Assignment 00

AERO 455 - CFD for Aerospace Applications

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Given: 15th January 2022
Due: 5pm EST on 28th January 2022

1 The Momentum Equations

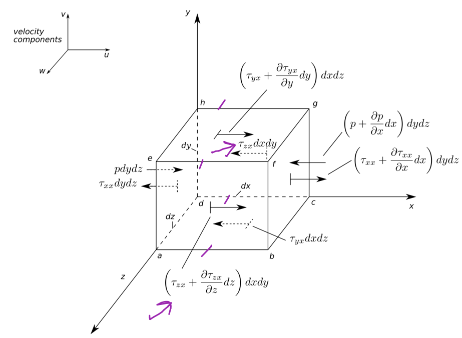


Figure 1.1: Infinitesimal fluid element with forces in the x-direction.

1.1 The x-momentum equation

The infinitesimal fluid element shown in figure 1.1 was used in the fluid dynamics review of lecture 03 to derive the differential x-momentum equation in non-conservation form for a moving viscous fluid element.

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_{b,x}. \tag{1.1}$$

1. Derive the differential x-momentum equation in *conservation* form starting from equation 1.1 by expanding the substantial derivative and using divergence vector identities to make the continuity equation appear so that it can be eliminated. Comment all your steps. (5 points)

2. Assuming a Newtonian fluid expand the force terms from the previous part using the dynamic viscosity and rate of strain tensor and explain what each term means. (10 points)

3. Derive the *integral* x-momentum equation in *conservation* form by integrating the differential x-momentum equation in conservation form derived in the previous part. Use the divergence theorem and comment all your steps. (5 points)

4. What is the difference between the substantial and local time derivatives and when should you use them? (2.5 points)

5. Which form of the momentum equations results directly from using a *fixed finite control volume*? (2.5 points)

1.2 The y-momentum equation in non-conservation form

1. Draw the infinitesimal fluid element shown in figure 1.1 for pressure and stress forces in the y-direction using the same sign convention from lecture 03. Explain what each force term means. (10 points)

2. Assuming a Newtonian fluid expand the force terms from the previous part using the dynamic viscosity and rate of strain tensor. (5 points)

3. Derive the y-momentum equation in *non-conservation* form based on the infinitesimal fluid element drawn in the previous part and following the procedure from lecture 03. Comment all your steps. (10 points)

1.3 The z-momentum equation in non-conservation form

1. Draw the infinitesimal fluid element shown in figure 1.1 for pressure and stress forces in the z-direction using the same sign convention from lecture 03. Explain what each force term means. (10 points)

2. Assuming a Newtonian fluid expand the force terms from the previous part using the dynamic viscosity and rate of strain tensor. (5 points)

3. Derive the z-momentum equation in *non-conservation* form based on the infinitesimal fluid element drawn in the previous part and following the procedure from lecture 03. Comment all your steps. (10 points)

2 The Energy Equation

In the lecture 03 of the fluid dynamics review we derived the total energy equation 2.1, with the total energy being the sum of the internal energy e and the mechanical energy $\dfrac{|\mathbf{v}|^2}{2}$.

$$
\begin{aligned}
\rho \frac{D}{Dt}\left(e + \frac{|\mathbf{v}|^2}{2}\right) = \rho \dot{q} &+ \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) \\
&- \frac{\partial(up)}{\partial x} - \frac{\partial(vp)}{\partial y} - \frac{\partial(wp)}{\partial z} \\
&+ \frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z} \\
&+ \frac{\partial(v\tau_{xy})}{\partial x} + \frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z} \\
&+ \frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z} \\
&+ \rho \mathbf{f} \cdot \mathbf{v}.
\end{aligned} \tag{2.1}
$$

1. Explain what each term in equation 2.1 means. (5 points)
2. Derive the differential *mechanical energy* equation in *non-conservation* form by multiplying the differential x-, y- and z-momentum equations in non-conservation form with the corresponding x-, y- and z-components of velocity respectively (u, v, w). You will need rewrite the time derivative terms in the multiplied momentum equations so as to obtain the substantial derivatives of $\dfrac{u^2}{2}$, $\dfrac{v^2}{2}$ and $\dfrac{w^2}{2}$. Keep in mind that $|\mathbf{v}|^2 = u^2 + v^2 + w^2$. Comment all your steps. (10 points)

