CFD for Aerospace Applications

Incompressible Inviscid Flow and Panel Methods

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Recap of the previous lecture

- We used the principle of conservation of mass to derive the differential non-conservative continuity equation for a compressible fluid,

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \mathbf{v} = 0.$$

- This lead us to the differential continuity equation for an incompressible fluid which is divergence-free (solenoidal),

$$\nabla \cdot \mathbf{v} = 0.$$

- Then we derived the integral continuity equation in conservation form for a compressible fluid,

$$\frac{\partial}{\partial t}\oiiint_V \rho dV + \oiint_S \rho\mathbf{v} \cdot \mathbf{dS} = 0,$$

and used the divergence theorem to derive its differential form,

$$\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\mathbf{v}) = 0.$$

Incompressible Inviscid Flow

- We used Newton's 2nd law $\mathbf{F} = m\mathbf{a}$ to derive the differential x-, y- and z-components of the momentum equation in non-conservation form for a compressible fluid,

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_{b,x}.$$

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho f_{b,y}.$$

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial z} + \frac{\partial \tau_{yz}}{\partial z} + \frac{\partial \tau_{zz}}{\partial z} + \rho f_{b,z}.$$

Incompressible Inviscid Flow

- Finally we used the 1st law of thermodynamics to derive the differential total energy equation in non-conservation form,

$$
\begin{aligned}
\rho \frac{D}{Dt}\left(e + \frac{|\mathbf{v}|^2}{2}\right) = \rho \dot{q} &+ \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) \\
&- \frac{\partial(up)}{\partial x} - \frac{\partial(vp)}{\partial y} - \frac{\partial(wp)}{\partial z} \\
&+ \frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z} \\
&+ \frac{\partial(v\tau_{xy})}{\partial x} + \frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z} \\
&+ \frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z} \\
&+ \rho \mathbf{f} \cdot \mathbf{v}.
\end{aligned}
$$

end of recap of the previous lecture

Incompressible inviscid flow

- In this week's lectures we will consider the numerical analysis of incompressible inviscid flow.
- Incompressible inviscid flow is a simplified theoretical state which can be very useful for analyzing incompressible flow, even at high Reynolds number, and without flow separation.
- In particular it can give good results for pressure.
- When the flow is also irrotational, it becomes a potential flow.
- We will also study the panel method which is the simplest possible numerical technique for analyzing such flow.

Incompressible Inviscid Flow

- Incompressible flow is constant density flow, so an infinitesimal element with constant mass moving along a streamline will also have constant volume.
- In the 2nd lecture we found that the divergence of velocity is equal to the time rate of change of element volume per volume. Since volume is constant it lead to,

$$\nabla \cdot \mathbf{v} = 0. \tag{1}$$

- We also get that from the differential continuity equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = \nabla \cdot \rho \mathbf{v} = \rho \nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{v} = 0. \tag{2}$$

Incompressible Inviscid Flow

- Equation 1 is in fact the continuity equation for incompressible flow.
- Since the flow is inviscid the viscous normal and shear stresses will be zero.
- The momentum equation becomes,

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{f}_b. \tag{3}$$

Here, p is the hydrostatic pressure and $\mathbf{f}_b$ the body force per unit mass.

- The total energy equation becomes,

$$\rho\frac{D}{Dt}\left(e + \frac{|\mathbf{v}|^2}{2}\right) = \rho\dot{q} + \nabla\cdot(k\nabla T) - \nabla\cdot(p\mathbf{v}) + \rho\mathbf{f}\cdot\mathbf{v}. \tag{4}$$

Incompressible Inviscid Flow

Incompressible inviscid irrotational flow

- If the fluid element does not rotate as it moves along the streamline then the flow is called *irrotational*.
- In this case the velocity can be expressed as the gradient of a scalar potential function we will denote here by ϕ,

$$\mathbf{v} = \nabla\phi. \tag{5}$$

- Taking the divergence of equation 5 and using equation 2 we get the Laplace equation for the velocity potential,

$$\nabla^2\phi = 0. \tag{6}$$

- This shows that inviscid, incompressible, irrotational flow (also called potential flow) is governed by the Laplace equation.

Incompressible Inviscid Flow

- Equation 6 is linear, which means that the solution of a sum of ϕ-like functions is equal to the sum of their individual solutions.
- Another implication of linearity is that a complicated irrotational, incompressible and inviscid flow pattern can be simulated by combining a number of basic flows that are also irrotational, incompressible and inviscid.
- We will examine three such basic flows in the next few slides, and then use them to simulate more complex flow patterns.
- These are the uniform flow, the source/sink flow, and the vortex flow.

Incompressible Inviscid Flow

Uniform flow

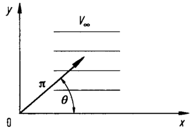

Figure 1: Uniform flow.

A uniform flow has constant velocity, as in figure 1 with velocity v_∞ in the x-direction. We find its potential function by solving the Laplace equation,

$$(v_\infty, 0) = \nabla\phi = (\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}).$$

$$\frac{\partial\phi}{\partial y} = 0 \Rightarrow \phi = \phi(x) + A \quad \text{where } A \text{ is a constant.}$$

$$\frac{\partial\phi}{\partial x} = v_\infty \Rightarrow \boxed{\phi = v_\infty x} \quad \text{with } A = 0.$$

We can also express the result in polar coordinates using $x = rcos\theta$, which yields $\phi = v_\infty rcos\theta$.

