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Aufgabe 1:

$$\dot{x}(t) = f_1(x) = \begin{pmatrix} -4 & -2 \\ 1 & -2 \end{pmatrix} x(t) + \begin{pmatrix} 4\cdot\sin(t) \\ \sin(t) - 2\cdot\cos(t) \end{pmatrix}, \quad x(0) = x_0$$

$$\dot{x}(t) = f_2(x) = \begin{pmatrix} -400 & 0 \\ 1 & -2 \end{pmatrix} x(t) + \begin{pmatrix} 400\cdot\sin(t) \\ \sin(t) - 2\cdot\cos(t) \end{pmatrix}, \quad x(0) = x_0.$$

a) für $f_1(\underline{x})$: $\det(\lambda\cdot\underline{\underline{I}} - \underline{\underline{A}}) \overset{!}{=} 0 \Rightarrow \begin{vmatrix} \lambda+4 & 2 \\ -1 & \lambda+2 \end{vmatrix} \overset{!}{=} 0$

$$\Rightarrow (\lambda+4)(\lambda+2)+2 \overset{!}{=} 0 \Rightarrow \begin{cases} \lambda_1 = -3+i \\ \lambda_2 = -3-i \end{cases}$$

$$\Rightarrow T_{max} = \frac{2\pi}{|Im(\lambda_i)|} = 2\pi \qquad T_{min} = \frac{1}{|Re(\lambda_i)|} = \frac{1}{3}$$

$$\Rightarrow \text{Steifigkeit koeffizient}: \frac{T_{max}}{T_{min}} = 6\pi$$

für $f_2(\underline{x})$: $\det(\lambda\cdot\underline{\underline{I}} - \underline{\underline{A}}) \overset{!}{=} 0 \Rightarrow \begin{vmatrix} \lambda+400 & 0 \\ -1 & \lambda+ \end{vmatrix} \overset{!}{=} 0$

$$\Rightarrow (\lambda+400)(\lambda+2) \overset{!}{=} 0 \Rightarrow \begin{cases} \lambda_1 = -400 \\ \lambda_2 = -2 \end{cases}$$

$$\Rightarrow T_{max} = \frac{1}{|Re(\lambda_2)|} = \frac{1}{2} \qquad T_{min} = \frac{1}{|Re(\lambda_1)|} = \frac{1}{400}$$

$$\Rightarrow \text{Steifigkeit koeffizient}: \frac{T_{max}}{T_{min}} = 200 \Rightarrow \underline{\text{Steif ODE}}$$

b) autonome Probleme $\Rightarrow$ gewöhnliche DG

$\Rightarrow \hat{\underline{x}} = \begin{pmatrix} \underline{x} \\ t \end{pmatrix}$ $\quad \underline{\underline{A}}$ ist ein 2x2 Matrix $\Rightarrow \underline{x}$ ist ein 2x1 Vektor

$$\Rightarrow \hat{\underline{x}} = \begin{pmatrix} x_1 \\ x_2 \\ t \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ mit } \hat{\underline{x}}_0 = \begin{pmatrix} 0.1 \\ 5 \\ 0 \end{pmatrix}$$

$$\Rightarrow \dot{\hat{\underline{x}}} = \hat{f}_1(\hat{\underline{x}}) = \begin{bmatrix} -4 & -2 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 4\cdot\sin(x_3) \\ \sin(x_3) - 2\cos(x_3) \\ 1 \end{pmatrix}$$

$$\Rightarrow \dot{\hat{\underline{x}}} = \hat{f}_2(\hat{\underline{x}}) = \begin{bmatrix} -400 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 400\cdot\sin(x_3) \\ \sin(x_3) - 2\cos(x_3) \\ 1 \end{pmatrix}$$

c) explizite Eulerverfahren

$$\hat{\underline{x}}^1 = \hat{\underline{x}}^0 + h \cdot \dot{\hat{\underline{x}}} = \hat{\underline{x}}_0 + h \cdot \hat{f}_1(\hat{\underline{x}}_0)$$

