CHAPTER 6

Derive differential Continuity, Momentum and Energy equations form Integral equations for control volumes.

Simplify these equations for 2-D steady, isentropic flow with variable density

CHAPTER 8

Write the 2 –D equations in terms of velocity potential reducing the three equations of continuity, momentum and energy to one equation with one dependent variable, the velocity potential.

CHAPTER 11

Method of Characteristics exact solution to the 2-D velocity potential equation.

Gauss's Theorem - Divergence Theorem

transforms a surface integral into a volume integral

$$\oiint_S (\vec{V})dS = \iiint_{vol} (\nabla \vec{V})d\,vol \quad \text{where: } (\vec{V}) \text{ is a vector}$$

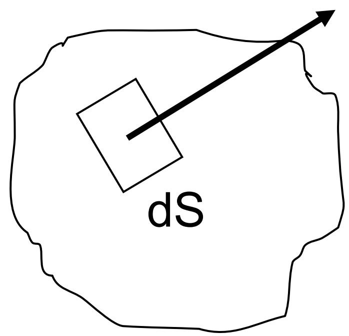
$$\oiint_S (a)dS = \iiint_{vol} (\nabla a)d\,vol \quad \text{where: } (a) \text{ is a scalar}$$

Gradient $\quad \nabla = \frac{\partial(\;)}{\partial x}\vec{i} + \frac{\partial(\;)}{\partial y}\vec{j} + \frac{\partial(\;)}{\partial z}\vec{k}$

∇ of a vector is a scalar

∇ of a scalar is a vector

CONTINUITY EQUATION CONSERVATIVE INTEGRAL FORM



$\vec{V}$, velocity vector

control volume
open thermodynamic system
region in space

$$-\oiint_S \rho \vec{V}\, dS \quad \text{net of mass leaving the control volume.}$$

(by convention mass inflow is +)

$$\frac{\partial}{\partial t}\iiint_{vol} \rho \; d\,vol \quad \text{change in mass inside the control volume}$$

$$\oiint_S \left(\rho \vec{V}\right) dS = \frac{\partial}{\partial t}\iiint_{vol} \rho \; d\,vol \quad \text{Continuity Equation in integral (conservative) form}$$

CONTINUITY EQUATION CONSERVATIVE INTEGRAL FORM

Gauss's Theorm transforms a surface integral into a volume integral

$$\oiint_S \vec{V} dS = \oiiint_{vol} \nabla \vec{V} d\ vol \qquad \text{where, } \nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

Δ (control volume mass) = net mass outflow

$$\frac{\partial}{\partial t} \oiiint_{vol} \rho \ d\ vol = -\oiint_S \left(\rho \vec{V}\right) dS$$

by convention mass inflow is +.

applying Gauss's Theorm to the net mass outflow term,

CONTINUITY EQUATION CONSERVATIVE INTEGRAL FORM

$$\oiiint_{vol} \frac{\partial}{\partial t}\rho \ \ d\,vol \ = -\oiiint_{vol} \nabla(\rho\vec{V}) \ \ d\,vol$$

$$\frac{\partial \rho}{\partial t} + \nabla(\rho\vec{V}) = 0 \quad (6.50)$$

unsteady, 3 - D, any fluid, variable density

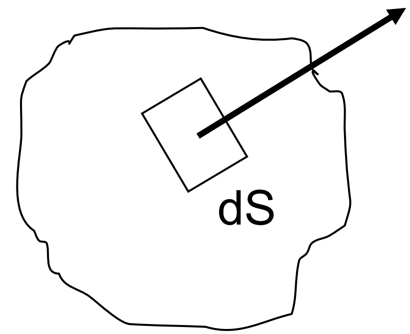
$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho\,u)}{\partial x} + \frac{\partial(\rho\,v)}{\partial y}\frac{\partial(\rho\,w)}{\partial z} = 0$$

substituting, $\dfrac{\partial(\rho\,u)}{\partial x} = u\dfrac{\partial \rho}{\partial x} + \rho\dfrac{\partial u}{\partial x}$ in x, y and z

$$\frac{\partial \rho}{\partial t} + \left(u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} \right) + \left(\rho\frac{\partial u}{\partial x} + \rho\frac{\partial v}{\partial y} + \rho\frac{\partial w}{\partial z} \right) = 0$$