Incompressible Inviscid Flow

Source and sink flow

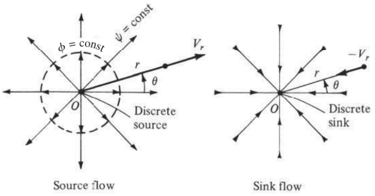

Figure 2: Source and sink flow.

- A source flow is one where the streamlines are radial lines emanating from an origin (point O in figure 2), and the velocity inversely proportional to the distance from the origin $v_r \propto 1/r$.
- A sink flow is one where the streamlines go towards the origin.
- The radial velocity in a source/sink flow is zero everywhere.

Incompressible Inviscid Flow

- Incompressible flow is possible everywhere in a source/sink flow except at the origin where $v_r \propto 1/r \to \infty$, so $\nabla \cdot \mathbf{v} \to \infty$, which becomes a singularity.
- Source/sink flow is irrotational and you can verify for yourself that $\nabla \times \mathbf{v} = 0$.
- The implication of the origin O being a singularity is it should be considered a *discrete* source or sink, with the radial flow *induced* and not flowing from it.
- This is a common approach, fundamental to many theoretical solutions of incompressible flow.
- To evaluate the potential function of source flow we follow the same procedure as uniform flow , except we use polar coordinates with the grad function $\nabla = (\dfrac{\partial}{\partial r}, \dfrac{1}{r}\dfrac{\partial}{\partial \theta})$ in two dimensions.

Incompressible Inviscid Flow

$$(v_r, 0) = \nabla\phi \Rightarrow v_r = \frac{\partial \phi}{\partial r} = \frac{A}{r}. \quad (7)$$

- This leads to $\phi = A\ln(r) + B$ where A and B are constants.
- $\dfrac{\partial \phi}{\partial \theta} = 0$ is already accounted for with B that we can safely set to 0.
- The constant A can be determined from volume flow rate considerations for a line source of length l perpendicular to the page in figure 2, as shown in figure 3 in three-dimensional perspective.

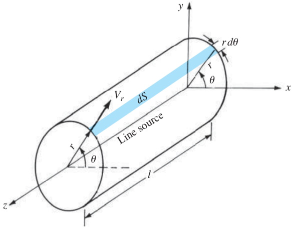

Figure 3: Line source.

Incompressible Inviscid Flow

- The line l along the z-axis in figure 3 is made of a number of discrete sources including O.
- The flow is two-dimensional in any plane perpendicular to the z-axis.
- The elemental mass flow $d\dot{m}$ across surface $\mathbf{dS} = (rd\theta)l$ is,

$$d\dot{m} = \rho \mathbf{v} \cdot \mathbf{dS} = \rho v_r (rd\theta) l. \tag{8}$$

- Integrating over the whole surface to get the mass flow rate $\dot{m}$,

$$\dot{m} = \int_0^{2\pi} \rho v_r (rd\theta) l = 2\pi r l \rho v_r. \tag{9}$$

Incompressible Inviscid Flow

- Dividing the result of equation 9 by density to get the volume flow rate $\dot{V}$,

$$\dot{V} = \frac{\dot{m}}{\rho} = 2\pi r l v_r. \tag{10}$$

- In turn the volume flow rate per unit length Λ, also called *source strength*, is,

$$\Lambda = \frac{\dot{V}}{l} = 2\pi r v_r,$$

$$\Rightarrow v_r = \frac{\Lambda}{2\pi r}. \tag{11}$$

- We can see that a positive value of Λ induces source flow and a negative value induces sink flow.

Incompressible Inviscid Flow

- Comparing v_r from the bottom line of equation 11 to equation $v_r = A/r$ from equation 7 we can deduce $A = \dfrac{\Lambda}{2\pi}$.
- This in turn allows us to rewrite the potential function of a source/sink flow as,

$$\boxed{\phi = \frac{\Lambda}{2\pi}\ln(r)}. \tag{12}$$

Incompressible Inviscid Flow

Vortex flow

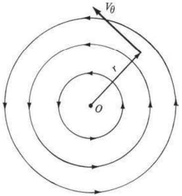

Figure 4: Vortex flow.

- Vortex flow is the polar opposite of source/sink flow, with streamlines going around the origin in circles, where $v_r = 0$ and $v_\theta = c/r$ with c constant.

Incompressible Inviscid Flow

- Vortex flow is identically incompressible ($\nabla \cdot \mathbf{v} = 0$) and irrotational ($\nabla \times \mathbf{v} = 0$) at every point except the origin where $r \to 0$. You can verify this yourself.
- The constant c can be evaluated by taking the circulation around the path of a given streamline of radius r,

$$\Gamma = -\oint \mathbf{v} \cdot \mathbf{ds} = -v_\theta (2\pi r), \tag{13}$$

which results in,

$$v_\theta = -\frac{\Gamma}{2\pi r}. \tag{14}$$

- Since $v_\theta = c/r$ we get,

$$\boxed{c = -\frac{\Gamma}{2\pi}}. \tag{15}$$

- The circulation Γ is also called the *strength* of the vortex, has the same units as Λ for source/sink flow and is also a volume flow rate per unit length.