$$= \begin{pmatrix} 0.1 \\ 5 \\ 0 \end{pmatrix} + 0.1 \cdot \left[\begin{bmatrix} -4 & -2 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0.1 \\ 5 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \cdot \sin(0) \\ \sin(0) - 2\cos(0) \\ 1 \end{pmatrix} \right] = \begin{pmatrix} -0.94 \\ 3.81 \\ 0.1 \end{pmatrix}$$

$$\hat{\underline{x}}^2 = \hat{\underline{x}}^1 + h \cdot \hat{f}_1(\hat{\underline{x}}^1)$$

$$= \begin{pmatrix} -0.94 \\ 3.81 \\ 0.1 \end{pmatrix} + 0.1 \cdot \left[\begin{bmatrix} -4 & -2 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} -0.94 \\ 3.81 \\ 0.1 \end{pmatrix} + \begin{pmatrix} 4 \cdot \sin(0.1) \\ \sin(0.1) - 2\cos(0.1) \\ 1 \end{pmatrix} \right] = \begin{pmatrix} -1,326 \\ 2,754 \\ 0,2 \end{pmatrix}$$

	$\hat{f}_1$			$\hat{f}_2$		
Schritt	x_1	x_2	t	x_1	x_2	t
0	0.1	5	0	0.1	5	0
1	-0,94	3.81	0.1	-3.9	3,81	0.1
2	-1,326	2,754	0.2	152,1	2,458	0,2
3	-1,3464	1,8706	0.3	-5931.9	16,9764	0,3

f_2 ist nicht geeignet für Schrittweite $h = 0.1$

da $\Delta t \leq \frac{1}{20} T_{min} = \frac{1}{8000}$ s sein soll

d)

Schritt	x_1	x_2	t
0	0.1	5	0
100	$6,5332 \cdot e^{-24}$	3,9116	0.1
200	$4.2683 \cdot e^{-46}$	3,0205	0.2
300	$2.7825 \cdot e^{-68}$	2,2910	0.3

Schrittweit reduziert

$\Rightarrow$ x_1 ist konvergenz

aber Rechenaufwand erhöht

e) Nullstellenproblem für implizite Eulerverfahren

$$F(\hat{\underline{x}}^k) = \hat{\underline{x}}^k + h \cdot \hat{f}_2(\hat{\underline{x}}^{k+1}) - \hat{\underline{x}}^{k+1} = 0$$

$$\Rightarrow = \hat{\underline{x}}^k + \begin{bmatrix} -400h-1 & 0 & 0 \\ h & -2h-1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \hat{\underline{x}}^{k+1} + h \cdot \begin{pmatrix} 400 \sin(x_3^{k+1}) \\ \sin(x_3^k) - 2\cos(x_3^{k+1}) \\ 1 \end{pmatrix}$$

Der Newton-Verfahren:

$$\hat{\underline{x}}^{neu} = \hat{\underline{x}}^{alt} - \left(\frac{\partial F(\hat{\underline{x}}^{alt})}{\partial \hat{\underline{x}}^{k+1}}\right)^{-1} \cdot F(\hat{\underline{x}}^{alt}) + \cancel{O}(F^2(x^{alt}))$$

$$\frac{\partial \hat{F}(\underline{x}^{alt})}{\partial \hat{\underline{x}}^{k+1}} = \begin{bmatrix} -400h-1 & 0 & 400h \cdot \cos(x_3^{alt}) \\ h & -2h-1 & h \cdot \cos(x_3^{alt}) + 2h \cdot \sin(x_3^{alt}) \\ 0 & 0 & -1 \end{bmatrix}$$

$$\text{mit } h = 0.1 \Rightarrow \frac{\partial \hat{F}(\underline{x}^{alt})}{\partial \hat{\underline{x}}^{k+1}} = \begin{bmatrix} -41 & 0 & 40 \cdot \cos(x_3^{alt}) \\ 0.1 & -1.2 & 0.1 \cos(x_3^{alt}) + 0.2 \sin(x_3^{alt}) \\ 0 & 0 & -1 \end{bmatrix}$$

