Take-home Midterm

AERO 455 - CFD for Aerospace Applications

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Given: 11th March 2022
Due: 5pm EST on 25th March 2022
Total points: 160/130
No extensions!

1 Fluid Dynamics (45 points)

This part tests you on the content of lectures 01 to 04.

1.1 The substantial derivative (9 points)

Answer the following questions.

1. What is the formula of the substantial derivative? *(2 points)*
2. Explain what each part means and give real-life examples. *(5 points)*
3. When is it appropriate to use the substantial derivative? *(2 points)*

1.2 Conservation Forms (6 points)

Answer the following questions.

1. When is it appropriate to use a transport equation in conservation form? *(2 points)*
2. Give an example. *(2 points)*
3. Which fluid model leads directly to a conservation form and why? *(2 points)*

1.3 The Total Energy Equation (20 points)

1.3.1 The total energy moving infinitesimal element (10 points)

Answer the following questions.

1. Draw the total energy moving infinitesimal element with fluxes in the y-direction and explain the meaning of each term. Assume a Newtonian stress model and thermal conduction for surface heat transfer. *(5 points)*

2. Repeat the same exercise for fluxes in the z-direction. *(5 points)*

1.3.2 The integral conservative total energy equation (10 points)

Derive the integral conservative total energy equation and use the divergence theorem where appropriate to transform volume integrals into surface integrals. Do not assume a particular form for the viscous stresses and surface heat transfer.

1.4 Potential Flow (10 points)

Answer the following questions.

1. Which panel method studied in class is used for non-lifting flow and which flow property can you get from it? *(2.5 points)*

2. Which panel method studied in class is used for lifting flow and which flow property can you get from it? *(2.5 points)*

3. Which panel method studied in class leads to an ill-defined algebraic system and how do you fix that in practice? *(2.5 points)*

4. What is the Kutta-Joukowski theorem and which panel method uses it? *(2.5 points)*

2 Finite Differencing, Stability and the Finite Volume Method (85 points)

All questions below relate to lectures 05, 06 and 07.

2.1 Finite Differencing and Stability (30 points)

Consider the second order accurate central difference scheme of the first derivative using a two-point stencil,

$$\frac{\partial q_i}{\partial x} = \frac{q_{i+1} - q_{i-1}}{2\Delta x} + O(\Delta x^2). \tag{2.1}$$

Consider also the second order accurate backward difference scheme of the first derivative using a three-point stencil,

$$\frac{\partial q_i}{\partial x} = \frac{3q_i - 4q_{i-1} + q_{i-2}}{2\Delta x} + O(\Delta x^2). \tag{2.2}$$

The 1D linear advection equation of a scalar q is given by,

$$\frac{\partial q}{\partial t} + a\frac{\partial q}{\partial x} = 0, \tag{2.3}$$

where a is a constant advection velocity.

1. Using a linear Neumann stability analysis, determine whether 2.3 is stable when using the scheme of equation 2.1 to approximate the first spatial derivative and under which conditions. Use a first order

forward difference in time for the transient term and an explicit formulation. You may assume an exponential time variation or a generic one, as was shown in lecture 07, slides 36 to 50. *(10 points)*

2. What is the applicable form of the Courant number here and what is its physical meaning compared to the grid velocity as shown in slides 54 to 56 of lecture 07? *(5 points)*

3. Repeat the previous exercises in 1 and 2 using the second order scheme from equation 2.2 for the first derivative. Compare your results to what you got with the central scheme. *(15 points)*

2.2 The Finite Volume Method (10 points)

Answer the following questions.

1. What is the Riemann problem? *(2.5 points)*

2. What is the advantage of the finite volume method over the finite difference method? *(2.5 points)*

3. What is flux upwinding? *(2.5 points)*

4. What is the MUSCL scheme and what is its order of accuracy? How does it maintain stability? *(2.5 points)*

2.3 Coding (45 points)

The Python script provided with the midterm is a finite difference discretization of the 1D linear advection equation using a first order forward difference two-point scheme in time for the transient term, and a first order backward difference two-point scheme in space for the advection term. The advection velocity is $a = 1$ and the domain length is $L = 1$. At the end of the script the solution of the advection of a step function is plotted against the x-axis.

2.3.1 2nd order accurate central differencing (20 points)

1. Add the second order accurate central two-point scheme of equation 2.1 to the script and plot its results on the same plot as the first-order backward difference two-point scheme at time $t = 0.75$. *(10 points)*

2. Keeping the Courant number limitations of both schemes in mind, can you refine the time step and mesh spacing until you your results stop varying at time $t = 0.75$? This is called time step and mesh independence. *(10 points)*

2.3.2 2nd order accurate backward differencing (20 points)

1. Add the second order accurate backward three-point scheme of equation 2.2 to the script and plot its results on the same plot as the first order scheme of the first derivative at time $t = 0.75$. *(10 points)*

2. Keeping the Courant number limitations of both schemes in mind, can you refine the time step and mesh spacing until you your results stop varying at time $t = 0.75$? Which time step and mesh spacing achieve time step and mesh independence? *(10 points)*

2.3.3 Comparing all three schemes (5 points)

Plot the solutions of all three schemes on the same plot at time $t = 0.75$ and answer the following questions.

3. Which of the three schemes exhibits the most oscillations and why? *(2.5 points)*

4. Which well-known stability theorem about the order of accuracy of a stable linear scheme applies to your results? *(2.5 points)*

3 Bonus Problem (30 points)

The 1D linear advection-diffusion equation is given by,

$$\frac{\partial q}{\partial t} + a\frac{\partial q}{\partial x} = \beta\frac{\partial^2 q}{\partial x^2}, \tag{3.1}$$

where $a = 1$ is a constant advection velocity and $\beta = 10^{-3}$ is a constant diffusion coefficient.

1. Modify the linear advection script from problem 2.3 by adding the following 2nd order accurate three-point central difference approximation of the second derivative,

$$\frac{\partial^2 q}{\partial x^2} = \frac{q_{i-1} - 2q_i + q_{i+1}}{\Delta x^2} + O(\Delta x^2). \tag{3.2}$$

   Since the diffusion operator is a second derivative you will need to add another boundary condition at the outlet of the domain. Use a zero-gradient boundary condition, which can be implemented by setting the solution value of the last point of the domain equal to that of the preceding point. Obviously, you will not need to solve for the solution at the last point anymore. (*15 points*)

2. Using the 2nd order accurate scheme of equation 2.2 for the advection term, simulate the transport of the same step function as in problem 2.3. Keep refining the time step and mesh spacing until you reach time step and mesh independence at time $t = 0.75$. Show it by plotting the results of successive time and space refinements on the same plot and mention which time-step and mesh spacing achieved time step and mesh independence. (*5 points*)

3. Once you have reached time step and mesh independence, run the script again for $\beta = 10^{-5}$, 10^{-4}, 10^{-2}, 10^{-1} and plot your results on the same plot for $t = 0.75$ with the baseline solution at $\beta = 10^{-3}$. *(5 points)*

4. How do your results for each β value compare to the baseline solution at $\beta = 10^{-3}$? Which solution has the most oscillations and what do you attribute that to? *(5 points)*