CFD for Aerospace Applications

Turbulence

Dr. Ziad Boutanios

Concordia University - MIAE

March 15, 2022

Introduction

- So far we have only considered laminar fluid transport, where fluid elements flow along their streamlines and do not move laterally.
- You can compare it to traffic where cars always stay in their lanes.
- Laminar flow is not so common in nature and does not last long even when it occurs.
- Laminar flow is enforced by viscosity, which dampens flow fluctuations that try to push the fluid elements across streamlines.
- At some point the fluctuation kinetic energy grows so large that it overcomes viscous damping and the fluctuations grow into chaotic features leading to turbulence.

Figure 1: Schlieren photograph of the transition from laminar to turbulent flow regime of a candle thermal plume (by Gary Settles, CC license).

- Figure 1 shows a laminar to turbulent regime transition in the thermal plume rising from a lit candle, with obvious chaotic features.
- Turbulence is the rule in real life and the overwhelming majority of flow you will be dealing with will be turbulent.
- In this course we will cover turbulence modelling for CFD.

Turbulence origin and characteristics

- Turbulence originates from the non-linear terms in the momentum equations where small linear fluctuations can become coupled.
- Its onset is typically monitored with the Reynolds number Re, which is the ratio of inertial forces F_i to viscous forces F_v.

$$\left.\begin{aligned} F_i &= \rho U^2 L_r^2 \\ F_v &= \mu U L_r \end{aligned}\right\} \Rightarrow \frac{F_i}{F_v} = \frac{\rho U L_r}{\mu} = Re,$$

  where L_r is a reference length of the flow.
- These coupled fluctuations eventually grow into chaotic fluctuations, which lead to the formation of larger vortices, also called eddies.

- The largest eddies are the most energetic and unstable, and break down into smaller eddies, leading to a scale hierarchy of eddies.
- The smallest eddy size is the so-called Kolmogorov length scale where viscous dissipation acts and is defined as,

$$\eta = \left(\frac{\nu^3}{\epsilon}\right)^{1/4}. \quad (1)$$

  Here, $\nu = \mu/\rho$ is the fluid kinematic viscosity, μ the dynamic viscosity, ρ the density and ϵ the energy dissipation rate.
- The corresponding time and velocity scales are respectively, $\tau_\eta = (\nu/\epsilon)^{1/2}$ and $u_\eta = (\nu\epsilon)^{1/4}$.

- Energy production happens at the largest eddy scales, and dissipation at the smallest.
- This energy cascade is shown in figure 2.

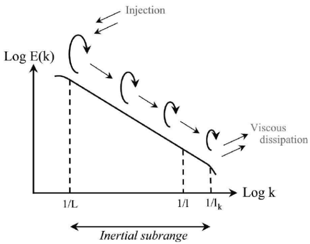

Figure 2: Turbulent energy cascade plotted against wavenumber (by Aakash30jan, CC license).

Turbulence characteristics:

Irregularity Turbulent flows are highly irregular, necessitating statistical treatment.

Increased mixing The increased kinetic energy available in turbulent flow compared to laminar flow promotes heat transfer as well as mixing of fluid and species (turbulent heat transfer and mixing coefficients are normally 10x larger than their laminar counterparts).

Rotationality Turbulent flows have non-zero vorticity and a strong 3D vortex generation mechanism called vortex stretching. It is this mechanism that is behind the energy cascade as the stretching reduces the vortex size.

Dissipation In the absence of a persistent energy supply turbulence dissipates quickly since the viscous shear stresses convert kinetic energy into internal energy. Large vortices stretch and cascade into smaller ones with the viscous shear stress more and more effective at the smaller scales, until it overcomes kinetic energy completely at the Kolmogorov scale and breaks down the smallest eddies.

- Vortex tubes formation and shedding is shown in figure 3.
- The vortex tubes stretch and break down into smaller tubes, and so on to the smallest scales and viscous dissipation.

