1.1.1 $$u_{i+1} = u_i + \Delta x \frac{\partial u_i}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 u_i}{\partial x^2} + \frac{\Delta x^3}{6}\frac{\partial^3 u_i}{\partial x^3} + O(\Delta x^4) \quad (1)$$

$$u_{i+2} = u_i + 2\Delta x \frac{\partial u_i}{\partial x} + \frac{(2\Delta x)^2}{2}\frac{\partial^2 u_i}{\partial x^2} + \frac{(2\Delta x)^3}{6}\frac{\partial^3 u_i}{\partial x^3} + O(\Delta x^4) \quad (2)$$

rearrange (1) divide by Δx
$$\frac{\partial u_i}{\partial x} = \frac{u_{i+1} - u_i}{\Delta x} - \frac{\Delta x}{2}\frac{\partial^2 u_i}{\partial x^2} - \frac{\Delta x^2}{6}\frac{\partial^3 u_i}{\partial x^3} + O(\Delta x^3) \quad (3)$$

rearrange (2)
$$\frac{\partial u_i}{\partial x} = \frac{u_{i+2} - u_i}{2\Delta x} - \Delta x \cdot \frac{\partial^2 u_i}{\partial x^2} - \frac{2\Delta x^2}{3}\frac{\partial^3 u_i}{\partial x^3} + O(\Delta x^3) \quad (4)$$

condition 1 : $2a + b = 0$
condition 2 : $a + b = 1$
$\left.\right\}$ $b = 2$, $a = -1$

combine (3)&(4) multiply by a&b
$$\frac{\partial u_i}{\partial x} = \frac{-u_{i+2} + 4u_{i+1} - 3u_i}{2\Delta x} + O(\Delta x^2)$$

1.1.2 $$u_{i-1} = u_i - \Delta x \frac{\partial u_i}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 u_i}{\partial x^2} - \frac{\Delta x^3}{6}\frac{\partial^3 u_i}{\partial x^3} + O(\Delta x^4)$$

$$u_{i-2} = u_i - 2\Delta x \frac{\partial u_i}{\partial x} + \frac{(2\Delta x)^2}{2}\frac{\partial^2 u_i}{\partial x^2} - \frac{(2\Delta x)^3}{6}\frac{\partial^3 u_i}{\partial x^3} + O(\Delta x^4)$$

$$\frac{\partial u_i}{\partial x} = \frac{u_i - u_{i-1}}{\Delta x} + \frac{\Delta x}{2}\frac{\partial^2 u_i}{\partial x^2} - \frac{\Delta x^2}{6}\frac{\partial^3 u_i}{\partial x^3} + O(\Delta x^3)$$

$$\frac{\partial u_i}{\partial x} = \frac{u_i - u_{i-2}}{2\Delta x} + \Delta x \frac{\partial^2 u_i}{\partial x^2} - \frac{2\Delta x^2}{3} + O(\Delta x^3)$$

condition 1: $2a + b$
condition 2: $a + b$
$\left.\right\}$ $b = 2$, $a = -1$

$$\frac{\partial u_i}{\partial x} = \frac{3u_i - 4u_{i-1} + u_{i-2}}{2\Delta x} + O(\Delta x^2)$$

$$\frac{2}{3} \quad \frac{4\,\cancel{8}\Delta x^{\cancel{3}}}{6 \cdot \cancel{2}\Delta x^2} \qquad \frac{4\Delta x^2}{2} \qquad 2\Delta x^2$$

1.2.1 $u_{i+1} = u_i + \Delta x \frac{\partial u_i}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 u_i}{\partial x^2} + \frac{\Delta x^3}{6}\frac{\partial^3 u_i}{\partial x^3} + O(\Delta x^4)$

$u_{i+2} = u_i + 2\Delta x \frac{\partial u_i}{\partial x} + \frac{(2\Delta x)^2}{2}\frac{\partial^2 u_i}{\partial x^2} + \frac{(2\Delta x)^3}{6}\frac{\partial^3 u_i}{\partial x^3} + O(\Delta x^4)$

$\frac{\partial^2 u_i}{\partial x^2} = \frac{2(u_{i+1} - u_i)}{\Delta x^2} - \frac{2}{\Delta x}\frac{\partial u_i}{\partial x} - \frac{\Delta x}{3}\frac{\partial^3 u_i}{\partial x^3} + O(\Delta x^2)$

$\frac{\partial^2 u_i}{\partial x^2} = \frac{(u_{i+2} - u_i)}{2\Delta x^2} - \frac{1}{\Delta x}\frac{\partial u_i}{\partial x} - \frac{2\Delta x}{3}\frac{\partial^3 u_i}{\partial x^3} + O(\Delta x^2)$

