CFD for Aerospace Applications

Fluid Dynamics Review

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Fluid Dynamics Review

What's a fluid?

- First things first, what is a fluid?
  - It is a continuum description of a liquid or a gas at scales orders of magnitude larger than the molecular scales. The continuum description means that the material is continuously distributed and completely fills the space it occupies. It can also be subdivided continually into infinitesimal elements with the same properties as the bulk material.
- A fluid can be approximated as a continuum when its Knudsen number $Kn < 0.01$.

$$Kn = \frac{\lambda}{L}, \tag{1}$$

where λ is the mean free path between atoms and molecules, and L is a representative macroscopic physical length scale.

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What's a fluid?

- An additional physical property exclusive to fluids is the following,

  > A fluid is any material with zero shear modulus (usually liquids or gases). As a result fluids cannot resist shear forces and will continuously change in shape under shear. This continuous change in shape leads to fluid motion, *flow*, which is a characteristic of fluids (e.g. water, air, honey, gasoline, hydrogen, oil, blood, etc.).

- This property is especially obvious in the way we calculate shear stress in fluids.

Fluid Dynamics Review

What's a fluid?

- Shear stress in solids,

$$\tau_s = G\gamma,$$

where G is the shear modulus, $\gamma = \frac{\delta l}{l_0}$ the strain, δl the deformation, and l_0 the initial length. γ is a static parameter precluding motion.

- Shear stress in fluids,

$$\tau_f = \mu\dot{\gamma},$$

where μ is the dynamic viscosity and $\dot{\gamma} = \frac{d\gamma}{dt}$ is the strain rate, the derivative of strain with respect to time, which is a velocity implying motion.

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Theories of fluid motion

- A fluid comprises huge numbers of atoms and molecules (particles) moving at the microscopic level.
- One can run molecular dynamics simulations of such motions with high accuracy but this approach is currently intractable at the macroscopic level since it requires computational resources of a supercomputer for even the simplest cases.
- For dilute gases where the average distance between particles is much larger than the particle size one can resort to statistical averaging, leading to the kinetic theory of gases.

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Theories of fluid motion

- Kinetic theory is much more difficult to apply to dense gases and liquids, the subject of most real-life applications, due the much smaller distances between particles and their disorderly state. This a major research subject in kinetic theory of gases and liquids.
- Important advances have been made during the last 20 years with the Lattice-Boltzmann method but exhibits some non-physical features and is still not as generalizable as the widely used Navier-Stokes equations, which will be discussed next.

Fluid Dynamics Review

Fundamental principles of fluid flow

- Fluid dynamics is a subdiscipline of fluid mechanics concerned with the study of fluids in motion, or fluid flow. It is a very wide field including other disciplines such as aerodynamics and hydrodynamics.
- Fluid dynamics is governed by three fundamental equations; the continuity, momentum and energy equations. These equations are mathematical statements of the following fundamental principles,
  (1) conservation of mass,
  (2) conservation of momentum $\mathbf{F} = \frac{d(m\mathbf{v})}{dt}$ (Newton's 2nd law) where $\mathbf{F}$ is the force acting on a fluid element and $\mathbf{v}$ is the fluid velocity vector,
  (3) conservation of energy, or the 1st law of thermodynamics.

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Fundamental principles of fluid flow

- Collectively these three conservation equations as applied to fluid flow are known as the Navier-Stokes (NS) equations, which were originally the three momentum conservation equations only.
- Computational Fluid Dynamics (CFD) is based on the numerical solution of the NS equations, which can be derived in different forms with minor consequences on general fluid dynamics theory and experiment.
- However, the form used in CFD can be critical and might lead to oscillations in the solution or even instability.
- It is therefore important to derive the NS equations in order to highlight their differences and similarities and understand their implications on CFD algorithms.
- CFD is mostly concerned with two main forms, the conservation and non-conservation forms.

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Fundamental principles of fluid flow: conservation and non-conservation forms

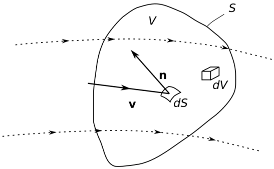

Figure 1: A fixed finite control volume.

- Consider an arbitrary fixed finite control volume depicted in figure 1, with volume V, enclosing surface S, and surface normal $\mathbf{n}$ pointing out of the control volume by convention.
- The local time rate of change of a property ϕ within the control volume is $\partial\phi/\partial t$, while its flux is $f(\phi) = \phi\mathbf{u}$.
- An increase (decrease) in ϕ corresponds to $\partial\phi/\partial t > 0$ ($\partial\phi/\partial t < 0$).
- Flow entering (leaving) the control volume corresponds to $\mathbf{u} \cdot \mathbf{n} < 0$ ($\mathbf{u} \cdot \mathbf{n} > 0$).