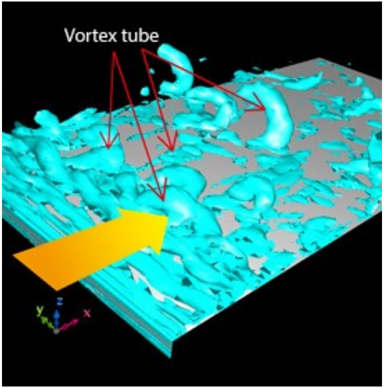

Figure 3: Vortex tubes in a turbulent flow over a flat plate.

Turbulence modelling

- Let us start by deriving the transport equations for a turbulent flow.
- For simplicity, consider an incompressible fluid with constant density ρ and dynamic viscosity μ flowing with velocity magnitude U over a body with reference length L_r.
- The continuity and momentum equations for the instantaneous incompressible flow are,

$$\nabla \cdot \mathbf{u} = 0, \tag{2}$$

and

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathbf{u}\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u}. \tag{3}$$

- Solving equations 2 and 3 directly requires resolving all turbulence scales in the flow from the largest integral scales to the smallest viscous Kolmogorov scale.
- This will require a mesh size of the order $O(Re^{9/4})$, which for a typical aerospace flow with a Reynolds number of at least $\approx 10^6$ translates into a mesh with 10^{13} points.
- This is called Direct Numerical Simulations (DNS), which is prohibitive for industrial applications.
- In reality we are mostly interested in the *effects* of turbulence on the mean flow rather than the turbulence itself.
- Therefore it is convenient to analyze the flow in two parts, an average component and a fluctuating component.
- The most common decomposition technique is Reynolds averaging, which allows decomposing a flow variable into an average component and a fluctuating component.

Reynolds averaging

- Two Reynolds averaging methods are used, one involving time averaging and the other ensemble averaging.
- Time averaging applies to steady base flow and leads to the Reynolds Averaged Navier-Stokes (RANS) equations.
- Ensemble averaging applies to transient base flow and leads to the Unsteady Reynolds Averaged Navier-Stokes (URANS) equations.

The RANS equations

- The Reynolds decomposition of the instantaneous velocity $\mathbf{u}$ consists of decomposing it into a time-averaged component $\mathbf{U}$ and fluctuating component $\mathbf{u}'$,

$$\mathbf{u} = \mathbf{U} + \mathbf{u}', \tag{4}$$

where,

$$\overline{\mathbf{u}} = \mathbf{U} = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \mathbf{u}(\mathbf{x}, t) dt. \tag{5}$$

- Furthermore, Reynolds averaging assumes statistically steady turbulence where time-averaged turbulent fluctuations vanish,

$$\overline{\mathbf{u}'} = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} \mathbf{u}' dt = 0. \tag{6}$$

- The time period T is picked large enough to satisfy equation 6 for turbulent fluctuations of all variables.
- Both components of a typical example are shown in figure 4.

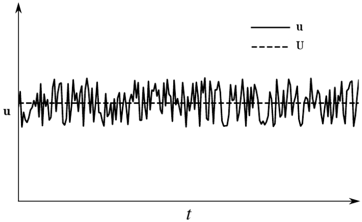

Figure 4: Instantaneous and time averaged velocity measurements.

- Performing a Reynolds decomposition and time average of equations 2 and 3 leads to the Reynolds Averaged Navier-Stokes equations,

$$\nabla \cdot \mathbf{U} = 0, \tag{7}$$

and

$$\nabla \cdot \mathbf{U}\mathbf{U} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 \mathbf{U} - \nabla \cdot \overline{\mathbf{u}'\mathbf{u}'}. \tag{8}$$

Note that $\dfrac{\partial \mathbf{U}}{\partial t} = 0$ by definition since **U** is a time average, therefore the average flow is steady.

- The new term $\nabla \cdot \overline{\mathbf{u}'\mathbf{u}'}$ on the RHS is Reynolds stress tensor due to non-linear coupling of the turbulent fluctuations.
- The whole point of the RANS approach is to model this term using a turbulence model.

