CFD for Aerospace Applications

Fluid Dynamics Review

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Recap of the previous lecture

- A fluid is a continuum description of a liquid or a gas ($Kn < 0.01$) with zero shear modulus, hence no resistance to shear forces.
- Shear stress in a fluid is proportional to the rate of strain, a velocity-like parameter highlighting flow characteristics.
- Fluid dynamics is governed by the three fundamental principles of conservation of mass, momentum and energy, collectively known as the Navier-Stokes (NS) equations.

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Recap of the previous lecture

- The specific form of the equations has important numerical consequences (oscillations, stability, etc.) on CFD simulations due to their associated mathematical properties.
- In CFD we are especially concerned with the conservation and non-conservation forms of the NS equations, the integral conservation form of the transport equation of a property ϕ being defined as,

$$\iiint_V \frac{\partial \phi}{\partial t} dV + \oiint_{\partial V} \phi \mathbf{u} \cdot \mathbf{n} dS = 0, \tag{1}$$

and its differential conservation form defined as,

$$\frac{\partial \phi}{\partial t} + \nabla \cdot \phi \mathbf{u} = 0. \tag{2}$$

- All other forms are in non-conservation form by definition.

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Recap of the previous lecture

- Derivation and analysis of the NS equations starts with simplified models of the flow, the most widely used being the finite control volume and the infinitesimal fluid element, either fixed or moving.
- The forms of transport equation directly derived from either model are shown in table 1 below.

	Finite Control Volume		Infinitesimal Fluid Element	
	Fixed	Moving	Fixed	Moving
Form	conservation	non-conservation	conservation	non-conservation
Direct Equation Type	Integral	Integral	Differential	Differential

Table 1: Summary of fluid models.

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Recap of the previous lecture

- The substantial derivative is the time rate of change for moving fluid models.
- The substantial derivative operator is defined as,

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla). \tag{3}$$

  It consists of a local partial derivative $(\frac{\partial}{\partial t})$ and a convective derivative $(\mathbf{u} \cdot \nabla)$.
- The local derivative is the rate of change felt or measured in a fixed location.
- The convective derivative is the rate of change felt or measured moving with the flow.
- The substantial derivative is the physical equivalent of the mathematical formalism of the total derivative.

- The divergence of velocity is the time rate of change of volume a *moving fluid element* per unit volume,

$$\nabla \cdot \mathbf{v} = \frac{1}{\delta V} \frac{D(\delta V)}{Dt}. \tag{4}$$

- It is a statement of compressibility and replaces the continuity equation for incompressible flow.

end of recap of the previous lecture

Conservation of mass

- We will now derive the conservation of mass principle for both a moving infinitesimal fluid element and a fixed finite control volume.
- This will highlight the differences between the conservation and non-conservation forms of the equations.

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Fundamental principles of fluid flow: conservation of mass, moving infinitesimal fluid element

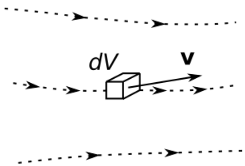

Figure 1: Moving infinitesimal fluid element.

- Consider the moving infinitesimal fluid element in figure 1, having a volume δV and a mass $\delta m = \rho \delta V$. Since mass is conserved and the element is moving the time rate of change of mass of the element is zero as expressed by the substantial derivative,

$$\frac{D(\delta m)}{Dt} = \frac{D(\rho \delta V)}{Dt} = 0. \quad (5)$$

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Fundamental principles of fluid flow: conservation of mass, moving infinitesimal fluid element

- Applying the chain rule we get,

$$\frac{D(\rho \delta V)}{Dt} = \delta V \frac{D\rho}{Dt} + \rho \frac{D\delta V}{Dt} = 0$$
$$\Rightarrow \frac{D\rho}{Dt} + \rho \left[\frac{1}{\delta V} \frac{D\delta V}{Dt} \right] = 0.$$

- The term in brackets is equal to $\nabla \cdot \mathbf{v}$ as we have seen in the previous lecture, which leads to the differential continuity equation in *non-conservation* form,

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0. \tag{6}$$

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Fundamental principles of fluid flow: conservation of mass, moving infinitesimal fluid element

- As explained in the previous lectures,
    1. The *infinitesimal fluid element* leads directly to a *PDE form*.
    2. The model *moving with the flow* leads to the *non-conservation form*.
- For incompressible flow and constant density the substantial derivative term in equation 6 reduces to zero and the density multiplier of the divergence term can be dropped since $\rho \neq 0$, which leads to the differential continuity equation for incompressible flow in non-conservation form,

$$\nabla \cdot \mathbf{v} = 0. \tag{7}$$

- This result highlights the fact that constant density leads to divergence-free (solenoidal) flow and constant volume, otherwise known as *incompressibility*.