Incompressible Inviscid Flow

- Our sign convention in the polar coordinates is that positive circulation corresponds to clockwise rotation and vice-versa.
- This is confirmed in equation 14 where positive circulation leads to negative v_θ and clockwise rotation.
- To find the potential function of a vortex flow we just need to integrate its velocity components,

$$
\begin{aligned}
v_r &= 0 = \frac{\partial \phi}{\partial r} \Rightarrow \phi = f(\theta) \text{ or a constant}, \\
v_\theta &= -\frac{\Gamma}{2\pi r} = \frac{1}{r}\frac{\partial \phi}{\partial \theta} \Rightarrow \frac{\partial \phi}{\partial \theta} = -\frac{\Gamma}{2\pi}, \\
\Rightarrow \phi &= -\frac{\Gamma}{2\pi}\theta + c.
\end{aligned}
$$

- The constant c can be safely set to 0 without loss of generality and our final result is,

$$\boxed{\phi = -\frac{\Gamma}{2\pi}\theta}. \tag{16}$$

Incompressible Inviscid Flow

Summary of basic flows

Type	**Velocity**	ϕ
Uniform in x-direction	$u = v_\infty$	$v_\infty x$
Source/sink	$v_r = \frac{\Lambda}{2\pi r}$	$\frac{\Lambda}{2\pi}\ln(r)$
Vortex	$v_\theta = -\frac{\Gamma}{2\pi r}$	$-\frac{\Gamma}{2\pi}\theta$

Table 1: Summary of basic flow velocities and potential functions.

The panel method

- The panel method was very popular in the 70s and 80s and remains useful today for quick analysis.
- It is often referred to as *computational aerodynamics* although it is a sub-discipline of CFD.
- We will examine the source panel method, which is used for non-lifting flows, and the vortex panel method, which is used for lifting flows.
- Finally we stress that a panel is a straight linear segment in 2D applications, and a flat quadrilateral in 3D applications.
- In what follows we consider 2D applications with straight linear segments for panels.

Incompressible Inviscid Flow

The source panel method

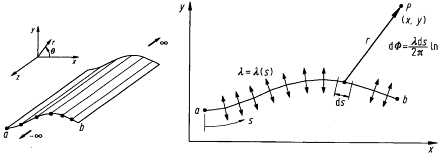

Figure 5: A source sheet.

- consider a single line source with λ its strength infinitesimally small.
- Now imagine an infinite number of such line sources stacked together to form a sheet as in figure 5, a so-called source sheet.
- In figure 5 the source lines are stacked perpendicular to the slide so they appear as a single edge on the right side of the figure.
- Source sheets are surfaces of 3D models, so we define source and vortex strength as volume flow rate per unit area (per unit length × unit depth) unlike volume flow rate per unit length for 2D models and linear panels considered earlier.

Incompressible Inviscid Flow

- Let's stick to one line source (2D) and let s be the distance along the edge, and $\lambda(s)$ the source strength per unit length along s.
- Here $\lambda(s)$ is defined as the volume flow rate per unit length, in the s-direction and unit depth since we are working in 2D.
- The strength of an infinitesimal section ds is then λds and this small section can be treated as a distinct source of strength λds.
- Now consider a point $P(x, y)$ in the flow at a distance r from ds, with an infinitesimal potential $d\phi$ induced at P,

$$d\phi = \frac{\lambda ds}{2\pi} \ln r. \tag{17}$$

- The complete velocity potential at P induced by the entire source sheet from a to b is then,

$$\boxed{\phi(x, y) = \int_a^b \frac{\lambda ds}{2\pi} \ln r.} \tag{18}$$

Incompressible Inviscid Flow

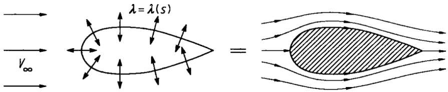

Figure 6: Superposition of uniform flow and a source sheet on an airfoil.

- Next we consider an arbitrary 2D airfoil in a flow with freestream velocity v_∞ as depicted in figure 6.
- The surface of the given body is covered with a source sheet of strength $\lambda(s)$, varying such that its combined action with the uniform flow makes the airfoil surface a streamline.
- We need to find the appropriate $\lambda(s)$ distribution and we will do this numerically.

Incompressible Inviscid Flow

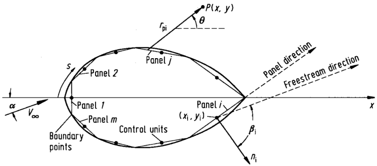

Figure 7: Source panel distribution over the surface of an airfoil.

- In figure 7 we approximate the airfoil with n straight panels. The source strength λ is constant within each panel but can vary from panel to panel, so we have λ_i with $1 \leq i \leq n$.
- This subdivision is called a *discretization* of the airfoil.
- The source panel method consists in calculating λ_i such that the airfoil surface coincides with a flow streamline.
- This latter condition is called a *boundary condition* in CFD and we deal with such conditions in every CFD technique and how they are imposed varies per flow and method.