$$\Rightarrow \hat{\underline{x}}^{neu} = \hat{\underline{x}}^{alt} - \begin{bmatrix} -41 & 0 & 40 \cdot \cos(x_3^{alt}) \\ 0.1 & -1.2 & 0.1 \cos(x_3^{alt}) + 0.2 \sin(x_3^{alt}) \\ 0 & 0 & -1 \end{bmatrix}^{-1} \cdot F(\hat{\underline{x}}^{alt})$$

$$\text{mit } \hat{\underline{x}}^{alt} = \hat{\underline{x}}^{0} = \begin{pmatrix} 0.1 \\ 5 \\ 0 \end{pmatrix}$$

$$F(\hat{\underline{x}}^{alt}) = \hat{\underline{x}}_0 + \begin{bmatrix} -41 & 0 & 0 \\ 0.1 & -1.2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \hat{\underline{x}}^0 + 0.1 \cdot \begin{pmatrix} 400 \sin(0) \\ \sin(0) - 2\cos(0) \\ 1 \end{pmatrix}$$

$(\hat{\underline{x}}^{k} = \hat{\underline{x}}^{k+1} = \hat{\underline{x}}_0)$

$$= \begin{pmatrix} 0.1 \\ 5 \\ 0 \end{pmatrix} + \begin{bmatrix} -41 & 0 & 0 \\ 0.1 & -1.2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{pmatrix} 0.1 \\ 5 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -0.2 \\ 0.1 \end{pmatrix} = \begin{pmatrix} -4 \\ -1.19 \\ 0.1 \end{pmatrix}$$

$$\Rightarrow \hat{\underline{x}}^{neu} = \begin{pmatrix} 0.1 \\ 5 \\ 0 \end{pmatrix} - \begin{bmatrix} -41 & 0 & 40 \\ 0.1 & -1.2 & 0.1 \\ 0 & 0 & -1 \end{bmatrix}^{-1} \begin{pmatrix} -4 \\ -1.19 \\ 0.1 \end{pmatrix}$$

$$= \begin{pmatrix} 0.1 \\ 4.0167 \\ 0.1 \end{pmatrix}$$

$$\dot{x}(t) = \sin\left(\frac{4}{3}t + 2\right) + \frac{1}{4}x(t) + \frac{2}{5}, \quad x(0) = x_0 = 0$$

Aufgabe 2:

a): Annahme: $\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} t \\ x(t) \end{pmatrix}$

$\Rightarrow \underline{f}(\underline{y}) = \begin{pmatrix} \dot{t} \\ \dot{x}(t) \end{pmatrix} = \begin{pmatrix} 1 \\ \sin(\frac{4}{3}y_1 + 2) + \frac{1}{4}y_2 + \frac{2}{5} \end{pmatrix}$ mit $\underline{y}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

explizite Euler-Verfahren: $x_{k+1} = x_k + h \cdot f(x_k)$

$\Rightarrow \underline{y}_1 = \underline{y}_0 + h \cdot \underline{f}(\underline{y}_0)$

$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 1 \\ \sin(2) + \frac{2}{5} \end{pmatrix} = \begin{pmatrix} 2 \\ 2.6186 \end{pmatrix}$

b): implizite Euler-Verfahren

$x_{k+1} = x_k + h \cdot f(x_{k+1})$

$\Rightarrow \underline{y}_1 = \underline{y}_0 + h \cdot \underline{f}(\underline{y}_1)$

$\begin{pmatrix} y_{1,1} \\ y_{1,2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + h \cdot \begin{pmatrix} 1 \\ \sin(\frac{4}{3}y_{1,1} + 2) + \frac{1}{4}y_{1,2} + \frac{2}{5} \end{pmatrix}$

$\Rightarrow \begin{pmatrix} y_{1,1} \\ y_{1,2} \end{pmatrix} = \begin{pmatrix} 2 \\ -2.3958 \end{pmatrix}$

c) Heun-Verfahren:

$s_1 = f(x_k)$, $s_2 = f(x_k + h_k s_1)$, $x_{k+1} = x_k + \frac{h_k}{2}(s_1 + s_2)$

$\underline{s}_1 = \underline{f}(\underline{y}_0)$, $\underline{s}_2 = \underline{f}(\underline{y}_0 + h \cdot \underline{s}_1)$, $\underline{y}_1 = \underline{y}_0 + \frac{h}{2}(\underline{s}_1 + \underline{s}_2)$