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Fundamental principles of fluid flow: conservation of mass, fixed finite control volume

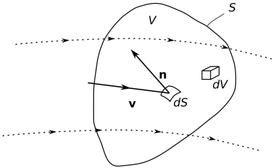

Figure 2: Fixed finite control volume.

- We now derive the continuity equation for an the arbitrary fixed finite control volume in figure 2 having volume V enclosed by a control surface S. Consider a point on the control surface with flow velocity $\mathbf{v}$ and elemental surface area $\mathbf{dS}$. Let dV be an elemental volume inside the finite control volume.

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Fundamental principles of fluid flow: conservation of mass, fixed finite control volume

- Conservation of mass on the finite control volume states that the net mass flow through control surface S should be balanced by the time rate of change of mass inside the control volume.
- The net mass flow through S is,

$$\oiint_S \rho \mathbf{v} \cdot \mathbf{dS}. \tag{8}$$

- The time rate of change of mass inside V is,

$$\frac{\partial}{\partial t} \oiiint_V \rho dV. \tag{9}$$

- Notice the use of the partial local time derivative since we are dealing with a fixed control volume and reference frame (Eulerian).

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Fundamental principles of fluid flow: conservation of mass, fixed finite control volume

- Adding them up leads to the integral form of the continuity equation in conservation form,

$$\frac{\partial}{\partial t}\oiiint_V \rho dV + \oiint_S \rho \mathbf{v} \cdot \mathbf{dS} = 0. \tag{10}$$

- Using the divergence theorem on the surface term and moving the time derivative inside the integral since it does not depend on volume,

$$\oiiint_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v})\right] dV = 0. \tag{11}$$

- Since the control volume is arbitrary the equation above means that the integrand must be zero which leads to the differential continuity equation in conservation form,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{12}$$

Conservation of momentum

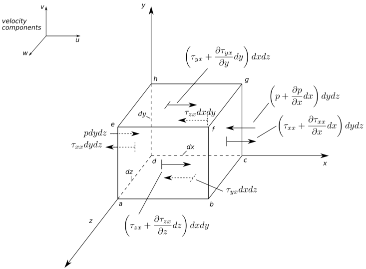

Figure 3: A moving infinitesimal fluid element with forces in the x-direction.

- We will now derive the momentum equation for a moving infinitesimal fluid element based on Newton's second law, $\mathbf{F} = m\mathbf{a}$, where $\mathbf{F}$ is the force on the element, m its mass and $\mathbf{a}$ its acceleration.

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Fundamental principles of fluid flow: conservation of momentum

- Newton's second law is a vector relation with x, y and z scalar relations. We will derive the x-component which corresponds to,

$$F_x = ma_x. \tag{13}$$

- We start with the forces on the LHS of equation 13, which fall in two categories, body forces and surface forces.
  - *Body forces*, like the gravitational, electric and magnetic forces. They act at a distance and directly on the volumetric mass of the element. They usually are potential forces.
  - *Surface forces*, which are due only to pressure on one hand and shear and normal stresses on the other. They act on the surface of the fluid element.
- Typically the body force will be gravitational denoted by $\mathbf{F_b} = m\mathbf{g}$ where $\mathbf{g}$ is the gravitational acceleration. Since the volume of the fluid element is $dV = dxdydz$ the body force would become,

$$\mathbf{F_b} = \rho \mathbf{g} dxdydz. \tag{14}$$

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Fundamental principles of fluid flow: conservation of momentum

- Here we will stick with a general formulation of the body forces,

$$\mathbf{F_b} = m\mathbf{f_b}, \tag{15}$$

  where $\mathbf{f_b}$ is the body force per unit mass.

- As for the stress forces we restrict ourselves to a Newtonian fluid with stresses proportional to the strain rate. Newtonian fluids are a great approximation for most aerospace applications.
- The stress tensor is denoted here by $\boldsymbol{\tau}$, and its individual components by τ_{ij}. This is a stress acting in the j-direction on a face normal to the i-axis (e.g. τ_{zx} in figure 3 is acting in the x-direction on a face normal to the z-axis, or face abfe).
- We introduce an additional sign convention, that positive increases in velocity components occur in the positive directions of the axes.

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Fundamental principles of fluid flow: conservation of momentum

- Take for example face abcd in the xz plane. According to our sign convention the u velocity component just below the face is smaller than on it since we are moving in the negative y-direction. Therefore the derivative $\partial u/\partial y < 0$.
- On the other hand, the velocity component v is constant on face abcd, which is parallel to the xz plane. So $\partial v/\partial x = 0$ on abcd and the shear stress component τ_{yx} is only proportional to $\partial u/\partial y < 0$. Therefore the force vector points in the negative x-direction.
- Similarly on face efgh, the u velocity component above the face is larger than on it since we are moving in the positive y-direction. Therefore the normal derivative of velocity is positive and the corresponding shear stress component is positive, and the force vector points in the positive x-direction.