3. Derive the differential *internal energy* equation in *non-conservation* form by substracting the differential mechanical energy equation obtained in the previous part from equation 2.1. Comment all your steps. (5 points)

4. Derive the differential *internal energy* equation in *conservation* form by expanding the substantial derivative and using divergence vector identities to make the continuity equation appear so it can be eliminated. Comment all your steps. (5 points)

1.1.1) $$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_{b,x} \quad [1.1.1]$$

Expanding Substantial derivative

$$\rho \frac{Du}{Dt} = \rho \frac{\partial u}{\partial t} + \rho \vec{V} \cdot \nabla u \quad [1.1.2]$$

We can expand the following derivative

$$\frac{\partial (\rho u)}{\partial t} = \rho \frac{\partial u}{\partial t} + u \frac{\partial \rho}{\partial t}$$

Solving for this we get

$$\rho \frac{\partial u}{\partial t} = \frac{\partial (\rho u)}{\partial t} - u \frac{\partial \rho}{\partial t} \quad [1.1.3]$$

Using the vector identity for the divergence of a scalar multiplied by a vector we get

$$\nabla \cdot (\rho u \vec{V}) = u \nabla \cdot (\rho \vec{V}) + (\rho \vec{V}) \cdot \nabla u$$

or

$$\rho \vec{V} \cdot \nabla u = \nabla \cdot (\rho u \vec{V}) - u \nabla \cdot (\rho \vec{V}) \quad [1.1.5]$$

Substituting [1.1.3] & [1.1.4] into [1.1.2] we get

$$\rho \frac{Du}{Dt} = \frac{\partial (\rho u)}{\partial t} - u \frac{\partial \rho}{\partial t} - u \nabla \cdot (\rho \vec{V}) + \nabla \cdot (\rho u \vec{V}]$$

factoring this we get

$$\rho \frac{Du}{Dt} = \frac{\partial (\rho u)}{\partial t} - u \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) \right] + \nabla \cdot (\rho u \vec{V}) \quad [1.1.6]$$

We can write the continuity equation as follows

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0$$

As such, in order to satisfy continuity, [1.1.6] reduces to the following.

$$\rho \frac{Du}{Dt} = \frac{\partial (\rho u)}{\partial t} + \nabla \cdot (\rho u \vec{V}) \quad [1.1.7]$$

Plugging [1.1.7] into [1.1.1] we get

$$\frac{\partial(\rho u)}{\partial t} + \nabla \cdot (\rho u \vec{V}) = -\frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_{b,x} \quad [1.1.8]$$

Here we can see the X-Momentum equatinon in conservation form

1.1.2) Assuming a Newtonian fluid we get the following for the stress tensors

$$\tau_{xx} = \lambda \nabla \cdot \vec{V} + 2\mu \frac{\partial u}{\partial x} \quad [1.1.9a]$$

$$\tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\right] \quad [1.1.9b]$$

$$\tau_{zx} = \mu\left[\frac{\partial w}{\partial y} + \frac{\partial v}{\partial y}\right] \quad [1.1.9c]$$

Where μ is viscosity & λ is bulk viscosity equal to $-\frac{2}{3}\mu$

Plugging [1.1.9a], [1.1.9b], & [1.1.9c] into [1.1.8] & expanding we get the complete Navier-Stokes equation in the x-direction in conservation form

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} + \frac{\partial(\rho uw)}{\partial z} = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial x}\left(\lambda \nabla \cdot \vec{V} + 2\mu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] + \rho F_{bx}$$

$-\frac{\partial P}{\partial x}$ is the force acting on the infinitesimal fluid element from the fluid pressure acting upon the walls of the volume

$\frac{\partial}{\partial x}\left(\lambda \nabla \cdot \vec{V} + 2\mu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right]$ represent the viscous forces acting upon the fluid element from the shear strain imposed on the walls of the fluid. This strain is caused by the velocity gradient found in the fluid, causing particles to want to pull each other along

$\rho F_{b,x}$ represents the body forces acting upon the fluid element. These forces are external forces such as gravity, or electromagnetic forces.

$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} + \frac{\partial(\rho uw)}{\partial z}$ represent the acceleration forces acting upon the fluid.