MONENTUM EQUATION CONSERVATIVE INTEGRAL FORM

$$F = \frac{d(mV)}{dt} = \text{change in momentum}$$



$\vec{V}$, velocity vector

Forces

Body Force $\displaystyle \iiint_{vol} \rho\, f\, d\,vol,$

where f is the body force constant

Pressure Force $\displaystyle -\oiint_{S} p\, dS$

Viscous Force $\displaystyle \oiint_{S} \tau\, dS$

control volume

open thermodynamic system

region in space

Momentum change inside the volume $\displaystyle \oiint_{S} \left(\rho\, \vec{V}\, dS\right)\vec{V}$

Change of Momentum with time $\displaystyle \iiint_{vol} \frac{\partial\left(\rho\, \vec{V}\right)}{\partial t}\, d\,vol$

$$\text{MomentumChange} = \text{Body Force} + \text{Pressure Force} + \text{Viscous Force}$$

$$\oiint_{S} \left(\rho\, \vec{V}\, dS\right)\vec{V} + \iiint_{vol} \frac{\partial\left(\rho\, \vec{V}\right)}{\partial t}\, d\,vol = \iiint_{vol} \rho\, f\, d\,vol - \oiint_{S} p\, dS + \oiint_{S} \tau\, dS$$

MONENTUM EQUATION CONSERVATIVE INTEGRAL FORM

$$\oiint_S \left(\rho\,\vec{V}\,d\vec{S}\right)\vec{V} + \iiint_{vol} \frac{\partial\left(\rho\,\vec{V}\right)}{\partial t}\,d\,vol = \iiint_{vol} \rho\,f\,d\,vol - \oiint_S p\,dS + \oiint_S \tau\,dS$$

using Gauss' s Therom (6.1),

$$\oiint_S \vec{A}\,d\vec{S} = \iiint_{vol} \left(\nabla A\right) d\,vol \quad \text{and} \quad \oiint_S (a)\,dS = \iiint_{vol} (\nabla a)\,d\,vol$$

to convert the three surface integrals to volume inteagrals.

$$\iiint_{vol} \nabla\left(\rho\,\vec{V}\right)\vec{V}\,d\,vol + \iiint_{vol} \frac{\partial\left(\rho\,\vec{V}\right)}{\partial t}\,d\,vol = \iiint_{vol} \rho\,f\,d\,vol - \iiint_{vol} \nabla p\,d\,vol + \iiint_{vol} \nabla\tau\,d\,vol$$

differentiating,

$$\frac{\partial\left(\rho\,\vec{V}\right)}{\partial t} = -\nabla p - \nabla\left(\rho\,\vec{V}\right)\vec{V} - \nabla\tau + \rho\,f$$

MOMENTUM EQUATIONS

unsteady, 3D, any fluid, variable density

$$\frac{\partial\left(\rho\,\vec{V}\right)}{\partial t} = -\nabla p - \nabla\left(\rho\,\vec{V}\right)\vec{V} - \nabla\tau + \rho\,f$$

$$\frac{\partial}{\partial t}\rho u = -\frac{\partial p}{\partial x} - \left(\rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \rho w\frac{\partial u}{\partial z}\right) - \left(\frac{\partial}{\partial x}\tau_{xx} + \frac{\partial}{\partial y}\tau_{yx} + \frac{\partial}{\partial z}\tau_{zx}\right) + \rho f_x$$

$$\frac{\partial}{\partial t}\rho v = -\frac{\partial p}{\partial y} - \left(\rho u\frac{\partial v}{\partial x} + \rho v\frac{\partial v}{\partial y} + \rho w\frac{\partial v}{\partial z}\right) - \left(\frac{\partial}{\partial x}\tau_{xy} + \frac{\partial}{\partial y}\tau_{yy} + \frac{\partial}{\partial z}\tau_{zy}\right) + \rho f_y$$