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Fundamental principles of fluid flow: conservation of momentum

- Note that the shear stress component on face efgh is defined relative to τ_{yx} the shear stress component on face abcd as,

$$\tau_{yx,efgh} = \tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy.$$

- This approach is used for all shear and normal stress components.
- The hydrostatic pressure p is defined the same way on parallel faces and the sign convention used is that pressure forces are always compressive acting towards the face from outside the element.
- Recovering forces from stress and pressure components is a simple matter of multiplying by the face area (e.g. $dxdz$ for faces abcd and efgh).

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Fundamental principles of fluid flow: conservation of momentum

- Taking all the above into consideration we can write,

$$
\begin{aligned}
F_x &= \left[p - \left(p + \frac{\partial p}{\partial x}dx\right)\right]dydz \\
&+ \left[\left(\tau_{xx} + \frac{\partial \tau_{xx}}{\partial x}dx\right) - \tau_{xx}\right]dydz \\
&+ \left[\left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}dy\right) - \tau_{yx}\right]dxdz \\
&+ \left[\left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}dz\right) - \tau_{zx}\right]dxdy \\
&= \left(-\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)dxdydz \qquad (16)
\end{aligned}
$$

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Fundamental principles of fluid flow: conservation of momentum

- Now we move to the acceleration term on the RHS of equation 13.
- The mass of the fluid element is fixed and equal to,

$$m = \rho dxdydz \tag{17}$$

- As for the acceleration since we are following a moving fluid element it is determined using the substantial derivative,

$$a_x = \frac{Du}{Dt}. \tag{18}$$

- Combining equations 15, 16, 18 and 17 and eliminating *dxdydz* on both sides we get the x-component of the momentum equation,

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_{b,x}. \tag{19}$$

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Fundamental principles of fluid flow: conservation of momentum

- The y and z-components of the momentum equation are derived the same way,

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho f_{b,y}. \tag{20}$$

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial z} + \frac{\partial \tau_{yz}}{\partial z} + \frac{\partial \tau_{zz}}{\partial z} + \rho f_{b,z}. \tag{21}$$

- You will derive them as part of your first assignment following the same steps shown here.

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Fundamental principles of fluid flow: conservation of momentum

- Note that equations 19, 20 and 21 are partial differential equations obtained directly by applying Newton's second law on an infinitesimal fluid element.
- Since the fluid element is moving the equations are also in non-conservation form.
- Obtaining the conservation form is straightforward and done by expanding the substantial derivative, using the chain rule and the continuity equation in conservation form.
- This will also be on the first assignment.

Conservation of energy

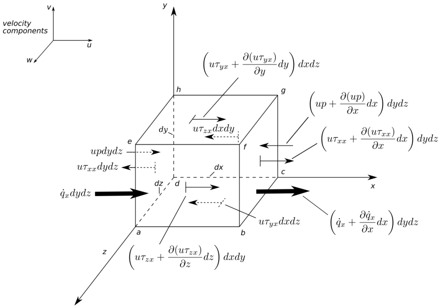

Figure 4: A moving infinitesimal fluid element with energy fluxes in the x-direction.

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Fundamental principles of fluid flow: conservation of energy

- Figure 4 depicts a moving infinitesimal fluid element with energy fluxes in the x-direction, and is the energy equivalent of the forces fluid element of Figure 3 that we used to derive the momentum equation.
- Figure 4 uses the same conventions as Figure 3.
- We will use figure 4 to derive the energy equation, a statement of *the first law of thermodynamics, which states that the rate of change of energy in a fluid element equals the net flux of heat into the element plus the rate of work done by body and surface forces.*
- We designate the three components above as,

A = the rate of change of energy in a fluid element
B = the net flux of heat into the element
C = the rate of work done by body and surface forces

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Fundamental principles of fluid flow: conservation of energy

- Let's start by evaluating C. It can be shown that the rate of work done by a force $\mathbf{f}$ acting on a moving body is equal to the dot product of the force and the velocity vector $\mathbf{v}$ of the moving body,

$$\rho \mathbf{f} \cdot \mathbf{v} dxdydz. \tag{22}$$

- For the pressure and shear forces in the x-direction shown the rate of work would be the force multiplied by the x-component of velocity, as shown explicitly in figure 4 as energy fluxes.

