CFD for Aerospace Applications

The Finite Volume Method

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Introduction

- In the previous lecture we introduced the finite difference method and used it to derive very useful approximations of derivatives with arbitrary accuracy.
- The finite difference method is applied at points and requires grids where the points can be found by structured indexing, the so-called structured grids.
- This is an important limitation of finite differencing since most engineering geometries of interest are far too complex to be discretized with a structured grid.
- This prompts the need for methods that are not limited to pointwise evaluation on structured grids.
- This need is met with the finite volume and finite element methods, both integral methods and subsets of the general weighted residual method.
- In this course we focus on the finite volume method.

The Finite Volume Method

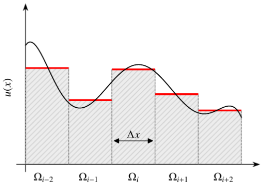

Figure 1: 1D finite volume discretization of a solution function on a linear domain.

- Consider figure 1 where the linear domain Ω is subdivided in Ω_i cells.
- Instead of evaluating the solution function $q(x)$ at points as in the finite difference method, the finite volume method evaluates $q(x)$ within each cell, where it is taken constant (red lines).
- This is called the Godunov method.
- The index i is not usually associated with a certain spatial order or structure. It simply designates the number of the cell in a generally *unstructured* grid.

Derivation of the finite volume method

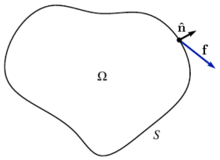

Figure 2: An arbitrary volume of undefined shape.

- Consider the fixed, constant and arbitrary volume Ω of undefined shape and having surface area S, depicted in figure 2.
- We will apply an integral conservation principle on Ω and use it to derive the finite volume method.

The Finite Volume Method

- Let $\mathbf{q}$ be a vector of conserved variables in Ω such as mass, momentum and energy. It could be for example,

$$\mathbf{q} = \begin{bmatrix} \rho \\ u \\ v \\ w \\ e \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \rho \\ u \\ e \end{bmatrix} \tag{1}$$

- The principle of integral conservation between the local change in $\mathbf{q}$ and its vector of surface fluxes *per unit area* $\mathbf{f}(\mathbf{q})$ can be written as,

$$\int_{\Omega} \frac{\partial \mathbf{q}}{\partial t} d\Omega + \int_{S} \mathbf{f}(\mathbf{q}) \cdot \hat{\mathbf{n}} ds = 0, \tag{2}$$

where $\hat{\mathbf{n}}$ is the outward unit normal vector at S.

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- Let us now divide the volume Ω into n cells.
- We can rewrite equation 2 in each cell i with volume Ω_i where $\mathbf{q}$ constant in each cell,

$$\int_{\Omega_i} \frac{\partial \mathbf{q_i}}{\partial t} d\Omega + \int_{S_i} \mathbf{f}(\mathbf{q_i}) \cdot \hat{\mathbf{n}} ds = 0. \tag{3}$$

- Since $\mathbf{q_i}$ is constant in cell i we can integrate directly over the volume Ω_i giving us,

$$\int_{\Omega_i} \frac{\partial \mathbf{q_i}}{\partial t} d\Omega = \frac{\partial \mathbf{q_i}}{\partial t} \Omega_i. \tag{4}$$

- The surface integral term can be discretized as,

$$\int_{S} \mathbf{f}(\mathbf{q}_i) \cdot \hat{\mathbf{n}} ds = \sum_{j=1}^{m} \mathbf{f}_j(\mathbf{q}_i) \cdot \hat{\mathbf{n}}_j s_j, \tag{5}$$

where we simply add up the flux contributions at every face of cell i. Here m is the number of faces of cell i and $\mathbf{f}_j$ is the flux per unit area at face j.

The Finite Volume Method

- Equation 3 can now be written in discrete form as,

$$\frac{\partial \mathbf{q_i}}{\partial t}\Omega_i + \sum_{j=1}^{m} \mathbf{f}_j(\mathbf{q}_i) \cdot \hat{\mathbf{n}}_j s_j = 0. \tag{6}$$

- Dividing by Ω_i and moving the flux term to the LHS we get the final form of the finite volume discretization of equation 3,

$$\frac{\partial \mathbf{q_i}}{\partial t} = -\frac{1}{\Omega_i}\sum_{j=1}^{m} \mathbf{f}_j(\mathbf{q}_i) \cdot \hat{\mathbf{n}}_j s_j. \tag{7}$$

- For a 1D problem like figure 1 where $\Omega_i = \Delta x_i$, unit normals are pointing out of Ω_i and we can take $s_j = 1$, equation 7 becomes,

$$\frac{\partial \mathbf{q_i}}{\partial t} = -\frac{\mathbf{f_{i+1/2}} - \mathbf{f_{i-1/2}}}{\Delta x_i}.$$

Indices $i - 1/2$ and $i + 1/2$ denote the left and right faces of cell Ω_i.