1.1.3) Looking at equation [1.1.8] we have

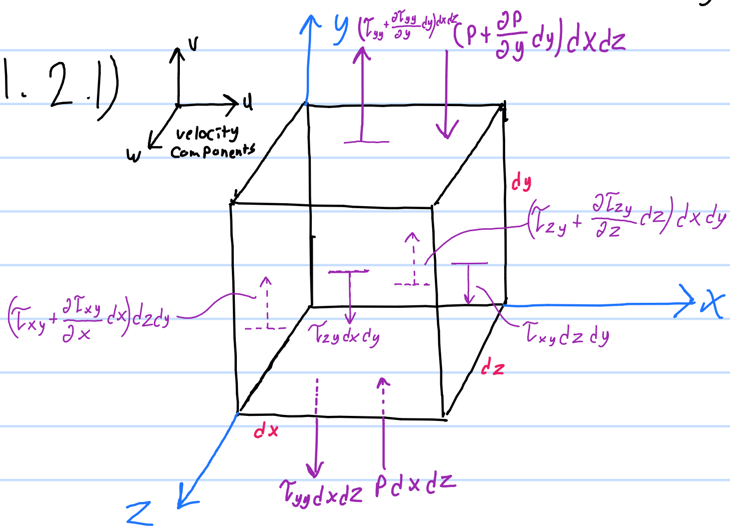
$$\frac{\partial(\rho u)}{\partial t} + \nabla \cdot (\rho u \vec{V}) = -\frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_{b,x} \quad [1.1.8]$$

This assignment is already way too long for 2% of my grade so I'll take the L here.

1.1.4) $\frac{D}{Dt}$ represents the substantial derivative, while $\frac{\partial}{\partial t}$ represents the local derivative. The substantial derivative describes the time rate of change that follows a moving element. The local time derivative on the other hand describes the time rate of change for a fixed point. The substantial derivative is simply the summation of the local & convective derivative. The local derivative should be used when the property of interest is to be observed at a fixed point. The substantial derivative however should be used when the property of interest for the entire flow field is being analyzed.

1.1.5) When the momentum equations are derived using a fixed finite control volume we obtain the integral conservation form of the momentum equations.

1.2.1)



- $Pdxdz$ & $(P+\frac{\partial P}{\partial y}dy)dxdz$ represent the hydrostatic pressure forces acting on the fluid element.

- $\tau_{zy}dxdy$ & $\tau_{xy}dydz$ represent the shear stresses acting upon the fluid while $(\tau_{zy}+\frac{\partial \tau_{zy}}{\partial z}dz)dxdy$ & $(\tau_{xy}+\frac{\partial \tau_{xy}}{\partial x}dx)dydz$ represent the shear forces defined relative to the first two shear forces.

- $\tau_{yy}dxdz$ represents the normal stress acting upon the fluid element, while $(\tau_{yy}+\frac{\partial \tau_{yy}}{\partial y}dy)dxdz$ represents the normal stress relative to the first stress

1.2.2) Assuming a Newtonian fluid we get the following for the stress tensors

$$\tau_{yy}=\lambda\nabla\cdot\vec{V}+2\mu\frac{\partial v}{\partial y},\quad \tau_{zy}=\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right),\quad \tau_{xy}=\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)$$

Expanding these we get the following (note pressure is ignored as it doesn't change)

$$\tau_{yy}dxdz=\left(\lambda\nabla\cdot\vec{V}+2\mu\frac{\partial v}{\partial y}\right)dxdz$$
$$\left(\tau_{yy}+\frac{\partial \tau_{yy}}{\partial y}dy\right)dxdz=\left[\lambda\nabla\cdot\vec{V}+2\mu\frac{\partial v}{\partial y}+\frac{\partial}{\partial y}\left(\lambda\nabla\cdot\vec{V}+2\mu\frac{\partial v}{\partial y}\right)dy\right]dxdz$$

$$\tau_{xy}dzdy=\left[\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]dzdy$$
$$\left(\tau_{xy}+\frac{\partial \tau_{xy}}{\partial x}dx\right)dzdy=\left[\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial x}\left(\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right)dx\right]dzdy$$

$$\tau_{zy}dxdy=\left[\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right]dxdy$$
$$\left(\tau_{zy}+\frac{\partial \tau_{zy}}{\partial z}dz\right)dxdy=\left[\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)+\frac{\partial}{\partial z}\left(\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right)dz\right]dxdy$$