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Fundamental principles of fluid flow: conservation of energy

- The net rate of work done by pressure forces in the x-direction on faces adhe and bcgf is,

$$\left[up - \left(up + \frac{\partial(up)}{\partial x}dx\right)dydz\right] = -\frac{\partial(up)}{\partial x}dxdydz. \tag{23}$$

- The net rate of work done by forces in the x-direction on faces abcd and efgh is,

$$\left[\left(u\tau_{yx} + \frac{\partial(u\tau_{yx})}{\partial y}dy\right)dxdz - u\tau_{yx}\right] = \frac{\partial(u\tau_{yx})}{\partial y}dxdydz. \tag{24}$$

- Similar expressions can be obtained for the remaining surface forces in the x-direction and they all add up to C_x the x-component of C,

$$C_x = \left[-\frac{\partial(up)}{\partial x} + \frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z}\right]dxdydz. \tag{25}$$

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Fundamental principles of fluid flow: conservation of energy

- A similar analysis gives us the y and z-components of C and it all adds up to the following,

$$
\begin{aligned}
C = \Bigg[& -\left(\frac{\partial(up)}{\partial x} + \frac{\partial(vp)}{\partial y} + \frac{\partial(wp)}{\partial z}\right) \\
& + \frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z} \\
& + \frac{\partial(v\tau_{xy})}{\partial x} + \frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z} \\
& + \frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z} \\
& + \rho\mathbf{f}\cdot\mathbf{v}\Bigg] dxdydz.
\end{aligned} \tag{26}
$$

- Note that the pressure terms on the first line are simply $\nabla \cdot (p\mathbf{v})$.

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Fundamental principles of fluid flow: conservation of energy

- Now we evaluate B the net flux of heat into the fluid element.
- The heat flux is due to,
  1. Volumetric heating such as thermal radiation.
  2. Surface heat transfer due to temperature gradients like thermal conduction.
- Defining $\dot{q}$ as the rate of volumetric heating per unit mass we can define the volumetric heating as,

$$\rho \dot{q} dxdydz. \tag{27}$$

- Defining $\dot{q}_x$ as the rate of heat transfer by thermal conduction in the x-direction and based on figure 4 we can determine the net heat transferred by thermal conduction in the x-direction across faces adhe and bcgf (thick black arrows) as,

$$\left[\dot{q}_x - \left(\dot{q}_x + \frac{\partial \dot{q}_x}{\partial x} dx \right) \right] dydz = -\frac{\partial \dot{q}_x}{\partial x} dxdydz. \tag{28}$$

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Fundamental principles of fluid flow: conservation of energy

- Similar expressions to equation 28 for the y and z-directions across the other faces add up to the total heat transfer by conduction,

$$-\left(\frac{\partial \dot{q}_x}{\partial x} + \frac{\partial \dot{q}_y}{\partial y} + \frac{\partial \dot{q}_z}{\partial z}\right) dxdydz. \tag{29}$$

- Summing up equations 27 and 29 leads to an expression for B,

$$B = \left[\rho \dot{q} - \left(\frac{\partial \dot{q}_x}{\partial x} + \frac{\partial \dot{q}_y}{\partial y} + \frac{\partial \dot{q}_z}{\partial z}\right)\right] dxdydz. \tag{30}$$

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Fundamental principles of fluid flow: conservation of energy

- Using $\dot{q}_x = -k\partial T/\partial x$, $\dot{q}_y = -k\partial T/\partial y$ and $\dot{q}_z = -k\partial T/\partial z$ where k is the thermal conduction coefficient and T temperature we get,

$$B = \left[\rho\dot{q} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\right] dxdydz. \qquad (31)$$

- Finally term A denotes the time rate of change of energy of the fluid element, which is the sum of the element's internal energy and kinetic energy per unit mass,

$$A = \rho\frac{D}{Dt}\left(e + \frac{|\mathbf{v}|^2}{2}\right) dxdydz. \qquad (32)$$

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Fundamental principles of fluid flow: conservation of energy

- Substituting equations 32, 31 and 26 into the first principle of thermodynamics leads to the final form of the total energy equation in non-conservation form as expected for a moving fluid element,

$$
\begin{aligned}
\rho \frac{D}{Dt}\left(e + \frac{|\mathbf{v}|^2}{2}\right) = \rho \dot{q} &+ \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) \\
&- \frac{\partial(up)}{\partial x} - \frac{\partial(vp)}{\partial y} - \frac{\partial(wp)}{\partial z} \\
&+ \frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z} \\
&+ \frac{\partial(v\tau_{xy})}{\partial x} + \frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z} \\
&+ \frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z} \\
&+ \rho \mathbf{f} \cdot \mathbf{v}. \qquad (33)
\end{aligned}
$$

Fluids Dynamics Review

Fundamental principles of fluid flow: conservation of energy

- One can also derive a mechanical energy equation and an internal energy equation.
- Explanations for how to go about it will be provided in your first assignment where you will be asked to derive the mechanical and internal energy equations.
- The momentum and total energy equations derived here are for viscous flow and include several viscous terms. Inviscid flow on the other hand does include viscous effects and is described by the Euler equations. These can be derived from the Navier-Stokes equations by simply neglecting viscous terms.