The URANS equations

- Another Reynolds averaging technique that leads to a transient averaged flow uses the so-called *ensemble* averaging.
- The ensemble average is defined as the average over the realizations of $\mathbf{u}$,

$$\langle \mathbf{u} \rangle = \frac{1}{N} \sum_{k=1}^{N} \mathbf{u}_k. \tag{9}$$

  Here, N is the number of realizations of $\mathbf{u}$.
- Using the same Reynolds decomposition as before,

$$\mathbf{u} = \mathbf{U} + \mathbf{u}',$$

- Since the ensemble average is associative we have,

$$\begin{aligned} \langle \mathbf{u} \rangle &= \langle \mathbf{U} \rangle + \langle \mathbf{u}' \rangle \\ &= \langle \mathbf{U} \rangle \end{aligned}$$

  since $\langle \mathbf{u}' \rangle = 0$ for a large enough N.

- This leads to,

$$\langle \mathbf{u} \rangle = \langle \mathbf{U} \rangle. \tag{10}$$

- Both components of a typical ensemble averaged instantaneous measurement of velocity are shown in figure 5.

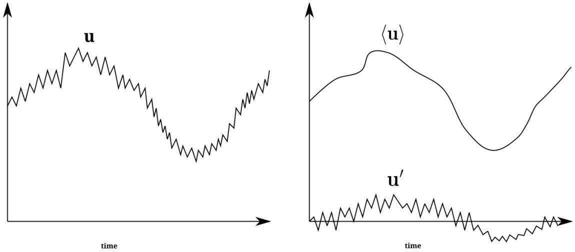

Figure 5: Instantaneous (left) and ensemble averaged (right) velocity measurements.

- Performing a Reynolds decomposition and ensemble average of equations 2 and 3 leads to the Unsteady Reynolds Averaged Navier-Stokes equations,

$$\nabla \cdot \langle \mathbf{U} \rangle = 0, \tag{11}$$

and

$$\frac{\partial \langle \mathbf{U} \rangle}{\partial t} + \nabla \cdot \langle \mathbf{U} \rangle \langle \mathbf{U} \rangle = -\frac{1}{\rho} \nabla P + \nu \nabla^2 \langle \mathbf{U} \rangle - \nabla \cdot \langle \mathbf{u}' \mathbf{u}' \rangle. \tag{12}$$

Note that $\frac{\partial \langle \mathbf{U} \rangle}{\partial t}$ now appears in the momentum equation since we are ensemble averaging and the base flow is unsteady.

- The new term $\nabla \cdot \langle \mathbf{u}' \mathbf{u}' \rangle$ on the RHS is Reynolds stress tensor due to non-linear coupling of the turbulent fluctuations.
- This term is modelled as done in the RANS approach using a turbulence model.

Turbulence modelling

- Numerous turbulence models are available in the literature (roughly 200), each for a specific application.
- The most accurate RANS turbulence model is the Reynolds Stress Model (RSM), which solves a differential transport equation for each of the nine components of the Reynolds stress tensor.
- We will restrict our discussion to the simpler Eddy viscosity turbulence models where the Reynolds stress tensor is modelled based on the Boussinesq hypothesis.
- Eddy viscosity models assume isotropic turbulence with all turbulent fluctuations equal, that is $u'_x = u'_y = u'_z = u'$.
- Isotropic turbulence means we only have one turbulent stress component in the Reynolds stress tensor, that is $\overline{u'^2}$.
- Accordingly, the turbulent kinetic energy becomes,

$$k = \frac{1}{2}\left(u'^2_x + u'^2_y + u'^2_z\right) = \frac{3}{2}u'^2 \tag{13}$$