$$\frac{\partial}{\partial t}\rho w = -\frac{\partial p}{\partial z} - \left(\rho u\frac{\partial w}{\partial x} + \rho v\frac{\partial v}{\partial y} + \rho w\frac{\partial w}{\partial z}\right) - \left(\frac{\partial}{\partial x}\tau_{xz} + \frac{\partial}{\partial y}\tau_{xz} + \frac{\partial}{\partial z}\tau_{zz}\right) + \rho f_z$$

restricting the momentum equation to Newtonian fluids for which the fluids stress is a linear function of the rate of deformation of the fluid - the change of velocity with distance.

for 1 D, $\tau = \mu \dfrac{du}{dx}$

$$\tau_{xx} = -2\mu \frac{\partial u}{\partial x} + \frac{2}{3}\mu\left(\nabla \vec{V}\right) \qquad \tau_{xy} = \tau_{yx} = -\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\tau_{yy} = -2\mu \frac{\partial v}{\partial x} + \frac{2}{3}\mu\left(\nabla \vec{V}\right) \qquad \tau_{yz} = \tau_{zy} = -\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\tau_{zz} = -2\mu \frac{\partial w}{\partial x} + \frac{2}{3}\mu\left(\nabla \vec{V}\right) \qquad \tau_{xy} = \tau_{yx} = -\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

ENERGY EQUATION CONSERVATIVE INTEGRAL FORTM

$$\text{First Law}\;\; Q = \Delta E + W = \Delta E + W_{shaft} + W_{viscous} + W_{pressure} + W_{body}$$

$$\text{Work} = \text{Force} \times \text{Velocity}$$

$$W_{shaft} = 0$$

$$\text{Work}_{pressure} \quad -\oiint_S (p\, dS)\vec{V}$$

$$\text{Work}_{body} \quad \iiint_{Vol} (\rho\, f\, d\, vol)\vec{V}$$

$$\text{Work}_{viscous} \; -\oiint_S (\tau\, dS)\vec{V}$$

$$\text{Net Energy into control volume} \qquad \oiint_S (\rho\, V\, dS)\left(e + \frac{V^2}{2}\right)$$

$$\text{Change in energy inside the control volume} \quad \frac{\partial}{\partial t}\iiint_{Vol} \rho\left(e + \frac{V^2}{2}\right) d\, vol$$

$$\text{Heat addition} \quad \oiint_S \vec{q}\, dS$$

$$\text{Internal energy,} \quad U = c_v T$$

First Law $Q = \Delta E + W = \Delta E + W_{shaft} + W_{viscous} + W_{pressure} + W_{body}$

$$Q = \Delta E_{\substack{\text{net in}\\ \text{control}\\ \text{volume}}} + \Delta E_{\substack{\text{change in}\\ \text{control volume}}} \quad W_{shaft} + W_{viscous} + W_{pressure} + W_{body}$$

$$Q = \oiint_S \left(\rho\,\vec{\nabla}\,dS\right)\left(e + \frac{V^2}{2}\right) + \frac{\partial}{\partial t}\iiint_{vol} \rho\left(e + \frac{V^2}{2}\right) d\,vol + -\oiint_S (\tau\,dS)\vec{V} - \oiint_S \left(p\,\vec{V}\right) dS + \iiint_{vol} (\rho\,f\,d\,vol)\vec{V} = (2.20a)$$

$$\frac{\partial}{\partial t}\rho\left(c_v T + \frac{V^2}{2}\right) = -\left(\nabla \rho\,\vec{\nabla}\left(c_v T + \frac{V^2}{2}\right)\right) - \nabla \bullet q - \nabla \bullet p\vec{V} - \nabla \bullet (\tau \bullet \vec{V}) + \rho(g \bullet \vec{V})$$

