

Expect the Unexpected

Problem 1) (Stefanica) Given a cube, you can jump from one vertex to a neighboring vertex with equal probability. Assume you start from a certain vertex. What is the expected number of jumps to reach the opposite vertex?

Problem 2) (The Coupon Collector Problem) Each unit of time, a random integer in $[n]$ is chosen uniformly at random. What is the expected time until all n numbers in $[n]$ are seen?

Problem 3) (Stefanica) Select numbers uniformly distributed between 0 and 1, one after another until you selected one greater than the previous one you selected.

- (a) On average, how many numbers have you selected?
- (b) What is the average value of the smallest number you have selected?

Problem 4) (Stefanica) A stick of unit length is dropped on the ground and breaks into two pieces. What is the expected length of the longest piece?

Problem 5) Alice and Bob have a and b dollars respectively. Each unit of time they flip a fair coin and if the result is heads, Bob gives Alice one dollar, otherwise Alice gives Bob one dollar. The game ends when either Alice or Bob is broke. What is the expected number of turns until the game ends?

And now for some for some variety!

Problem 6) (HMC Putnam Seminar) A farmer, who sells grain, has a set of four weights that each weigh an integer number of pounds and a fair balance. She claims that she can weigh any integer number of pounds of grain up to a maximum of N using just these weights. What is the maximum value of N and what should the weights be? *Challenge: Given n weights what is the maximum value of N and what should the weights be?*

Hints:

1. Try labeling the vertices and letting E_i be the expected number of jumps to finish from vertex i .
2. Suppose you have collected k so far. How long until you get a new one?
3. No hint. Haven't tried it yet!
4. Draw a picture of the sample space!
5. Martingale theory, markov chains, or a recursion could be helpful here :)
6. How many configurations are possible? Also, experiment with smaller N