1.2.3) first we take Newtons second law in the y directions

$$F_y = m a_y$$

We will start with the forces on the left hand side. We have two types of forces

- Body forces such as gravitational or electromagnetic. These are represented as $F_{b,y}$. We say $F_{b,y} = m f_{b,y}$ where $f_{b,y}$ is force per unit mass

- Surface forces, which can be seen on the fluid element drawn for 1.2.1. We can expand these as follows,

$$F_{s,y} = \left[p - \left(p + \frac{\partial p}{\partial y}dy\right)\right]dx\,dz + \left[\left(\tau_{yy} + \frac{\partial \tau_{yy}}{\partial y}dy\right) - \tau_{yy}\right]dx\,dz + \left[\left(\tau_{zy} + \frac{\partial \tau_{zy}}{\partial z}dz\right) - \tau_{zy}\right]dy\,dx + \left[\left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x}dx\right) - \tau_{xy}\right]dz\,dy$$

$$F_{s,y} = \left[\cancel{p} - \left(\cancel{p} + \frac{\partial p}{\partial y}dy\right)\right]dx\,dz + \left[\left(\cancel{\tau_{yy}} + \frac{\partial \tau_{yy}}{\partial y}dy\right) - \cancel{\tau_{yy}}\right]dx\,dz + \left[\left(\cancel{\tau_{zy}} + \frac{\partial \tau_{zy}}{\partial z}dz\right) - \cancel{\tau_{zy}}\right]dy\,dx + \left[\left(\cancel{\tau_{xy}} + \frac{\partial \tau_{xy}}{\partial x}dx\right) - \cancel{\tau_{xy}}\right]dz\,dy$$

We can cancel opposite terms to get the following

$$F_{s,y} = \left[-\frac{\partial p}{\partial y} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \frac{\partial \tau_{xy}}{\partial x}\right]dx\,dy\,dz$$

The total force will be $F_{s,y} + F_{b,y}$

Now we move to the right hand side of Newtons 2nd Law
The mass of the fluid element is simply

$$m = \rho\, dx\,dy\,dz$$

The acceleration is simply

$$a_y = \frac{Dv}{Dt}$$

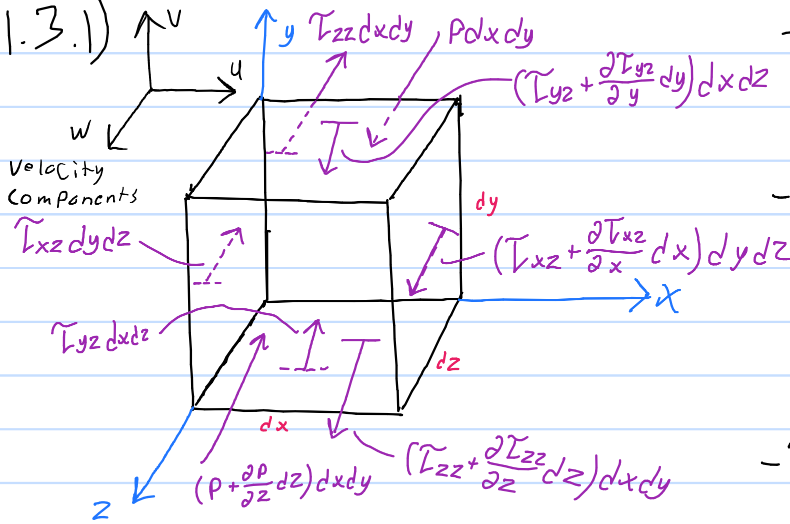
Combining all these equations we get

$$\rho\, \cancel{dx\,dy\,dz} \frac{Dv}{Dt} = \left[-\frac{\partial p}{\partial y} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \frac{\partial \tau_{xy}}{\partial x} + \rho f_{b,y}\right]\cancel{dx\,dy\,dz}$$

Dividing by $dx\,dy\,dz$ we get

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \frac{\partial \tau_{xy}}{\partial x} + \rho f_{b,y}$$

1.3.1)



- $Pdxdy$ & $(p+\frac{\partial p}{\partial z}dz)dxdy$ represent the hydrostatic pressure forces acting on the fluid element.