Condition 1: $a + 2b = 0$
Condition 2: $a + b = 1$
$a = 2$, $b = -1$

$$2\Delta x^2 \qquad \frac{8\Delta x^3}{6 \cdot 2\Delta x^2}$$

$\frac{\partial^2 u_i}{\partial x^2} = \frac{u_{i+2} - 2u_{i+1} + u_i}{\Delta x^2} + O(\Delta x^2)$

1.2.2 $u_{i-1} = u_i - \Delta x \frac{\partial u_i}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 u_i}{\partial x^2} - \frac{\Delta x^3}{6}\frac{\partial^3 u_i}{\partial x^3} + O(\Delta x^4)$

$u_{i-2} = u_i - 2\Delta x \frac{\partial u_i}{\partial x} + \frac{(2\Delta x)^2}{2}\frac{\partial^2 u_i}{\partial x^2} - \frac{(2\Delta x)^3}{6}\frac{\partial^3 u_i}{\partial x^3} + O(\Delta x^4)$

$\frac{\partial^2 u_i}{\partial x^2} = \frac{2(u_{i-1} - u_i)}{\Delta x^2} + \frac{2}{\Delta x}\frac{\partial u_i}{\partial x} + \frac{\Delta x}{3}\frac{\partial^3 u_i}{\partial x^3} + O(\Delta x^2)$

$\frac{\partial^2 u_i}{\partial x^2} = \frac{u_{i-2} - u_i}{2\Delta x^2} + \frac{1}{\Delta x}\frac{\partial u_i}{\partial x} + \frac{2\Delta x}{3}\frac{\partial^3 u_i}{\partial x^3} + O(\Delta x^2)$

Condition 1: $a + 2b = 0$
Condition 2: $a + b = 1$
$a = 2$, $b = -1$

$\frac{\partial^2 u_i}{\partial x^2} = \frac{-u_{i-2} + 4u_{i-1} - 3u_i}{2\Delta x^2} + O(\Delta x^2)$

1.3 I followed the method provided in the slides. The sum of all coefficients for all answers is zero.

2.1 1) a lower order upwind discretization of the linear advection equation is quite stable but not very accurate. To improve this accuracy, we use a higher order which is less stable

2) Godunov's theorem states:

Linear numerical schemes for solving partial differential equations, having the property of not generating new extrema, can be at most first order accurate

This embodies the monotone scheme

3) The Iserles barrier theorem states:

The maximum order of accuracy O(max) of a stable semi-discrete advection scheme with 'p' upwind points and d downwind point is

$$O(max) = \min(p+d,\ 2p,\ 2d+2)$$

The natural conclusion to draw is that upwind biased schemes could perform better than pure upwind schemes to attain stable higher order accuracy

4) The Total Variation Diminishing scheme defines a discrete solution

q_i^t with $i\,[0,n]$ at time t as $TV(q^t) = \sum_{i=1}^{n} |q_{i+1}^t - q_i^t|$

and a continuous variable $q(x,t)$ $TV(q) = \int_a^b |(\partial q/\partial x)\cdot dx$

The total variation of a variable q demonstrates the oscillation in a function

5) a total variation diminishing scheme is a scheme that satisfies

$$TV(q^{t+1}) \leq TV(q^t)$$

6) a MUSCL scheme is a finite volume method that can provide stable & highly accurate results. The motivation behind it is that the scheme can be used for flows with discontinuities. The order of accuracy of its original version by van Leer is second order

2.2 1) The well-posedness of an initial value problem is the stability of the IVP of a PDE

for the L^2-norm of a discretized function q $\|q\|^2 = \sqrt{\Delta x \sum_{n=-\infty}^{\infty} |q_n|^2}$

this is well-posed if for any time $T>0$, there is a constant C_T so that $q(x,t)$ satisfies

$$\|q(x,t)\|^2 \leq C_T \|q(x,0)\|^2 \quad \text{for } t \in [0,T]$$

2) The consistency of a FDS is how well the chosen scheme corresponds to the PDE it is supposed to represent.

a finite difference scheme $P_{\Delta x}^{\Delta t} q = f$ is consistent with the PDE $Pq = f$ to order (r,s) if for any smooth function $q(x,t)$

$$Pq - P_{\Delta x}^{\Delta t} q \to O(\Delta x^r, \Delta t^s) \quad \text{as} \quad \Delta x, \Delta t \to 0$$

3) The convergence property of a FDS tells us how close the numerical solution is to the exact one

4) The sources of numerical error are discretization error and roundoff error

5) Roundoff error is the result of a computer repetitively rounding off numbers to some significant figure as a result of the floating point approximation.

2.3 1) The Neumann stability method is a linear approach

2) A transient explicit scheme has solution values which are only from current and previous time levels

3) A transient implicit scheme has solution values belonging to the next time level

4) The Courant number tells you how many times the diffusion transport velocity is to the speed at which numerical information travels across the entire cell, or grid velocity

For linear diffusion to be stable, $C_d < 0.5$
For linear advection to be stable, $C_a \leq 1$

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