The Finite Volume Method

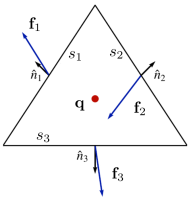

Figure 3: A triangular cell with fluxes and centre variable.

- We see that the RHS is a second-order approximation of the cell centre flux and a first-order approximation of face fluxes.
- The LHS of equation 7 can be approximated with a suitable finite differencing scheme, with $\mathbf{q}_i$ taken at the cell centre.
- For the RHS we simply add up the contributions of all faces of cell i where fluxes are defined at the cell faces as shown in the triangular 2D cell of figure 3, a typical unstructured cell, highlighting the applicability of the finite volume to unstructured grids as well.

The Finite Volume Method

The Riemann problem

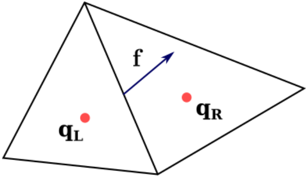

Figure 4: The Riemann problem for two adjacent triangular cells.

- We derived the finite volume method for one cell and illustrated it in figure 3.
- However, for two adjacent cells sharing the same face there can be only one flux.
- This is illustrated in figure 4 where the flux at the common face depends on the cell centre variables in the left and right cells,

$$\mathbf{f} = \mathbf{f}(\mathbf{q}_L, \mathbf{q}_R). \tag{8}$$

The Finite Volume Method

- Matching the fluxes at the common faces is called the Riemann problem.
- The type of function used for the flux in equation 8 depends on the equation solved.
- We will revisit the three fundamental equations introduced in the finite difference lecture, develop finite volume schemes for them and show examples of flux functions for the Riemann problem.

The Finite Volume Method

The 1D linear advection equation

- The 1D linear advection equation for the variable **q** is,

$$\frac{\partial \mathbf{q}}{\partial t} + \alpha \frac{\partial \mathbf{q}}{\partial x} = 0. \tag{9}$$

- Multiplying the equation above by $d\Omega$, integrating over cell i and using the divergence theorem we get,

$$\int_{\Omega_i} \frac{\partial \mathbf{q}_i}{\partial t} d\Omega + \int_{S_i} \alpha \mathbf{q}_i \cdot \hat{\mathbf{n}} ds = 0. \tag{10}$$

- Dividing the equation above by the cell volume Ω_i and moving the surface integral to the RHS we get the finite volume representation of the linear advection equation,

$$\frac{\partial \mathbf{q}_i}{\partial t} = -\frac{1}{\Omega_i} \int_{S_i} \alpha \mathbf{q}_i \cdot \hat{\mathbf{n}} ds. \tag{11}$$

The Finite Volume Method

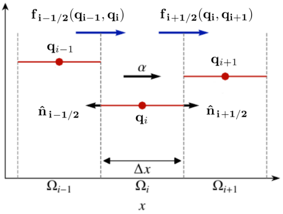

Figure 5: The finite volume grid of the 1D linear advection problem.

- Comparing equation 11 to equation 7 we can deduce the flux function as,

$$\mathbf{f}_i = \alpha \mathbf{q}_i. \tag{12}$$

- The face fluxes for the finite volume discretization of the 1D linear advection problem are shown in figure 5 where face variables such as fluxes and face normals are defined with fractional indices (e.g. $i - 1/2$, $i + 1/2$).
- The face on the side the flow comes from is called the upwind face (here $i - 1/2$), and the face on the side the flow goes to is called the downwind face (here $i + 1/2$).

The Finite Volume Method

The upwind scheme

- In figure 5 we set $\alpha > 0$, so the flow is from left to right.
- It means that any property or variable advected by the flow (which we call information) will be transported from left to right.
- A natural choice of flux functions for the cells shown in figure 5 would be,

$$\mathbf{f}_{i-1/2} = \alpha \mathbf{q}_{i-1} \tag{13}$$

and,

$$\mathbf{f}_{i+1/2} = \alpha \mathbf{q}_{i}. \tag{14}$$

- Let's apply the fluxes given by equations 13 and 14 to the finite volume discretization of the linear advection equation given by equation 11.