Incompressible Inviscid Flow

- So how do we enforce the flow streamline boundary condition? We start with a simple model of the flow around the airfoil, a point $P(x, y)$ at a distance r_{pj} from any point on panel j.
- the velocity potential $\Delta\phi_j$ induced at P by the panel j is,

$$\Delta\phi_j = \frac{\lambda_j}{2\pi}\int_j \ln r_{pj} ds_j. \tag{19}$$

- Accordingly, the potential induced at P by all panels on the airfoil surface is the sum of the individual $\Delta\phi_j$,

$$\phi(P) = \sum_{j=1}^{n}\Delta\phi_j = \sum_{j=1}^{n}\frac{\lambda_j}{2\pi}\int_j \ln r_{pj} ds_j. \tag{20}$$

- In equations 19 and 20 the distance r_{pj} is given by,

$$r_{pj} = \sqrt{(x - x_j)^2 + (y - y_j)^2}, \tag{21}$$

where (x_j, y_j) are coordinates *anywhere* on the jth panel.

Incompressible Inviscid Flow

- The choice of the location (x_j, y_j) on panel j is quite important and here is the best one that transpired over the years.
- Since P is arbitrary let's put it at the control point at the centre of the a certain panel and designate it with index i. From now on index i designates point P at the control point of panel i. The need to differentiate will become clear soon.
- Equations 20 and 21 become,

$$\phi(x_i, y_i) = \sum_{j=1}^{n} \frac{\lambda_j}{2\pi} \int_j \ln r_{ij} ds_j, \tag{22}$$

and

$$r_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}. \tag{23}$$

- Equation 22 represents the contribution of all panels to the potential at the control point of the ith panel.

Incompressible Inviscid Flow

- Keep in mind that the source potential function $\phi(x_i, y_i)$ induces flow normal to the panel itself at the control point (x_i, y_i).
- However, since the airfoil surface is a streamline then the sum of the contribution of the potential function and the freestream should add up to zero normal velocity.
- The normal contribution of the freestream velocity can be easily calculated,

$$v_{\infty,n}\big|_i = \mathbf{v}_\infty \cdot \mathbf{n}_i = v_\infty \cos\beta_i. \tag{24}$$

Here, $\mathbf{n}_i$ is the normal to the panel and β_i the angle between $\mathbf{v}_\infty$ and $\mathbf{n}_i$.

Incompressible Inviscid Flow

- The normal component of velocity induced at (x_i, y_i) by source panel j is,

$$v_{ij,n} = \frac{\partial}{\partial n_i}[\phi_j(x_i, y_i)]. \tag{25}$$

- When the derivative in equation 26 is carried out, r_{ij} appears in the denominator and $r_{ii} = 0$. This singularity arises from the contribution of panel i to itself. Fortunately it can be shown that this contribution to the derivative is $\lambda_i/2$ for $i = j$.
- Summing up the contributions of all panels to the normal velocity at (x_i, y_i),

$$v_{ij,n} = \frac{\lambda_i}{2} + \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{\lambda_j}{2\pi} \int_j \frac{\partial}{\partial n_i}(\ln r_{ij}) ds_j. \tag{26}$$

- The body streamline boundary condition states that,

$$v_{\infty,n}|_i + v_{i,n} = 0. \tag{27}$$

Incompressible Inviscid Flow

- Replacing equations 24 and 26 in equation 27,

$$v_\infty \cos\beta_i + \frac{\lambda_i}{2} + \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{\lambda_j}{2\pi} \int_j \frac{\partial}{\partial n_i}(\ln r_{ij}) ds_j = 0. \tag{28}$$

- Setting the value of the integral to I_{ij} we rewrite the equation above as,

$$v_\infty \cos\beta_i + \frac{\lambda_i}{2} + \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{\lambda_j}{2\pi} I_{ij} = 0. \tag{29}$$

- Equation 29 is a linear algebraic equation for the n λ unknowns, where the value of the integral depends only on known panel geometries.

Incompressible Inviscid Flow

- We can write this equation for each panel, giving us n algebraic equations for n unknowns, which is a well-posed system that can easily be solved numerically for all λ values.
- The obtained λ values correspond approximately to a streamline coinciding with the airfoil surface.
- They can be used to derive the velocities at the panel control points, which are always tangential since it is the only possibility with inviscid methods.
- Accuracy can be improved by adding more panels and more closely representing the airfoil surface, especially at high curvature areas such as the leading edge. The panels can have different lengths.
- Next we will derive the velocity tangent to the surface at each control point, and we will use that velocity to derive the pressure.

Incompressible Inviscid Flow

- Just like the normal velocity the tangential velocity has two contributions, one from the freestream and the other induced from the panel source strengths.
- Let s be the distance along the airfoil body surface, measured positive from front to rear. The tangential velocity contribution of the freestream at the control point of the ith panel is,

$$v_{\infty,s}\big|_i = v_\infty \sin\beta_i. \tag{30}$$

- The tangential velocity $v_{s,i}$ at the control point of the ith panel is,

$$v_{s,i} = \frac{\partial\phi}{\partial s} = \sum_{j=1}^{n} \frac{\lambda_j}{2\pi} \int_j \frac{\partial}{\partial s_i}(\ln r_{ij}) ds_j. \tag{31}$$

- Notice we do not have special treatment for the tangential contribution of panel i to itself since the panel source can only induce volume flow in a direction normal to itself.