$\underline{s}_1 = \begin{pmatrix} 1 \\ 1,3093 \end{pmatrix}$ $\underline{s}_2 = \begin{pmatrix} 1 \\ 0,0557 \end{pmatrix}$ $\underline{y}_1 = \begin{pmatrix} 2 \\ 1,3650 \end{pmatrix}$

d) Runge-Kutta-Verfahren

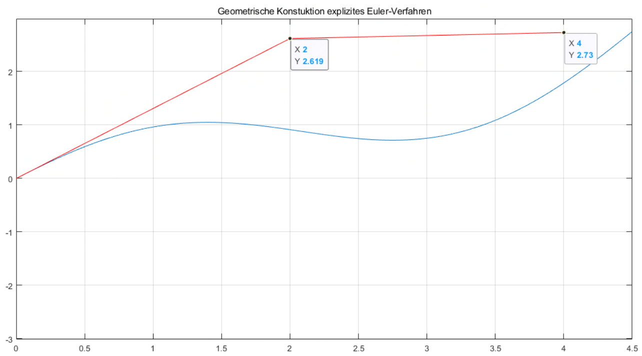
$\underline{s}_1 = \underline{f}(\underline{y}_0)$, $\underline{s}_2 = \underline{f}(\underline{y}_0 + \frac{h}{2} \cdot \underline{s}_1)$, $\underline{s}_3 = \underline{f}(\underline{y}_0 + \frac{h}{2}\underline{s}_2)$, $\underline{s}_4 = \underline{f}(\underline{y}_0 + h \cdot \underline{s}_3)$

$\underline{y}_1 = \underline{y}_0 + \frac{h}{6} \cdot \underline{f}(\underline{s}_1 + 2\underline{s}_2 + 2\underline{s}_3 + \underline{s}_4)$

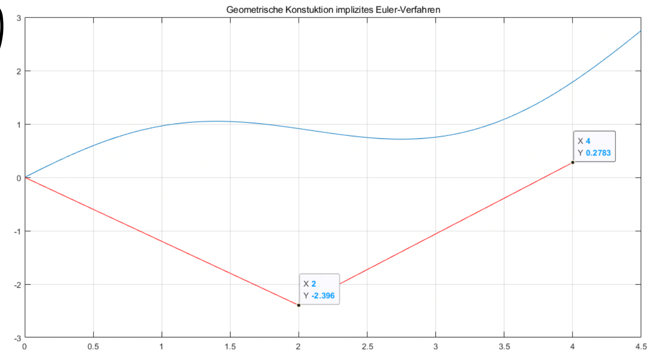
$$\Rightarrow \quad \underline{s}_1 = \begin{pmatrix} 1 \\ 1{,}3093 \end{pmatrix} \qquad \underline{s}_2 = \begin{pmatrix} 1 \\ 0{,}5368 \end{pmatrix} \qquad \underline{s}_3 = \begin{pmatrix} 1 \\ 0{,}5436 \end{pmatrix} \qquad \underline{s}_4 = \begin{pmatrix} 1 \\ -0{,}4271 \end{pmatrix}$$

$$\underline{y}_1 = \begin{pmatrix} 2 \\ 0{,}8810 \end{pmatrix}$$

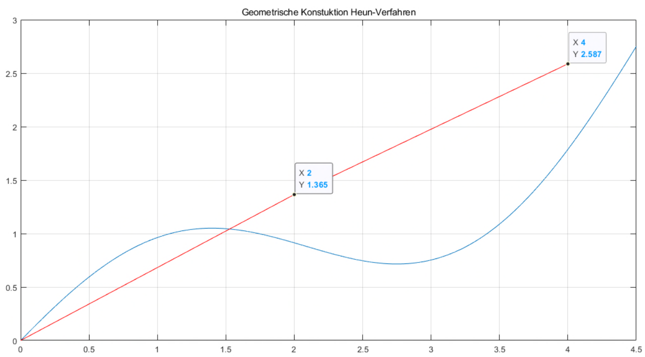
a)



b)



c)



d)