- $\tau_{xz}dydz$ & $\tau_{yz}dxdz$ represent the shear stresses acting upon the fluid while $(\tau_{xz}+\frac{\partial \tau_{xz}}{\partial x}dx)dydz$ & $(\tau_{yz}+\frac{\partial \tau_{yz}}{\partial y}dy)dxdz$ represent the shear forces defined relative to the first two shear forces.

- $\tau_{zz}dxdy$ represents the normal stress acting upon the fluid element, while $(\tau_{zz}+\frac{\partial \tau_{zz}}{\partial z}dz)dxdy$ represents the normal stress relative to the first stress

1.3.2) Assuming a Newtonian fluid we get the following for the stress tensors

$$\tau_{zz}=\lambda\nabla\cdot\vec{V}+2\mu\frac{\partial w}{\partial z},\quad \tau_{xz}=\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right),\quad \tau_{yz}=\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)$$

Expanding these we get the following (note pressure is ignored as it doesn't change)

$$\tau_{zz}dxdy=(\lambda\nabla\cdot\vec{V}+2\mu\frac{\partial w}{\partial z})dxdy$$
$$(\tau_{zz}+\frac{\partial \tau_{zz}}{\partial z}dz)dxdy=\left[\lambda\nabla\cdot\vec{V}+2\mu\frac{\partial w}{\partial z}+\frac{\partial}{\partial z}(\lambda\nabla\cdot\vec{V}+2\mu\frac{\partial w}{\partial z})dz\right]dxdy$$

$$\tau_{xz}dydz=\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]dydz$$
$$(\tau_{xz}+\frac{\partial \tau_{xz}}{\partial x}dx)dydz=\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial x}\left(\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right)dx\right]dydz$$

$$\tau_{yz}dxdz=\left[\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right]dxdz$$
$$(\tau_{yz}+\frac{\partial \tau_{yz}}{\partial y}dy)dxdz=\left[\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)+\frac{\partial}{\partial y}\left(\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right)dy\right]dxdz$$

1.3.3) Starting with Newton's 2^{nd} law in the z-direction

$$F_z=ma_z$$

We start with the forces on the left hand side. These are divided into two seperate types of forces, body & surface forces.

Body forces are external forces acting on the fluid element such as gravity or electromagnetic forces and are represented as $F_{b,z}$. We say $F_{b,z}=mf_{b,z}$ where m is the element mass $\rho dxdydz$ & $f_{b,z}$ is the force per unit mass.

Surfaces forces can be seen in the diagram from 1.3.1. We expand these as follows.

$$F_{s,z} = \left[P - \left(P + \frac{\partial P}{\partial z} dz\right)\right] dx\,dy + \left[\left(\tau_{zz} + \frac{\partial \tau_{zz}}{\partial z} \partial z\right) - \tau_{zz}\right] dx\,dy + \left[\left(\tau_{xz} + \frac{\partial \tau_{xz}}{\partial x} dx\right) - \tau_{xz}\right] dy\,dz + \left[\left(\tau_{yz} + \frac{\partial \tau_{yz}}{\partial y} dy\right) - \tau_{yz}\right] dx\,dz$$

We can cancel opposite terms as follows

$$F_{s,z} = \left[\cancel{P} - \left(\cancel{P} + \frac{\partial P}{\partial z} dz\right)\right] dx\,dy + \left[\left(\cancel{\tau_{zz}} + \frac{\partial \tau_{zz}}{\partial z} \partial z\right) - \cancel{\tau_{zz}}\right] dx\,dy + \left[\left(\cancel{\tau_{xz}} + \frac{\partial \tau_{xz}}{\partial x} dx\right) - \cancel{\tau_{xz}}\right] dy\,dz + \left[\left(\cancel{\tau_{yz}} + \frac{\partial \tau_{yz}}{\partial y} dy\right) - \cancel{\tau_{yz}}\right] dx\,dz$$

from this we get the following

$$f_{s,z} = \left[-\frac{\partial P}{\partial z} + \frac{\partial \tau_{zz}}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y}\right] dx\,dz\,dy$$

The total force will be $F_{b,z} + F_{s,z}$ or

$$f_z = \left[-\frac{\partial P}{\partial z} + \frac{\partial \tau_{zz}}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \rho f_{b,z}\right] dx\,dy\,dz$$