Incompressible Inviscid Flow

- Having determined the two contributions we can finally derive the expression of the velocity at the panel control point,

$$\boxed{v_i = v_\infty \sin\beta_i + \sum_{j=1}^{n} \frac{\lambda_j}{2\pi} \int_j \frac{\partial}{\partial s_i}(\ln r_{ij}) ds_j.} \tag{32}$$

- This is a simple post-processing exercise.
- The pressure coefficient at control point i can be obtained from Bernouilli's equation,

$$p_\infty + \frac{1}{2}\rho v_\infty^2 = p_i + \frac{1}{2}\rho v_i^2. \tag{33}$$

- Defining the pressure coefficient as $c_{p,i} = (p_\infty - p_i)/\frac{1}{2}\rho v_\infty^2$ we get,

$$\boxed{c_{p,i} = 1 - \left(\frac{v_i}{v_\infty}\right)^2.} \tag{34}$$

Incompressible Inviscid Flow

- But what about accuracy?
- Recall that λ_i the strength of the ith panel is in fact volume flow rate per unit length. Therefore its actual strength is $\lambda_i s_i$ where s_i is the length of the panel, is equivalent to volume/mass flow rate contributed by the panel.
- Since the airfoil body is neither adding nor removing mass from the flow the sum of all $\lambda_i s_i$ should be zero,

$$\sum_{i=1}^{n} \lambda_i s_i = 0. \tag{35}$$

- The residual of equation 35 is a direct measure of the accuracy of the source panel method and can be used to judge if a certain panel size/distribution is accurate enough.
- One can also use equation 35 to drive successive refinements of the panel discretization of the airfoil until desired accuracy has been achieved.

Incompressible Inviscid Flow

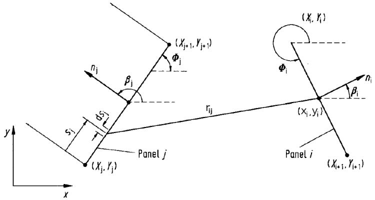

Figure 8: Generic panel geometry for evaluating source panel method integrals.

- We need to evaluate the integrals I_{ij} and $\int_j \frac{\partial}{\partial s_i}(\ln r_{ij}) ds_j$ and we can do this geometrically based on the panel configuration shown in figure 8.
- There, (x_i, y_i) are the coordinates of the control point of panel i to the right, and (x_j, y_j) are the running coordinates over panel j to the left.
- (X_i, Y_i) and (X_{i+1}, Y_{i+1}) are the boundary point coordinates of panel i, and the same applies to panel j with (X_j, Y_j) and (X_{j+1}, Y_{j+1}).

Incompressible Inviscid Flow

- The angles between the x-axis and normal vectors n_i and n_j are β_i and β_j, respectively.
- The integral I_{ij} is taken over the entire jth panel at control point i,

$$I_{ij} = \int_j \frac{\partial}{\partial n_i}(\ln r_{ij}) ds_j, \tag{36}$$

with $r_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \quad \Rightarrow$

$$\begin{aligned}
\frac{\partial}{\partial n_i}(\ln r_{ij}) &= \frac{1}{r_{ij}} \frac{\partial r_{ij}}{\partial n_i} \\
&= \frac{1}{r_{ij}} \frac{1}{2} \left[(x_i - x_j)^2 + (y_i - y_j)^2\right]^{-1/2} \\
&\quad \left[2(x_i - x_j)\frac{dx_i}{dn_i} + 2(y_i - y_j)\frac{dy_i}{dn_i}\right] \\
&= \frac{(x_i - x_j)\cos\beta_i + (y_i - y_j)\sin\beta_i}{(x_i - x_j)^2 + (y_i - y_j)^2}.
\end{aligned} \tag{37}$$

Incompressible Inviscid Flow

- More so, the angles Φ_i and Φ_j in figure 8 are measured counter-clockwise from the x-axis to the bottom of each panel. Therefore,

$$\beta_i = \Phi_i + \frac{\pi}{2},$$

and,

$$\sin\beta_i = \cos\Phi_i, \quad (38)$$

$$\cos\beta_i = -\sin\Phi_i. \quad (39)$$

- We also have from figure 8,

$$x_j = X_j + s_j\cos\Phi_j, \quad (40)$$

and,

$$y_j = Y_j + s_j\sin\Phi_j. \quad (41)$$

Incompressible Inviscid Flow

- Substituting equations 37, 38, 39, 40 and 41 into equation 36 we get,

$$I_{ij} = \int_0^{s_j} \frac{Cs_j + D}{s_j^2 + 2As_j + B} ds_j, \tag{42}$$

where,

$$
\begin{aligned}
A &= -(x_i - X_j)\cos\Phi_j - (y_i - Y_j)\sin\Phi_j,\\
B &= (x_i - X_j)^2 + (y_i - Y_j)^2,\\
C &= \sin(\Phi_i - \Phi_j),\\
D &= (y_i - Y_j)\cos\Phi_i - (x_i - X_j)\sin\Phi_i,\\
s_j &= \sqrt{(X_{j+1} - X_j)^2 + (Y_{j+1} - Y_j)^2}.
\end{aligned}
$$