The Finite Volume Method

- First we transform the surface integral into a discrete summation over the cell faces while keeping in mind that $S_{i-1/2} = S_{i+1/2} = 1$, $\Omega_i = \Delta x_i S_{i-1/2} = \Delta x_i S_{i+1/2}$, $\hat{n}_{i-1/2} = -1$ and $\hat{n}_{i+1/2} = 1$,

$$\frac{\partial \mathbf{q}_i}{\partial t} = -\frac{1}{\Omega_i}\int_{S_i} \alpha \mathbf{q}_i \cdot \hat{\mathbf{n}} ds$$

$$\Rightarrow \frac{\partial \mathbf{q}_i}{\partial t} = -\frac{1}{\Delta x_i}\sum_{j=1-1/2}^{1+1/2} \mathbf{f_j} \cdot \hat{\mathbf{n}}_\mathbf{j} S_j. \tag{15}$$

- Plugging the flux expressions and surface values into the equation above yields,

$$\frac{\partial \mathbf{q}_i}{\partial t} = -\alpha \frac{\mathbf{q_i} - \mathbf{q_{i-1}}}{\Delta x_i} + O(\Delta x) \tag{16}$$

The Finite Volume Method

- Equation 16 is identical to the first-order finite difference result from the previous lecture.
- Going further we discretize the transient term on the LHS using the first-order forward differencing (also known as the Euler method) and recover the same result as we did with finite differencing,

$$\boxed{\mathbf{q_i^{t+1}} = \mathbf{q_i^t} - \frac{\alpha \Delta t}{\Delta x}(\mathbf{q_i^t} - \mathbf{q_{i-1}^t}) + O(\Delta t, \Delta x).} \tag{17}$$

- We see here that an upwind finite volume scheme is equivalent to a first-order finite difference scheme.

The Finite Volume Method

- Equation 17 expresses $\mathbf{q_i^{t+1}}$ as a function of the variables in the current time ($\mathbf{q_i^t}$ and $\mathbf{q_{i-1}^t}$) and is called an explicit scheme.
- In an explicit scheme the variables in the next time step can be solved for directly since they depend on known variables from the current time step.
- If on the other hand we had variables from the next time step ($\mathbf{q_i^{t+1}}$ and $\mathbf{q_{i-1}^{t+1}}$) the scheme would be called implicit.
- In an implicit scheme the variables in the next time step must be solved for by inverting a matrix since they depend on other unknown variables from the next time step.

The Finite Volume Method

- We designate the multiplier on the RHS by,

$$\boxed{Co = \frac{\alpha \Delta t}{\Delta x}.} \tag{18}$$

- This is a non-dimensional velocity called the Courant (or CFL) number and is a important measure of explicit scheme stability.
- In other words it is a numerical stability dimensionless number, much like the Reynolds or Mach number used for fluid dynamics.
- For stable explicit schemes one must always have a courant number $Co < 1$.

The Finite Volume Method

The central scheme

- Here the flux is defined as the average of the upwind and downwind values.

$$\mathbf{f_{i-1/2}} = \frac{\alpha \mathbf{q_{i-1}} + \alpha \mathbf{q_i}}{2} \tag{19}$$

$$\mathbf{f_{i+1/2}} = \frac{\alpha \mathbf{q_i} + \alpha \mathbf{q_{i+1}}}{2}. \tag{20}$$

- Plugging the above fluxes into equation 15 we get,

$$\frac{\partial \mathbf{q}_i}{\partial t} = -\alpha \frac{\mathbf{q_{i+1}} - \mathbf{q_{i-1}}}{2\Delta x_i} + O(\Delta x^2). \tag{21}$$

- Using first-order forward finite difference for the transient term,

$$\boxed{\mathbf{q_i^{t+1}} = \mathbf{q_i^t} - \alpha \frac{\Delta t}{2\Delta x_i}(\mathbf{q_{i+1}^t} - \mathbf{q_{i-1}^t}) + O(\Delta t, \Delta x^2).} \tag{22}$$

The Finite Volume Method

- We see here that a finite volume scheme with central flux is equivalent to a second-order central finite difference scheme.
- Note that the Courant number of a central scheme is $Co = \alpha \Delta t / 2\Delta x$, which is half that of the upwind scheme.
- This is the cost of increased accuracy, we advance slower.