$$\frac{\partial}{\partial t}\rho\left(c_v T + \frac{V^2}{2}\right) = -\left(\frac{\partial}{\partial x} u\rho\left(c_v T + \frac{V^2}{2}\right) + \frac{\partial}{\partial y} w\rho\left(c_v T + \frac{V^2}{2}\right) + \frac{\partial}{\partial y} w\rho\left(c_v T + \frac{V^2}{2}\right)\right)$$

$$-\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right) - \left(\frac{\partial}{\partial x}\rho u + \frac{\partial}{\partial y}\rho v + \frac{\partial}{\partial z}\rho w\right)$$

$$-\left(\frac{\partial}{\partial x}\left(\tau_{xx} u + \tau_{xy} v + \tau_{xz} w\right) + \frac{\partial}{\partial y}\left(\tau_{yx} u + \tau_{yy} v + \tau_{yz} w\right) + \frac{\partial}{\partial z}\left(\tau_{zx} u + \tau_{zy} v + \tau_{zz} w\right)\right)$$

$$\rho c_v\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\right) = -\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right) - T\left(\frac{\partial p}{\partial T}\right)_p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$-\left(\tau_{xx}\frac{\partial u}{\partial x} + \tau_{yy}\frac{\partial v}{\partial y} + \tau_{zz}\frac{\partial w}{\partial z}\right) - \left(\tau_{xy}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \tau_{xz}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + \tau_{yz}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right)$$

EQUATION SUMMARY - 3D, viscous, variable density

CONTINUITY

$$\frac{\partial \rho}{\partial t}+\left(u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+w\frac{\partial \rho}{\partial z}\right)+\left(\rho\frac{\partial u}{\partial x}+\rho\frac{\partial v}{\partial y}+\rho\frac{\partial w}{\partial z}\right)=0$$

MOMENTUM – x, y, z directions

$$\frac{\partial}{\partial t}\rho u=-\frac{\partial p}{\partial x}-\left(\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}+\rho w\frac{\partial u}{\partial z}\right)-\left(\frac{\partial}{\partial x}\tau_{xx}+\frac{\partial}{\partial y}\tau_{yx}+\frac{\partial}{\partial z}\tau_{zx}\right)+\rho f_x$$

$$\frac{\partial}{\partial t}\rho v=-\frac{\partial p}{\partial y}-\left(\rho u\frac{\partial v}{\partial x}+\rho v\frac{\partial v}{\partial y}+\rho w\frac{\partial v}{\partial z}\right)-\left(\frac{\partial}{\partial x}\tau_{xy}+\frac{\partial}{\partial y}\tau_{yy}+\frac{\partial}{\partial z}\tau_{zy}\right)+\rho f_y$$

$$\frac{\partial}{\partial t}\rho w=-\frac{\partial p}{\partial z}-\left(\rho u\frac{\partial w}{\partial x}+\rho v\frac{\partial v}{\partial y}+\rho w\frac{\partial w}{\partial z}\right)-\left(\frac{\partial}{\partial x}\tau_{xz}+\frac{\partial}{\partial y}\tau_{xz}+\frac{\partial}{\partial z}\tau_{zz}\right)+\rho f_z$$

ENERGY

$$\rho c_v\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\right)=-\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z}\right)-T\left(\frac{\partial p}{\partial T}\right)_p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$$

$$-\left(\tau_{xx}\frac{\partial u}{\partial x}+\tau_{yy}\frac{\partial v}{\partial y}+\tau_{zz}\frac{\partial w}{\partial z}\right)-\left(\tau_{xy}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\tau_{xz}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\tau_{yz}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right)$$

<u>EQUATION SUMMARY</u> - 3D, viscous, variable density

CONTINUITY

$$\frac{\partial \rho}{\partial t}+\left(u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+w\frac{\partial \rho}{\partial z}\right)+\left(\boxed{\rho\frac{\partial u}{\partial x}+\rho\frac{\partial v}{\partial y}}+\rho\frac{\partial w}{\partial z}\right)=0$$