Incompressible Inviscid Flow

- Setting

$$E = \sqrt{B - A^2} = (x_i - X_j)\sin\phi_j - (y_i - Y_j)\cos\phi_j,$$

we get an expression for I_{ij} in equation 42 from standard tables of integrals,

$$I_{ij} = \frac{C}{2}\ln\left(\frac{s_j^2 + 2As_j + B}{B}\right) + \frac{D - AC}{E}\left(\tan^{-1}\frac{s_j + A}{E} - \tan^{-1}\frac{A}{E}\right). \quad (43)$$

- Equation 43 applies to any pair of panels.
- A similar procedure provides an expression for the velocity integral from equation 32,

$$\int_j \frac{\partial}{\partial s_i}(\ln r_{ij})ds_j = \frac{D - AC}{E}\ln\left(\frac{s_j^2 + 2As_j + B}{B}\right) - C\left(\tan^{-1}\frac{s_j + A}{E} - \tan^{-1}\frac{A}{E}\right). \quad (44)$$

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Source panel method example

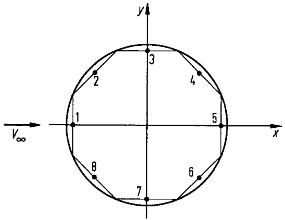

Figure 9: A cylinder subdivided with panels.

- We apply the source panel method to the cylinder depicted in figure 9, subdivided in 8 panels.
- It is found by experience that this number provides excellent accuracy.
- We need to calculate I_{ij} as given by equation 43 for each of the panels.

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- Take panel 4 as the *i*th panel and panel 2 as the *j*th panel, and let's calculate I_{42}.
- Assuming a unit radius we get,

$$X_j = -0.9239 \qquad X_{j+1} = -0.3827 \qquad Y_j = 0.3827$$
$$Y_{j+1} = 0.9239 \qquad \Phi_i = 315^\circ \qquad \Phi_j = 45^\circ$$
$$x_i = 0.6533 \qquad y_i = 0.6533$$

- Substituting these numbers in the formulas for A, B, C, D, E and s_j we get,

$$A = -1.3065 \qquad B = 2.5607 \qquad C = -1 \qquad D = 1.3065$$
$$s_j = 0.7654 \qquad E = 0.9239$$

- Inserting the above values into equation 43 we get,

$$I_{42} = 0.4018. \tag{45}$$

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- Similarly we obtain,

$$
\begin{aligned}
I_{41} &= 0.4074 & I_{43} &= 0.3528 & I_{45} &= 0.3528 \\
I_{46} &= 0.4018 & I_{47} &= 0.4074 & I_{48} &= 0.4084
\end{aligned}
$$

- Plugging the values above into equation 29 and keeping in mind that $\beta_i = 45^\circ$ we can write for panel 4,

$$
\begin{aligned}
0.4074\lambda_1 + 0.4018\lambda_2 + 0.3528\lambda_3 + \pi\lambda_4 + 0.3528\lambda_5 + 0.4018\lambda_6 \\
+ 0.4074\lambda_7 + 0.4084\lambda_8 = -0.7071 \times 2\pi v_\infty .
\end{aligned} \tag{46}
$$

- Equation 46 is a linear algebraic equation for the 8 unknown λ values. We can write similar equations for each of the other 7 panels to give us 8 equations to be solved for the unknown λ values.

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- After solving the 8 equations we get the final results,

$$\lambda_1/2\pi v_\infty = 0.3765 \qquad \lambda_2/2\pi v_\infty = 0.2662 \qquad \lambda_3/2\pi v_\infty = 0$$
$$\lambda_4/2\pi v_\infty = -0.2662 \qquad \lambda_5/2\pi v_\infty = -0.3765 \qquad \lambda_6/2\pi v_\infty = -0.2662$$
$$\lambda_7/2\pi v_\infty = 0 \qquad \lambda_8/2\pi v_\infty = 0.2662$$

- Note the symmetrical distribution of the λ values with respect to the centre of the cylinder. This is to be expected for non-lifting flow.
- Since the panel lengths are all equal the accuracy check of equation 35 reduces to,

$$\sum_{j=1}^{n} \lambda_j = 0. \tag{47}$$

- Plugging the obtained λ values in the equation above we see it is satisfied identically.

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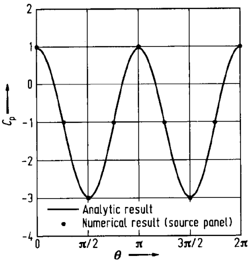

Figure 10: Comparison of the source panel method to analytical pressure coefficient over the non-lifting cylinder case.

- The tangential velocity at the control point of the *i*th panel is calculated using equation 32, with the integral given by equation 44 by plugging in the λ values and summing over the panels.
- The pressure coefficient is obtained by plugging the tangential velocities into equation 34 and compared against the analytical solution of the Laplace equation in figure 10. As verified earlier we have excellent accuracy.