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Fundamental principles of fluid flow: conservation and non-conservation forms

- Taking the integral balance of ϕ within the control volume we write,

$$\iiint_V \frac{\partial \phi}{\partial t} dV + \oiint_{\partial V} \phi \mathbf{u} \cdot \mathbf{n} dS = 0. \tag{2}$$

  Based on our sign convention increase (decrease) in ϕ within V is counteracted by opposite sign flux at the enclosing surface S.

- Using the divergence theorem on the surface integral term of equation 2 we obtain,

$$\iiint_V \left(\frac{\partial \phi}{\partial t} + \nabla \cdot \phi \mathbf{u} \right) dV = 0. \tag{3}$$

- Since the control volume is arbitrary the equation above means that the integrand must be zero, which leads to the transport equation in differential form,

$$\frac{\partial \phi}{\partial t} + \nabla \cdot f(\phi) = 0. \tag{4}$$

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Fundamental principles of fluid flow: conservation and non-conservation forms

- Equation 4 is a transport equation involving the local time rate of change and flux derivatives, which is defined as the differential conservation form. It is also called the divergence form due to the use of the divergence operator.
- Equation 2 is defined as the integral conservation form.
- Any transport equation that does not take the form of equations 4 or 2 is in non-conservation form.
- The sign convention used here will stay with us for the rest of the course.

Fluid Dynamics Review

Fundamental principles of fluid flow: flow models

- The general procedure of deriving the equations of fluid flow is the following:
    1. Choose the appropriate fundamental principles from physics, which in our case are the conservation of mass, momentum and energy.
    2. Apply these principles to a suitable model of the flow.
    3. Extract the mathematical equations representing the physical principles above.
- Unlike a solid body in translational motion with one velocity for all its parts, a squishy substance like a fluid will exhibit variable velocity over its parts. To properly visualize a fluid two models have emerged over time,
    1. the Finite Control Volume and,
    2. the Infinitesimal Fluid Element.

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Fundamental principles of fluid flow: the finite control volume

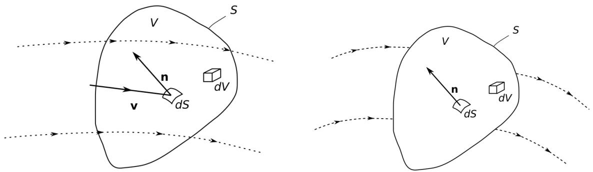

Figure 2: Left, fixed finite control volume. Right, moving finite control volume.

- A finite control volume V is a closed volume drawn within a finite region of the flow and surrounded by a control surface S which bounds it. It can be fixed in space with the flow streamlines passing through it (figure 2 left), or it can be moving with the fluid while containing the same fluid particles (figure 2 right).

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Fundamental principles of fluid flow: the finite control volume

- In both cases, it is a reasonably large finite region of the flow and the fundamental physical principles (conservation of mass, momentum and energy) are applied to the fluid inside the control volume, and to the fluid crossing the control surface if the control volume is fixed in space.
- The fluid flow equations *directly* obtained from the fixed control volume will be in *integral* form. Partial differential equations can be obtained from the integral form *indirectly*. Whether integral or differential both will be in *conservation* form.
- The fluid flow equations *directly* obtained from the moving control volume in either integral or partial differential form will be in *non-conservation* form.

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Fundamental principles of fluid flow: the infinitesimal fluid element

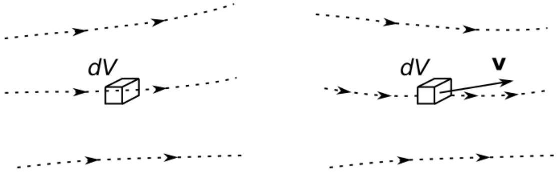

Figure 3: Left, fixed infinitesimal fluid element. Right, moving infinitesimal fluid element.

- Consider and infinitesimal fluid element with a differential volume dV in a general flow field as in figure 3. The fluid element is infinitesimal from the differential calculus sense and contains a large enough number of molecules to behave as a continuum.

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Fundamental principles of fluid flow: the infinitesimal fluid element

- The fluid element can be fixed in space (figure 3, left) with fluid flowing through it, or moving along a streamline at a velocity V (figure 3, right).
- Just like the finite control volume we apply the fundamental conservation principles to the fluid element instead of the entire flow field. This leads *directly* to the fundamental equations in *partial differential form*.
- Again, just like the finite control volume, the fixed fluid element will lead to the *conservation form* and the moving element will lead to the *non-conservation form*.