- The Boussinesq hypothesis is based on the analogy with laminar stresses being proportional to laminar velocity gradients multiplied by the laminar viscosity, $\boldsymbol{\tau} = \mu \nabla \mathbf{U}$.
- For tubulent flow we assume that the Reynolds stresses are proportional to the average velocity gradients and the turbulent viscosity μ_t, plus the turbulent kinetic energy and compressibility contributions,

$$\boldsymbol{\tau}' = -\rho\overline{\mathbf{u}'\mathbf{u}'} = 2\mu_t \mathbf{S} - \frac{2}{3}\mathbf{I}(\rho k + \mu_t \nabla \cdot \mathbf{U}). \tag{14}$$

  Here, $\mathbf{S}$ is the rate of strain tensor, $\mathbf{I}$ is the identity matrix and k the turbulent kinetic energy.
- Since we can readily calculate the average velocity gradients we only need to estimate the turbulent viscosity μ_t and the turbulent kinetic energy k.

- RANS eddy viscosity models are classified by the number of PDEs they solve to get μ_t and k.
- These range from 0-equation to 4-equation models, but we will not discuss 3- and 4-equation models here.
- Table 1 shows examples of 0- to 2-equation models with the variables solved for.

Number of eqs	Example	Variables
0	Algebraic Prandtl mixing length model	turbulent viscosity μ_t
1	1-equation Prandtl mixing length	μ_t and k
2	k-ϵ, k-ω, k-ω-SST, etc.	μ_t and k and ϵ, μ_t and k and ω, etc.

Table 1: 0- to 2-equation turbulence models with variables concerned.

The 0-equation Prandtl mixing length model

- The 0-equation Prandtl mixing length model uses dimensional arguments and analogy with the molecular mean free path to define the turbulent viscosity as,

$$\mu_t = \rho l_m^2 \left|\mathbf{S}\right| . \tag{15}$$

- l_m is the characteristic length over which a fluid element conserves its properties, and is taken constant or set using empirical correlations.
- This model is simple and robust.
- It predicts $\mu_t = 0$ when $\mathbf{S} = 0$, which is not physical.
- The 0-equation Prandtl mixing length model cannot properly account for turbulence history effects like advection/diffusion of turbulent energy.
- Like other 0-equation models, it is only appropriate for simple situations.

The 1-equation Prandtl mixing length model

- The 1-equation Prandtl mixing length model introduces the turbulent kinetic energy in the turbulent viscosity expression,

$$\mu_t = \rho l_m \sqrt{k}. \tag{16}$$

- An additional transport PDE for k is solved,

$$\frac{\partial(\rho k)}{\partial t} + \nabla \cdot (\rho k \mathbf{U}) = P_k - \epsilon + \nabla \cdot \left[\left(\mu + \frac{\mu_t}{\sigma_k} \right) \nabla k \right]. \tag{17}$$

  Here, $P_K = \boldsymbol{\tau}' \cdot \nabla \mathbf{U}$ is the production of turbulent kinetic energy, $C_D = 0.08$ is an empirical constant, $\sigma_k = 1$, and the turbulent dissipation rate $\epsilon = \rho C_D k^{3/2} / l_m$.
- This model is simple and as robust as its 0-equation version.
- It accounts for some turbulent kinetic energy history effects.
- It cannot properly account for variable turbulent energy dissipation rate.

The 2-equation $k-\epsilon$ turbulence model

- The $k-\epsilon$ turbulence model is easily the most popular, with many versions developed for different applications and to address different shortcomings.
- Here, we will look at the standard version of the $k-\epsilon$ turbulence model.
- The standard $k-\epsilon$ turbulence model defines the turbulent viscosity as,

$$\mu_t = \rho C_\mu \frac{k^2}{\epsilon}. \tag{18}$$

  Here, $C_\mu = 0.09$.
- Comparing equations 16 and 18 we can readily derive an expression for the mixing length of the $k-\epsilon$ turbulence model,

$$l_m = C_\mu \frac{k^{3/2}}{\epsilon}. \tag{19}$$

- Equation 19 tells us that the $k-\epsilon$ turbulence model implies a variable mixing length that depends on k and ϵ, as opposed to constant or empirical correlations for 0- and 1-equation models.
- The $k-\epsilon$ turbulence model consists of a transport PDE for k identical to equation 17, and a transport PDE for ϵ.
- The transport PDE for ϵ is,