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The vortex panel method

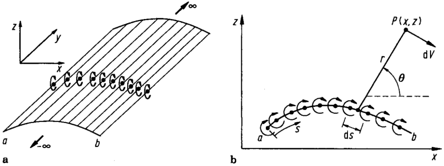

Figure 11: A vortex sheet.

- We now move to the vortex panel method, normally used for lifting flow applications.
- Just like we did for the source sheet, consider a sheet made of an infinite number of straight vortex filaments stacked side by side (figure 11). The strength of each filament is infinitesimally small.

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- With s being the distance measured along the vortex sheet in edge view, we define $\gamma(s)$ as the strength of the vortex sheet per unit length along s. The strength of an infinitesimal portion ds is then γds, and this small section can be treated as an individual vortex.
- Consider point P in the flow located at a distance r from ds. The infinitesimal vortex with strength γds induces a velocity potential $d\Phi$ at P where,

$$d\Phi = -\frac{\gamma ds}{2\pi}\theta. \tag{48}$$

- The velocity potential at P due to the entire vortex sheet from a to b is then,

$$\Phi = -\frac{1}{2\pi}\int_a^b \theta\gamma ds. \tag{49}$$

- Similarly the circulation around the vortex sheet can be calculated as,

$$\Gamma = \int_a^b \gamma ds. \tag{50}$$

Incompressible Inviscid Flow

- An interesting and useful property of vortex sheets is that the tangential velocity component above an below the sheet experiences a jump given by,

$$\gamma = u_1 - u_2. \tag{51}$$

- A direct consequence of equation 51 is that $\gamma = 0$ at the trailing edge of an airfoil, which is one form of the Kutta condition fixing the value of circulation around an airfoil with a sharp trailing edge. More so, the circulation around the sheet is related to the lift force L through the Kutta-Joukowski theorem,

$$L = \rho_\infty \mathbf{v}_{infty} \Gamma. \tag{52}$$

- Since the vortex panel method can provide the amount of circulation it can be used to calculate lift on lifting bodies.
- This is demonstrated in detail in the next slides.

Incompressible Inviscid Flow

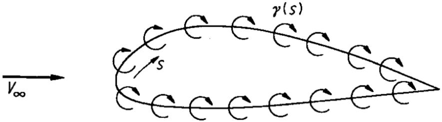

Figure 12: An arbitrary airfoil covered with a vortex sheet.

- Consider an arbitrary airfoil covered with a vortex sheet as in figure 12. To calculate the lift generated by this airfoil we will need to find the distribution of circulation $\gamma(s)$ that makes the airfoil surface coincide with a flow streamline, much like we did for the source panel method.
- We use the same type of airfoil discretization as in figure 7 that we used for the source panel method, with constant vortex strength per unit length $\gamma(s)$ per panel.
- We have n panels with unknown vortex strengths per unit length $\gamma_1, \ldots, \gamma_j, \ldots, \gamma_n$.

Incompressible Inviscid Flow

- We need to solve for the unknown γ_j such that the airfoil surface coincides with a flow streamline and the Kutta condition is satisfied ($\gamma_{TE} = 0$).
- As explained with the source panel method, our boundary condition is that the normal component of velocity is zero at the control point of each panel.
- Let P be a point located at (x, y) in the flow and r_{pj} the distance from any point on panel j to P. Based on equation 16 he velocity potential $\Delta\phi_j$ induced at P by panel j is,

$$\Delta\phi_j = -\frac{1}{2}\int_j \theta_{pj}\gamma_j ds_j, \tag{53}$$

where γ_j is constant over panel j and can be taken out of the integral. The angle θ_{pj} is given by,

$$\theta_{pj} = \tan^{-1}\frac{y - y_j}{x - x_j}. \tag{54}$$

Incompressible Inviscid Flow

- The potential due to all panels at point P is,

$$\phi(P) = \sum_{j=1}^{n} \phi_j = -\sum_{j=1}^{n} \frac{\gamma_j}{2\pi} \int_j \theta_{pj} ds_j. \tag{55}$$

- Since P is an arbitrary point we can put it at the control point (x_i, y_i) of panel i, just like we did for the source panel method, which leads to,

$$\theta_{ij} = \tan^{-1} \frac{y_i - y_j}{x_i - x_j} \tag{56}$$

and,

$$\phi(x_i, y_i) = \sum_{j=1}^{n} \phi_j = -\sum_{j=1}^{n} \frac{\gamma_j}{2\pi} \int_j \theta_{ij} ds_j. \tag{57}$$

- Equation 57 is the contribution of all panels to the potential function at the control point of panel i.