Now we do the right hand side of newtons second law

for the mass we have $m = \rho\, dx\,dy\,dz$, while acceleration is represented by

$$a_z = \frac{Dw}{Dt}$$

Combining all these equations we get

$$\rho\, dx\,dy\,dz \frac{Dw}{Dt} = \left[-\frac{\partial P}{\partial z} + \frac{\partial \tau_{zz}}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \rho f_{b,z}\right] dx\,dy\,dz$$

Dividing by $dx\,dy\,dz$ we get

$$\rho \frac{Dw}{Dt} = -\frac{\partial P}{\partial z} + \frac{\partial \tau_{zz}}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \rho f_{b,z}$$

2.1) $$\underbrace{\rho \frac{D}{Dt}\left(e + \frac{|V|^2}{2}\right)}_{A} = \overbrace{\rho \dot{q} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)}^{B}$$
$$- \frac{\partial (up)}{\partial x} - \frac{\partial (vp)}{\partial y} - \frac{\partial (wp)}{\partial z} + \frac{\partial (u\tau_{xx})}{\partial x} + \frac{\partial (u\tau_{yx})}{\partial y} + \frac{\partial (u\tau_{zx})}{\partial z}$$
$$+ \frac{\partial (v\tau_{xy})}{\partial x} + \frac{\partial (v\tau_{yy})}{\partial y} + \frac{\partial (v\tau_{zy})}{\partial z} + \frac{\partial (w\tau_{xz})}{\partial x} + \frac{\partial (w\tau_{yz})}{\partial y}$$
$$+ \frac{\partial (w\tau_{zz})}{\partial z} + \rho f \cdot V \qquad [2.1.1]$$

We can divide this equaxion into 3 parts, A, B, & C.

Part A is the left hand side of the equation $\rho \frac{D}{Dt}\left(e + \frac{|V|^2}{2}\right)$

This part describes the rate of change of energy inside the fluid element. In this equation e represents the fluid elements internal energy per unit mass while $\frac{|V|^2}{2}$ represents the fluid elements kinetic energy per unit mass

Part B is found on the right hand side & is represented by

$$\rho \dot{q} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)$$

In this $\rho \dot{q}$ is the volumetric heating per unit mass

$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)$ on the other hand represents the total conductive heat transfer across each face where k is the thermal conduction coefficient.

C is represented by the rest of the eauation, & describes the rate of work done by body & surface forces.

$\rho f \cdot \vec{V}$ is the rate of work done by body forces

$-\frac{\partial (up)}{\partial x} - \frac{\partial (vp)}{\partial y} - \frac{\partial (wp)}{\partial z}$ is the rate of work done by pressure forces

$\frac{\partial (u\tau_{zx})}{\partial z} + \frac{\partial (v\tau_{xy})}{\partial x} + \frac{\partial (v\tau_{yy})}{\partial y} + \frac{\partial (v\tau_{zy})}{\partial z} + \frac{\partial (w\tau_{xz})}{\partial x} + \frac{\partial (w\tau_{yz})}{\partial y} + \frac{\partial (w\tau_{zz})}{\partial z}$ represents the rate of work done by forces from shear & normal stresses.

2.2) first we write out all of the momentum equations

$$\rho \frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_{b,x} \quad [2.2.1a]$$

$$\rho \frac{Dv}{Dt} = -\frac{\partial P}{\partial y} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \frac{\partial \tau_{xy}}{\partial x} + \rho f_{b,y} \quad [2.2.1b]$$

$$\rho \frac{Dw}{Dt} = -\frac{\partial P}{\partial z} + \frac{\partial \tau_{zz}}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \rho f_{b,z} \quad [2.2.1c]$$

Now we multiply [2.2.1a] by u, [2.2.1b] by v, & [2.2.1C] by w

$$\rho \frac{D(\frac{u^2}{2})}{Dt} = -u\frac{\partial P}{\partial x} + u\frac{\partial \tau_{xx}}{\partial x} + u\frac{\partial \tau_{yx}}{\partial y} + u\frac{\partial \tau_{zx}}{\partial z} + \rho u f_{b,x} \quad [2.2.2a]$$

$$\rho \frac{D(\frac{v^2}{2})}{Dt} = -v\frac{\partial P}{\partial y} + v\frac{\partial \tau_{yy}}{\partial y} + v\frac{\partial \tau_{zy}}{\partial z} + v\frac{\partial \tau_{xy}}{\partial x} + \rho v f_{b,y} \quad [2.2.2b]$$

$$\rho \frac{D(\frac{w^2}{2})}{Dt} = -w\frac{\partial P}{\partial z} + w\frac{\partial \tau_{zz}}{\partial z} + w\frac{\partial \tau_{xz}}{\partial x} + w\frac{\partial \tau_{yz}}{\partial y} + \rho w f_{b,z} \quad [2.2.2c]$$