$$\frac{\partial(\rho\epsilon)}{\partial t} + \nabla\cdot(\rho\epsilon\mathbf{U}) = C_{1\epsilon}\frac{\epsilon}{k}P_k - C_{2\epsilon}\rho\frac{\epsilon^2}{k} + \nabla\cdot\left[\left(\mu + \frac{\mu_t}{\sigma_\epsilon}\right)\nabla\epsilon\right]. \qquad (20)$$

Here, the constants are $C_{1\epsilon} = 1.44$, $C_{2\epsilon} = 1.92$, and $\sigma_\epsilon = 1.3$.

- The standard $k - \epsilon$ turbulence model assumes fully turbulent flow and is not appropriate for flow with laminar parts and transitions to turbulence.
- It is however a pretty robust model with reasonable accuracy for a wide range of turbulent flows, which makes it a very popular turbulence model.
- It is overly diffusive and inaccurate in stagnation regions and near no-slip walls.
- Its constants require tweaking for different applications.
- The standard $k - \epsilon$ turbulence model requires near-wall treatment as well, and this treatment differs between low-Reynolds and high-Reynolds versions.

Wall treatment of the $k-\epsilon$ model

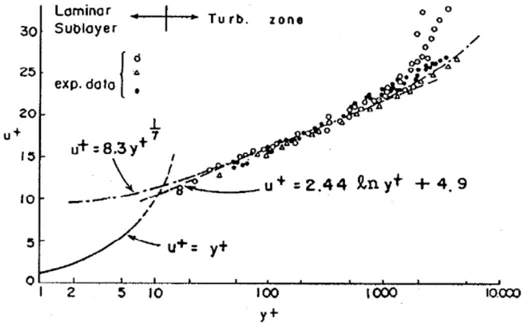

Figure 6: Structure of the turbulent boundary layer forming at no-slip walls.

- The wall normal height $y^+ = u_\tau y/\nu$, where y is the height above the wall, $u_\tau = \sqrt{\tau_w/\rho}$ is the wall friction velocity, and τ_w is the wall shear stress.
- The velocity in wall coordinates is defined as $u^+ = u/u_\tau$.

- In figure 6 we see three distinct regions:

1. A viscous laminar sublayer at $y^+ < 11$ where $u^+ = y^+$ and the flow is dominated by viscosity.
2. A buffer layer for $11 < y^+ < 60$ where viscosity and turbulence are equally important.
3. A fully turbulent layer for $y^+ > 60 <$ where viscosity plays no role and the flow profile is logarithmic, $u^+ = 2.44 ln(y^+) + 4.9$.

- The $k-\epsilon$ turbulence model has two ways to handle the near-wall region, called the low-Reynolds and high-Reynolds approaches.

The low-Reynolds standard $k-\epsilon$ model

- The computational mesh is fine enough to resolve the laminar sublayer, with the first mesh point located at $1 < y^+ < 11$ (see figure 7, left).
- Damping functions are required for some of the model constants to recover correct behaviour of k and ϵ near the wall.
- We set $C_{2\epsilon} = f_2 C_{2\epsilon}$ and $C_\mu = f_\mu C_\mu$, where

$$f_2 = 1 - 0.3e^{-Re_t^2}, \tag{21}$$

and

$$f_\mu = e^{-3.4/(1+0.02Re_t)^2}. \tag{22}$$

Here $Re_t = \dfrac{\rho k^2}{\mu\epsilon}$.

- This approach is very accurate near the wall but requires very fine meshes and a high number of cells.
- It is the advisable approach when accurate estimates of gradients and stresses at the wall are required.

