a) Sea $x_1(t) = A e^{j\omega_0 t}$, $x_2(t) = B e^{j5\omega_0 t}$, $\omega_0 = \frac{2\pi}{T}$

$T, A, B \in \mathbb{R}^+$

Determinar la distancia entre las dos señales.

Para encontrar la distancia resolvemos $\lim_{T\to\infty} \frac{1}{T}\int_T |x_1(t) - x_2(t)|^2\, dt$

Expandimos $|x_1(t) - x_2(t)|^2 = x_1^2(t) - 2x_1(t)\,x_2(t) + x_2^2(t)$

$$\lim_{T\to\infty} \frac{1}{T}\int_T \left(|x_1(t)|^2 - 2|x_1(t)\,x_2(t)| + |x_2(t)|^2\right) dt$$, repartimos la integral

$$\lim_{T\to\infty} \frac{1}{T}\left[\int_T |x_1(t)|^2\, dt - 2\int_T |x_1(t)\,x_2(t)|\, dt + \int_T |x_2(t)|^2\, dt\right]$$

$$\lim_{T\to\infty} P_{x_1} - \frac{2}{T}\int_T x_1(t)\,x_2(t) - P_{x_2}$$

donde $P_{x_1} = \frac{1}{T}\int_T |x_1(t)|^2\, dt = \frac{1}{T}\int_T \left(A e^{j\omega_0 t}\right)^2 dt$

$$\frac{1}{T}\int_T (A)^2 \left(e^{j\omega_0 t}\right)^2 dt$$, $\left(e^{j\omega_0 t}\right)^2 = e^{j\omega_0 t}\left(e^{j\omega_0 t}\right)^*$

$$\frac{A^2}{T}\int_T e^{j\omega_0 t}\left(e^{-j\omega_0 t}\right) dt$$, $e^a e^b = e^{a+b}$

$$\frac{A^2}{T}\int_T e^{j\omega_0 t - j\omega_0 t}\, dt = \frac{A^2}{T}\int_T e^0\, dt$$ evaluamos desde $-T/2$ a $T/2$

$$= \frac{A^2}{T}\, t\Big|_{-T/2}^{T/2} = \frac{A^2}{T}\left(T/2 - (-T/2)\right) = \frac{A^2}{T}\, T$$

Por lo tanto $P_{x_1} = A^2$

Ahora $P_{x_2} = \frac{1}{T}\int_T |x_2(t)|^2 dt = \frac{1}{T}\int_T \left|B e^{j5\omega_0 t}\right|^2 dt$

$\frac{1}{T}\int_T B^2 (e^{js\omega_0 t})^2 dt$, $(e^{js\omega_0 t})^2 = e^{js\omega_0 t}(e^{js\omega_0 t})^*$

$$\frac{B^2}{T}\int_T e^{js\omega_0 t} e^{-js\omega_0 t} dt = \frac{B^2}{T}\int_T dt = \frac{B^2}{T} t\Big|_{-T/2}^{T/2} = \frac{B^2}{T}(T/2-(-T/2))$$

$P_{x_2} = B^2$

Ahora: $-\frac{2}{T}\int x_1(t)\, x_2(t)\, dt = -\frac{2}{T}\int_T A e^{j\omega_0 t} B e^{js\omega_0 t} dt$

$$-\frac{2AB}{T}\int_T e^{j\omega_0 t + js\omega_0 t} dt = -\frac{2AB}{T}\int_T e^{j6\omega_0 t} dt$$

realizamos la sustitución $u = j6\omega_0 t$, $du = j6\omega_0 dt$

$dt = \frac{du}{j6\omega_0}$, Calculamos los nuevos límites de integración

teniendo en cuenta que los originales son $\int_{-T/2}^{T/2}$

Si $t = -T/2$, Si $t = T/2$ reemplazamos $\omega_0 = \frac{2\pi}{T}$

$u = j6\omega_0(-T/2)$ $u = j6\omega_0(T/2)$

$u = j6\frac{2\pi}{T}\left(-\frac{T}{2}\right)$ $u = j6\frac{2\pi}{T}\left(\frac{T}{2}\right)$