2D steady incompressible, inviscid

MOMENTUM – x, y, z directions

$$\frac{\partial}{\partial t}\rho u=\boxed{-\frac{\partial p}{\partial x}-\left(\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}\right.}\left.+\rho w\frac{\partial u}{\partial z}\right)-\left(\frac{\partial}{\partial x}\tau_{xx}+\boxed{\frac{\partial}{\partial y}\tau_{yx}}+\frac{\partial}{\partial z}\tau_{zx}\right)+\rho f_x$$

$$\frac{\partial}{\partial t}\rho v=-\frac{\partial p}{\partial y}-\left(\rho u\frac{\partial v}{\partial x}+\rho v\frac{\partial v}{\partial y}+\rho w\frac{\partial v}{\partial z}\right)-\left(\frac{\partial}{\partial x}\tau_{xy}+\frac{\partial}{\partial y}\tau_{yy}+\frac{\partial}{\partial z}\tau_{zy}\right)+\rho f_y$$

$$\frac{\partial}{\partial t}\rho w=-\frac{\partial p}{\partial z}-\left(\rho u\frac{\partial w}{\partial x}+\rho v\frac{\partial v}{\partial y}+\rho w\frac{\partial w}{\partial z}\right)-\left(\frac{\partial}{\partial x}\tau_{xz}+\frac{\partial}{\partial y}\tau_{xz}+\frac{\partial}{\partial z}\tau_{zz}\right)+\rho f_z$$

ENERGY

$$\rho c_v\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\right)=-\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z}\right)-T\left(\frac{\partial p}{\partial T}\right)_p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$$

$$-\left(\tau_{xx}\frac{\partial u}{\partial x}+\tau_{yy}\frac{\partial v}{\partial y}+\tau_{zz}\frac{\partial w}{\partial z}\right)-\left(\tau_{xy}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\tau_{xz}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\tau_{yz}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right)$$

BOUNDARY LAYER Prandtl 1904

Divide a flow into two regions according to the forces that prevail

BOUNDARY LAYER

thin layer near wall

viscous forces as improtant as interial forces

$\frac{\partial u}{\partial y}$ large, $\tau = \mu \frac{\partial u}{\partial y}$ very large

ignore traverse momentum equations

2 – D incompresible boundary layer equations,

$$u \frac{\partial u}{\partial y} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{dp}{dx} + \frac{1}{\rho} \frac{\partial \tau_{yx}}{dy}$$

$$u \frac{\partial u}{\partial y} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{dp}{dx} + \frac{\mu}{\rho} \frac{\partial^2 u}{dy^2}$$

$$\frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y} = 0$$

FREE STREAM

$\tau = 0, \ \mu = 0,$

Potential Flow

isentropic, frictionless

irrotational,

uniform and parallel

EQUATION SUMMARY - 3D, viscous, variable density

CONTINUITY

$$\frac{\partial \rho}{\partial t}+\left(u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+w\frac{\partial \rho}{\partial z}\right)+\left(\rho\frac{\partial u}{\partial x}+\rho\frac{\partial v}{\partial y}+\rho\frac{\partial w}{\partial z}\right)=0$$

MOMENTUM $-$ x, y, z directions

$$\frac{\partial}{\partial t}\rho u=-\frac{\partial p}{\partial x}-\left(\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}+\rho w\frac{\partial u}{\partial z}\right)-\left(\frac{\partial}{\partial x}\tau_{xx}+\frac{\partial}{\partial y}\tau_{yx}+\frac{\partial}{\partial z}\tau_{zx}\right)+\rho f_x$$

$$\frac{\partial}{\partial t}\rho v=-\frac{\partial p}{\partial y}-\left(\rho u\frac{\partial v}{\partial x}+\rho v\frac{\partial v}{\partial y}+\rho w\frac{\partial v}{\partial z}\right)-\left(\frac{\partial}{\partial x}\tau_{xy}+\frac{\partial}{\partial y}\tau_{yy}+\frac{\partial}{\partial z}\tau_{zy}\right)+\rho f_y$$

$$\frac{\partial}{\partial t}\rho w=-\frac{\partial p}{\partial z}-\left(\rho u\frac{\partial w}{\partial x}+\rho v\frac{\partial v}{\partial y}+\rho w\frac{\partial w}{\partial z}\right)-\left(\frac{\partial}{\partial x}\tau_{xz}+\frac{\partial}{\partial y}\tau_{xz}+\frac{\partial}{\partial z}\tau_{zz}\right)+\rho f_z$$