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Fundamental principles of fluid flow: summary of fluid models

	Finite Control Volume		Infinitesimal Fluid Element	
	Fixed	Moving	Fixed	Moving
Form	conservation	non-conservation	conservation	non-conservation
Direct Equation Type	Integral	Integral	Differential	Differential

Table 1: Summary of fluid models.

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Fundamental principles of fluid flow: comments on the conservation and non-conservation forms

- In theoretical and experimental fluid dynamics it is generally irrelevant which form of the Navier-Stokes equations we deal with, as one can easily be obtained from the other, without much effect on the results.
- However, the conservation and non-conservation forms have different mathematical behaviour with important consequences on some CFD cases and proper care must be taken as to which form is used and how it is implemented in CFD software.
- In fact, the conservation and non-conservation form terminology has arisen primarily the field of CFD due to their importance to numerical analysis.
- The comments on the different forms of the Navier-Stokes equations will become more clear once we derive the equations and start applying to numerical analysis.

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Fundamental principles of fluid flow: the substantial derivative

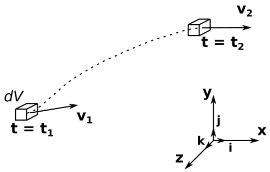

Figure 4: Fluid element moving with the flow field.

- The substantial derivative is ubiquitous in fluid dynamics, and while being very important it is often misunderstood.
- We consider an infinitesimal fluid element moving in a transient flow field with a density $\rho(x, y, z, t)$ and a velocity $\mathbf{v}(x, y, z, t)$,

$$\begin{aligned}\mathbf{v}(x, y, z, t) = u(x, y, z, t)\mathbf{i} \\ + v(x, y, z, t)\mathbf{j} + w(x, y, z, t)\mathbf{k}\end{aligned}$$

Fluid Dynamics Review

Fundamental principles of fluid flow: the substantial derivative

- The fluid element in figure 4 flows from point 1 at time t_1 with velocity $\mathbf{v}_1 = \mathbf{v}(x_1, y_1, z_1, t_1)$ and density $\rho_1 = \rho(x_1, y_1, z_1, t_1)$ to point 2 at time t_2 with velocity $\mathbf{v}_2 = \mathbf{v}(x_2, y_2, z_2, t_2)$ and density $\rho_2 = \rho(x_2, y_2, z_2, t_2)$.
- Using a Taylor expansion around point 1 and rearranging we can write,

$$\begin{aligned}\rho_2 - \rho_1 = &\left(\frac{\partial \rho}{\partial x}\right)_1 (x_2 - x_1) + \left(\frac{\partial \rho}{\partial y}\right)_1 (y_2 - y_1) \\ &+ \left(\frac{\partial \rho}{\partial z}\right)_1 (z_2 - z_1) + \left(\frac{\partial \rho}{\partial t}\right)_1 (t_2 - t_1) \\ &+ (\text{ higher order terms })\end{aligned}$$

Fluid Dynamics Review

Fundamental principles of fluid flow: the substantial derivative

- dividing left and right sides by $(t_2 - t_1)$ and neglecting higher order terms,

$$\frac{\rho_2 - \rho_1}{t_2 - t_1} = \left(\frac{\partial \rho}{\partial x}\right)_1 \left(\frac{x_2 - x_1}{t_2 - t_1}\right) + \left(\frac{\partial \rho}{\partial y}\right)_1 \left(\frac{y_2 - y_1}{t_2 - t_1}\right) + \left(\frac{\partial \rho}{\partial z}\right)_1 \left(\frac{z_2 - z_1}{t_2 - t_1}\right) + \left(\frac{\partial \rho}{\partial t}\right)_1 \tag{5}$$

- The left hand side of equation 5 is the average time rate of change of density of the fluid element moving from point 1 to 2.
- In the limit of t_2 approaching t_1 we define the substantial derivative as,

$$\frac{D\rho}{Dt} \equiv \lim_{t_2 \to t_1} \left(\frac{\rho_2 - \rho_1}{t_2 - t_1}\right)$$

Fluid Dynamics Review

Fundamental principles of fluid flow: the substantial derivative

- Here, the substantial derivative $\dfrac{D\rho}{Dt}$ is the instantaneous time rate of change of density as the fluid element moves through space, from point 1 to 2. It is the time rate of change we see/measure as we follow the moving fluid element with our eyes.
- In comparison $\frac{\partial \rho}{\partial t}$ is the *local* instantaneous time rate of change at a fixed point in space. It is the time rate of change we see/measure as we focus on the fixed fluid element with the flow passing through it.