The Finite Volume Method

The blended scheme

- The blended scheme is one that allows blending the upwind and central schemes.
- Take for example the upwind face where the upwind and central fluxes are defined respectively as,

$$\mathbf{f_u} = \alpha \mathbf{q_{i-1}},$$

and

$$\mathbf{f_c} = \frac{\alpha}{2}(\mathbf{q_{i-1}} + \mathbf{q_i}).$$

- The blended flux function is then defined as,

$$\mathbf{f_b} = (1 - \phi)\mathbf{f_u} + \phi \mathbf{f_c}, \tag{23}$$

where ϕ is a blending scalar coefficient.

The Finite Volume Method

- $\phi = 0$ recovers the first-order upwind flux function, and $\phi = 1$ recovers the second-order central flux function.
- $0 < \phi < 1$ recovers an intermediate accuracy blended flux function.
- In practice one calculates ϕ based on flow variables indicative of flow stability in order to use it as a switching function preserving stability and accuracy.
- In unstable regions of the flow one can switch to first-order upwind flux.
- In stable regions of the flow one can switch to second-order central flux.

The Finite Volume Method

The conservative inviscid Burgers equation

- For the Burgers equation the solution vector $\mathbf{q}$ is limited to the velocity components,

$$\mathbf{q} = [u \; v \; w]^T. \tag{24}$$

- The generalized inviscid Burgers equation for the generic variable $\mathbf{q}$ is written as,

$$\frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} \cdot \nabla \mathbf{q} = 0. \tag{25}$$

- The non-conservative advective term above can be converted to a divergence form using the identity below,

$$\mathbf{q} \cdot \nabla \mathbf{q} = \nabla \cdot \mathbf{q}\mathbf{q} - \mathbf{q} \nabla \cdot \mathbf{q}. \tag{26}$$

- Using equation 26 in equation 25 we get the conservative inviscid Burgers equation ,

$$\frac{\partial \mathbf{q}}{\partial t} + \nabla \cdot \mathbf{q}\mathbf{q} - \mathbf{q} \nabla \cdot \mathbf{q} = 0. \tag{27}$$

The Finite Volume Method

- We have seen in the finite difference lecture that the solution of the conservative inviscid Burgers equation is highly directional and tends to develop a discontinuous shock-wave like feature.
- This suggests that an upwind flux is the appropriate choice.
- The upwind flux functions in the x-direction at the common faces to the cells become,

$$\mathbf{f_{i-1/2}} = \mathbf{q_{i-1}} \mathbf{q_{i-1}}, \tag{28}$$

and

$$\mathbf{f_{i+1/2}} = \mathbf{q_i} \mathbf{q_i}. \tag{29}$$

- Let's consider the x-component of equation 27,

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = 0,$$

which can also be written as

$$\frac{\partial u}{\partial t} = -\frac{1}{2}\frac{\partial u^2}{\partial x} - v\frac{\partial u}{\partial y} - w\frac{\partial u}{\partial z}. \tag{30}$$

The Finite Volume Method

- The upwind flux function of equation 30 in the x-direction is clearly then,

$$f(u) = \frac{1}{2}u^2.$$

- Let us assume the flow is one-dimensional so $v = w = 0$. Equation 30 becomes,

$$\frac{\partial u}{\partial t} = -\frac{\partial f(u)}{\partial x}$$

- Multiplying the equation above by $d\Omega_i = \Delta x_i S_i$ in 1D and integrating like we did for the linear advection problem,

$$\begin{aligned}\frac{\partial u_i}{\partial t}\Delta x_i S_i &= -\int_{\Omega_i}\frac{\partial f(u)}{\partial x}d\Omega_i\\ &= -\sum_{j=i-1/2}^{i+1/2} f(u)\hat{n}_j S_j.\end{aligned}$$

The Finite Volume Method

- Keeping in mind that $S_i = S_{i-1/2} = S_{i+1/2} = 1$, $\hat{n}_{i-1/2} = -1$ and $\hat{n}_{i+1/2} = 1$ we can approximate the transient term with a forward difference first derivative and get,

$$\frac{u_i^{t+1} - u_i^t}{\Delta t} = -\frac{1}{2\Delta x_i}\left(\left(u_i^t\right)^2 - \left(u_{i-1}^t\right)^2\right) + O(\Delta t, \Delta x). \tag{31}$$

- Isolating u_i^{t+1} on the LHS we recover a result identical to the finite difference discretization of the 1D conservative inviscid Burgers equation,

$$u_i^{t+1} = u_i^t - \frac{\Delta t}{2\Delta x_i}\left(\left(u_i^t\right)^2 - \left(u_{i-1}^t\right)^2\right) + O(\Delta t, \Delta x). \tag{32}$$

The Finite Volume Method

The linear diffusion equation

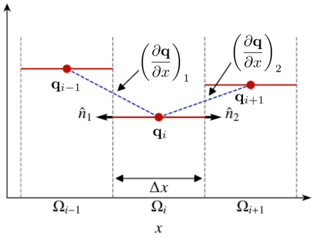

Figure 6: The finite volume grid of the 1D linear diffusion problem.

