbol{q}_0 + \frac{\sin t\theta}{\sin\theta}\boldsymbol{q}_1$$

$$\cos\theta = \boldsymbol{q}_0 \cdot \boldsymbol{q}_1$$



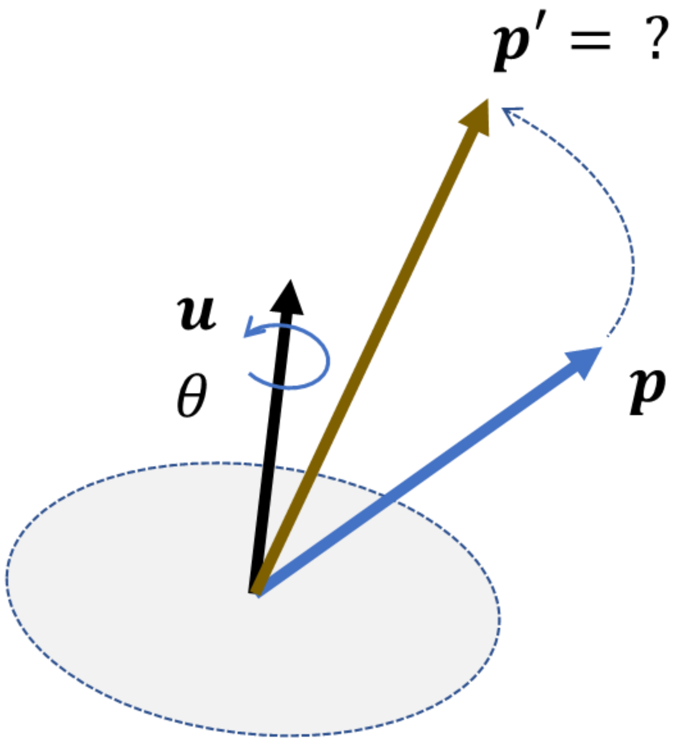
[回] Quaternions

Rotations can be represented by unit quaternions

$$\boldsymbol{q} = \begin{bmatrix} w \\ \boldsymbol{v} \end{bmatrix} = \left[\cos\frac{\theta}{2}, \boldsymbol{u}\sin\frac{\theta}{2}\right]$$

Representation is not unique

$\boldsymbol{q}, -\boldsymbol{q}$ represent the same rotation



- Easy to compose? ✓ Need normalization, hard to manipulate,
- Easy to apply? ✓ Quaternion multiplication
- Easy to interpolate? ✓ SLERP, need to deal with singularities
- No Gimbal lock ✓

Questions?