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Fundamental principles of fluid flow: the substantial derivative

- We also have in equation 5,

$$\lim_{t_2 \to t_1} \left(\frac{x_2 - x_1}{t_2 - t_1} \right) \equiv u,$$

$$\lim_{t_2 \to t_1} \left(\frac{y_2 - y_1}{t_2 - t_1} \right) \equiv v,$$

$$\lim_{t_2 \to t_1} \left(\frac{z_2 - z_1}{t_2 - t_1} \right) \equiv w.$$

- We can now rewrite the substantial derivative of density as,

$$\frac{D\rho}{Dt} = u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \frac{\partial \rho}{\partial t}. \tag{6}$$

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Fundamental principles of fluid flow: the substantial derivative

- Since $\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$ we define the substantial derivative operator,

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla) \tag{7}$$

- From equation 7 we see that the substantial derivative is the sum of $\partial/\partial t$ the *local derivative*, and $\mathbf{u} \cdot \nabla$ the *convective derivative*.
- The local derivative is the temperature change you would feel if you were standing still and the temperature changed locally.
- The convective derivative is the temperature change you would feel if you were moving through a spatial temperature gradient, like an open hallway connecting a heated room to the freezing winter outdoors.

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Fundamental principles of fluid flow: the substantial derivative

- Let's look at this from a mathematical angle and calculate the total derivative of $\rho(x, y, z, t)$ using the chain rule,

$$d\rho = \frac{\partial \rho}{\partial x}dx + \frac{\partial \rho}{\partial y}dy + \frac{\partial \rho}{\partial z}dz + \frac{\partial \rho}{\partial t}dt,$$

  divide by dt and since $u = dx/dt$, $v = dy/dt$, $w = dz/dt$ we get,

$$\frac{d\rho}{dt} = u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \frac{\partial \rho}{\partial t}. \tag{8}$$

- Equation 8 is the same as equation 6. Therefore the substantial derivative is simply the physical equivalent of the mathematical formalism of total derivative with respect to time. However, the initial discussion is necessary to highlight the physical meaning of the substantial derivative and its components.

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Fundamental principles of fluid flow: summary of the substantial derivative

- The substantial derivative is the time rate of change on a *moving* fluid element and can be calculated for any flow property or parameter (density, pressure, velocity, etc.) by applying the following operator,

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla). \tag{9}$$

- It consists of a local derivative and a convective derivative.
- It is the physical equivalent of the mathematical formalism of the total derivative with respect to time.

Fluid Dynamics Review

Fundamental principles of fluid flow: physical meaning of the divergence of velocity

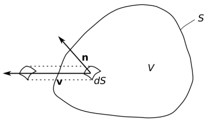

Figure 5: Moving control volume V.

- The velocity divergence $\nabla \cdot \mathbf{v}$ appears frequently in the transport equations of fluid dynamics and is quite significant for conservation of mass as we shall see later. This warrants special consideration of its physical meaning.
- Consider the moving control volume in figure 5, with fixed mass, variable volume V and control surface S as it moves through regions of different density.

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Fundamental principles of fluid flow: physical meaning of the divergence of velocity

- The change in volume ΔV due to the movement of an infinitesimal surface element dS, with surface normal $\mathbf{n}$, moving at velocity $\mathbf{v}$ over a time increment Δt is,

$$\Delta V = (\mathbf{v}\Delta t)\cdot\mathbf{n}dS = (\mathbf{v}\Delta t)\cdot\mathbf{dS} \quad \text{where } \mathbf{dS} = \mathbf{n}dS. \tag{10}$$

- As $dS \to 0$, the time rate of change of the control volume over Δt is,

$$\frac{DV}{Dt} = \frac{1}{\Delta t}\oiint_S (\mathbf{v}\Delta t)\cdot\mathbf{dS} = \oiint_S \mathbf{v}\cdot\mathbf{dS}. \tag{11}$$

- Using the divergence theorem on the right hand side of the equation above,

$$\frac{DV}{Dt} = \oiiint_V (\nabla\cdot\mathbf{v})dV. \tag{12}$$

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Fundamental principles of fluid flow: physical meaning of the divergence of velocity

- Assume the control volume is shrunk to δV so small (infinitesimal) that $\nabla \cdot \mathbf{v}$ is constant and we can move it out of the integral,

$$\frac{D(\delta V)}{Dt} = (\nabla \cdot \mathbf{v}) \oiiint_V d(\delta V) = (\nabla \cdot \mathbf{v}) \delta V. \tag{13}$$

- Rearranging the result above leads to,

$$\nabla \cdot \mathbf{v} = \frac{1}{\delta V} \frac{D(\delta V)}{Dt}. \tag{14}$$

- Equation 14 is our final result and it means that the divergence of velocity is the time rate of change of a *moving fluid element* per unit volume.