ENERGY

$$\rho c_v\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\right)=-\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z}\right)-T\left(\frac{\partial p}{\partial T}\right)_p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$$
$$-\left(\tau_{xx}\frac{\partial u}{\partial x}+\tau_{yy}\frac{\partial v}{\partial y}+\tau_{zz}\frac{\partial w}{\partial z}\right)-\left(\tau_{xy}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\tau_{xz}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\tau_{yz}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right)$$

2-D, steady, inviscid (isentropic), variable density

CONTINUITY

$$\left(u\frac{d\rho}{dx}+v\frac{d\rho}{dy}\right)+\left(\rho\frac{du}{dx}+\rho\frac{dv}{dy}\right)=0$$

$$\frac{\partial(\,)}{\partial t}=0$$

$$\frac{\partial(\,)}{\partial z}=0$$

$$w=0$$

$$\tau=0$$

$\mathrm{MOMENTUM} - x, y, z$ directions

$$\frac{\partial p}{\partial x}=-\left(\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}\right)$$

$$\frac{\partial p}{\partial y}=-\left(\rho u\frac{\partial v}{\partial x}+\rho v\frac{\partial v}{\partial y}\right)$$

ENERGY

$$\rho c_v\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=-\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}\right)-T\left(\frac{\partial p}{\partial T}\right)_p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

<u>VELOCITY POTENTIAL</u> – reduce to one equation

$\oint_C \vec{V}d\vec{l} = 0$ for irrotational flow

isentropic, $\tau = 0, \mu = 0$

$\vec{V}d\vec{l}$ is independent of path

an exact differential,

dependent only on position

exact differential $d(\ ) = \frac{\partial(\ )}{\partial x}dx + \frac{\partial(\ )}{\partial y}dy$

$$d(\vec{V}d\vec{l}) = \frac{\partial(\vec{V}d\vec{l})}{\partial x}dx + \frac{\partial(\vec{V}d\vec{l})}{\partial y}dy$$

$$d(\vec{V}d\vec{l}) = u\ dx\ \vec{i} + u\ dy\ \vec{j}$$

by comparison, $u = \frac{\partial(\vec{V}d\vec{l})}{\partial x}$, $v = \frac{\partial(\vec{V}d\vec{l})}{\partial y}$

u and v are functions

of the same scalar qauntity,

define as Φ, velocity potential function

$$u = \frac{\partial(\Phi)}{\partial x}, v = \frac{\partial(\Phi)}{\partial y}$$

CHECK : Greens Theorem, $\oint_C \rightarrow \oiint_S$

$$\oint_C \vec{V}\ d\vec{l} = \oiint_S \left(\frac{\partial v}{\partial x} - \frac{\partial v}{\partial x} \right) dxdy = 0$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial x}$$

substituting, $u = \frac{\partial(\Phi)}{\partial x}, v = \frac{\partial(\Phi)}{\partial y}$

$$\frac{\partial^2(\Phi)}{\partial x dy} = \frac{\partial^2(\Phi)}{\partial y dx}$$