Incompressible Inviscid Flow

- We are dealing with inviscid flow so the component of velocity normal to the surface of the airfoil must be zero. This component is due to two contributions, one from the freestream given by equation 24 in the source panel analysis,

$$V_{\infty,n}\big|_i = \mathbf{V}_\infty \cdot \mathbf{n}_i = V_\infty \cos\beta_i,$$

and the other by each vortex panel j given by equation 25,

$$V_{ij,n} = \frac{\partial}{\partial n_i}[\phi_j(x_i, y_i)].$$

- Combining equations 25 and 57 to recover the contribution of all panels at i we get,

$$V_n = -\sum_{j=1}^{n} \frac{\gamma_j}{2\pi} \int_j \frac{\partial \theta_{ij}}{\partial n_i} ds_j. \tag{58}$$

Incompressible Inviscid Flow

- Summing up the normal components due to the freestream and the panels we get,

$$V_\infty \cos\beta_i - \sum_{j=1}^{n} \frac{\gamma_j}{2\pi} \int_j \frac{\partial \theta_{ij}}{\partial n_i} ds_j = 0. \tag{59}$$

- Just like we have seen with the source panel method we end up with an algebraic equation with n unknows being the values of the vortex strengths γ_j at all panels, which represents our boundary condition at the airfoil surface.
- When we write this equation for all panels we get n equations for the n unknowns that can be easily solved numerically.
- For convenience we designate the integral in equation 59 by J_{ij} and the equation is now written as,

$$V_\infty \cos\beta_i - \sum_{j=1}^{n} \frac{\gamma_j}{2\pi} J_{ij} = 0. \tag{60}$$

Incompressible Inviscid Flow

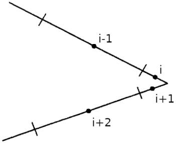

Figure 13: Vortex panels at the trailing edge of an airfoil.

- So far we have discussed the vortex panel method like we did the source panel method. This is the point where the discussions diverge.
- We mentioned previously that the Kutta condition, $\gamma_{TE} = 0$ must be enforce at the trailing edge of an airfoil. In order to do this numerically we must have the two control points i and $i+1$ in figure 13 as close as possible to the TE and enforce,

$$\gamma_i = -\gamma_{i+1}, \tag{61}$$

so that both circulation add up to 0 at the TE, which enforces the Kutta condition.

Incompressible Inviscid Flow

- This sounds simple enough but we end up with $n+1$ equations for n unknowns, which is an over-determined system.
- We circumvent this problem by solving equation 60 at $n-1$ points only. The way to do this is to solve at all points except $i+1$, and set the circulation at the $i+1$ to the negative of the circulation at i that we solved for. We could also interchange both points as it does not matter as long as the panels are small enough at the TE.
- In practice this is an additional boundary condition imposed at one point.
- So having calculated all circulation values at the control points of all panels we still have to calculate the tangential velocity at the airfoil surface and the pressure coefficients.
- We could do it the long way by differentiating the potential function with respect to the tangential direction like we did for the source panel method.
- Luckily, there is an easier method using the circulation jump across a vortex sheet.

Incompressible Inviscid Flow

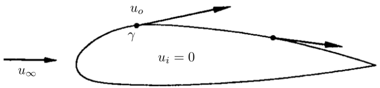

Figure 14: Airfoil with inner and outer velocities.

- Equation 51 relates the jump in circulation above and below a vortex sheet to the tangential velocity,

$$\gamma = u_o - u_i,$$

where u_o is the tangential velocity outside the airfoil and above the vortex sheet, and u_i inside the airfoil below the vortex sheet as depicted in figure .

- Since the velocity inside the airfoil is $u_i = 0$ we simply recover $u_o = \gamma$ the circulation.
- At this point all that is left is to calculate the pressure coefficients using Bernouilli's law (equation 33).

Incompressible Inviscid Flow

- To calculate the lift force L generated by the airfoil (which is the whole point of using a vortex panel method in the first place) we need calculate the total circulation Γ due to all the panels,

$$\Gamma = \sum_{j=1}^{n} \gamma_j s_j. \tag{62}$$

- Thanks to the Kutta-Joukowski theorem,

$$L = \rho_\infty v_\infty \Gamma.$$

- This concludes the vortex panel analysis.

Incompressible Inviscid Flow

- This version of the vortex panel method is called a 1st order method because it uses a constant value of γ over each panel and is meant to give a general idea.
- It is sensitive to the number of panels used, their sizes and distribution so there is a degree of arbitrariness to it.
- There is also the need to ignore one of the points to have a determined system, and different choices of points to ignore can yield different results.
- The method also suffers from oscillations in the γ distribution due to numerical inaccuracies.
- These issues will be studied in more detail in the future lectures when we study modern approaches to CFD such as the finite volume method.

Incompressible Inviscid Flow

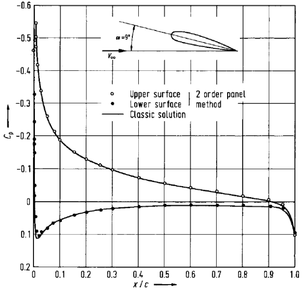

Figure 15: Pressure coefficient distribution over a NACA 0012 airfoil for a 2nd order vortex panel method.

- These problems can be overcome by using higher-order distributions of γ. For example, the 2nd order method uses a linear variation of γ over the panel with matching values to neighbouring panels at their common boundaries. This obviously requires solving for the boundary values, adding more unknowns to the problem.
- A 2nd order pressure coefficient result of a NACA 0012 airfoil is shown in figure 15, compared to the theoretical solution.

Incompressible Inviscid Flow

- For your next assignment you will write two python codes, one for a source panel method and the other for a vortex panel method.
- You will use the codes calculate pressure coefficient distributions for classical problems admitting theoretical solutions and compare to these.
- You will also test the accuracy of the codes for different panel distributions.
- The assignment details will be provided this week on Moodle.