$u = -j6\pi$ $u = j6\pi$

$$-\frac{2AB}{T}\int_{-j6\pi}^{j6\pi}\frac{e^u}{j6\frac{2\pi}{T}}du = \frac{\frac{-2AB}{T}}{\frac{j62\pi}{T}}\left[e^{j6\pi} - e^{-j6\pi}\right]$$

$$\frac{-2AB}{j12\pi}\left[(\cos(6\pi) + j\operatorname{Sen}(6\pi)) - (\cos(6\pi) - j\operatorname{Sen}(6\pi))\right]$$

$$\frac{-2AB}{j12\pi}\left[(1+j0)-(1-j0)\right]$$

$$\frac{-2AB}{j12\pi}\left[1-1\right] = \frac{-2AB}{j12\pi}\left[0\right] = 0$$

$$\frac{2}{T}\int_T x_1(t)\,x_2(t) = 0$$

Entonces $\displaystyle\lim_{T\to\infty} \frac{1}{T}\int_T |x_1(t)-x_2(t)|^2\,dt$

$$= \lim_{T\to\infty} P_{x_1} - \frac{2}{T}\int_T x_1(t)\,x_2(t) - P_{x_2} = A^2 - 0 + B^2$$

$$\overline{P_{x_1-x_2}} = A^2 + B^2$$

b) $X(t) = 3\cos(1000\pi t) + 5\sin(2000\pi t) + 10\cos(11000\pi t)$ $F_s = 5000$

Tomamos la frecuencia angular mayor entre las tres señales.

$\omega_1 = 1000\pi$, $\omega_2 = 2000\pi$, $\omega_3 = 11000\pi$; $\omega = 2\pi F$

$F_1 = \frac{1000\pi}{2\pi}$, $F_2 = \frac{2000\pi}{2\pi}$, $F_3 = \frac{11000\pi}{2\pi}$

$F_1 = 500$, $F_2 = 1000$, $F_3 = 5500$

En este caso $\omega_3 = 11000\pi$, partiendo de $\omega = 2\pi F = \frac{2\pi}{T}$
despejamos T.

$T = \frac{2\pi}{\omega} = \frac{2\pi}{11000\pi} = \frac{1}{5500}$ ahora despejamos F.

$F = \frac{1}{T} = \frac{1}{\frac{1}{5500}} = 5500$ hz

La relación de Nyquist nos dice: $F_s \geq 2F$

$5000 \geq 2(5500) \rightarrow 5000 \geq 11000$ No se cumple

La frecuencia de muestreo del conversor análogo digital no será suficiente para representar adecuadamente la señal.

Discreticemos la señal

Como una computadora no puede recibir infinitos valores de tiempo continuos, realizamos el cambio $t = nt_s$, pero $t_s = \frac{1}{F_s}$ entonces $t = \frac{n}{F_s}$, por el conversor análogo digital $F_s = 5000$ Hz

reemplazamos $t = \frac{n}{5K}$ en la señal original

$$x\left[\frac{n}{F_s}\right] = 3\cos\left(\frac{1K\pi}{5K}n\right) + 5\sin\left(\frac{2K\pi}{5K}n\right) + 10\cos\left(\frac{11K\pi}{5K}n\right]$$

$$x\left[\frac{n}{F_s}\right] = 3\cos\left(\frac{\pi}{5}n\right) + 5\sin\left(\frac{2\pi}{5}n\right) + 10\cos\left(\frac{11\pi}{5}n\right)$$

la señal $10\cos\left(\frac{11\pi}{5} n\right)$ donde $\Omega = \frac{11}{5}\pi$ es una copia porque $\frac{11}{5}\pi$ está fuera del rango $[-\pi, \pi]$ para llegar al original hacemos $\Omega_{original} = \Omega_{copia} - 2\pi$

$$\Omega_{original} = \frac{11}{5}\pi - 2\pi = \frac{\pi}{5}$$

por lo tanto la señal discretizada es:

$$x\left[\frac{n}{Fs}\right] = 3\cos\left(\frac{\pi}{5} n\right) + 5\sin\left(\frac{2\pi}{5} n\right) + 10\cos\left(\frac{\pi}{5} n\right)$$

$$x\left[\frac{n}{Fs}\right] = 13\cos\left(\frac{\pi}{5} n\right] + 5\sin\left(\frac{2\pi}{5} n\right)$$