The High-Reynolds standard $k-\epsilon$ model

- This is also called the Wall Function approach.
- Instead of refining the mesh all the way within the viscous sublayer, we start at the base of the turbulent layer $y^+ > 60$ (see figure 7, right).
- A slip condition with $u \neq 0$ is imposed at the wall to account for wall shear stress.
- Within the first cell above the wall the velocity is computed from the logarithmic profile $u^+ = 2.44 ln(y^+) + 4.9$.
- Other wall functions are available in the literature for rough walls, adverse pressure gradients, non-equilibrium flow, etc.
- This approach is less accurate at the wall and is advisable when one needs accurate turbulent estimates in the outer flow, not at the wall.

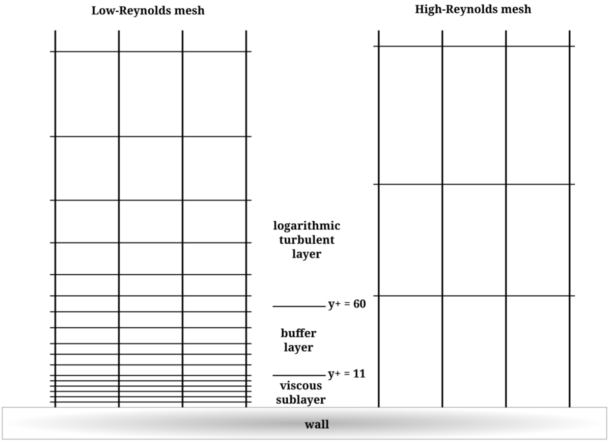

Figure 7: Examples of a low-Reynolds mesh (left) and a high-Reynolds mesh (left) close to the wall.

Other 2-equation turbulence models

- The $k-\epsilon$ models suffers from considerable issues close to no-slip walls, mostly due to inaccurate modelling of the ϵ equation.
- The $k-\omega$ model was developed to remedy such issues, where $\omega = \epsilon/k$ is called the specific dissipation rate.
- The $k-\omega$ model is better suited to wall-bounded flow (like channel flow).
- However, the $k-\omega$ model suffers from its own issues like high sensitivity to boundary conditions.
- Over time, a hybrid model called the $k-\omega$–SST model was developed as a combination of both $k-\omega$ and $k-\epsilon$ models.
- The $k-\omega$–SST model uses $k-\omega$ close to the wall, and $k-\epsilon$ away from it, with blending functions in the intermediate regions.
- It has since become the standard 2-equation model and is available in low-Reynolds and high-Reynolds modes.

Summary

- Turbulence originates from the non-linear terms in the momentum equations where small linear fluctuations can become coupled.
- It consists of a hierarchic eddy structure spanning a range from the largest integral scales to the smallest viscous scales (Kolmogorov scale).
- The onset of turbulence is usually monitored with the flow Reynolds number.
- Turbulence is characterized by irregularity, increased mixing, rotationality, and dissipation.
- Dissipation is the ultimate result of vortex stretching at the largest integral scales that progressively leads to smaller and smaller vortices.

- Turbulence modelling ranges from the most accurate and costly Direct Numerical Simulation that resolves all the scales, to the least accurate and most affordable eddy viscosity models that resolve some of the largest scales.
- Eddy viscosity models are used on RANS (steady) and URANS (transient) equations, and range from 0-equation to 4-equation formulations.
- They can provide reasonable estimates of the *effects* of turbulence that are relevant to engineering applications.

- Eddy viscosity models like $k-\epsilon$ and $k-\omega$–SST are available in low-Reynolds and high-Reynolds mode.
- The low-Reynolds model is the more costly model and resolves the flow all the way to the viscous sublayer, which requires a mesh with a minimum $y^+ < 11$ at the wall.
- Low-Reynolds models require the use of damping functions to recover physical behaviour of k and ϵ at the wall.
- The high-Reynolds model is the less costly model and does not resolve the viscous sublayer and buffer layer, which requires a mesh with a minimum $y^+ > 60$ at the wall.
- High-Reynolds models require the use of wall functions at the wall to simulate the effects of wall shear.