CONTINUITY EQUATION 2 - D steady, inviscid, varable density

$$\left(u\frac{d\rho}{dx}+v\frac{d\rho}{dy}\right)+\left(\rho\frac{du}{dx}+\rho\frac{dv}{dy}\right)=0$$

continuty equation in terms of velocity potential

$$\text{substitute:}\quad u=\frac{\partial(\Phi)}{\partial x}=\Phi_x,\qquad \frac{du}{dx}=\frac{\partial^2(\Phi)}{\partial x^2}=\Phi_{xx}$$

$$v=\frac{\partial(\Phi)}{\partial y}=\Phi_x,\qquad \frac{dv}{dy}=\frac{\partial^2(\Phi)}{\partial x^2}=\Phi_{xx}$$

$$\frac{\partial(\Phi)}{\partial x}\frac{d\rho}{dx}+\frac{\partial(\Phi)}{\partial y}\frac{d\rho}{dy}+\rho\frac{\partial^2(\Phi)}{\partial x^2}+\rho\frac{\partial^2(\Phi)}{\partial x^2}=0$$

$$\Phi_x\frac{d\rho}{dx}+\Phi_x\frac{d\rho}{dy}+\rho\Phi_{xx}+\rho\Phi_{yy}=0$$

2 variables, ρ and Φ
density will be eliminated by the momentum equations.

MOMENTUM EQUATIONS

multipy x direction by dx

$$-\frac{\partial p}{\partial x}dx = \rho u\frac{\partial u}{\partial x}dx + \rho v\frac{\partial u}{\partial y}dx$$

since for irrotational flow,

$$\frac{du}{dy} = \frac{dv}{dx}$$

$$-\frac{\partial p}{\partial x}dx = \rho u\frac{\partial u}{\partial x}dx + \rho v\frac{\partial v}{\partial x}dx$$

substitute :

$$u = \frac{\partial(\Phi)}{\partial x} = \Phi_x$$

$$\frac{du}{dx} = \frac{\partial^2(\Phi)}{\partial x^2} = \Phi_{xx}$$

$$v = \frac{\partial(\Phi)}{\partial y} = \Phi_y$$

$$\frac{dv}{dy} = \frac{\partial^2(\Phi)}{\partial x^2} = \Phi_{yx}$$

$$-\frac{\partial p}{\partial x}dx = \rho\left(\Phi_x\Phi_{xx} + \Phi_y\Phi_{yx}\right)$$

for the y direction equation,

$$-\frac{\partial p}{\partial y}dy = \rho\left(\Phi_x\Phi_{xy} + \Phi_y\Phi_{yy}\right)$$

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$

$$\partial \rho = \left(\frac{\partial p}{a^2}\right)$$

$$\frac{\partial \rho}{\partial x} = \frac{1}{a^2}\frac{\partial p}{\partial x}$$

$$-\frac{\partial \rho}{\partial x} = \frac{\rho}{a^2}\left(\Phi_x \Phi_{xx} + \Phi_y \Phi_{yx}\right)$$

$$-\frac{\partial \rho}{\partial y} = \frac{\rho}{a^2}\left(\Phi_x \Phi_{xy} + \Phi_y \Phi_{yy}\right)$$

substituting into the contuinuity equation,

$$-\frac{\partial\rho}{\partial x}=\frac{\rho}{a^2}\left(\Phi_x\Phi_{xx}+\Phi_y\Phi_{yx}\right)$$

$$-\frac{\partial\rho}{\partial y}=\frac{\rho}{a^2}\left(\Phi_x\Phi_{xy}+\Phi_y\Phi_{yy}\right)$$

$$\left(1-\frac{\Phi_x^2}{a^2}\right)\Phi_{xx}+\left(1-\frac{\Phi_y^2}{a^2}\right)\Phi_{yy}-2\frac{\Phi_x\Phi_y}{a^2}\qquad(8.17,\text{for }2-\text{D})$$

$$\left(1-\frac{\Phi_x^2}{a^2}\right)\Phi_{xx}+\left(1-\frac{\Phi_y^2}{a^2}\right)\Phi_{yy}-2\frac{\Phi_x\Phi_y}{a^2}\Phi_{yy} \qquad (8.17,\text{for } 2-\text{D})$$