Noting that $u^2+v^2+w^2=|\vec{V}|^2$ and $\vec{V}=\langle u,v,w\rangle$ & $\vec{f}=\langle f_x, f_y, f_z\rangle$ we can add equation [2.2.2a], [2.2.2b] & [2.2.2c] to get

$$\rho \frac{D(\frac{|V|^2}{2})}{Dt} = -u\frac{\partial P}{\partial x} - v\frac{\partial P}{\partial x} + u\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) \quad [2.2.3]$$

$$+ v\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right)$$

$$+ w\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right) + \rho \vec{f}\cdot\vec{V}$$

2.3) We can subtract equation 2.2.3 from equation 2.1.1 we obtain the following

$$\rho \frac{De}{Dt} = \rho\dot{q} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)$$

$$-p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \tau_{xx}\frac{\partial u}{\partial x} + \tau_{yx}\frac{\partial u}{\partial y} + \tau_{zx}\frac{\partial u}{\partial z}$$

$$+ \tau_{xy}\frac{\partial v}{\partial x} + \tau_{yy}\frac{\partial v}{\partial y} + \tau_{zy}\frac{\partial v}{\partial z} + \tau_{xz}\frac{\partial w}{\partial x} + \tau_{yz}\frac{\partial w}{\partial y} + \tau_{zz}\frac{\partial w}{\partial z}$$

This is the energy equation in terms of internal energy in non-conservation form

We can take the fact that $\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$ & $\tau_{yz} = \tau_{zy}$ to rewrite the above equation as

$$\rho \frac{De}{Dt} = \rho\dot{q} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)$$

$$-p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \tau_{xx}\frac{\partial u}{\partial x} + \tau_{yy}\frac{\partial v}{\partial y} + \tau_{zz}\frac{\partial w}{\partial y}$$

$$+ \tau_{yx}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \tau_{zx}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + \tau_{zy}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \quad [2.3.1]$$

2.4) from the definition of substantial derivative

$$\rho\frac{De}{Dt} = \rho\frac{\partial e}{\partial t} + \rho\vec{V}\cdot\nabla e \quad [2.4.1]$$

But we also have $\frac{\partial(\rho e)}{\partial t} = \rho\frac{\partial e}{\partial t} + e\frac{\partial \rho}{\partial t}$

or $\rho\frac{\partial e}{\partial t} = \frac{\partial(\rho e)}{\partial t} - e\frac{\partial \rho}{\partial t} \quad [2.4.2]$

We also have $\nabla\cdot(\rho e\vec{V}) = e\nabla\cdot(\rho\vec{V}) + \rho\vec{V}\cdot\nabla e$ which can be simplified to $\rho\vec{V}\cdot\nabla e = \nabla\cdot(\rho e\vec{V}) - e\nabla\cdot(\rho\vec{V})$ [2.4.3]

We can substitute [2.4.2] & [2.4.3] into [2.4.1] we get

$$\rho\frac{De}{Dt} = \frac{\partial(\rho e)}{\partial t} - e\left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{V})\right] + \nabla\cdot(\rho e\vec{V}) \quad [2.4.4]$$

The term in the square brackets must be 0 to satisfy continuity, so we get

$$\rho\frac{De}{Dt} = \frac{\partial(\rho e)}{\partial t} + \nabla\cdot(\rho e\vec{V}) \quad [2.4.5]$$

Substituting [2.4.5] into [2.3.1] we get

$$\begin{aligned}\frac{\partial(\rho e)}{\partial t} + \nabla\cdot(\rho e\vec{V}) = \rho\dot{q} &+ \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) \quad [2.4.6]\\ &- p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \tau_{xx}\frac{\partial u}{\partial x} + \tau_{yy}\frac{\partial v}{\partial y} + \tau_{zz}\frac{\partial w}{\partial y}\\ &+ \tau_{yx}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \tau_{zx}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + \tau_{zy}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\end{aligned}$$

This equation represents the internal energy equation in conservation form.