- The final example is the 1D linear diffusion problem in figure 6.
- We recall that the transient linear diffusion equation is,

$$\frac{\partial \mathbf{q}}{\partial t} - \beta \nabla^2 \mathbf{q} = 0. \tag{33}$$

The Finite Volume Method

- The diffusion coefficient β in equation 33 is taken constant but it can just as easily be variable.
- We can see that the flux in equation 33 has the form,

$$\mathbf{f}(\mathbf{q}) = -\beta \nabla \mathbf{q}. \tag{34}$$

- We repeat the same procedure as before by multiplying equation 33 by $d\Omega_i$ and integrating over Ω_i while using the flux function of equation 34,

$$\int_{\Omega_i} \frac{\partial \mathbf{q}}{\partial t} d\Omega_i = \int_{\Omega_i} \nabla \cdot \mathbf{f}(\mathbf{q}) d\Omega_i. \tag{35}$$

- Using the same manipulations as before result in the following semi-discrete equation,

$$\frac{\partial \mathbf{q}}{\partial t} = \frac{1}{\Delta x_i} \sum_{j=i-1/2}^{i+1/2} \mathbf{f}(\mathbf{q_j}) \cdot \hat{\mathbf{n}}_\mathbf{j}. \tag{36}$$

The Finite Volume Method

- The flux at the cell faces is a first-derivative of the variable **q** and needs to be computed using finite differencing.
- For constant β diffusion is isotropic so upwind flux will not be appropriate. We therefore use second-order central differencing,

$$\mathbf{f}(\mathbf{q_{i-1/2}}) = -\beta \frac{\mathbf{q_i} - \mathbf{q_{i-1}}}{\Delta x_i} + O(\Delta x^2). \tag{37}$$

$$\mathbf{f}(\mathbf{q_{i+1/2}}) = -\beta \frac{\mathbf{q_{i+1}} - \mathbf{q_i}}{\Delta x_i} + O(\Delta x^2). \tag{38}$$

- We also use first-order forward differencing in time,

$$\frac{\partial \mathbf{q}}{\partial t} = \frac{\mathbf{q_i^{t+1}} - \mathbf{q_i^{t}}}{\Delta t} + O(\Delta t).$$

The Finite Volume Method

- Using the above discretizations in equation 36 while keeping in mind the directions of the unit normal vectors we get the final expression for the finite volume approximation of the solution at the next time step,

$$\mathbf{q_i^{t+1}} = \mathbf{q_i^t} + \beta \Delta t \frac{\left(\mathbf{q_i^{t+1}} - 2\mathbf{q_i} + \mathbf{q_{i-1}}\right)}{\Delta x_i^2} + O(\Delta t, \Delta x^2). \tag{39}$$

- The above approximation of $\mathbf{q_i^{t+1}}$ is identical to the one derived for linear diffusion by finite differencing alone and is also first-order in time and second-order in space.

Summary of the finite volume method

- The finite volume method is based on conservation transport equations of the following form,

$$\frac{\partial \mathbf{q}}{\partial t} + \nabla \cdot \mathbf{f} = 0,$$

  where $\mathbf{f}(\mathbf{q})$ is the flux of $\mathbf{q}$.

- Upon integration over cell volume Ω_i once recovers the semi-discrete finite volume formulation,

$$\frac{\partial \mathbf{q}}{\partial t} = -\frac{1}{\Omega_i} \sum_{j=1}^{m} \mathbf{f_j} \cdot \mathbf{\hat{n}_j} s_j,$$

  where the sum is taken over the faces of cell i, $\mathbf{\hat{n}_j}$ and s_j being the outward unit normal and surface area of face j respectively.

The Finite Volume Method

- At the faces one solves a Riemann problem to match the face flux in both adjacent cells.
- It was shown that the finite volume method recovers a finite difference discretization over structured grids while conserving fluxes to first-order accuracy at the faces and second-order accuracy at the cell centre, which is called the Godunov method.
- Unlike the finite difference method, since the finite volume method is integral it can also be used on unstructured grids.
- Transport equations that do not have the conservative flux forms shown can be manipulated into them with the remaining terms treated separately.