substituting, $u=\Phi_x, v=\Phi_y$

$$(1-\frac{u^2}{a^2})\Phi_{xx}+\left(1-\frac{v^2}{a^2}\right)\Phi_{xy}-2\frac{uv}{c^2}\Phi_{yy}=0 \qquad (11.5)$$

exact differentials for $\frac{\partial\Phi}{dx}$ and $\frac{\partial\Phi}{dy}$,

$$d\left(\frac{\partial\Phi}{dx}\right)=\frac{\partial^2\Phi}{\partial x^2}dx+\frac{\partial^2\Phi}{\partial x\partial y}dy=\Phi_{xx}dx+\Phi_{xy}dy=du \qquad (11.6)$$

$$d\left(\frac{\partial\Phi}{dy}\right)=\frac{\partial^2\Phi}{\partial x dy}dx+\frac{\partial^2\Phi}{\partial y\partial y}dy=\Phi_{xy}dx+\Phi_{yy}dy=dv \qquad (11.7)$$

3 simultaneous linear equations in Φ

$$\left(1-\frac{u^2}{a^2}\right)\Phi_{xx} - 2\frac{uv}{c^2}\Phi_{xy} + \left(1-\frac{v^2}{a^2}\right)\Phi_{yy} = 0$$

$$\Phi_{xx}dx + \Phi_{xy}dy + 0 = du$$

$$\Phi_{xy}dx + 0 + \Phi_{yy}dy = dv$$

$$\Phi_{xy} = \frac{\begin{vmatrix} (1-\frac{u^2}{a^2}) & 0 & (1-\frac{v^2}{a^2}) \\ dx & du & 0 \\ 0 & dv & dy \end{vmatrix}}{\begin{vmatrix} (1-\frac{u^2}{a^2}) & -2\frac{uv}{a^2} & (1-\frac{v^2}{a^2}) \\ dx & du & 0 \\ 0 & dv & dy \end{vmatrix}} = \frac{N}{D}$$

Φ_{xy} is indeterminate,

on a characteristic of the solution,

When both N and D are 0

Φ_{xy} is indeterminate.

$N = 0$ defines the characteristic

of the solution, $C = f(x, y)$.

$D = 0$ defines properties along

the charactersitic.

$$N, \text{numerator} = 0$$

$$(1-\frac{u^2}{a^2})\,dudy + (1-\frac{v^2}{a^2})dxdv = 0$$

$$\frac{du}{dv} = \frac{(1-\frac{u^2}{a^2})}{(1-\frac{v^2}{a^2})}\left(\frac{-\frac{uv}{a^2} \pm \sqrt{\frac{u^2+v^2}{a^2}-1}}{1-\frac{u^2}{a^2}}\right)$$

$$d\theta = \pm\sqrt{M^2-1}\,\frac{dV}{V}$$

$$\int d\theta \text{ is the Prandtl - Meyer Function}$$

$$\theta + \upsilon(M) = K_{-}\text{along}C_{-}\text{characteristic}$$

$$\theta - \upsilon(M) = K_{+}\text{along}C_{+}\text{characteristic}$$

$$D, \text{denominator} = 0$$

$$(1-\frac{u^2}{a^2})(dy)^2 + 2\frac{uv}{a^2}dxdy + (1-\frac{v^2}{a^2})(dx)^2 = 0$$

$$(1-\frac{u^2}{a^2})\left(\frac{dy}{dx}\right)^2 + 2\frac{uv}{a^2}\left(\frac{dy}{dx}\right) + (1-\frac{v^2}{a^2}) = 0$$

$$\left(\frac{dy}{dx}\right)_{\text{characteristic}} = \frac{-\frac{uv}{a^2} \pm \sqrt{\frac{u^2+v^2}{a^2}-1}}{1-\left(\frac{u^2}{a^2}\right)}$$

$$\left(\frac{dy}{dx}\right)_{\text{characteristic}} = \frac{-\frac{uv}{a^2} \pm \sqrt{(M^2-1)-1}}{1-\left(\frac{u^2}{a^2}\right)}$$