Semester Finale

Problem 1) (Matic) There are M green apples and N red apples in a basket. We take apples out randomly one by one until all the apples left in the basket are red. What is the probability that the moment we stop the basket is empty?

Problem 2) (Matic) A fair coin is tossed n times. What is the expected product of the number of heads and the number of tails? *Challenge: What is the coin is biased and lands on heads with probability p.*

Problem 3) (Matic) You roll a fair n-sided die repeatedly and sum the outcomes. What is the expected number of rolls until the sum is a multiple of n for the first time?

Problem 4) (Stefanica) The number 2^{29} has nine digits, all different. Without computing 2^{29}, what is the missing digit?

Problem 5) When is the first time after 12pm that the minute and the hour hands meet again?

Problem 6) What is the value of i^i?

Problem 7) (Amazon) Let S_5 be the set of all $n = 5$ digit numbers that use each digit in $\{1, 2, 3, 4, 5\}$ exactly once. What is the sum all the numbers in S_5? *Challenge: What about for general $n \in [9]$?*

And now for some variety!

Problem 8) Thanks to Julius Barth for this neat problem! There is a dragon terrorizing a village of n people. Every day the dragon comes and eats one of the villagers until no more villagers are left. However, th villagers have poison pills available, and every day i some number of villagers $x(i)$ will take a poison pill. If the dragon eats a poisoned villager the dragon will die along with the poisoned villagers on that day. If the dragon eats a non-poisoned villager, it survives, but all of the villagers die. How many villagers should take the poison pill each day to maximize the expected number of surviving villagers?

Hints:

1. Consider the number of sequences where all apples are removed and a green apple is removed last. There is also a nice argument that involves finding disjoint events.
2. Given the number of heads, what can you say about the number of tails?
3. Consider the current sum modulo n.
4. Consider the number modulo 9
5. Use symmetry!
6. Is there another way to express i?
7. Try smaller n and look for a pattern.
8. Try coding it and look for a pattern!